Online Appendix for Dynamic Scheduling in a Many-Server Multi-Class System: Role of Customer Impatience in Large Systems

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This online appendix consists of three sections: Section A discusses how our approach can be extended to the case of class-dependent service rates, Section B proves the main results in the paper (Theorem 1 and Propositions 1-5) and Section C proves lemmas used in Section B.

A Extension to Class-Dependent Service Rates

When service rates are class-dependent, we can no longer describe the system dynamics with a single-dimensional state descriptor approximated by a single-dimensional diffusion process (see Gurvich and Whitt, 2009). Instead, we need to keep track of the queue-length process for each class to describe the system dynamics, and hence we require a $K$-dimensional diffusion process to approximate these dynamics. This results in the DCP formulation yielding a partial differential equation (PDE) as the HJB equation. Our objective in this subsection is to derive this HJB equation PDE and describe the solution to the DCP. The formulation in this section generalizes that in Harrison and Zeevi (2004) from exponential to general abandonment, and also changes their objective from infinite-horizon discounted-cost to long-run-average cost.

Let $\mu_k$ be the class $k$ service rate and define $\nu := \sum_{k \in K} \frac{a_k}{\mu_k}$. Note that $\nu$ is $\frac{1}{\mu}$ if $\mu_k = \mu$ for all $k \in K$ as in our previous sections and it is necessary to have at least $\nu \lambda$ servers to guarantee no abandonment in a system without randomness. Similar to the equation (3), we assume

$$N = \nu \lambda + \beta \sqrt{\lambda}$$

for some $\beta \in \mathbb{R}$. To proceed, define $z_k := \frac{a_k}{\nu \mu_k}$ for $k \in K$. Then, $z_k$ represents the contribution of class $k$ to the entire workload in the system and we can interpret $z_k N$ as the number of servers

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dedicated to class $k$ in a system with no stochastic fluctuation. Similar to (8), let

$$X_k(t) := Q_k(0) + Z_k(0) - z_k N + A_k(t) - S_k(t) - R_k(t)$$

be the centered number of class $k$ customers in the system. Observe that $Q_k(t) = X_k(t) - (Z_k(s) - z_k N)$, so that

$$X_k(t) \geq Z_k(s) - z_k N. \quad (1)$$

Using the same strong approximation argument and the hazard rate approximation of $R_k$ (both from Section 4.1), we have

$$X_k(t) = X_k(0) - a_k \beta \sqrt{\lambda} t - \int_0^t \left[ \mu_k (Z_k(s) - z_k N) + \lambda m_k \left( \frac{X_k(s) - (Z_k(s) - z_k N)}{\lambda} \right) \right] ds + \sqrt{2a_k \lambda B_k(t)} + \epsilon_\lambda(t), \quad (2)$$

where $\epsilon_\lambda$ is the error term from the aforementioned approximations. In the above display, $Z_k(s) - z_k N$ is our decision that controls how many more (or less) servers we allocate to class $k$ relative to the ideal amount ($z_k N$) in order to accommodate the stochastic fluctuations in the system. Furthermore, note that under any non-idling policy, we have

$$\sum_{k \in \mathcal{K}} (Z_k(t) - z_k N) = \left( \sum_{k \in \mathcal{K}} X_k(t) \right) \wedge 0. \quad (3)$$

We approximate $X_k$ by $\hat{X}_k$ which is the (weak) solution to the following SDE:

$$\hat{X}_k(t) = \hat{X}_k(0) - a_k \beta \sqrt{\lambda} t - \int_0^t \left[ \mu_k \hat{X}_k(s) + \lambda m_k \left( \frac{\hat{X}_k(s) - u_k(\hat{X}_k(s))}{\lambda} \right) \right] ds + \sqrt{2a_k \lambda W_k(t)}, \quad (4)$$

where $u := (u_1, \ldots, u_K)$ is our decision in the DCP formulation. As in the earlier sections, we focus on stationary controls so that $u$ is only a function of $\hat{X} := (\hat{X}_1, \ldots, \hat{X}_K)$, and cannot vary with time. Motivated by (1) and (2), the set of controls that we consider is

$$\mathcal{A}(x) := \left\{ u \in \mathbb{R}^K : u_k \leq x_k, \sum_{k \in \mathcal{K}} u_k = \left( \sum_{k \in \mathcal{K}} x_k \right) \wedge 0 \right\}. \quad (5)$$

Let $\mathcal{A}(\infty)$ be the collection of $K$-dimensional functions $u$ such that $u(x) \in \mathcal{A}(x)$ for $x \in \mathbb{R}^K$. As
in Section 4.1, we approximate the original objective (1) by

\[ V(u) := \lim_{t \to \infty} \mathbb{E} \left[ \sum_{k \in K} r_k \frac{\lambda}{t} \int_0^t m_k \left( \frac{\hat{X}_k(s) - u_k(\hat{X}(s))}{\lambda} \right) ds \right] \]

for \( u \in \mathcal{A}(\infty) \). Our objective in the DCP formulation is to find \( u^* \in \mathcal{A}(\infty) \) such that

\[ V(u^*) = \lim_{t \to \infty} \mathbb{E} \left[ \sum_{k \in K} r_k \frac{\lambda}{t} \int_0^t m_k \left( \frac{\hat{X}_k^*(s) - u_k^*(\hat{X}^*(s))}{\lambda} \right) ds \right] = \inf_{u \in \mathcal{A}(\infty)} V(u), \]

where \( \hat{X}_k^* \) weakly solves (3) under the control \( u = u^* \). The following result characterizes the HJB equation (which is a PDE), and the solution to the DCP.

**Proposition A.1.** Suppose there exists a twice continuously differentiable function \( w^*: \mathbb{R}^K \to \mathbb{R} \) having bounded partial derivatives and constant \( \kappa^* < \infty \) that jointly solve

\[
\sum_{k \in K} a_k \lambda \frac{\partial^2 w^*(x)}{\partial x_k^2} + \inf_{u \in \mathcal{A}(x)} \sum_{k \in K} \left\{ \lambda r_k m_k \left( \frac{x_k - u_k}{\lambda} \right) - \frac{\partial w^*(x)}{\partial x_k} \left( \lambda m_k \left( \frac{x_k - u_k}{\lambda} \right) + \mu_k u_k + \frac{a_k \beta \sqrt{\lambda}}{\nu} \right) \right\} = \kappa^*.
\]

(5)

Assume that for \( \hat{X} \) that weakly solves (3) under any control \( u \in \mathcal{A}(\infty) \),

\[ \lim_{t \to \infty} \frac{\mathbb{E}[w^*(\hat{X}(t))]}{t} = 0. \]

Then, \( \kappa^* = \inf_{u \in \mathcal{A}(\infty)} V(u) \) and, assuming the infimum is achieved in (5), the optimal control, \( u^* \), is given by

\[ u^*(x) = \arg \min_{u \in \mathcal{A}(x)} \sum_{k \in K} \left\{ \lambda r_k m_k \left( \frac{x_k - u_k}{\lambda} \right) - \frac{\partial w^*(x)}{\partial x_k} \left( \lambda m_k \left( \frac{x_k - u_k}{\lambda} \right) + \mu_k u_k + \frac{a_k \beta \sqrt{\lambda}}{\nu} \right) \right\} \]

for \( x \in \mathbb{R}^K \).

We prove this result at the end of Section B.

Note that the PDE in (5) is not amenable to solving directly. So, we propose a simplification. In particular, we propose to approximate the system by setting the service rate of each class at the average weighted service across all classes, i.e.,

\[ \frac{1}{\mu} = \sum_{k \in K} \frac{a_k}{\mu_k}. \]
Then, we can solve the DCP for the class-independent service rate case as in the earlier sections of this paper. We performed a numerical study comparing the performance of this policy with that of the best threshold control in Figure 4, and found that the same insights carry through: the threshold policy is sub-optimal and further that our proposed policy dominates it.

B Proofs of Main Results

Proof of Theorem 1:

Proof of part (i). In the case when $\beta \geq 0$ and the hazard rate function of the patience time distribution of each class is increasing, the existence part of this theorem is equivalent to Theorem 4.2 of Kim and Ward (2013). A careful examination of the proofs in Kim and Ward (2013), which considers a single server setting and hence the domain of the HJB equation is $\mathbb{R}^+$ instead of $(-\infty, d)$, indicates that that proof does not rely on the structure of the hazard rate function. So, the proof of part (i) for $\beta \geq 0$ can be completed by following Kim and Ward (2013) and we focus on the case when $\beta < 0$ for proving the existence of the HJB solution.

To prove the existence, consider a family of ODEs indexed by $\kappa \geq 0$:

$$
\lambda v'(x) + \left(\sqrt{\mu} [x^+] - \beta \sqrt{\lambda} \right) \sqrt{\mu} v(x) + \lambda \min_{q \in A} \phi(x, v(x), q) = \kappa \text{ such that } \sup_{x \leq 0} |v(x)| < \infty. \quad (6)
$$

We obtain (6) by modifying the boundary condition $\sup_{x < d} |v(x)| < \infty$ in (5). The following lemma proves the existence of a unique $v : (-\infty, d) \rightarrow \mathbb{R}$ that solves (6), which is denoted by $v_\kappa$, and provides useful properties of $v_\kappa$.

Lemma B.1. For each $\kappa \geq 0$, the ODE (6) has a unique solution $v_\kappa$. $v_\kappa(x)$ is jointly continuous in $(\kappa, x) \in \mathbb{R}^+ \times (-\infty, d)$, and is non-negative and strictly increasing in $x \leq 0$. Further, $v_{\kappa_1}(x) > v_{\kappa_2}(x)$ for all $x \in (-\infty, d)$ and $\kappa_1 > \kappa_2$ and $v_\kappa'(x)$ is continuous in $x \in (-\infty, d)$.

To proceed, let $r := \min\{r_1, \ldots, r_K\}$ and

$$
\mathcal{D} := \left\{ \kappa \geq 0 : \sup_{x \in [0, d]} v_\kappa(x) < r \right\}
$$

and $\mathcal{U} := \mathcal{D}^c \cap \mathbb{R}^+$. Given the properties of $\{v_\kappa\}_{\kappa \geq 0}$ in Lemma B.1, we prove the theorem by showing that there exists a unique $\kappa \in (0, \infty)$ such that $\sup_{x \in (-\infty, d)} |v_\kappa(x)| \leq r$ and $v_\kappa(x)$ is increasing in $x \in (-\infty, d)$. To do so, we need the following lemmas.

Lemma B.2. Neither $\mathcal{U}$ nor $\mathcal{D}$ is empty and $\sup \mathcal{D} = \inf \mathcal{U} \in (0, \infty)$.

Lemma B.3. Suppose there exists $x_1 \geq 0$ such that $v_\kappa'(x_1) < 0$. Then, $v_\kappa(x)$ is strictly decreasing in $x > x_1$.

Lemma B.4. $\lim_{x \rightarrow d} v_\kappa(x)$ is either $-\infty$, $r$, or $\infty$.  

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Lemma B.5. If $\kappa \in D$, then $\lim_{x \to d} v_{\kappa}(x) = -\infty$.

Lemma B.6. If $\kappa \in U$, then $\lim_{x \to d} v_{\kappa}(x) \geq r$.

Lemma B.7. $\min_{q \in A} \phi(x, w, q)$ is jointly continuous in $(x, w) \in (0, d) \times \mathbb{R}$.

By Lemma B.2, $\kappa^* := \inf U \in (0, \infty)$. Then, the continuity of $v_{\kappa}$ in $\kappa$ derived in Lemma B.1 and Lemmas B.5-B.6 imply that $\kappa^* \in U$. Also, $\sup_{x \in [0, d]} v_{\kappa^*}(x) = r$ follows by the continuity of $v_{\kappa}$ in $\kappa$. It only remains to show that $v'_{\kappa^*}(x) \geq 0$ for all $x \in [0, d)$. Suppose there exists $\bar{x} \in [0, d)$ such that $v'_{\kappa^*}(\bar{x}) < 0$. Because $\sup_{x \geq 0} v_{\kappa^*}(x) = r$, we know that $v_{\kappa^*}(\bar{x}) \leq r$. This implies that $\lim_{x \to \bar{x}} v_{\kappa^*}(x) < r$ by Lemmas B.3 and B.4. However, this contradicts to Lemma B.6 because $\kappa^* \in U$. Hence $v'_{\kappa^*}(x) \geq 0$ for all $x \in [0, d)$. Finally, we use Lemma B.4 to prove the uniqueness of the HJB solution by showing that for any $\kappa > \kappa^*$, $\lim_{x \to d} v_{\kappa}(x) > r$. Suppose on the contrary that there exists $\kappa > \kappa^*$ such that $\lim_{x \to d} v_{\kappa}(x) = r$. Then, for this $\kappa$, by Lemma B.7 and by noting $\lim_{x \to d} v'_{\kappa}(x) = 0$, we establish
$$\beta \sqrt{\lambda \mu r} = \kappa$$
by taking $\lim_{x \to d}$ on the both hand sides of (6). But, this is a contradiction because by repeating the limiting argument we reach $\beta \sqrt{\lambda \mu r} = \kappa^*$ but $\kappa^* < \kappa$.

Proof of part (ii). The following lemma is useful in this proof.

Lemma B.8. For the diffusion process, $Z$, defined in (14) under any $q \in A(\infty)$, we have
$$\lim_{t \to \infty} \frac{E[|Z(t)|]}{t} = 0.$$

To show the optimality of $q^*$, we show
$$\kappa^* = \lim_{T \to \infty} \frac{E}{T} \left[ \sum_{k \in K} r_k \frac{\lambda T}{\int_0^T m_k \left( [Z^*(t)]^+ q_k^* ([Z^*(t)]^+) \right) dt} \right], \quad (8)$$
$$\kappa^* \leq \limsup_{T \to \infty} \frac{E}{T} \left[ \sum_{k \in K} r_k \frac{\lambda T}{\int_0^T m_k \left( [Z(t)]^+ q_k ([Z(t)]^+) \right) dt} \right], \quad (9)$$
for any $q \in A(\infty)$ and $Z$ defined in (14). To proceed, let $u(x) := \int_{-\infty}^x v_{\kappa^*}(y) dy$. By using Ito’s lemma on $u(Z^*(t))$ and $u(Z(t))$, we achieve
$$\kappa^* = \frac{E[u(Z^*(T))]}{T} + \frac{E}{T} \left[ \sum_{k \in K} r_k \frac{\lambda}{\int_0^T m_k \left( [Z^*(t)]^+ q_k^* ([Z^*(t)]^+) \right) dt} \right], \quad (10)$$
$$\kappa^* \leq \frac{E[u(Z(T))]}{T} + \frac{E}{T} \left[ \sum_{k \in K} r_k \frac{\lambda}{\int_0^T m_k \left( [Z(t)]^+ q_k ([Z(t)]^+) \right) dt} \right]. \quad (11)$$
Then, by using the same argument given early in this paragraph, we conclude that $q$ most linearly in $x$ as shown in part (i) that $\sup_{x \in (-\infty, d)} v_{k^*}(x) \leq r$. 

**Proof of Proposition 1:**

Recall the definitions of $J_1$, $J_2$, $J_3$, and $C$ given in the algorithm presented in Section 5. Observe that when $m_k(x) = \gamma_k x$ for some $\gamma_k > 0$, the optimization $\min_{q \in A} \phi(x, v^*(x), q)$ for $x > 0$ reduces to

$$\min_{q \in A} x \sum_{k \in K} (r_k - v^*(x)) \gamma_k q_k.$$ 

Hence,

$$q_k^*(x) = 1 \text{ if and only if } (r_k - v^*(x)) \gamma_k < (r_j - v^*(x)) \gamma_j \text{ for all } j \neq k, \quad (12)$$

which is equivalent to $(r_k \gamma_k - r_j \gamma_j) < (\gamma_k - \gamma_j) v^*(x)$ for all $j \neq k$. This implies that when $J_1 = \{ K \}$, $q_k^*(x) = 0$ for any $k \leq K - 1$ and $x > 0$ (recall we have $r_1 \gamma_1 \geq r_2 \gamma_2 \geq \cdots \geq r_K \gamma_K$), so that $q^*$ is a static priority with $q_k^*(x) = 1$ for $x > 0$.

Assume $J_1$. Relabel the class indices so that $J_1 = \{1, 2, \ldots, J_1\}$ with $\gamma_1 r_1 \geq \gamma_2 r_2 \geq \cdots \geq \gamma_{J_1} r_{J_1}$ and $\gamma_1 > \gamma_2 > \cdots > \gamma_{J_1}$. We now show that $q_k^*(x) = 0$ for all $x > 0$ and $k \in J_1 \setminus J_2$ (that is $k \notin C$). Suppose on the contrary that $q_k^*(x) = 1$ for some $x \in \mathbb{R}$ and $k \in J_1 \setminus J_2$. Since $\gamma_k - \gamma_j > 0$ for $j > k$ and $\gamma_k - \gamma_j < 0$ for $j < k$, from (12),

$$\frac{r_k \gamma_k - r_j \gamma_j}{\gamma_k - \gamma_j} < v^*(x) \quad \text{for } j \in \{k + 1, \ldots, J_1\}$$

and

$$\frac{r_k \gamma_k - r_j \gamma_j}{\gamma_k - \gamma_j} > v^*(x) \quad \text{for } j \in \{1, \ldots, k - 1\}.$$ 

This implies

$$\frac{r_k \gamma_k - r_{J_1} \gamma_{J_1}}{\gamma_k - \gamma_{J_1}} < \frac{r_k \gamma_k - r_1 \gamma_1}{\gamma_k - \gamma_1}.$$

However, this is a contradiction because $(\gamma_k, r_k \gamma_k) \notin C$ implies that

$$\frac{r_{J_1} \gamma_{J_1} - r_k \gamma_k}{\gamma_1 - \gamma_k} < \frac{r_{J_1} \gamma_{J_1} - r_1 \gamma_1}{\gamma_1 - \gamma_1} < \frac{r_k \gamma_k - r_{J_1} \gamma_{J_1}}{\gamma_k - \gamma_{J_1}}.$$ 

Next, we show that for any $j \in J_2 \setminus J_3$, $q_j^*(x) = 0$ for all $x > 0$. Let $V$ be the set of vertices of $\text{conv}(C)$. For such $j$, it is straightforward to check that there must exist two points in $V$, namely $(r_i, r_i \gamma_i)$ and $(r_n, r_n \gamma_n)$, such that $i < n$

$$\frac{r_i \gamma_i - r_j \gamma_j}{\gamma_i - \gamma_j} < \frac{r_i \gamma_i - r_n \gamma_n}{\gamma_i - \gamma_n} < \frac{r_j \gamma_j - r_n \gamma_n}{\gamma_j - \gamma_n}.$$ 

Then, by using the same argument given early in this paragraph, we conclude that $q_j^*(x) = 0$ for
To complete the proof, relabel the class indices so that \( J_3 = \{1, 2, \ldots, J_3\} \) with \( \gamma_1 r_1 \geq \gamma_2 r_2 \geq \cdots \geq \gamma_{J_3} r_{J_3} \) and \( \gamma_1 > \gamma_2 > \cdots > \gamma_{J_3} \). Recall from Step 4 of the algorithm in Section 5 that

\[
0 = T_{J_3+1} < T_{J_3} < \cdots < T_2 < T_1 = \infty.
\]

So, if \( v^* (x) \in (T_{j+1}, T_j) \) for some \( j \in \{1, \ldots, J_3\} \), then for such \( x > 0 \), we have

\[
v^* (x) < \frac{r_k \gamma_k - r_j \gamma_j}{\gamma_k - \gamma_j} \quad \text{for all } k \in \{1, \ldots, j-1\},
v^* (x) > \frac{r_j \gamma_j - r_k \gamma_k}{\gamma_j - \gamma_k} \quad \text{for all } k \in \{j+1, \ldots, J_3\}.
\]

From the above display, we can deduce \( q^*_j (x) = 1 \) by (12). Finally, such \( q^* \) is a threshold control or a static priority because \( v^* (x) \) is increasing in \( x \) by Theorem 1. □

Proof of Proposition 2:

Before proving the proposition, it should be noted that for each \( q \in \mathcal{A}(\infty) \), there exists \( d(q) \) such that the diffusion process under \( q \) will never reach states larger than or equal to \( d(q) \). Also, we have \( d(q) \leq d := \lambda \sum_{k \in \mathcal{K}} a_k d_k \) (recall the interior of the domain of \( G_k \) is \((0, d_k)\)) for any \( q \in \mathcal{A}(\infty) \). This is because \( \sum_{k \in \mathcal{A}} m_k (\frac{x+q_k}{\lambda}) = \infty \) for any \( x \geq d \) and any \( q \in \mathcal{A} \), implying there is drift of \(-\infty\) when the state reaches \( d \).

We first show that under the condition stated in the proposition, \( q^* \) is neither a static priority nor a threshold control. Then, we complete the proof by establishing that when \( q^* \) is neither a static priority nor a threshold control, the cost achieved under any static priority or threshold control is strictly greater than the optimal cost \( \kappa^* \).

For the first part, we draw a contradiction by assuming that \( q^* \) is either a threshold control or a static priority. Recall that \( q^* \) solves

\[
q^* (x) = \arg \min_{q \in \mathcal{A}} \left\{ \sum_{k \in \mathcal{K}_n} (r_k - v^* (x)) m_k \left( \frac{x+q_k}{\lambda} \right) + \sum_{k \in \mathcal{K} \setminus \mathcal{K}_n} (r_k - v^* (x)) m_k \left( \frac{x+q_k}{\lambda} \right) \right\}
\]

for \( x \in \mathbb{R} \). From (13) and by the assumption on \( \mathcal{K}_n \) in the statement of the proposition, if \( q^* \) is either a threshold or a static priority, then there must exist \( a > 0 \), such that:

\[
\text{for any } x \in (0, a), \quad \sum_{k \in \mathcal{K}_n} q^*_k (x) = 1, \quad \text{while} \quad \sum_{k \in \mathcal{K} \setminus \mathcal{K}_n} q^*_k (x) = 0.
\]

This implies that for any \( x \in (0, a) \) there exists \( k \in \mathcal{K}_n \) such that \( q^*_k (x) = 1 \). However, this is a
contradiction because for such \( x \) and \( k \), 1 is never the optimizer of \( \inf_{q \in [0,1]} z(q) \) where

\[
z(q) := (r_k - v^*(x)) m_k \left( \frac{x(q)}{\lambda} \right) + (r_j - v^*(x)) m_j \left( \frac{x(1-q)}{\lambda} \right),
\]

for any \( j \in \mathcal{K}_n \backslash \{k\} \) as \( z'(0) < 0 \) and \( z'(1) > 0 \).

For the second part, let \( \kappa_q \) be the objective function value for (15) and \( Z_q \) be the diffusion process in (14) under \( \bar{q} \) which is either a threshold control or static priority. We want to prove that \( \kappa_{\bar{q}} > \kappa^* \). Suppose on the contrary that \( \kappa_{\bar{q}} = \kappa^* \). Recall \( u(x) = \int_{-\infty}^{x} v^*(y) \, dy \). By applying Ito’s lemma on \( u(Z_q(t)) \), we have

\[
\frac{\mathbb{E}[u(Z_q(t)) - u(Z_q(0))]}{t} + \mathbb{E} \left[ \sum_{k \in \mathcal{K}} r_k \frac{\lambda}{t} \int_0^t m_k \left( \frac{[Z_q(s)]^+ \bar{q}_k ([Z_q(s)]^+)}{\lambda} \right) \, ds \right] = \mathbb{E} \left[ \frac{1}{t} \int_0^t H(Z_q(s), \bar{q}) \, ds \right],
\]

where for \( z \in (-\infty, d) \) and \( q \in \mathcal{A} \)

\[
H(z, q) := \lambda v^{*d}(z) + v^*(z) \left( -\beta \sqrt{\mu \lambda + \mu |z|} \right) + \lambda \sum_{k \in \mathcal{K}} (r_k - v^*(z)) m_k \left( \frac{[z]^+ \bar{q}_k ([z]^+)}{\lambda} \right).
\]

By Lemma B.8, the \( \limsup_{t \to \infty} \) of the first term in the left hand side of (15) is 0. Also note that the \( \limsup_{t \to \infty} \) is \( \kappa^* \) (this follows from Theorem 1 and the supposition that \( \kappa_{\bar{q}} = \kappa^* \)), so that the \( \limsup_{t \to \infty} \) of the second term in the left hand side of (15) is \( \kappa^* \). Hence, we have

\[
\kappa^* = \limsup_{t \to \infty} \mathbb{E} \left[ \frac{1}{t} \int_0^t H(Z_q(s), \bar{q}) \, ds \right] \\
\geq \liminf_{t \to \infty} \mathbb{E} \left[ \frac{1}{t} \int_0^t H(Z_q(s), \bar{q}) \, ds \right] \\
\overset{(a)}{=} \mathbb{E} \left[ \liminf_{t \to \infty} \frac{1}{t} \int_0^t H(Z_q(s), \bar{q}) \, ds \right],
\]

where (a) follows by Fatou’s lemma. Here, we can use Fatou’s lemma because

\[
H(Z_q(s), \bar{q}) \overset{(a)}{=} H(Z_q(s), q^*) \overset{(b)}{=} \kappa^* > 0.
\]

In (17), (a) follows because \( q^*(x) \) is an optimizer of \( \arg\min_{q \in \mathcal{A}} (r_k - v^*(x)) m_k (\frac{z(q)}{\lambda}) \) for \( x > 0 \), (b) follows because \( (v^*, \kappa^*) \) solves (5), and (c) follows because of Lemma B.2.

To proceed, note that in the proof of Lemma B.8 ((35) and (36)), we establish that \( \mathbb{E}[T_0] < \infty \) and \( \mathbb{E}[\tau] < \infty \), where \( T_0 := \inf \{ t \geq 0 : Z(t) = 0 \} \) and \( \tau \) is the length of i.i.d regenerative cycles in which the diffusion process departs from 0, reaches some constant greater than 0 but less than \( d(q) \), and returns back to 0. Also, note that by the first equality in (16), we know that for any
\( \epsilon > 0 \) there exists \( T < \infty \) such that
\[
\mathbb{E} \left[ \frac{1}{t} \int_0^t H(Z_q(s), \bar{q}) \, ds \right] < \kappa^* + \epsilon. \tag{18}
\]

Jointly, \( \mathbb{E}[T_0] < \infty, \mathbb{E}[\tau] < \infty, \) and (18) are sufficient to satisfy conditions stated in Theorem 2.3 of Sigman (2011). Hence, we can use Theorem 2.5 of Chapter 13 of Sigman (2011) to conclude that there exists \( Z^*_q, \) a random variable governed by the stationary distribution of \( Z_q \) and that
\[
\mathbb{E} \left[ \liminf_{t \to \infty} \frac{1}{t} \int_0^t H(Z_q(s), \bar{q}) \, ds \right] = \mathbb{E} \left[ H(Z^*_q, \bar{q}) \right]. \tag{19}
\]

Putting (16), (18), and (19), we have
\[
\kappa^* \geq \mathbb{E} \left[ H(Z^*_q, \bar{q}) \right] \geq \mathbb{E} \left[ H(Z^*_q, q^*) \right] = \kappa^*.
\]

Thus, we obtain \( \mathbb{E} \left[ H(Z^*_q, \bar{q}) \right] = \kappa^* \). Further, from (17), we have \( H(Z^*_q, \bar{q}) \geq \kappa^* \). This implies \( H(Z^*_q, \bar{q}) = \kappa^* = H(Z^*_q, q^*) \) almost surely, and hence
\[
\bar{q}^* = \arg \min_{q \in A} \phi(Z^*_q, v^*(Z^*_q), q) \tag{20}
\]
almost surely. To complete the proof of the proposition, suppose we can show that
\[
\mathbb{P}(Z^*_q \in (\delta_1, \delta_2)) > 0 \tag{21}
\]
for any \( 0 < \delta_1 < \delta_2 < a \) where \( a \) is the constant defined prior to (14). Then, (20) would be a contradiction because we have already argued earlier in the proof (the third paragraph) that \( \bar{q}(x) \) is not an optimizer for \( \min_{x \in A} \phi(x, v^*(x), q) \) for all \( x \in (0, a) \). Therefore, we would obtain \( \kappa_{\bar{q}} > \kappa^* \), which would complete the proof. Thus, we next establish (21). Let
\[
\hat{\mu}(x) := -\beta \sqrt{\mu \lambda} + \mu [x]^- - \lambda \sum_{k \in K} \mu_k \left( \frac{[x]^+}{\lambda} \right)
\]
for \( x \in \mathbb{R} \). It is straightforward to see there exists a differentiable function \( \underline{\mu} \) and \( x_0 < 0 \) such that \( \underline{\mu}(x) \leq \hat{\mu}(x) \) and \( \underline{\mu}(x) = [x]^- \) for all \( x < x_0 \). Also, let \( Z \) and \( \bar{Z} \) be diffusion processes that (weakly) solve
\[
Z(t) = Z(0) + \int_0^t \underline{\mu}(Z(s)) \, ds + \sqrt{2\lambda} \bar{W}(t)
\]
and
\[
\bar{Z}(t) = \bar{Z}(0) - \beta \sqrt{\mu \lambda} t + \mu \int_0^t [\bar{Z}(s)]^- \, ds + \sqrt{2\lambda} \bar{W}(t),
\]
respectively, for two independent standard brownian motions \( \bar{W} \) and \( \bar{W} \). Assume \( Z(0) = \bar{Z}(0) = 0 \).
$Z(0)$. Then, we have
\[
Z(t) \leq Z_q(t) \leq \bar{Z}(t)
\]
and
\[
\mathbb{P}(Z_q^* \in (\delta_1, \delta_2)) \geq \mathbb{P}(\{Z^* < \delta_2\} \cap \{Z^* > \delta_1\}) = \mathbb{P}(\bar{Z}^* < \delta_2) \mathbb{P}(Z^* > \delta_1),
\]
where $\bar{Z}^*$ and $Z^*$ are random variables governed by the stationary distributions for $Z$ and $\bar{Z}$, respectively. So, to prove $\mathbb{P}(\bar{Z}^* < \delta_2) > 0$, it suffices to show that $\mathbb{P}(\bar{Z}^* < \delta_2) \mathbb{P}(Z^* > \delta_1) > 0$.

Because $\bar{Z}$ is a diffusion process with a piece-wise linear drift and $\beta \geq 0$, we can directly use Browne and Whitt (1995) to show that $\mathbb{P}(\bar{Z}^* < \delta_2) > 0$.

To show $\mathbb{P}(Z^* > \delta_1) > 0$, we use a similar argument from the proof of Proposition 6.1 of Reed and Ward (2008) that calculates the stationary distribution for diffusion processes with differentiable infinitesimal drifts. (Because $\beta \geq 0$, no additional conditions on the hazard rate structure are needed; otherwise, when $\beta < 0$, this proof requires that the same conditions assumed in the proposition in Reed and Ward (2008) to ensure positive recurrence are satisfied.) To proceed, define an operator $A$ such that
\[
(Af)(y) = \mu(y) f'(y) + \lambda f''(y)
\]
for any twice-differentiable function $f$ and $y \in \mathbb{R}$. Then, using the argument from Reed and Ward (2008), the stationary distribution of $Z$, which we denote by $p$, is the solution of
\[
\lambda p''(x) - p'(x) \mu(x) - p(x) \mu'(x) = 0 \tag{22}
\]
for $x \in (-\infty, \infty)$. (Because Reed and Ward (2008) consider diffusion processes that are reflected at the origin, the counter part of the operator $A$ in Reed and Ward (2008) is defined for $x \geq 0$ and so does for the counter part of $p(x)$.) Since
\[
p(x) = \frac{\exp\left(\frac{1}{\lambda} \int_{-\infty}^{x} \mu(y) \, dy\right)}{\int_{-\infty}^{\infty} \exp\left(\frac{1}{\lambda} \int_{-\infty}^{x} \mu(y) \, dy\right) \, dx}
\]
solves (22) and $\int_{-\infty}^{\infty} p(x) \, dx = 1$, we have
\[
\mathbb{P}(Z^* > \delta_1) = \frac{\int_{\delta_1}^{\infty} \exp\left(\frac{1}{\lambda} \int_{-\infty}^{x} \mu(y) \, dy\right) \, dx}{\int_{-\infty}^{\infty} \exp\left(\frac{1}{\lambda} \int_{-\infty}^{x} \mu(y) \, dy\right) \, dx} > 0,
\]
which completes the proof of the proposition. \qed
Proof of Proposition 3:

Proof of part (i). Observe that \( \left( \frac{r_1 - v^*(x)}{r_2 - v^*(x)} \right) \frac{m_1(x/\lambda)}{x/\lambda} \) in (23) is strictly decreasing in \( x \) and converges to 0 as \( x \) grows. This is because (i) \( \frac{m_1(x/\lambda)}{x/\lambda} \) is decreasing in \( x \) when \( h_1(x) \) is decreasing in \( x \), and (ii) \( \frac{r_1 - v^*(x)}{r_2 - v^*(x)} \) is strictly decreasing in \( x \) and converges to 0 as \( x \) grows by Theorem 1. Also, observe that by (24), we have

\[
\lim_{x \to \infty} \left( \frac{r_1 - v^*(x)}{r_2 - v^*(x)} \right) \frac{m_1(x/\lambda)}{x/\lambda} > \gamma_2.
\]

Therefore, \( q^* = q_1^1 \) for some \( T \in (0, \infty) \) as there must exists a unique \( T \in (0, \infty) \) such that \( \left( \frac{r_1 - v^*(T)}{r_2 - v^*(T)} \right) \frac{m_1(T/\lambda)}{T/\lambda} = \gamma_2 \), \( \left( \frac{r_1 - v^*(x)}{r_2 - v^*(x)} \right) \frac{m_1(x/\lambda)}{x/\lambda} > \gamma_2 \) for \( x < T \), and \( \left( \frac{r_1 - v^*(x)}{r_2 - v^*(x)} \right) \frac{m_1(x/\lambda)}{x/\lambda} < \gamma_2 \) otherwise.

If (24) is not satisfied, then \( \left( \frac{r_1 - v^*(x)}{r_2 - v^*(x)} \right) \frac{m_1(x/\lambda)}{x/\lambda} < \gamma_2 \) for all \( x > 0 \), and therefore \( q^* = q_2^2 \).

Proof of part (ii). The proof of this part is identical to part (i) and is omitted. \( \square \)

Proof of Proposition 4:

First, observe that

\[
\lim_{x \to 0} \left( \frac{r_1 - v^*(x)}{r_2 - v^*(x)} \right) \frac{m_1(x/\lambda)}{x/\lambda} > \gamma_2
\]

because \( (r_1 - v^*(0)) h_1(0) > (r_2 - v^*(0)) \gamma_2 \). Therefore, there must exist \( T_1 \in (0, \infty) \) such that \( q^*(x) = (0, 1) \) for \( x \in (0, T_1) \) by (23). Also, observe that

\[
\lim_{x \to \infty} \left( \frac{r_1 - v^*(x)}{r_2 - v^*(x)} \right) \frac{m_1(x/\lambda)}{x/\lambda} = \infty.
\]

This is because \( \lim_{x \to \infty} \frac{m_1(x/\lambda)}{x/\lambda} > 0 \) and \( \lim_{x \to \infty} \frac{r_1 - v^*(x)}{r_2 - v^*(x)} = \infty \) by the condition \( r_1 > r_2 \) which implies \( \lim_{x \to \infty} v^*(x) = r_2 \) by Theorem 1. Hence, there must exist \( T_2 < \infty \) such that \( T_1 \leq T_2 \) and \( q^*(x) = (0, 1) \) for \( x \in [T_2, \infty) \) by (23).

The above paragraph establishes the structure of an optimal control of the DCP (16) and also the fact that any threshold control cannot solve the HJB (5). Then, using the proof of the second part of Proposition 2, we can conclude that the objective function value of (15) under any threshold control is strictly greater than \( \kappa^* \), establishing the sub-optimality of threshold control for the DCP (16). \( \square \)

Proof of Proposition 5:

Let

\[
c(q) := (r_1 - v^*(x)) a_1 \lambda \int_0^{xq_1(a_1 \lambda)} h_1(u) du + (r_2 - v^*(x)) x \gamma_2 (1 - q)
\]
for $q \in [0,1]$. It is straightforward to check that $c(q)$ is convex in $q$ when $h_1(x)$ is increasing. Observe that
\[ c'(q) = (r_1 - v^*(x)) \frac{xq_1}{a_1 \lambda} - (r_2 - v^*(x)) x \gamma_2, \]
so that $q \in [0,1]$ that minimizes $c(q)$ is given by $\min \{ 1, \frac{\lambda_0}{x} h_1^{-1} \left( \frac{r_2 - v^*(x)}{r_1 - v^*(x)} \gamma_2 \right) \}$. Therefore, $q^*$ satisfies (26). Observe that $q^* \neq q_1^*$ and $q^* \neq \{ q_2^2 \}$ for $i \in \{1,2\}$ and $T \in (0,\infty)$ because $\frac{\lambda_0}{x} h_1^{-1} \left( \frac{r_2 - v^*(x)}{r_1 - v^*(x)} \gamma_2 \right)$ takes values greater than 0. In fact the only $x < \infty$ such that $\frac{\lambda_0}{x} h_1^{-1} \left( \frac{r_2 - v^*(x)}{r_1 - v^*(x)} \gamma_2 \right) = 0$ is a unique solution (if exists) of
\[ \frac{r_2 - v^*(x)}{r_1 - v^*(x)} \gamma_2 = h_1(0). \]

Under (i), which is $r_1 \geq r_2$, $\frac{\lambda_0}{x} h_1^{-1} \left( \frac{r_2 - v^*(x)}{r_1 - v^*(x)} \gamma_2 \right)$ is decreasing in $x$ and converges to 0 as $x$ grows by Theorem 1. Hence $q^* \neq q_2^2$. Under (ii), which is $r_1 \geq r_2$ and $G_k$ has the bounded domain, $\frac{\lambda_0}{x} h_1^{-1} \left( \frac{r_2 - v^*(x)}{r_1 - v^*(x)} \gamma_2 \right)$ is smaller than 1 for large $x$. Hence $q^* \neq q_2^2$.

To show that the cost under threshold or static control is higher than $\kappa^*$, let
\[ \mu(x) := -\beta \sqrt{x \lambda} + \mu(x) - \lambda \left\{ a_k \int_0^{x^+q_1(x)} h_1(u) \, du + \frac{\gamma_2 u^+ q_2(x)}{x} \right\} \]
for any $x \in \mathbb{R}$ and $q(x) = (q_1(x), q_2(x)) \in A$. It is straightforward to check that $\lim_{x \to \infty} \mu(x) = \infty$. Then, we can find $\underline{\mu}(x)$ and $\bar{\mu}(x)$ that are differentiable,
\[ \underline{\mu}(x) \leq \mu(x) \leq \bar{\mu}(x), \]
\[ \lim_{x \to \infty} \underline{\mu}(x) = \lim_{x \to \infty} \bar{\mu}(x) = \infty, \text{ and } \lim_{x \to -\infty} \mu(x) = \lim_{x \to -\infty} \bar{\mu}(x) = \infty. \]

So, by defining $Z$ and $\bar{Z}$ be the (weak) solutions of
\[ Z(t) = Z(0) + \int_0^t \underline{\mu}(Z(s)) \, ds + \sqrt{2 \lambda W(t)} \]
\[ \bar{Z}(t) = Z(0) + \int_0^t \bar{\mu}(Z(s)) \, ds + \sqrt{2 \lambda \bar{W}(t)} \]
for independent standard Brownian motions $W$ and $\bar{W}$, we can use the same bounding argument from the proof of Proposition 2 to show both threshold control and static priority admit a higher cost than $\kappa^*$.
Proof of Proposition A.1:

Given the existence of a twice continuously differentiable function $w^*$ stated in the proposition, we apply Ito’s lemma to $w^* (\hat{X} (t))$ to get

$$
\frac{w^* (\hat{X} (t))}{t} + \sum_{k \in K} r_k \lambda \int_0^t m_k \left( \frac{\dot{X}_k (s) - u_k (\hat{X} (s))}{\lambda} \right) ds
$$

$$
= \sum_{k \in K} \frac{1}{t} \int_0^t \left( - \frac{\partial w^* (\hat{X} (t))}{\partial x_k} \lambda m_k \left( \frac{\dot{X}_k (s) - u_k (\hat{X} (s))}{\lambda} \right) + \mu_k u_k (\hat{X} (s)) + \frac{a_k \beta \sqrt{X}}{\nu} \right) ds
$$

$$
+ \sum_{k \in K} \frac{1}{t} \int_0^t \sqrt{2a_k \lambda} \frac{\partial w^* (\hat{X} (s))}{\partial x_k} dW_k (s)
$$

$$
\geq \kappa^* + \sum_{k \in K} \frac{1}{t} \int_0^t \sqrt{2a_k \lambda} \frac{\partial w^* (\hat{X} (s))}{\partial x_k} dW_k (s),
$$

where the last inequality follows from the property of $w^*$ and $\kappa^*$ as stated in the proposition. By assumption, $w^*$ has bounded partial derivatives, which implies

$$
\mathbb{E} \left[ \int_0^t \sqrt{2a_k \lambda} \frac{\partial w^* (\hat{X} (s))}{\partial x_k} dW_k (s) \right] = 0.
$$

(23)

Also by assumption

$$
\lim_{t \to \infty} \mathbb{E} \left[ \frac{w^* (\hat{X} (t))}{t} \right] = 0.
$$

(24)

Then, we have $V (u) \geq \kappa^*$ for any $u \in A (\infty)$.

Similar to the above, we can also apply Ito’s lemma to $w^* (\hat{X}^* (t))$ to find

$$
\frac{w^* (\hat{X}^* (t))}{t} + \sum_{k \in K^*} r_k \lambda \int_0^t m_k \left( \frac{\dot{X}_k^* (s) - u_k^* (\hat{X}^* (s))}{\lambda} \right) ds
$$

$$
= \kappa^* + \sum_{k \in K^*} \frac{1}{t} \int_0^t \sqrt{2a_k \lambda} \frac{\partial w^* (\hat{X}^* (s))}{\partial x_k} dW_k^* (s).
$$

Since (23) and (24) hold for $\hat{X} = \hat{X}^*$, we have $V (u^*) = \kappa^*$, proving the optimality of $u^*$. \qed
C Proofs of Lemmas

We will use the following result in this section.

Lemma C.9. \( \min_{q \in A} \phi(x, w, q) \) is decreasing in \( w \in \mathbb{R} \) for all \( x \in (0, d) \) and is increasing in \( x \in (0, d) \) for all \( w \in \mathbb{R} \).

We omit the details of the proof of this lemma: The first claim of this lemma is a special case of Lemma 6.2 in Kim and Ward (2013) (\( \theta = 0 \) therein) and the second claim of this lemma can be proved by using the similar argument in the proof of Lemma 6.2 in Kim and Ward (2013).

Proof of Lemma B.1:

First, we prove that (6) has a unique solution for \( x \leq 0 \) and this solution satisfies conditions stated in the lemma. If \( x \leq 0 \), the ODE in (6) becomes

\[
\lambda v'(x) - \left( \sqrt{\mu x + \beta \sqrt{\lambda}} \right) \sqrt{\mu} v(x) = \kappa
\]

because \( \min_{q \in A} \phi(x, v(x), q) = 0 \) for all \( x \leq 0 \). Up to the value of \( v(0) \), the above ODE has a unique solution

\[
v(x) = \exp \left( \frac{\sqrt{\mu}}{2} \left( \frac{\sqrt{\mu} x^2 + 2 \frac{\beta}{\sqrt{\lambda} x}}{2} \right) \right) \left( v(0) - \int_x^0 \kappa \exp \left( - \frac{\sqrt{\mu}}{2} \left( \frac{\sqrt{\mu} y^2 + 2 \frac{\beta}{\sqrt{\lambda} y}}{2} \right) \right) dy \right).
\]

(26)

For each \( \kappa \), we then use the boundary condition in (6) to pin down \( v(0) \). For the boundary condition in (6) to hold, it is necessary that \( \lim_{x \to -\infty} v(x) < \infty \), which requires

\[
v(0) = \int_{-\infty}^0 \frac{\kappa}{\lambda} \exp \left( - \frac{\sqrt{\mu}}{2} \left( \frac{\sqrt{\mu} y^2 + 2 \frac{\beta}{\sqrt{\lambda} y}}{2} \right) \right) dy
\]

(27)

because \( \lim_{x \to \infty} \exp \left( \frac{\sqrt{\mu}}{2} \left( \frac{\sqrt{\mu} x^2 + 2 \frac{\beta}{\sqrt{\lambda} x}}{2} \right) \right) = \infty \). Under (27), \( v(x) \) in (26) is given by

\[
v(x) = \frac{\kappa}{\lambda} \sqrt{\frac{\mu}{\lambda}} \Phi \left( \frac{\sqrt{\mu}}{\lambda} \left( x + \frac{\beta}{\sqrt{\mu}} \right) \right),
\]

(28)

where \( \Phi \) denotes the cdf of the standard normal distribution. It is straightforward to check that the function in the right hand side satisfies conditions stated in the lemma. So, we have now proved that the ODE in (6) has a unique solution for \( x \leq 0 \) and this solution satisfies conditions stated in the lemma. To complete the proof of this lemma, we now consider \( x > 0 \). Note that (6) is equivalent to

\[
\lambda v'(x) - \beta \sqrt{\mu} v(x) + \lambda \min_{q \in A} \phi(x, v(x), q) = \kappa, \text{ such that } v(0) = \frac{\kappa}{\lambda} \sqrt{\frac{\mu}{\lambda}} \Phi' \left( \frac{\beta}{\sqrt{\mu}} \right)
\]

(29)
for which we know the existence of the unique solution that is jointly continuous in $(\kappa, x) \in \mathbb{R}^+ \times \mathbb{R}^+$ by Lemma 4.1 of Kim and Ward (2013). By connecting $v(x)$ that solves (25) for $x \leq 0$ and that solves (29) for $x \geq 0$, we now have the unique solution of (6) for each $\kappa > 0$. Let us call this solution $v_\kappa$. Note that $v'_\kappa(x)$ is continuous in $x \in (-\infty, d)$: $v'_\kappa(x)$ is continuous in $x < 0$ because of (28), is continuous in $x > 0$ by Lemma C.9 and (29), and $\lim_{x \uparrow 0^+} v'_\kappa(x) = \lim_{x \downarrow 0^+} v'_\kappa(x)$ by (25) and (29).

To prove that $v_\kappa$ is increasing in $\kappa$, observe from (29) that we have $v_{\kappa_1}(0) > v_{\kappa_2}(0)$ for $\kappa_1 > \kappa_2$. Suppose $\{x > 0 : v_{\kappa_2}(x) = v_{\kappa_1}(x)\}$ is not empty and let $y$ be the infimum of this set. Then, because $v_{\kappa_2}(x) < v_{\kappa_1}(x)$ for all $x \in (0, y)$, we must have $v'_{\kappa_1}(y) < v'_{\kappa_2}(y)$ and hence $\lambda v'_{\kappa_1}(y) - \kappa_1 < \lambda v'_{\kappa_2}(y) - \kappa_2$. However, this is a contradiction because

\[
\lambda v'_{\kappa_1}(y) - \kappa_1 = \beta \sqrt{\lambda \mu v_{\kappa_1}(y)} - \lambda \min_{q \in A} \phi(y, v_{\kappa_1}(y), q) \quad \text{(i)}
\]

\[
\geq \beta \sqrt{\lambda \mu v_{\kappa_2}(y)} - \lambda \min_{q \in A} \phi(y, v_{\kappa_2}(y), q)
\]

\[
= \lambda v'_{\kappa_2}(y) - \kappa_2,
\]

where (i) follows because $v_{\kappa_1}(y) = v_{\kappa_2}(y)$ by the definition of $y$. □

Proof of Lemma B.2:

Recall from the proof of Lemma B.1 that $v_\kappa(0) = \frac{\xi}{\mu} \sqrt{\frac{\lambda}{\Phi(\beta)}}$ converges to $\infty$ as $\kappa$ grows, $U$ is non-empty. This also implies that $\inf U < \infty$

Let us now show that $0 \in \mathcal{D}$ and hence $\mathcal{D}$ is non-empty. Suppose $0 \in U$ and define $x_0 := \inf \{x \geq 0 : v_0(x) = r\}$. To proceed, let $\kappa := \arg \min_{k \in K} \{r_k\}$ and observe that for $x \in (0, d)$

\[
\min_{q \in A} \phi(x, r, q) = \min_{q \in A} \left\{ \sum_{k \neq \kappa} (r_k - r) m_k \left( \frac{xq_k}{\lambda} \right) + (r - r) m_{\kappa} \left( \frac{xq_k}{\lambda} \right) \right\} = 0, \quad (30)
\]

where the last equality follows by setting $q_k = 0$ for $k \neq \kappa$ and $q_\kappa = 1$. Then, by (29), we have

\[
\lambda v'_0(x_0) = \beta \sqrt{\lambda \mu r}.
\]

Because $\beta < 0$, the above equation display implies that $v'_0(x_0) < 0$. This is a contradiction because $v_0(0) = 0$ by (28) which requires $v'_0(x_0) \geq 0$.

To prove $\sup \mathcal{D} = \inf U$, it suffices to show that if $\kappa_1 \in U$, then $\kappa \in U$ for all $\kappa > \kappa_1$ and if $\kappa_1 \in \mathcal{D}$, then $\kappa \in D$ for all $\kappa < \kappa_1$. However, this is obvious because $v_\kappa$ is increasing in $\kappa$ by Lemma B.1.

We now show that $\inf U > 0$. To see this note that $v_\kappa(0) < r$ for any $\kappa < r \sqrt{\mu \Phi(\beta)}$. Hence, in order for such $\kappa$ to belong to $U$, we must have $v'_\kappa(x_r) > 0$ where $x_r := \inf \{x > 0 : v_\kappa(x) \geq r\}$. 

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To \( v'_\kappa(x) > 0 \), we must have \( \kappa > -\beta \sqrt{\lambda \mu r} \) because \( \lambda v'_\kappa(x) = \kappa + \beta \sqrt{\lambda \mu r} \) by (29) and (30). Since \( \beta < 0 \), \( \inf \mathcal{U} > 0 \).

**Proof of Lemma B.3:**

If \( v_\kappa(x) \) is not strictly decreasing in \( x > x_1 \), there must exist \( x_1 < \tilde{x}_1 < \hat{x}_1 \) such that \( v_\kappa(\tilde{x}_1) = v_\kappa(\hat{x}_1) \) and \( v'_\kappa(\tilde{x}_1) \leq 0 \leq v'_\kappa(\hat{x}_1) \). From (29), we have

\[
\lambda v'_\kappa(\tilde{x}_1) = \kappa + \beta \sqrt{\lambda \mu v_\kappa(\tilde{x}_1)} - \lambda \min_{q \in \mathcal{A}} \phi(\tilde{x}_1, v_\kappa(\tilde{x}_1), q)
\]

where \((a)\) follows by Lemma C.9. However, this is a contradiction as we already have \( v'_\kappa(\hat{x}_1) \geq v'_\kappa(\tilde{x}_1) \). Hence \( v_\kappa(x) \) is strictly decreasing in \( x > x_1 \).

**Proof of Lemma B.4:**

We prove \( \lim_{x \to d} v_\kappa(x) \) is either \(-\infty\), \( r \) or \( \infty \). Suppose on the contrary \( \lim_{x \to d} v_\kappa(x) \) is a finite number that is not \( r \). Suppose first

\[
\lim_{x \to d} v_\kappa(x) \in (-\infty, r),
\]

so that \( \lim_{x \to d}(r_k - v_\kappa(x)) > 0 \) for all \( k \in \mathcal{K} \). Also observe that for any \( q \in \mathcal{A} \), we have \( \lim_{x \to d} m_k\left(\frac{xq(x)}{a_k \lambda}\right) = \infty \) for at least one \( k \in \mathcal{K} \). Therefore, we have

\[
\lim_{x \to d, q(x) \in \mathcal{A}} \min_{x \to d, q(x) \in \mathcal{A}} \phi(x, v_\kappa(x), q(x)) = \lim_{x \to d, q(x) \in \mathcal{A}} \lambda \sum_{k \in \mathcal{K}} (r_k - v_\kappa(x)) m_k\left(\frac{xq_k(x)}{a_k \lambda}\right) = \infty
\]

However, this is a contradiction because the first two terms in (29) converge to finite numbers.

Suppose on the other hand, \( \lim_{x \to d} v_\kappa(x) \in (r, \infty) \). Then, we have \( \lim_{x \to d} (r_k - v^*(x)) < 0 \) and similarly to the above, we draw a contradiction because

\[
\lim_{x \to d, q(x) \in \mathcal{A}} \min_{x \to d, q(x) \in \mathcal{A}} \phi(x, v_\kappa(x), q(x)) = -\infty.
\]

**Proof of Lemma B.5:**

Suppose \( v'_\kappa(x) \geq 0 \) for all \( x \in [0, d) \). Then, by the definition of \( \mathcal{D} \), \( v_\kappa(x) \) converges from below to some constant \( c < r \) as \( x \to d \). But, this is a contradiction with the same reasoning used in the first
paragraph of the proof of Lemma B.4. Therefore, there must exist \( \bar{x} \) such that \( v'_\kappa(\bar{x}) < 0 \). Given the existence of such \( \bar{x} \), we can use Lemma B.3 to conclude that \( \lim_{x \to d} v_\kappa(x) = -\infty \) as \( v_\kappa(x) \) cannot converge to \( r \) as \( x \) grows for \( \kappa \in \mathcal{D} \) by the definition of \( \mathcal{D} \).

**Proof of Lemma B.6:**

Define \( U_1 := \{ \kappa \in U : v_\kappa(0) < r \} \). Because \( v_\kappa(0) \) is increasing in \( \kappa \), we have \( \sup U_1 = \inf U \setminus U_1 \).

Hence to complete the proof, it suffices to prove that \( \lim_{x \to d} v_\kappa(x) \geq r \) for \( \kappa \in U_1 \) because \( v_\kappa(x) \) is increasing in \( \kappa \) for \( x \) by Lemma B.1.

Fix \( \kappa_1 \in U_1 \) and define \( x_1 := \inf \{ x \geq 0 : v_{\kappa_1}(x) = r \} \). Then, we must have \( v'_{\kappa_1}(x_1) > 0 \). Suppose there exists \( x_2 > x_1 \) such that \( v_\kappa(x_2) = r \). Since \( \min_{q \in A} \phi(x,r,q) = 0 \) by (30), we immediately see from (29) that \( v'_{\kappa_1}(x_1) = v'_{\kappa_1}(x_2) > 0 \). Therefore, for any \( x > x_1 \), we must have \( v_\kappa(x) \geq r \) and \( \lim_{x \to d} v_{\kappa_1}(x) \geq r \) as desired. \( \square \)

**Proof of Lemma B.7:**

This lemma is a special case of the first part of the proof of Lemma 4.1 in Kim and Ward (2013) (\( \theta = 0 \) therein). So, we omit the proof. \( \square \)

**Proof of Lemma B.8:**

The proof of this lemma is similar to that of Proposition 4.2 in Kim and Ward (2013). The main difference is that in our paper the diffusion process is not reflected at the origin and can take negative values.

To complete the proof of the lemma, it suffices to prove that

\[
\lim_{t \to \infty} \frac{\mathbb{E}[Z(t)^2]}{t^2} = 0, \tag{32}
\]

because by Jensen’s inequality, we have

\[
\frac{\mathbb{E}[|Z(t)|]}{t} \leq \sqrt{\frac{\mathbb{E}[Z(t)^2]}{t^2}}.
\]

To proceed, note that \( Z \) is delayed regenerative process which is composed of the initial cycle during which the process \( Z \) reaches 0 from \( z \) (recall that we assume \( Z(0) = z \)) that is followed by the i.i.d. regenerative cycles that we describe now. Fix \( \bar{z} \in (\max\{z,0\}, d(q)) \) (recall that \( d(q) \) is the upper bound of \( Z \) under \( q \in A(\infty) \)). Then, in each of these i.i.d. cycles, the process first hits \( \bar{z} \) from 0 and reaches back to 0 from \( \bar{z} \). We denote the length of the initial cycle and the length of an i.i.d. cycle by \( T_0 \) and \( \tau \), respectively. Given that \( Z \) is a delayed regenerative process, the argument
in page 27 of Kim and Ward (2013) shows that it is sufficient to prove

\[
\frac{\mathbb{E} \left[ \int_0^t Z(s) \, ds \right]}{t} \rightarrow \frac{\mathbb{E} \left[ \int_{T_0}^{T_0+\tau} Z(s) \, ds \right]}{\mathbb{E}\left[\tau\right]} \quad \text{as } t \to \infty
\]  

(33)
to establish (32).

To prove (33), we first establish

(i) \( T_0 < \infty \) almost surely;

(ii) \( \mathbb{E}\left[\tau\right] < \infty \);

(iii) \( \mathbb{E} \left[ \int_{T_0}^{T_0+\tau} |Z(s)| \, ds \right] < \infty \);

(iv) \( \mathbb{E} \left[ \int_0^{T_0} |Z(s)| \, ds \right] < \infty \).

Once (i)-(iv) are established, (33) follows from Theorem 2.3 of Chapter 13 of Sigman (2011). This is because (i)-(iv) verify conditions listed in that theorem: (i) and (ii) jointly imply that \( Z \) is positive recurrent, (iii) implies that the condition (which is \( \mathbb{E} \left[ \int_{T_0}^{T_0+X_1} |f(X(s))| \, ds \right] < \infty \) in Theorem 2.3 of chapter 13 of Sigman (2011)) is satisfied, and (iv) implies that (7) in Sigman (2011) is satisfied.

To establish (i)-(iv) above, let \( T_{a,b} \) be the time that it takes for the diffusion process \( Z \) to reach \( b \) if the process initiated at \( a \) for \( a,b \in (-\infty, d(q)) \), and define

\[
\mathbb{E}_w[\cdot] := \mathbb{E}[\cdot | Z(0) = w]
\]
for any \( w \in (-\infty, d(q)) \). Then, we can prove (i)-(iv) by respectively using

\[
\begin{align*}
\mathbb{E} [T_0] &\overset{(a)}{=} \mathbb{E}_z [T_{z,0}] \\
&= \mathbb{E}_z \left[ \int_0^{T_{z,0}} 1 \, ds \right]; \\
\mathbb{E} [\tau] &\overset{(b)}{=} \mathbb{E}_0 [T_{0,z}] + \mathbb{E}_z [T_{z,0}] \\
&= \mathbb{E}_0 \left[ \int_0^{T_{0,z}} 1 \, ds \right] + \mathbb{E}_z \left[ \int_0^{T_{z,0}} 1 \, ds \right]; \\
\mathbb{E} \left[ \int_{T_0}^{T_0+\tau} |Z(s)| \, ds \right] &\overset{(c)}{\leq} \sqrt{\mathbb{E} \left[ \int_{T_0}^{T_0+\tau} Z(s)^2 \, ds \right]} \leq \sqrt{\mathbb{E} \left[ \int_0^{T_{0,z}} Z(s)^2 \, ds \right]} + \mathbb{E}_z \left[ \int_0^{T_{z,0}} Z(s)^2 \, ds \right]; \\
\mathbb{E} \left[ \int_0^{T_0} |Z(s)| \, ds \right] &\overset{(d)}{\leq} \sqrt{\mathbb{E} \left[ \int_0^{T_{0,z}} Z(s)^2 \, ds \right]} + \mathbb{E}_z \left[ \int_0^{T_{z,0}} Z(s)^2 \, ds \right]; \\
\mathbb{E}_w \left[ \int_0^{T_{w,z}} Z(s)^k \, ds \right] &< \infty \text{ for } w \in \{0, z\} \text{ and } k \in \{0, 2\}, \quad (35) \\
\mathbb{E}_z \left[ \int_0^{T_{z,0}} Z(s)^k \, ds \right] &< \infty \text{ for } k \in \{0, 2\}, \quad (36)
\end{align*}
\]

In the above display of equations, (a)-(f) follow because:

(a) This is obvious as \( Z(0) = z \).

(b) Let \( T_1 := \inf \{ t \geq T_0 : Z(t) = \bar{z} \} \) and \( T_2 = \inf \{ t \geq T_1 : Z(t) = 0 \} \). Then, we have \( \tau = (T_1 - T_0) + (T_2 - T_1) \) where \( \mathbb{E} [T_1 - T_0] = \mathbb{E}_0 [T_{0,z}] \) and \( \mathbb{E} [T_2 - T_1] = \mathbb{E}_z [T_{z,0}] \).

(c) This is by Jensen’s inequality.

(d) We can use a similar argument to that in (b) to establish this.

(e) This is by Jensen’s inequality.

(f) Let \( \bar{T}_1 := \inf \{ t \geq T_0 : Z(t) = \bar{z} \} \) and \( \bar{T}_2 := \inf \{ t \geq \bar{T}_1 : Z(t) = 0 \} \). Then,

\[
\mathbb{E} \left[ \int_0^{T_0} Z(s)^2 \, ds \right] \leq \mathbb{E} \left[ \int_0^{\bar{T}_1} Z(s)^2 \, ds \right] + \mathbb{E} \left[ \int_{\bar{T}_1}^{\bar{T}_2} Z(s)^2 \, ds \right] = \mathbb{E}_z \left[ \int_0^{T_{z,0}} Z(s)^2 \, ds \right] + \mathbb{E}_z \left[ \int_0^{T_{z,0}} Z(s)^2 \, ds \right].
\]

Using the right hand sides in (34), we can finish the proof the lemma by showing

\[
\begin{align*}
\mathbb{E}_w \left[ \int_0^{T_{w,z}} Z(s)^k \, ds \right] &< \infty \text{ for } w \in \{0, z\} \text{ and } k \in \{0, 2\}, \quad (35) \\
\mathbb{E}_z \left[ \int_0^{T_{z,0}} Z(s)^k \, ds \right] &< \infty \text{ for } k \in \{0, 2\}, \quad (36)
\end{align*}
\]
To show (35) and (36), it is useful to define an operator

\[ Au(x) := \lambda u''(x) + f(x) u'(x), \]

for any twice differentiable function \( u : (-\infty, d(q)) \to \mathbb{R} \), where

\[ f(x) := -\beta \sqrt{\mu \lambda} + \mu [x] - \lambda \sum_{k \in K} m_k \left( \frac{[x]^+ q_k ([x]^+)}{\lambda} \right) \]

for \( q \in A(\infty) \).

**Establishing (35).** For \( k \in \{0, 2\} \), define for \( x \in (-\infty, d(q)) \)

\[ r(x) := \int_{-\infty}^{x} -\frac{1}{\lambda} y^k \exp \left( -\int_{y}^{x} \frac{1}{\lambda} f(z) \, dz \right) \, dy \]

\[ u(x) := \int_{\tilde{z}}^{x} r(y) \, dy. \]

Then, it is straightforward to check that \( u(x) \) is the solution of

\[ Au(x) = -x^k \]

such that \( u(\tilde{z}) = 0 \). By applying Ito’s lemma on \( u(Z(t)) \), we have

\[ u(Z(t \wedge T_{w,\tilde{z}})) = u(Z(0)) - \int_{0}^{t \wedge T_{w,\tilde{z}}} Z(s)^k \, ds + \lambda \int_{0}^{t \wedge T_{w,\tilde{z}}} u'(Z(s)) \, dW(s). \]

(Note that the function \( u \) might not be twice-continuously differentiable because \( q \) is not necessarily continuous. But as discussed in Section 4.7 of Harrison (1985), we can still apply Ito’s lemma.)

To proceed, we argue that \( u(x) \geq 0 \) for \( x \leq \tilde{z} \) and the stochastic integral is a martingale. The former is straightforward because \( u(\tilde{z}) = 0 \) and \( u'(x) < 0 \) for \( x \leq \tilde{z} \). For the latter, observe that for any \( x \leq 0 \), \( r(0) \leq r(x) \leq 0 \), which implies that \( u'(x) \) is uniformly bounded for \( x \leq 0 \) because \( r(0) > -\infty \). Also, because \( u' \) is continuous, \( u'(x) \) is bounded for \( x \in [0, \tilde{z}] \). Therefore as long as \( w < \tilde{z}, |u'(Z(s))| \) is bounded for \( s \in [0, t \wedge T_{w,\tilde{z}}] \) and \( \lambda \int_{0}^{t \wedge T_{w,\tilde{z}}} u'(Z(s)) \, dW(s) \) is a martingale. So, we have

\[ u(w) = E_w \left[ u(Z(t \wedge T_{w,\tilde{z}})) + \int_{0}^{t \wedge T_{w,\tilde{z}}} Z(s)^k \, ds \right] \geq E_w \left[ \int_{0}^{t \wedge T_{w,\tilde{z}}} Z(s)^k \, ds \right], \]

for \( w \in \{0, z\} \). By applying the monotone convergence theorem (we can do this because \( Z(s)^k \geq 0 \) for \( k \in \{0, 2\} \)), we reach

\[ E_w \left[ \int_{0}^{T_{w,\tilde{z}}} Z(s)^k \, ds \right] \leq u(w) < \infty \]
for \( w \in \{0, z\} \) and \( k \in \{0, 2\} \) and hence establish (35).

**Establishing** (36). Note that for the expectation \( \mathbb{E}_z \), \( Z(s) \geq 0 \) for \( s \in [0, T_{\bar{z},0}] \). Hence, we can repeat the same argument that establishes the equation (49) in the proof of Proposition 4.2 in Kim and Ward (2013) to complete the proof of (36).

□

**References**


