Optimal Commissions and Subscriptions in Networked Markets

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Two salient features of most online platforms are that they do not dictate the transaction prices, and use commissions/subscriptions for extracting revenues. We consider a platform that charges commission rates and subscription fees to sellers and buyers for facilitating transactions, but does not directly control the transaction prices, which are determined by the traders. Buyers and sellers are divided into types, and we represent the compatibility between different types using a bipartite network. Traders are heterogeneous in terms of their valuations, and different types have possibly different value distributions. The platform chooses commissions-subscriptions to maximize its revenues.

We provide a convex optimization formulation to calculate the revenue-maximizing commissions/subscriptions, and establish that, typically, different types should be charged different commissions/subscriptions depending on their network positions. We establish lower and upper bounds on the platform’s revenues in terms of the supply-demand imbalance across the network. Motivated by simpler schemes used in practice, we show that the revenue loss can be unbounded when all traders on the same side are charged the same commissions/subscriptions, and bound the revenue loss in terms of the supply-demand imbalance across the network. Charging only buyers or only sellers leads to a (bounded) revenue loss, even when different types on the same side can be charged differently. Under mild assumptions, we establish that a revenue-maximizing platform achieves at least 2/3 of the maximum achievable social welfare. Our results highlight the suboptimality of commonly used payment schemes, and showcase the importance of accounting for the compatibility between different user types.

Key words: Pricing and Revenue Management; Two-sided Market; Buyer-seller Networks; Sharing Economy

1. Introduction

Platforms facilitating the exchange of goods and services between individuals are prevalent: one can purchase goods from others on eBay, arrange accommodation through Airbnb, find temporary projects/workers on online labor markets such as Upwork. The revenue models favored by these platforms vary. For facilitating the transactions, some of these platforms charge a commission (a percentage of the total transaction amount) to agents participating in a transaction, while others charge
a subscription fee (a flat fee that users pay to gain access to the platform), or a combination of both. For example, most third-party sellers on Amazon pay a $39.99 monthly subscription fee plus per-item selling fees (which vary by category), whereas Airbnb charges a 3% commission to the property owners (hosts) and a 0%-20% commission to the travelers (guests) whenever a property is rented. Hence, often the revenues of these platforms depend not only on the chosen commissions/subscriptions, but also on the prices at which buyers/sellers choose to transact.

Most platforms do not dictate the transaction prices. Instead, buyers and sellers determine at which price the goods or services will be exchanged: hosts decide on the price per night for their properties on Airbnb, sellers set prices for their goods on Amazon, and freelancers set their hourly rates on online labor markets. These prices depend on seller/buyer characteristics as well as the amount of supply-demand in the market for comparable goods and services. For instance, on Airbnb the (reservation) value of potential guests (hosts) looking for short-term rentals (offering their properties) might depend on features such as the neighborhood and the number of bedrooms. Moreover, not all buyers and sellers on a platform are compatible with each other: a business traveler going to NYC is likely interested in renting a property in Manhattan and not in the Bronx, whereas a leisure traveler might be interested in both. The number of compatible hosts (guests) for each guest (host) type impacts the final transaction prices. Thus, the imbalance of supply-demand across different types complicates the choice of commissions and subscriptions for the platform.

The objective of this paper is to understand how platforms should design commission/subscription fees with the objective of maximizing their revenues in markets where not all buyers and sellers are compatible. Our contribution is threefold: (i) we introduce a stylized model capturing the main features of the platform’s problem and characterize the revenue-optimal commissions/subscriptions (which exploit the compatibility structure), (ii) we study the impact of the compatibility structure on the surplus of market participants (buyers, sellers, and platform), and (iii) we analyze the implications of using even simpler commission/subscription schemes.

In our model, buyers and sellers are divided into finitely many types, each of which has some mass of infinitesimal agents. Not all buyer and seller types are compatible, and the compatibility between these types is represented by a bipartite network: nodes on one side correspond to buyer types, nodes on the other side correspond to seller types, and edges capture compatibility between different types of buyers and sellers. Each seller has a unit of good/service to offer, and each buyer demands at most one unit of the good/service from a compatible seller. These compatibilities can capture taste differences (e.g., a buyer may be interested only in the types of goods/services a subset of the sellers offer), geographical restrictions (e.g., being able to provide/receive services only in certain neighborhoods

1 Source: https://services.amazon.com/selling/faq.html
or cities), or other sources of mismatch (e.g., a mismatch in the desired and available skills in online labor markets). Unlike in a matching setting, we assume that buyers do not have heterogeneous preferences over compatible sellers, and neither do sellers over compatible buyers. Thus, preferences of the buyers/sellers are summarized by the valuations for the good they demand/supply (which are drawn from known type-specific distributions) and by the compatibility network. The platform chooses subscription fees and commissions, which are possibly type-specific. To capture the fact that the platform does not dictate the prices, we assume that, after the commissions and subscriptions are chosen by the platform, the transaction prices and equilibrium trades are determined endogenously in a competitive equilibrium.

We then provide a tractable optimization problem for obtaining the optimal commission rates and the subscription fees in the networked market described above. The problem of choosing revenue-maximizing subscriptions/commissions admits a natural nonconvex optimization formulation. We provide a convex relaxation of this formulation (which can thus be efficiently solved) and show that this relaxation is tight. Using the optimal dual solution of this relaxed problem, optimal commissions/subscriptions can be constructed in a tractable way. We establish that the optimal commissions/subscriptions are not unique. In particular, revenues can be maximized by using only commissions or using only subscriptions. However, in general, optimal fees are type-dependent and both sides of the market must be charged. Notice that in the simpler setting where there is only one type of buyer and one type of seller, to maximize revenues it would suffice to charge payments only to one side. Thus, naively, the same result may be expected to hold in general settings as well. Our finding illustrates that taking into account the underlying compatibility network leads to significantly different insights, and is fundamental for a platform’s commission/subscription design problem.

We also shed light on the impact of the network structure on the average surplus of different buyer/seller types. Intuitively, the buyer types that are the “most supply constrained” are the ones whose average surplus is the lowest; the second most supply constrained buyers have the second lowest surplus, and so on. While this is intuitive, it is not a priori clear how to define the most supply constrained types in a setting where not all buyers and sellers are compatible with each other. We make this intuition precise, by providing an appropriate measure of seller scarcity that takes into account the network structure. Our result allows us to rank buyer types according to their average surplus (and we show that a similar result applies to seller types as well). Moreover, leveraging this result we also characterize how the revenues of the platform change as a function of the populations of seller/buyer types. In particular, we show that it is least profitable to expand the buyer types who have the lowest average surplus (as they are already the ones who are the most under-supplied), and that, it is most profitable to expand the seller types whose average surplus is the highest.
We then explore the impact of the network structure on the revenues of the platform. We show that networks that satisfy a weighted variant of Hall’s marriage condition (see, e.g., Schrijver (2003)) maximize the revenues among all networks with the same number of types, populations, and value distributions. Moreover, for any other network, the revenues of the platform can be lower bounded by measuring to what degree this condition is violated.

Motivated by the revenue schemes adopted by many real-world platforms, we study what happens if we restrict ourselves to using simpler commissions/subscriptions schemes. We show that if we require using the same commissions/subscriptions for all buyers and similarly for all sellers, the revenue loss can be unbounded when agent types have heterogeneous value distributions. However, if all buyer types have the same value distribution and so do all seller types, then the revenue loss can be lower bounded in terms of the supply-demand imbalance induced by the network structure. Surprisingly, we show that, in general, charging subscriptions/commissions to only one side of the market leads to much lower revenues than optimal, even when different types on the same side are charged differently. We illustrate these findings with an example motivated by Airbnb, which calibrates some of our model primitives with real data. It is worth highlighting that our observations are consistent with the trending practice of Amazon, Airbnb, and Alibaba, which charge heterogeneous commissions/subscriptions to sellers and buyers in their marketplaces.

Overall, our results shed light on the design of revenue-maximizing commissions/subscriptions for platforms and highlight the importance of explicitly taking into account the structure of the compatibility network for this purpose. We establish that doing so leads to qualitatively different insights into the optimal commission/subscription structures (e.g., it is no longer revenue-maximizing to have agents only on one side pay to use the platform). At the same time, the underlying network structure has a first-order impact on the revenues of the platform as well as the surplus of its users. While the focus of this paper is on the revenue maximization problem of the platform, it is also worth considering the welfare consequences of using revenue-maximizing commissions/subscriptions. We show that the welfare achieved by a platform using revenue-maximizing commissions/subscriptions is lower bounded under reasonable assumptions, and we provide bounds for different settings.

Select proofs are provided in the online appendix. Remaining technical details can be found in our technical report (see Birge et al. (2018)).

1.1. Literature Review

The seminal papers by Rochet and Tirole (2003, 2006) and Armstrong (2006) study how a platform should set commissions/subscriptions in two-sided markets by taking into account network externalities. This framework was later extended by Weyl (2010) to a fairly general setting where $M \geq 2$ different groups participate in the market. In these papers, the payoff of each agent is given by an
exogenously specified function of the number of participants in her own group as well as the other
groups. By contrast, in our setting, buyers and sellers trade at endogenously determined prices that
are influenced by the platform’s choice of commissions/subscriptions. Hence, the network external-
ities do not admit a closed-form expression in terms of the number of traders who participate in
the market. Moreover, the number of participants in different groups (types) are related through
nontrivial equilibrium constraints. This has two important implications. First, the optimal com-
misions/subscriptions no longer admit a direct characterization in terms of first order optimality
conditions in the revenue optimization problem. Second, our equilibrium conditions add a matching
element onto the existing models of revenue maximization in platforms, thereby contributing to one
of the main future research directions suggested by Weyl (2010). In our setting, we establish that the
problem of finding the revenue-maximizing commissions/subscriptions can be formulated in terms of
the marginal traders of each buyer/seller type, which is consistent with a similar observation by Weyl
(2010). In contrast to the previous literature, we provide a tractable convex optimization formulation
for obtaining the optimal commissions/subscriptions. Furthermore, our work explicitly considers a
compatibility network, and sheds light on the dependence of the platform’s revenues and the sur-
plus of market participants on the network structure. In addition, we contribute to this literature
by studying the limitations and scope of different commission/subscription schemes in networked
markets.

Our paper closely relates to the models of buyer-seller networks; see, e.g., Shapley and Shubik
(1971), Kranton and Minehart (2001), Kakade et al. (2004), Babaioff et al. (2009), Blume et al.
(2009). In these models, each node of an underlying (bipartite) network corresponds to a trader, and
edges of the network encode which agents can trade with which other agents. A recent and growing
literature has explored variants of these models in order to study intermediation and bargaining in
networked systems (e.g., Abreu and Manea (2012), Condorelli et al. (2017), Manea (2018)), competi-
tion in networked Cournot markets (e.g., Bimpikis et al. (2015), Lin et al. (2017), Cai et al. (2017)),
inefficiencies due to barriers to trade in networked markets (see Gofman (2014), Elliot (2015), Gof-
man (2017)). In a recent paper, Goyal (2017) provides a thorough review of this literature. Our
paper complements these works by exploring how a platform can influence the trading outcome by
appropriately designing commissions/subscriptions in a trading network. The recent literature has
also explored other controls that platforms can use to improve their operations. For instance, Baner-
jee et al. (2017) study how the platforms should control which sellers and buyers are visible to each
other, and provides algorithms for the solutions of the induced decision problems. By contrast, in this
paper we control only the commissions/subscriptions, and doing so leads to very different decision
problems for the platform.
Our work is also related to the burgeoning literature in operations management that studies service platforms. A branch of this literature has focused on decentralized markets, and used control levers other than pricing to influence the market outcomes (e.g., Allon et al. (2012), Kanoria and Saban (2017), Arnosti et al. (2014)). In our work, we also focus on a decentralized market but instead we use commissions/subscriptions to study the platform’s revenue maximization problem. Benjaafar et al. (2018) explore the design of commissions in a similar manner to our work, albeit in a setting with no underlying compatibility network – which plays a key role in our analysis and results. Among other application areas, this literature has also made a substantial impact on the operations of ride-sharing platforms (e.g., Banerjee et al. (2015), Cachon et al. (2017), Gurvich et al. (2016), Afeche et al. (2018), Taylor (2018), Tang et al. (2017), Ozkan and Ward (2017), Bimpikis et al. (2018), Cohen and Zhang (2017), Castro et al. (2018)). In our setting a key difference is that the transaction prices are determined endogenously, whereas this literature mainly assumes that these quantities are determined by the platform. Finally, in a recent work Hu and Zhou (2016) consider a setting where buyers and suppliers are also divided into types, and the value of the match between two agents depends on their types. In their setting, agents arrive dynamically, the platform has full control over matches, and the objective of the platform is to maximize welfare. By contrast, we study a static setting where matches occur in a decentralized fashion, and the platform seeks to maximize its revenues by optimally designing commissions/subscriptions.

2. Model and Preliminary Results

We consider a platform that provides a marketplace for buyers and sellers to trade with each other. Buyers and sellers are divided into finitely many types, denoted by $\mathcal{B} = \{1, \ldots, m\}$ and $\mathcal{S} = \{1, \ldots, n\}$ respectively. To capture taste differences or other trade frictions, we assume that not all types of buyers and sellers are compatible. Compatibility between types is represented using a compatibility network: an undirected bipartite graph $G(\mathcal{B} \cup \mathcal{S}, E)$, where the edge set $E$ defines the potential trading opportunities between sellers and buyers. Without loss of generality, we consider networks in which each node has degree at least one. We denote the set of neighbors of type $i \in \mathcal{B} \cup \mathcal{S}$ in $G(\mathcal{B} \cup \mathcal{S}, E)$ by $N_E(i)$. Similarly, we denote by $N_E(X)$ the set of all neighbors of types in set $X \subseteq \mathcal{B} \cup \mathcal{S}$ that do not belong to $X$, i.e., $N_E(X) := \{ j | (i,j) \in E, i \in X, j \notin X \}$.

Buyers/sellers are infinitesimal, and we denote the total mass of buyers of type $j \in \mathcal{B}$ by $b_j > 0$, and the total mass of sellers of type $i \in \mathcal{S}$ by $s_i > 0$. Each seller supplies an (infinitesimal) unit amount of an indivisible service or product and, similarly, each buyer seeks to purchase the same unit amount from a compatible seller. Thus, each buyer can transact with at most one seller, and vice versa. A buyer receives the same value from transacting with any compatible seller and, similarly, a seller has the same (reservation) value for transacting with any compatible buyer. Buyers of the same type are
heterogeneous in their values; for a type $j \in B$, the cumulative distribution function of their values is given by $F_{b_j} : [0, \bar{v}_{b_j}] \rightarrow [0, 1]$. Here the upper bound $\bar{v}_{b_j}$ on the support is such that $\bar{v}_{b_j} \in \mathbb{R}_+ \cup \{\infty\}$, where with some abuse of notation $\bar{v}_{b_j} = \infty$ captures distributions with unbounded support. Similarly, the (reservation) values for type-$i$ sellers are distributed according to $F_{s_i} : [0, \bar{v}_{s_i}] \rightarrow [0, 1]$, where $\bar{v}_{s_i} \in \mathbb{R}_+ \cup \{\infty\}$. We impose the following assumption on the value distributions $F_{s_i}(v)$ and $F_{b_j}(v)$.

**Assumption 1.** The value distributions are nonatomic. Furthermore, we assume that $F_{s_i}(v)$ and $F_{b_j}(v)$ are continuously differentiable and strictly increasing in $v \in (0, \bar{v}_{s_i})$ and $v \in (0, \bar{v}_{b_j})$ for all $i \in S$ and $j \in B$. Finally, we assume that (reservation) values have bounded means, i.e., $\int_0^{\bar{v}_{s_i}} 1 - F_{s_i}(x)dx < \infty$, $\int_0^{\bar{v}_{b_j}} 1 - F_{b_j}(x)dx < \infty$, for all $i \in S, j \in B$.

Observe that under this assumption, functions $F_{b_j}$ and $F_{s_i}$ are invertible. For every $j \in B$ and $i \in S$, let $F_{b_j}^{-1} : [0, 1] \rightarrow [0, \bar{v}_{b_j}]$ and $F_{s_i}^{-1} : [0, 1] \rightarrow [0, \bar{v}_{s_i}]$ denote the corresponding inverse functions, i.e.,

$$F_{b_j}^{-1}(F_{b_j}(x)) = x \text{ for } x \in [0, \bar{v}_{b_j}], \text{ and } F_{s_i}^{-1}(F_{s_i}(x)) = x \text{ for } x \in [0, \bar{v}_{s_i}).$$

To state our analytical results more conveniently, we extend the domains of the value distributions to $\mathbb{R}$: for every $j \in B$ we let $F_{b_j}(v) = 1$ for $v \geq \bar{v}_{b_j}$ and $F_{b_j}(v) = 0$ for $v \leq 0$, and similarly for sellers. (Note that the ranges of the inverse functions $F_{b_j}^{-1}$ and $F_{s_i}^{-1}$ are respectively restricted to $[0, \bar{v}_{b_j}]$ and $[0, \bar{v}_{s_i}]$, and hence these functions are well-defined despite the domain extension of $F_{b_j}$ and $F_{s_i}$.)

The platform can charge fees to buyers and sellers for facilitating transactions. In particular, we assume that the platform chooses commission rates (a percentage of the total transaction price) and subscription fees (lump-sum transfers to access the market, which are independent of the transaction amount). We assume that these commission rates and subscription fees are identical for all agents with the same type, but we allow them to be different across types. Formally, given a trading network $G(B \cup S, E)$ we denote by $(\gamma, \mu)$ with $\gamma, \mu \in \mathbb{R}^{\vert S \vert + \vert B \vert}$ the platform’s commission and subscription vectors, where respectively $\gamma_i^s$ ($\gamma_j^b$) represents the commission rate and $\mu_i^s$ ($\mu_j^b$) represents the subscription fee charged to type-$i$ sellers (type-$j$ buyers). We focus on nonnegative commissions/subscriptions, and denote the set of feasible commission rates by $\Gamma = \{ \gamma : \gamma_i^s \in [0, 1], \gamma_j^b \in [0, \infty), \forall i \in S, \forall j \in B \}$, and the set of feasible subscription fees by $\mathcal{U} = \{ \mu : \mu_i^s, \mu_j^b \in [0, \infty), \forall i \in S, \forall j \in B \}$.

To illustrate the effect of these subscriptions/commissions on agents’ utilities, suppose that seller $i$ offers her product/service at price $p$, and that buyer $j$ transacts with her. The buyer makes a payment of $p(1 + \gamma_j^b)$ to the platform for this transaction, and the seller receives $p(1 - \gamma_i^s)$. In addition, to transact through the platform the buyer (seller) makes a lump-sum transfer of $\mu_j^b$ ($\mu_i^s$) to the platform. If the buyer has value $v_b$ for the product and the seller has (reservation) value $v_s$, their utilities as a result of this transaction are $v_b - p(1 + \gamma_j^b) - \mu_j^b$ and $p(1 - \gamma_i^s) - \mu_i^s - v_s$, respectively.

The buyers and sellers can choose not to participate in the platform in which case their utility is normalized to zero. We say that a trader is utility-maximizing if, given commissions/subscriptions...
and prices, her trade maximizes her utility. In light of the above discussion, it can be seen that utility-maximizing buyers and sellers trade only if doing so results in nonnegative utility; and if a utility-maximizing buyer trades, she trades with a seller whose price is the lowest among all compatible sellers. We assume that all buyers and sellers are utility-maximizing.

In contrast to most of the recent literature on two-sided markets that either suppresses the role of prices or views prices as a decision of the platform, a novel feature of our model is that we allow prices to be formed endogenously. This is a key feature of many real-world platforms such as Airbnb, Upwork, eBay, etc., where the transaction prices are not dictated by the platform. To capture the endogenous nature of the prices formally, we focus on the competitive equilibria of the trading network, which we define next.

**Definition 1.** Given a commission-subscription pair \((\gamma, \mu) \in \Gamma \times \mathcal{U}\) chosen by the platform, a competitive equilibrium \((\mathbf{p}, \mathbf{x}, \mathbf{q}^s, \mathbf{q}^b)\) is a tuple that consists of a price vector \(\mathbf{p} \in \mathbb{R}^{\vert S\vert}\) (where \(p_i\) denotes the price of the products/services offered by sellers of type \(i \in S\)), supply and demand vectors \(\mathbf{q}^s \in \mathbb{R}^{\vert S\vert}, \mathbf{q}^b \in \mathbb{R}^{\vert B\vert}\) (where \(q_i^s/q_j^b\) represent the total quantities supplied/demanded by type-\(i\) sellers/type-\(j\) buyers), and a flow vector \(\mathbf{x} \in \mathbb{R}^{\vert E\vert}\) (where \(x_{ij}\) indicates the aggregate amount of products/services supplied by type-\(i\) sellers to type-\(j\) buyers), and that satisfies the following constraints:

\[
q_i^s = s_i F_{s_i} \left( (1 - \gamma_i^s) p_i - \mu_i^s \right), \quad \forall i \in S, \tag{2a}
\]

\[
q_j^b = b_j \left[ 1 - F_{b_j} \left( (1 + \gamma_j^b) \min_{\nu':(i',j') \in E} \{ p_{\nu'} \} + \mu_j^b \right) \right], \quad \forall j \in B, \tag{2b}
\]

\[
q_i^s = \sum_{j':(i,j') \in E} x_{ij'}, \quad q_j^b = \sum_{i':(i',j) \in E} x_{ij'}, \quad \forall i \in S, j \in B, \tag{2c}
\]

\[
x_{ij} \geq 0, \quad \forall (i, j) \in E; \quad x_{ij} = 0, \quad \forall i \notin \arg\min_{\nu':(i',j) \in E} \{ p_{\nu'} \}, \quad j \in B. \tag{2d}
\]

We denote by \(\mathcal{X}(\gamma, \mu)\) the set of competitive equilibria, i.e.,

\[
\mathcal{X}(\gamma, \mu) := \{ (\mathbf{p}, \mathbf{x}, \mathbf{q}^s, \mathbf{q}^b) : \text{ conditions (2a)–(2d) are satisfied} \}.
\]

Condition (2a) states that, given price \(p_i\), all type-\(i\) sellers with nonnegative utility (i.e., those with values of at most \((1 - \gamma_i^s)p_i - \mu_i^s\)) will participate and transact in the market; thus, the total mass of type-\(i\) sellers who participate and transact in the market is \(q_i^s = s_i F_{s_i} (1 - \gamma_i^s)p_i - \mu_i^s\). Similarly, all buyers of type \(j \in B\) with nonnegative surplus will participate in the market; these are buyers with values of at least \((1 + \gamma_j^b) \min_{\nu':(i',j') \in E} \{ p_{\nu'} \} + \mu_j^b\), as buyers find it optimal to transact with a compatible seller with the lowest price. Hence, consistent with Condition (2b), the total mass of type-\(j\) buyers who participate and transact in the market is given by \(q_j^b = b_j \left[ 1 - F_{b_j} (1 + \gamma_j^b) \min_{\nu':(i',j) \in E} \{ p_{\nu'} \} + \mu_j^b \right]\).

Condition (2c) is the market clearing condition: in an equilibrium, there should be a feasible allocation of goods such that each seller who is willing to sell her product is able to do so, and each buyer who demands a product is able to buy it from compatible sellers with the lowest price. Finally,
Condition (2d) ensures that the allocation of goods is consistent with buyers’ preferences, i.e., buyers transact only with sellers who offer the lowest price.

Observe that here we abstract away information frictions and implicitly assume that buyers and sellers have full information about the prices available on the platform. In addition, we restrict attention to all sellers of the same type offering the same price, but this restriction is without loss of generality in equilibrium. (If we allowed sellers of the same type to have different prices in an equilibrium, all compatible buyers would demand goods from sellers who offer the lowest price. Thus, the sellers with higher prices would not find a buyer, and the market would not clear.)

Next, we establish that, for any given commissions/subscriptions, an equilibrium exists and is essentially unique. That is, the mass of buyers (sellers) of each type transacting in any equilibrium is identical, and the equilibrium price of any seller type that is involved in some transactions is also identical, although there might exist several feasible flows that lead to the same market outcome. We leverage this uniqueness result in Section 3 in our study of the revenue maximization problem of the platform.

**Proposition 1.** For any \((\gamma, \mu) \in \Gamma \times \mathcal{U}\), there exists a vector \((p, x, q^s, q^b)\) such that \((p, x, q^s, q^b) \in \mathcal{X}(\gamma, \mu)\). Furthermore, given \((\gamma, \mu)\), all competitive equilibria share the same supply-demand vector \((q^s, q^b)\). In addition, each seller type that transacts nonzero quantities in these equilibria always offers the same prices; i.e., the vector \((p_i)_{i: q^s_i>0}\) is the same in all equilibria.

In Appendix A we show that for given commissions/subscriptions, an equilibrium can be constructed by solving a convex optimization problem. This result is analogous to the classic competitive equilibrium models where the allocation can be solved via a convex optimization problem and the price vector is determined by the corresponding dual variables; see, e.g., Shapley and Shubik (1971).

How should the platform choose commissions/subscriptions to maximize its revenues? We proceed by formulating the platform’s revenue maximization problem:

\[
V_{opt} = \max_{(\gamma, \mu, p, x, q^s, q^b)} \sum_{i,j:(i,j) \in E} (\gamma_i^p + \gamma_j^b)p_i x_{ij} + \sum_{i,j:(i,j) \in E} (\mu_i^s + \mu_j^b)x_{ij} \quad (3a)
\]

\[
s.t. \quad (p, x, q^s, q^b) \in \mathcal{X}(\gamma, \mu), \quad (3b)
\]

\[
(\gamma, \mu) \in \Gamma \times \mathcal{U}. \quad (3c)
\]

Consider a feasible solution \((\gamma, \mu, p, x, q^s, q^b)\) to this problem. Constraint (3c) ensures that \((\gamma, \mu)\) correspond to feasible commission-subscription vectors. Constraint (3b) implies that the tuple \((p, x, q^s, q^b)\) is an equilibrium under the vector of commissions/subscriptions \((\gamma, \mu)\). In the objective, the first term corresponds to the revenue obtained through the commissions in the aforementioned equilibrium, whereas the second term corresponds to the revenue due to subscription fees. In this
problem, the objective is not concave in the decision variables, and the set of feasible solutions is nonconvex (see Appendix EC.4.2 of the e-companion).

Note that, in principle for given \((\gamma, \mu)\), the equilibria in \(X(\gamma, \mu)\) could lead to different revenues for the platform. We conclude this section by an immediate corollary of Proposition 1 which establishes that this is never the case.

**Corollary 1.** For given \((\gamma, \mu) \in \Gamma \times \mathcal{U}\), all \((p, x, q^s, q^b) \in X(\gamma, \mu)\) yield the same revenue.

Hereafter, we denote by \(V(\gamma, \mu)\) the platform's revenues under commission-subscription pair \((\gamma, \mu)\).

### 3. Revenue Maximization via Commissions and Subscriptions

The nonconvexity of the platform's revenue maximization problem poses a potential challenge for finding the optimal commissions/subscriptions. In this section, we first establish that despite this nonconvexity, under mild assumptions, the optimal commissions/subscriptions can still be obtained in a tractable way by solving a convex relaxation of the platform’s revenue maximization problem (Section 3.1). We then explore the impact of the network structure on the surplus of different buyer/seller types and on the platform’s revenues (Section 3.2).

#### 3.1. Finding the Optimal Commissions and Subscriptions

Given an equilibrium \((p, x, q^s, q^b) \in X(\gamma, \mu)\), we refer to the buyer (seller) of type \(j \in B\) (\(i \in S\)) with the lowest (highest) valuation who trades as the marginal buyer (seller) of this type. It can be seen that the valuation of the type-\(j\) marginal buyer is given by

\[
v_{b_j}^m := F_{b_j}^{-1}(1 - q_{b_j}/b_j),
\]

and the valuation of the type-\(i\) marginal seller by

\[
v_{s_i}^m := F_{s_i}^{-1}(q_{s_i}/s_i).
\]

In equilibrium, marginal agents must have nonnegative surplus; otherwise they would prefer not to trade, thus violating the equilibrium conditions. This observation implies that the net transfer from the marginal type-\(j\) buyer to the rest of the market (that is, to the sellers and the platform) must be at most equal to her value \(v_{b_j}^m\). Similarly, the net transfer from the rest of the market to the marginal type-\(i\) seller must be at least \(v_{s_i}^m\). Since all trading agents of the same type make the same

\(^2\) In fact, it can be seen that, in general, the marginal seller (similarly buyer) has zero surplus, and the net transfer from the rest of the market to her is exactly \(v_{b_j}^m\). The only exception is when the value distributions are bounded, the corresponding equilibrium price \(p_i\) is higher than this bound, and all sellers of type \(i\) trade in the equilibrium \((p, x, q^s, q^b) \in X(\gamma, \mu)\). However, as we subsequently establish in Theorem 1, this will never occur under a revenue-maximizing vector of commissions/subscriptions.
payment, it follows that, in equilibrium, all trading type-j buyers pay at most $v_{b_j}$, and all trading type-i sellers receive at least $v_{s_i}$. The platform’s revenues can be expressed as the total payments by the buyers minus the total payments to the sellers. Hence, as per the previous discussion, in equilibrium $(p, x, q^s, q^b) \in X(\gamma, \mu)$ this quantity can be upper bounded by

$$h(q^s, q^b) := \sum_{j \in \mathcal{B}} F_{b_j}^{-1} \left( 1 - \frac{q^b_j}{b_j} \right) q^b_j - \sum_{i \in \mathcal{S}} F_{s_i}^{-1} \left( \frac{q^s_i}{s_i} \right) q^s_i. \quad (6)$$

Since for any given commissions/subscriptions $(\gamma, \mu)$ all corresponding equilibria induce the same revenue $V(\gamma, \mu)$ (Corollary 1), we conclude that $V(\gamma, \mu) \leq h(q^s, q^b)$ for any $(p, x, q^s, q^b) \in X(\gamma, \mu)$.

We provide our characterization of the optimal commissions/subscriptions under the following additional assumption on the value distributions, which we impose for the remainder of the paper.

**Assumption 2.** The functions $uF_{b_j}^{-1}(1 - u)$ and $-uF_{s_i}^{-1}(u)$ are strictly concave in $u \in [0, 1]$ for all $i \in \mathcal{S}$ and for all $j \in \mathcal{B}$.

Focusing momentarily on buyers, it can be seen that the concavity of $uF_{b_j}^{-1}(1 - u)$ is equivalent to the regularity of buyers’ value distributions. Regularity is a standard assumption in mechanism design that renders optimal mechanism design problems tractable. Since in our setting the market has two sides, in addition to the regularity of buyers’ valuations, we require sellers’ (reservation) valuations to satisfy a closely related condition. Assumption 2 holds for a broad class of distributions, including uniform, exponential, generalized Pareto, Weibull, and (truncated) normal distributions.

Note that, under Assumption 2, $h(q^s, q^b)$ is a concave function of its argument. Exploiting this concavity, we next provide a convex relaxation of the revenue maximization problem in (3):

$$\tilde{V}_{opt} = \max_{x, q^s, q^b} \sum_{j \in \mathcal{B}} F_{b_j}^{-1} \left( 1 - \frac{q^b_j}{b_j} \right) q^b_j - \sum_{i \in \mathcal{S}} F_{s_i}^{-1} \left( \frac{q^s_i}{s_i} \right) q^s_i \quad (7a)$$

$$\text{s.t.} \quad q^s_i - \sum_{j : (i, j) \in E} x_{ij} = 0, \quad \forall i \in \mathcal{S}, \quad (7b)$$

$$\sum_{i : (i, j) \in E} x_{ij} - q^b_j = 0, \quad \forall j \in \mathcal{B}, \quad (7c)$$

$$q^s_i \leq s_i, \quad \forall i \in \mathcal{S}, \quad (7d)$$

$$q^b_j \leq b_j, \quad \forall j \in \mathcal{B}, \quad (7e)$$

$$x_{ij} \geq 0, \quad \forall (i, j) \in E. \quad (7f)$$

This relaxation is obtained by first replacing the objective of (3) with the upper bound $h(q^s, q^b)$ on the revenues, and then relaxing constraint (3c) $(\gamma, \mu) \in \Gamma \times U$—as well as some of the equilibrium constraints in (3b). In particular, this formulation imposes equilibrium flow constraint (2c) (plus nonnegativity), ensures that equilibrium supply (demand) of each type is less than the mass of sellers (buyers) of the relevant type (as implied by (2a) and (2b)), and relaxes the remaining equilibrium...
The optimal dual variables corresponding to constraints that implement a relaxation of problem \( (\gamma, \mu) \) is a tractable convex optimization problem that can be interpreted as a min-cost network flow problem with convex edge costs captured in terms of

\[
-F_{b_j}^{-1} \left( 1 - q^b_j / b_j \right) q^b_j \quad \text{and} \quad F_{s_i}^{-1} (q^s_i / s_i) q^s_i.
\]

Note that, after replacing the objective and relaxing the constraints, the resulting problem maximizes revenues by searching over supply-demand vectors \((q^*, q^b)\) and a flow vector \((x)\), and neither the prices \(p\) nor the commissions/subscriptions \((\gamma, \mu)\) appear in the new formulation. That is, the formulation does not require a feasible solution \((q^*, q^b, x)\) to be supported in an equilibrium by some commissions/subscriptions and prices.

Optimization problem (3) is a tractable convex optimization problem that maximizes revenues by searching over supply-demand vectors \((x)\) and a price-flow vector \((p, x)\) such that \((p, x, q^*, q^b) \in X(\gamma, \mu)\).

While a priori it is not clear whether the allocation \((q^*, q^b)\) is implementable, Theorem 1 establishes that this is indeed tight and its solution readily yields the mass of buyers/sellers of each type that participate in the market under the optimal solution to (3).

As the objective function is strictly concave in \((q^*, q^b)\), all optimal solutions to (7) must share the same supply-demand vector \((q^*, q^b)\). Moreover, this strict concavity also implies that if there exists a feasible solution to (3) that achieves the same revenue as the optimal solution to (7), then the supply-demand vector in the solution to (3) must also be \((q^*, q^b)\). Thus, the relaxation of the platform’s revenue maximization problem in (7) is tight if and only if the allocation \((q^*, q^b)\) is implementable, i.e., there exist commissions/subscriptions \((\gamma, \mu)\) and a price-flow vector \((p, x)\) such that \((p, x, q^*, q^b) \in X(\gamma, \mu)\).

Theorem 1. Let \((x, q^*, q^b)\) denote an optimal solution to (7) and let \(\{\theta^s_i\}_i\) and \(\{\theta^b_j\}_j\) denote optimal dual variables corresponding to constraints (7b) and (7c). Then:

(i) Optimization problem (7) is a tight relaxation of problem (3), i.e., \(V_\text{opt} = \bar{V}_\text{opt}\). Moreover, any optimal solution to (3) must have supply-demand vectors equal to \((q^*, q^b)\).

(ii) The optimal commission-subscription vector that implements \((q^*, q^b)\) is not unique. In particular, one can construct commissions/subscriptions that maximize the platform’s revenues by using the optimal primal and dual variables of (4) as follows:

(a) Set \(\mu = 0\), and employ the following commission rates \((\gamma^s, \gamma^b) \in \Gamma:\)

\[
\gamma^s_i = 1 - \frac{1}{\theta^s_i} F_{s_i}^{-1} \left( \frac{\tilde{q}^s_i}{s_i} \right) \quad \text{for all } i \in S, \quad \gamma^b_j = \frac{1}{\theta^b_j} F_{b_j}^{-1} \left( 1 - \frac{\tilde{q}^b_j}{b_j} \right) - 1 \quad \text{for all } j \in B.
\]

(b) Set \(\gamma = 0\), and employ the following subscription fees \((\mu^s, \mu^b) \in \mathcal{U}:\)

\[
\mu^s_i = \theta^s_i - F_{s_i}^{-1} \left( \frac{\tilde{q}^s_i}{s_i} \right) \quad \text{for all } i \in S, \quad \mu^b_j = F_{b_j}^{-1} \left( 1 - \frac{\tilde{q}^b_j}{b_j} \right) - \theta^b_j \quad \text{for all } j \in B.
\]
Theorem 1 establishes that the revenue-maximizing outcome can be implemented with either commissions or subscriptions, and that the platform does not need to consider more complicated schemes that involve both. On the other hand, the optimal payment structure can be type-dependent; i.e., different agent types, even on the same side, are exposed to different commissions/subscriptions. Note that it is not clear what type of revenue loss should be expected if attention is restricted to offering identical commissions/subscriptions to all participants on one side of the market. We revisit this point in Section 4. Finally, the first part of the theorem shows that the upper bound on the revenue provided in (6) gives precisely the revenues of the platform for \((q^s, q^b) = (\bar{q}^s, \bar{q}^b)\). This in turn implies that under the optimal commissions/subscriptions all trading type-j buyers (type-i sellers) pay (receive) exactly \(v^m_{bj} (v^m_{si})\), and the corresponding marginal agents have zero surplus.

Figure 1 illustrates the optimal commissions/subscriptions characterized in Theorem 1 for a simple network. In this example, the platform finds it optimal to target different types with different commissions/subscriptions. It can also be seen that depending on their network position some types may be involved in more trades than others, and not all compatible types of buyers/sellers trade.

![Figure 1](image-url)

**Figure 1** Consider a network with 4 buyer and 4 seller types, with value distributions \(F_{si}(v) = F_{bj}(v) = 1 - \exp(-v^s)\), where \(i, j = 1, 2, 3, 4\) for all \(v \in [0, \infty)\). Denote the population vectors by \(s = (1, 1, 1, 1)\) and \(b = (2, 2, 2, 2)\). The revenue-maximizing commissions (or subscriptions) given in Theorem 1 as well as the corresponding mass of buyers/sellers of each type that trade are reported in the figure. The equilibrium prices for the sellers are \((p_1, p_2, p_3, p_4) = (0.8, 0.8, 0.99, 1.19)\). The flow associated with the highlighted edges is given by \(x_{11} = x_{21} = 0.23\), \(x_{32} = 0.32\), and \(x_{43} = x_{44} = 0.20\). All other edges have zero flow (trade) under the optimal solution.

**Remark:** The idea of finding the optimal transfers to the platform by formulating the platform’s problem over allocations (i.e., the mass of different types of agents who participate in the market) appeared previously in [Weyl (2010)](Weyl2010). That paper, unlike ours, does not shed light on when these transfers can be tractably computed. Moreover, in our setting, due to market clearing conditions the set of implementable allocations are further restricted through “flow constraints” (7b) and (7c), whereas no analogous restriction is present in [Weyl (2010)](Weyl2010). Finally, a key assumption in [Weyl (2010)](Weyl2010) is that the platform can choose a positive or negative transfer to any type of agent. This flexibility...
guarantees that, for any given allocation, there will exist transfers that implement this allocation, i.e., transfers such that the given allocation will indeed be the equilibrium allocation under those transfers. However, once the allowable commissions/subscriptions (and hence the transfers that can be charged by the platform) are restricted (e.g., to nonnegative values), it is no longer true that any allocation can be implemented, i.e., that given any allocation \((q^b, q^s)\) one can find commissions/subscriptions \((\gamma, \mu)\) such that \((q^b, q^s)\) are the equilibrium demand/supply under \((\gamma, \mu)\). (In fact, it can be shown that the set of implementable equilibrium supply-demand vectors constitutes a nonconvex subset of \(\mathbb{R}^{|B|+|S|}\)). Theorem 1 implies that for one to find optimal commissions/subscriptions, the restriction to nonnegative transfers is without loss of optimality. This is because \([7]\) remains a relaxation of \([3]\) even if nonnegativity of transfers is not imposed in the latter problem. This observation also implies that, unlike in the literature on two-sided markets (see, e.g., [Rochet and Tirole (2006)], in our setting the platform does not benefit from setting transfers below costs (which are assumed to be zero) on one side of the market.

3.2. Impact of the Network Structure

This section is devoted to understanding the impact of the compatibility network structure on the traders’ surplus (Section 3.2.1) and on the platform’s revenues (Section 3.2.2), when the platform employs optimal commissions/subscriptions. In order to isolate the effect of the network structure, throughout this section we conduct our analysis under the assumption that the value distributions are homogeneous on each side, i.e., \(F_s(v) = F_s(v)\) and \(F_b(j) = F_b(v)\) for all \(v \in \mathbb{R}, j \in B, i \in S\), but we still allow for populations of different sizes.

3.2.1. Agents’ Surplus and Network Positions. We first focus on the impact of the compatibility network structure on the average surplus of the different buyer/seller types under revenue-optimal commissions and subscriptions. We start by characterizing the average equilibrium surplus of a trader type in terms of the valuation of its marginal agents. Recall that, under a vector of optimal commissions and subscriptions, marginal buyers (sellers) have zero surplus in equilibrium, and all trading agents of the same type make the same payments to the platform as they purchase/sell the good at the same price and face the same commissions and subscriptions. In light of this, the equilibrium surplus of a buyer of type \(j \in B\) (seller of type \(i \in S\)) with valuation \(v\) is given by \(\max\{v - v^m_{bj}, 0\}\), where \(v^m_{bj}\) (\(v^m_{si}\)) denotes the valuation of the marginal buyer of type \(j\) (seller of type \(i\)). Consequently, in equilibrium, the aggregate surpluses of buyers of type \(j \in B\) can be given by \(b_j \int_{v^m_{bj}}^{v} (v - v^m_{bj})dF_b(j)(v)\), and those of sellers of type \(i \in S\) can be given by \(s_i \int_{0}^{v^m_{si}} (v^m_{si} - v)dF_s(i)(v)\). Finally, the average surplus of a type can be calculated by dividing its aggregate surplus by its population (e.g., \(\frac{\int_{v^m_{bj}}^{v} (v - v^m_{bj})dF_b(j)(v)}{|B|}\)).
for \( j \in \mathcal{B} \). Notice that, by Theorem 1, all optimal commission/subscription schemes lead to the same \((q^b, q^s)\), which in turn implies that all optimal schemes induce the same marginal agents and thus the same surplus for any given agent.

It can be readily seen from the above discussion that if the valuation of the marginal buyer (seller) of a given type is large, then the corresponding average surplus is small (large). Therefore, in the rest of the subsection we focus on understanding the impact of the compatibility network structure on the valuations of the marginal agents of different types. It is worth noting that, while it is possible to efficiently compute the valuation of the marginal agents for a given problem instance by solving problem (7), a priori the impact of the network structure on values of marginal agents of different types is not clear. Our next result provides an answer to this question.

**Theorem 2.** Given a network \( G(\mathcal{B} \cup \mathcal{S}, E) \) and a population profile \((s, b)\), define \( S^{(0)} = \mathcal{S}, B^{(0)} = \mathcal{B}, \) and \( E^{(0)} = E \). For \( t = 1, 2, \ldots, \) let \( B_t \) and \( S_t \) be iteratively defined as follows:

\[
B_t = \arg \min_{B \subseteq \mathcal{B}^{(t-1)}} \left( \sum_{i \in N_{E^{(t-1)}(B)}} s_i \right) / \left( \sum_{j \in B} b_j \right) \quad \text{and} \quad S_t = N_{E^{(t-1)}}(B_t),
\]

where \( \mathcal{S}^{(t)} = \mathcal{S}^{(t-1)} \setminus S_t, \mathcal{B}^{(t)} = \mathcal{B}^{(t-1)} \setminus B_t, \) and \( E^{(t)} = \{(i, j) \in E : i \in \mathcal{S}^{(t)} \text{ and } j \in \mathcal{B}^{(t)}\} \). If \( j_1 \in B_{t_1}, j_2 \in B_{t_2} \) (similarly \( i_1 \in S_{t_1}, i_2 \in S_{t_2} \)) with \( t_1 \leq t_2 \), then the marginal agent of type \( j_1 \) (\( i_1 \)) has weakly higher value than that of type \( j_2 \) (\( i_2 \)) under optimal commissions/subscriptions.

The key quantity for ranking the valuations of the marginal agents of different types is the scarcity of sellers experienced by different types of buyers. In particular, to obtain our ranking, we first identify an induced subgraph that consists of a set of buyers and their neighboring sellers such that the ratio of the sellers’ population to the buyers’ population is the smallest. Intuitively, the buyers in this subgraph are those that are the most supply constrained. Under optimal commissions/subscriptions, the valuations of the marginal buyers in this subgraph are the largest. We then remove these buyers/sellers from the network, and repeat this procedure for the induced subnetwork in order to identify a subset of types whose marginal agents have the second largest valuations and so on. We illustrate Theorem 2 in the following simple example.

**Example 1.** Consider the network in Figure 1 and recall that all seller types have a population of size one and all buyer types have a population of size two. The subset of buyers \( B \) for which

\(^3\) In fact, in the appendix we prove a stronger version of this result, where when \( t_1 < t_2 \) the value of the marginal agent of type \( j_1 \) is strictly higher than that of type \( j_2 \). A similar result holds on the seller side: there exists some \( t \) such that all seller types in \( S_t \) with \( t \leq \bar{t} \) have marginal values that equal to \( \bar{v}_{i_1} \), i.e., the upper bound of the value distribution of sellers. For \( t_1, t_2 > \bar{t} \) marginal agents of different seller types in \( S_{t_1} \) and \( S_{t_2} \) admit a strict ranking. Here \( \bar{t} \) is characterized in terms of the underlying value distributions. Moreover, except for cases where all sellers of a certain type trade under the optimal commissions/subscriptions (which necessitates bounded seller value distributions), we have \( \bar{t} = 0 \), and a strict ranking of all seller types can be obtained.
the quantity \( \sum_{i \in N_E(B)} s_i \) is the smallest is given by \( B_1 = \{3, 4\} \). Figure 1 shows that, under optimal commissions/subscriptions, we have \( q_j^b = \hat{q}_j^b < q_j^p \) for \( j \notin B_1 \). Given that all buyer types have the same populations, this in turn implies that the marginal agents of buyer types in \( B_1 \) have the highest valuations. This observation is consistent with Theorem 2. Similarly, \( S_t = N(B_1) = \{4\} \), and \( q_4^p \) is the highest among the seller types; thus the valuation of the type-4 marginal seller is the highest.

Even though the sets \( B_t \) admit a combinatorial characterization in Theorem 2, they can be computed efficiently by solving (7) and obtaining the vector \((q^*, q^p)\) of sellers/buyers who transact under the optimal commissions/subscriptions. As all seller (buyer) types have the same value distribution, the seller (buyer) types that have the highest (lowest) trade amount per unit of population \( q_j^p / s_i \) (\( q_j^b / b_i \)) have marginal agents with the highest valuations. These sets of agents correspond to \( S_t \) and \( B_t \), respectively. Similarly, the sets of agents \( S_t, B_t \) for all \( t \geq 1 \) can be derived from the optimal solution to (7).

We conclude this section by noting that the ranking of network components obtained in Theorem 2 is closely related to the concept of lexicographically optimal bases of polymatroids. The set given by \( P' = \{ y \in \mathbb{R}^{||S||} : \sum_{i \in S'} y_i \leq \rho(S'), \forall S' \subset S \} \), for a nondecreasing submodular (set) function \( \rho : 2^S \rightarrow \mathbb{R} \) defined on subsets of some finite set \( S \), is a polymatroid. We refer to \( y \in P' \) as a base of \( P' \), if \( \sum_{i \in S} y_i = \rho(S) \). For \( y \in \mathbb{R}^{||S||} \), let \( T(y) \) denote the vector obtained after the entries of \( y \) are arranged in increasing order. Given positive weights \( w = \{w_i\}_{i=1}^{||S||} \), a base \( y \) of \( P' \) is called lexicographically optimal with respect to \( w \), if \( T(\{y_i/w_i\}) \) is lexicographically greater than \( T(\{\hat{y}_i/w_i\}) \) for any base \( \hat{y} \) of \( P' \) (see Fujishige (1980)).

To see how this relates to our problem, consider the set \( P = \{ q^b : (x, q^*, q^b) \text{ satisfies (7b)-(7d) and (7f)} \} \). Intuitively, this set captures the feasible assignments of sellers to buyers, i.e., for \( q^b \in P \), the quantity \( q_j^b \) can be interpreted as the mass of sellers “assigned” to type-\( j \) buyers. Using Lemma 4.1 in Megiddo (1974), it can be shown that this set can equivalently be represented as follows:

\[
    P = \left\{ y : \sum_{j \in B} y_j \leq \sum_{i \in N_E(B)} s_i, \forall B \subset B, \; y_j \geq 0, \forall j \in B \right\},
\]

which immediately implies that \( P \) is a polymatroid. Define a vector \( \tilde{y} \) such that \( \tilde{y}_j = \frac{b_j}{\sum_{j' \in B_t} b_{j'}} \sum_{i \in S_t} s_i' \) for \( j \in B_t \), where \( B_t \) and \( S_t \) are defined as in Theorem 2. It can be shown that \( \tilde{y} \) is the lexicographically optimal base of \( P \) with respect to weight vector \( b \). In other words, the characterization in Theorem 2 reveals the lexicographically optimal base of a polymatroid \( P \) associated with

\(^4\) \( \rho \) is a nondecreasing submodular function if \( \rho(A) \leq \rho(B) \) for \( A \subset B \subset S \), and \( \rho(A) + \rho(B) \geq \rho(A \cup B) + \rho(A \cap B) \) for any \( A, B \subset S \).
the assignments of sellers to buyers. In fact, we exploit this connection in the proof of Theorem 2 by first reformulating (7) as an optimization problem over a polymatroid, and then leveraging the structural properties of optimal solutions of such problems to obtain our ranking result.

3.2.2. Network Structure and Optimal Revenue. We next discuss the impact of the network structure on the platform’s revenues. To obtain our results, we make use of the following variant of Hall’s marriage condition:

**Definition 2.** We say that a network \( G(B \cup S, E) \) with seller/buyer populations \((s, b)\) satisfies the weighted Hall’s marriage condition if

\[
\sum_{i \in N_E(B)} s_i \geq \frac{\sum_{i \in S} s_i}{\sum_{j \in B} b_j} \sum_{j \in B} b_j \quad \text{for all } B \subset B.
\]

Similarly, we say that the network satisfies the \( \varepsilon \)-marriage condition if for some \( \varepsilon \in [0, 1] \) we have

\[
\sum_{i \in N_E(B)} s_i \geq (1 - \varepsilon) \frac{\sum_{i \in S} s_i}{\sum_{j \in B} b_j} \sum_{j \in B} b_j \quad \text{for all } B \subset B.
\]

Hall’s marriage condition requires that \(|N_E(B)| \geq |B|\) for all \( B \subset B \). A bipartite network (which has two sides with equal cardinality) admits a perfect matching if and only if it satisfies Hall’s marriage condition (see, e.g., Schrijver (2003)). Intuitively, this condition implies that for any subset of nodes on one side, we have a sufficient number of nodes on the other side to “cover” them.

It can be seen that (9) is a weighted version of this condition. In particular, we assign a weight of \( \frac{b_j}{\sum_{l \in B} b_l} \) to each \( j \in B \) and of \( \frac{s_i}{\sum_{l \in S} s_l} \) to each \( i \in S \), and require the total weight for buyers in \( B \) to be smaller than the total weight of corresponding sellers \( N_E(B) \) on the other side of the market.

In our setting, this condition ensures that for any set of buyers \( B \), the ratio of total allowable supply to total demand of \( B \) (i.e., \( \sum_{i \in N_E(B)} s_i / \sum_{j \in B} b_j \)) is not smaller than the ratio of total supply to demand in the market (i.e., \( \sum_{i \in S} s_i / \sum_{j \in B} b_j \)). In other words, the supply in the market is “balanced,” and no set of buyers in the market is excessively deprived of it.

Given buyer/seller types \( B \) and \( S \), and population profile \((s, b)\), we denote by \( V_{\text{opt}}(E, s, b) \) the optimal revenue that can be obtained for the network \( G(B \cup S, E) \), i.e., the optimal objective of (3) for this network. Furthermore, we denote by \( V_{\text{max}}(s, b) \) the maximum revenue that can be obtained by any edge set \( E \), i.e.,

\[
V_{\text{max}}(s, b) := \max_{E \subseteq B \times S} V_{\text{opt}}(E, s, b).
\]

Using this notation, we provide a tight upper bound on the platform’s revenues that holds independently of the network structure, and a lower bound on the revenues of a given network structure.

**Theorem 3.** Fix the set of sellers \( S \), the set of buyers \( B \), and the population profile \((s, b)\). Let \( s_0 = \sum_{i \in S} s_i \) and \( b_0 = \sum_{j \in B} b_j \) denote the total populations of sellers and buyers, respectively. Then:
(i) \( V_{\max}(s,b) = b_0 \max_{r \leq \min\{1, \frac{s_0}{b_0}\}} \left[ F_b^{-1}(1-r) - F_s^{-1}\left(\frac{r}{s_0/b_0}\right)\right] r. \) Moreover, \( V_{\opt}(E,s,b) = V_{\max} \) if and only if \( G(\mathcal{B} \cup \mathcal{S},E) \) satisfies the weighted Hall’s marriage condition.

(ii) If \( G = (\mathcal{B} \cup \mathcal{S},E) \) satisfies the \( \varepsilon \)-marriage condition, then \( V_{\opt}(E,s,b) \geq (1-\varepsilon)V_{\max}(s,b) \).

Note that this result implies that all networks with the same total population \((s_0,b_0)\) of sellers and buyers (and even with a possibly different number of seller/buyer types), admit the same upper bound on the revenues (given by \( V_{\max}(s,b) \)). Moreover, this upper bound is achieved (with appropriate commissions/subscriptions) when the weighted Hall’s marriage condition holds. Intuitively, if the weighted Hall’s marriage condition holds, then every subset of buyer types is exposed to a sufficient mass of sellers, and in this case the revenues the platform can achieve (under optimal commissions/subscriptions) is larger than the revenues achievable under any network structure. When this condition fails to hold, quantifying the extent of its “failure” in terms of the \( \varepsilon \)-marriage condition, yields a lower bound on the optimal revenue. Qualitatively, this result suggests that lower revenues can be expected in networks where supply is distributed less evenly, e.g., when some buyers have access to a limited supply while others have access to abundant supply.

We next focus on a different question of practical relevance: if the platform could expand its market and increase the mass of traders of some type by a small amount (e.g., through ads), which types would improve the revenues most significantly and hence should be targeted? Interestingly, the ranking obtained in Theorem 2 provides an answer to this question.

Formally, let \( e_i \in \mathbb{R}^{|\mathcal{S}|} \) denote the \( i \)-th unit vector, and define \( \frac{\partial}{\partial s_i} V_{\opt}(E,s,b) := \lim_{\delta \to 0^+} \frac{V_{\opt}(E,s+\delta e_i,b)}{\delta} \) as the directional derivative of the platform’s revenues with respect to the size of the population of type-\( i \) sellers. Qualitatively, this quantity captures how sensitive the optimal revenue of the platform is to the size of the population of type-\( i \) sellers. Consider the set \( D_s := \{ \frac{\partial}{\partial s_i} V_{\opt}(E,s,b) \}_{i=1} \). For \( t = 1,2,\ldots \), we refer to the set of seller types \( i \) for which \( \frac{\partial}{\partial s_i} V_{\opt}(E,s,b) \) achieves the \( t \)-th largest value in \( D_s \) as the \( t \)-th-most profitable seller types to expand. We also similarly define the \( t \)-th-least profitable buyer types to expand.

**Corollary 2.** Fix population profile \((s,b)\) and network \( G(\mathcal{B} \cup \mathcal{S},E) \). Suppose that \( V_{\opt}(E,y_1,y_2) \) is differentiable\(^5\) with respect to \( y_1,y_2 \) at \((y_1,y_2)=(s,b)\). Let \( \mathcal{S}_t \) and \( \mathcal{B}_t \) be defined as in Theorem 3. Then, \( \mathcal{S}_t \) corresponds to the set of the \( t \)-th-most profitable seller types to expand, and \( \mathcal{B}_t \) corresponds to the set of the \( t \)-th-least profitable buyer types to expand.

Intuitively, if a buyer type is among the most supply constrained, then a small increase in the population of the compatible seller types could lead to the most significant improvement in the revenues

\(^5\) Notice that we assumed differentiability of \( V_{\opt}(E,y_1,y_2) \) at \((y_1,y_2)=(s,b)\). This condition generically holds, and allows us to obtain an elementary proof of the claim by using the envelope theorem. Similar results can be obtained after relaxing the differentiability condition by using the sensitivity analysis ideas from convex optimization; see Shapiro (1995).
of the platform. Note that the aforementioned set of sellers is given precisely by \( S_1 \). Corollary 2 supports this intuition, and shows that an (infinitesimal) increase in the population of sellers in \( S_1 \) leads to the highest improvement in the revenues of the platform. The result further allows for ranking different seller types as well as buyer types in terms of how sensitive the revenues of the platform are to their population sizes.

Consider a network that satisfies the weighted Hall’s marriage condition. It readily follows from (9) that for this network \( B_1 = B \) and \( S_1 = S \). Thus, Corollary 2 implies that for such networks, the impact of expanding any seller (similarly buyer) type (by an infinitesimal amount) on the revenues of the platform is identical. Similarly, Theorem 2 implies that all seller types have the same average surplus (similarly for the buyers). Recall that the weighted marriage condition captures when the supply in the market is balanced. Thus, these observations imply that, when the supply in the market is balanced, expanding any seller (buyer) type has the same impact on the platform’s revenues, and all seller (buyer) types have the same average surplus (under the optimal commissions/subscriptions). However, when the supply is not distributed in a balanced manner, different types end up with a different average surplus and the platform’s revenues may be less/more sensitive to sizes of certain populations (as shown in Theorem 2 and Corollary 2).

We conclude this section by providing a preorder to compare compatibility networks in terms of the revenues that can be extracted from them.

**Proposition 2.** Suppose that we have two networks \( G(\mathcal{B} \cup \mathcal{S}, E_1) \) and \( G(\mathcal{B} \cup \mathcal{S}, E_2) \) that differ only in their edge set, and let \((s, b)\) be the seller/buyer populations. Suppose that the networks satisfy

\[
\sum_{i \in N_{E_1}(B)} s_i \geq \sum_{i \in N_{E_2}(B)} s_i \quad \text{for all } B \subset \mathcal{B}.
\]

Then, \( V_{\text{opt}}(E_1, s, b) \geq V_{\text{opt}}(E_2, s, b) \).

Proposition 2 suggests a natural preorder on the network structures with the same seller/buyer populations, i.e., \( G(\mathcal{B} \cup \mathcal{S}, E_1) \succeq G(\mathcal{B} \cup \mathcal{S}, E_2) \) if \( \sum_{i \in N_{E_1}(B)} s_i \geq \sum_{i \in N_{E_2}(B)} s_i \) for all \( B \subset \mathcal{B} \). Thus, our result states that the revenues in a larger network (in terms of this preorder) are larger. Intuitively, this result implies that if every subset of buyer types has access to more sellers in one of the networks, then the platform is able to extract more revenue from that network. This result is consistent with Theorem 3 since the network that is larger in terms of the partial order would satisfy the \( \varepsilon \)-marriage condition with a smaller \( \varepsilon \). We illustrate this result in Figure 2.

### 4. Suboptimality of Simpler Commission/Subscription Schemes

The optimal commission/subscription scheme derived in Theorem 1 typically requires charging payments to both sides of the market and treating buyer/seller types differently in terms of their commissions/subscriptions. Consistent with these results, some platforms charge different commissions/subscriptions to different types of sellers. For instance, in Amazon and Alibaba (where third-party sellers also offer their products), commission rates depend on the types of products offered.
Figure 2 Denote the network on the left (right) by \( G_L \) (\( G_R \)) and its edge set by \( E_L \) (\( E_R \)). In both \( G_L \) and \( G_R \), every node has a unit population, and all types have uniform valuations. It can be checked that \( G_L \preceq G_R \). Proposition 2 implies that the platform can extract more revenue from the network on the left. In fact, despite the number of edges being the same, \( V_{\text{opt}}(E_L, s, b) = \frac{3}{8} \geq \frac{1}{3} = V_{\text{opt}}(E_R, s, b) \).

However, simpler commission/subscription schemes are also observed in practice. Until 2016, Upwork charged a 10% flat commission rate to freelancers and nothing to clients. Airbnb implements a slightly more complicated scheme by charging a flat commission rate of 3% to hosts and heterogeneous commissions rates of 0%-20% to guests.

Motivated by these examples, we explore how simpler and practically appealing schemes perform in terms of revenues relative to the optimal commissions/subscriptions. To that end, we first study a setting where the platform is restricted to charging the same commissions/subscriptions to all the agents on the same side of the market (Section 4.1). Then, we focus on settings where the platform can target different types with different commissions, but is restricted to charging payments only to one side of the market (Section 4.2). Finally, we illustrate our findings by focusing on a dataset that is representative of Airbnb bookings and explore how using different commission/subscription schemes impacts the revenues of the platform (Section 4.3).

4.1. Homogeneous Commissions and Subscriptions

We first consider the case where the platform charges the same commissions/subscriptions to all the agents on the same side of the market. Formally, we restrict attention to homogeneous subscription fees \( \mu = (\mu^s 1, \mu^b 1) \) and homogeneous commission rates \( \gamma = (\gamma^s 1, \gamma^b 1) \), and define the problem of maximizing revenues using homogeneous commissions/subscriptions as:

\[
V_h = \max_{(\gamma^s 1, \gamma^b 1) \in \Gamma, (\mu^s 1, \mu^b 1) \in \mathcal{U}} V(\gamma^s 1, \gamma^b 1, \mu^s 1, \mu^b 1),
\]

where, with some slight abuse of notation, \( V(\gamma^s 1, \gamma^b 1, \mu^s 1, \mu^b 1) = V(\gamma, \mu) \) denotes the platform’s revenues under the aforementioned commissions/subscriptions. Similarly, we denote by \( V_{h(\text{sub})} \) and \( V_{h(\text{com})} \) the optimal revenue of the platform from using only homogeneous subscriptions (i.e., setting \( \gamma^s = \gamma^b = 0 \)) and from using only homogeneous commissions (i.e., setting \( \mu^s = \mu^b = 0 \)), respectively.

Recall that in Section 3.1 we obtained optimal commissions/subscriptions by solving the convex relaxation of the platform’s revenue maximization problem in (7). This relaxation is no longer tight.
when we impose homogeneous commissions/subscriptions, as the fee homogeneity restriction is non-trivial. That is, in general, it is not possible to construct homogeneous commissions/subscriptions that support the optimal solution of the aforementioned relaxation.

Despite not being able to use the convex optimization problem of Section 3.1, an approximately optimal homogeneous commissions/subscriptions can be obtained a tractable way by defining a grid of over the four decision variables \((\mu^s, \mu^b, \gamma^s, \gamma^b)\), obtaining the equilibrium for any tuple of parameters by solving a convex optimization problem (see Appendix A), and computing and comparing the corresponding revenues.

In light of Theorem 1, it may be appealing to search only over commissions or only over subscriptions (as the theorem suggests that doing so is optimal when the platform is not restricted to using homogeneous commissions/subscriptions). Example 2 illustrates that this leads to strictly lower revenues for the platform in the context of homogeneous commissions/subscriptions.

**Example 2.** Consider a bipartite network with three seller types and three buyer types, with the edge set given by \(E = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 3)\}\). Sellers and buyers populations are \(s = (7, 12, 20)\) and \(b = (70, 24, 20)\). All seller types have a valuation distribution equal to \(F_s(v) = 1 - \exp(-v/40)\), and all buyer types have a valuation distribution equal to \(F_b(v) = 1 - \exp(-v/50)\). In this example, we obtain \(V_h/V_{opt} = 91.0\%\). Furthermore, if we use only commissions we have \(V_{h(com)}/V_{opt} = 90.8\%\), and if we use only subscriptions we have \(V_{h(sub)}/V_{opt} = 87.8\%\). \(\square\)

There are two important takeaways from the above example. First, homogeneous commissions/subscriptions are in general suboptimal and they can yield substantially lower revenues relative to the ones given in Theorem 1, even for very simple networks. Second, \(V_h\) cannot be implemented using only commissions or only subscriptions and, moreover, the revenues obtained by using only commissions or only subscriptions are different.

Given that restricting attention to homogeneous commissions/subscriptions is not without loss of optimality, it is of interest to understand how large the revenue loss can be. Proposition 3 addresses this question, by showing that in general this revenue loss can be arbitrarily bad.

**Proposition 3.** For any \(\varepsilon > 0\), there exists a problem instance such that \(V_h/V_{opt} < \varepsilon\).

Intuitively, when the platform is able to charge heterogeneous commissions/subscriptions, it has some freedom to manage supply and demand differently for different submarkets. To see this, consider a network that consists of two seller types and two connected components. Assume that the first seller type faces a set of buyers with larger value distributions (in the first-order stochastic dominance sense) relative to the set of buyers the other seller type faces. If the platform restricts attention to homogeneous commissions/subscriptions then either one component may end up with too much supply or the other may end up with too little supply, and both outcomes lead to a low
aggregate revenue. By discriminating these seller types by using different commissions/subscriptions, the platform can induce different supply levels in these different components and improve revenues.

In the proof of Proposition [3], we construct problem instances where the homogeneous commissions/subscriptions perform arbitrarily bad by considering settings where different types have very different value distributions. One might ask whether a similar result could be obtained if all types on the same side share the same valuation distribution. In fact, even though homogeneous commissions/subscriptions are still suboptimal in that setting (see Example [2] for an illustration), it is possible to obtain bounds on the revenue loss. In particular, by focusing on such problem instances, Proposition [4] identifies two settings where the revenue loss due to homogeneous commissions/subscriptions is bounded. This proposition as well as the rest of the results of this section focus on problem instances with a fixed network and a given population profile \((s, b)\). As such, when we state our results we suppress the dependence of revenues on these quantities (i.e., we use \(V_{\text{opt}}\) and \(V_{\text{max}}\) as opposed to \(V_{\text{opt}}(E, s, b)\) and \(V_{\text{max}}(s, b)\) – where \(V_{\text{max}}(s, b)\) is as defined in \([10]\)).

**Proposition 4.** Suppose that all types on the same side share the same valuation distribution. Then:

(i) If \(F_s(v)\) is concave in \(v \in [0, \bar{v}_s]\) and \(F_b(v)\) is convex in \(v \in [0, \bar{v}_b]\) and \(\bar{v}_b < \infty\), then for any network \(G(S \cup B, E)\), we have \(V_h \geq \frac{1}{2} V_{\text{opt}}\).

(ii) If the network satisfies the \(\varepsilon\)-marriage condition, then \(V_h \geq (1 - \varepsilon) V_{\text{max}}\).

This proposition provides lower bounds on the revenues achievable under the homogeneous commissions/subscriptions. In the first part, we show that at least half of the optimal revenue can be obtained when the valuation distributions of traders are well-behaved (e.g., exponential distribution for sellers, and uniform for buyers). Note that this bound holds for any network, i.e., it is independent of the network structure. On the other hand, in the second part we do not make any assumption on the distributions but instead we provide a bound that depends on the network structure. In particular, the second part of the proposition implies that if the network satisfies the weighted Hall’s marriage condition, then using homogeneous commissions/subscriptions is optimal. Moreover, when this condition does not hold, quantifying by how much it is violated (in terms of the \(\varepsilon\)-marriage condition) allows us to bound the revenue loss incurred by the platform. Note that, in this case, the bound is in terms of the upper bound \(V_{\text{max}}\) on revenues achievable for any network.

### 4.2. Charging Payments Only to One Side of the Market

We next restrict attention to commission/subscription schemes where the platform can charge only one side of the market, while possibly still discriminating agent types in terms of their commissions/subscriptions. Formally, let \(\Gamma^s = [0, 1]^n\), \(\Gamma^b = [0, \infty)^m\), \(U^s = [0, \infty)^n\), and \(U^b = [0, \infty)^m\) denote
the space of feasible seller commission rates, buyer commission rates, seller subscription fees, and buyer subscription fees, respectively. We denote by $V_s$ ($V_b$) the maximum revenue that the platform can obtain by charging payments only to sellers (buyers). With some abuse of notation, these quantities are given as follows:

$$V_s = \max_{(\gamma^s, \mu^s) \in \Gamma^s \times U^s} V(\gamma^s, 0, \mu^s, 0),$$

$$V_b = \max_{(\gamma^b, \mu^b) \in \Gamma^b \times U^b} V(0, \gamma^b, 0, \mu^b).$$

(12)

It is worthwhile to mention that, as in the case of homogeneous commissions/subscriptions, we cannot rely on the relaxation in (7) to solve this problem. In other words, having zero commissions/subscriptions on one side is a nontrivial restriction and, under this restriction, it is not always possible to find commissions/subscriptions that support the optimal solution of the aforementioned relaxation. Thus, for our numerical studies, we approximately characterize the revenue under one-sided extraction by constructing a grid of commissions/subscriptions and, for each commission-subscription vector in this grid, we compute the equilibrium and the corresponding revenue.

We proceed by illustrating that charging payments only to one side of the market is not without loss of optimality. In particular, by focusing on a simple network structure, our next example shows that the revenue loss relative to the optimal commissions/subscriptions of Section 3.1 is nontrivial.

**Example 3.** Consider a complete bipartite network with three seller types and three buyer types with population vectors $s = (1, 1, 1)$ and $b = (2, 2, 2)$. Let the value distributions for type-$i$ sellers be $F_{s_i}(v) = 1 - \exp\left(-\frac{v}{\lambda^s_i}\right)$ where $k^s = (3, 2, 1)$ and $\lambda^s = (20, 1, 0.1)$, and let the value distributions for type-$j$ buyers be $F_{b_j}(v) = 1 - \exp\left(-\frac{v}{\lambda^b_j}\right)$ where $k^b = (3, 2, 1)$ and $\lambda^b = (1, 5, 10)$. In this example we obtain $V_b = 10.05$, $V_s = 8.84$ and $V_{opt} = 10.78$, and hence $\frac{V_s}{V_{opt}} = 81.9\%$ and $\frac{V_b}{V_{opt}} = 93.3\%$. □

This example also illustrates why charging payments to both sides of the market is strictly better than charging payments only to one side. To see this suppose that the platform charges payments only to sellers, i.e., $\mu^b = \gamma^b = 0$. Observe that in this example type-3 buyers have larger valuations in expectation than the other two buyer types (since $\lambda^b_3 = 10 \gg \max\{\lambda^b_2, \lambda^b_1\} = 1$). Thus, when the platform does not charge any payments to buyers, too many type-3 buyers demand to trade with sellers on the other side of the market. On the other hand, recall from Theorem [4] that at the revenue-maximizing outcome, the equilibrium supply-demand is uniquely defined. Hence, when the induced buyer demand is too high (under $\mu^b = \gamma^b = 0$), it may overshoot the optimal amount identified in Theorem [4] thereby leading to suboptimal revenues. By charging payments appropriately to the buyers, the platform can reduce the demand of type-3 buyers to the optimal levels. In other words, without charging commissions/subscriptions to both sides of the market, it may not be feasible to achieve optimal supply/demand levels. A similar argument also reveals the suboptimality of charging commissions/subscriptions only to the sellers.
In the previous subsection, we showed that when the platform is restricted to using homogeneous commissions/subscriptions, in general the revenue loss can be arbitrarily bad. We conclude this section by establishing that, in contrast to the case of homogeneous commissions/subscriptions, if the platform charges payments only to one side of the market, the revenue loss is always bounded. Furthermore, charging payments only to one side of the market is without loss of optimality when agents’ valuations admit identical distributions.

**Proposition 5.** Suppose that the platform can charge payments only to one side of the market. Then:

(i) If the platform can choose whether to charge buyers or sellers, then at least half of the optimal revenue can be obtained, i.e., \( \max\{V_s, V_b\} \geq \frac{1}{2} V_{opt} \).

(ii) If the value distributions of the buyers are homogeneous, i.e., \( F_{b_j}(v) = F_b(v) \) for all \( j \in B \), then \( V_s = V_{opt} \); if the value distributions of the sellers are homogeneous, i.e, \( F_{s_i}(v) = F_s(v) \) for all \( i \in S \), then \( V_b = V_{opt} \).

To establish the first part of the result we start with the optimal subscriptions of Theorem 1 and set payments of the buyers or the sellers equal to zero. We show that in at least one of the two cases, the induced subscription vector achieves at least half of the optimal revenues. Note that this result not only establishes that charging payments only to one side of the market yields approximate optimality, but also that it is sufficient to rely on subscriptions for this purpose. We also point out that the first part of the theorem does not make any assumptions on the network structure or the value distributions.

On the contrary, to establish the second part, we focus on settings where all agent types on one side of the market have the same value distributions, but we still allow for arbitrary distributions on the other side and for an arbitrary network structure. Theorem 1 shows that the supply-demand levels associated with any revenue optimal outcome are unique. Intuitively, in order to induce the optimal supply-demand levels the platform relies on commissions/subscriptions. When the platform is restricted to charging payments only to one side of the market, it is not possible to simultaneously achieve optimal supply-demand levels, which in turn leads to some revenue loss. Proposition 5(ii) implies that when one side has homogeneous value distributions, this conclusion changes. In this case, appropriate market prices induce the optimal trade levels on one side of the market, and the choice of commissions/subscriptions induce optimal trade levels on the other side.

### 4.3. Numerical Study

We now present an example motivated by Airbnb – an online marketplace that allows guests to rent homes for short stays from hosts. Focusing on some regions of the city of Chicago, we study the
impact that different commission/subscription schemes have on the revenues of the platform. It is worth clarifying that our objective in this section is not to provide a detailed empirical study, which is beyond the scope of this paper. We recognize that Airbnb has a large-scale complex operation, and our model does not capture all features of this operation. However, we believe that this is a useful setting to illustrate the key ideas and findings presented so far in a realistic and nontrivial networked market. To this end, we remind the reader that the primitives of our model are the bipartite network that captures the compatibility between different buyer/seller types, the populations of these types, and their value distributions. Whenever possible, we use publicly available Airbnb data\(^6\) to calibrate these primitives. We proceed by explaining how we obtained each primitive of the model.

First, we define the network. Agent types are characterized by geographical locations and the number of bedrooms of the supplied/demanded listings. In particular, we focus on three regions of Chicago: North Side (N), West Side (W), and Central (C) (see Appendix EC.8 in the e-companion for a map of neighborhoods contained in each region), which (according to the dataset) collectively contain more than 70% of the listings in the city. Within each region, we further restrict attention to listings for entire homes (as opposed to listings that offer a room in a home). Some of the listings only allow for long-term rentals (which in some cases require the minimum stay length to be greater than 30 days). For our study, we focus on listings that provide short-term rentals (the minimum length of stay requirement is no more than seven days). Each host (seller) type is defined by the region and the number of bedrooms in the listing: one bedroom (or studio), two bedrooms, three or more bedrooms. This gives us in total 9 host types, defined by the set \(S = \{N_1^a, N_2^a, N_3^a, C_1^a, C_2^a, C_3^a, W_1^a, W_2^a, W_3^a\}\). We also define a guest (buyer) type per location and number of bedrooms, which also gives us a total of nine types: \(B = \{N_1^b, N_2^b, N_3^b, C_1^b, C_2^b, C_3^b, W_1^b, W_2^b, W_3^b\}\). Here, \(N_2^b\) is the type that corresponds to hosts in the North Side whose listings have two bedrooms, and \(N_2^b\) is the type that corresponds to guests who primarily want to stay in the North Side and require at least two bedrooms (a similar interpretation applies to the remaining types). We assume that each guest can be compatible only with host types whose listings have at least as many bedrooms as the guest requires. We also assume that guests who would like to stay primarily in the West Side or the North Side are compatible with the hosts in Central Chicago (which contains the main tourist attractions of the city). The resulting compatibility network is depicted in Figure 3.

The hosts can black out some dates and not offer their property to guests on these dates. The dataset contains information on the pre-rental nights available for each host (which is the total number of available unbooked nights in the 365 days following the dayz on which data was scraped).

\(^6\) In particular, we use the 2016-2017 listing data in Chicago obtained from “http://insideairbnb.com/get-the-data.html”. The dataset has rich information on the host side, including listing locations, host information, room availability, listed price, cleaning fee, number of reviews, etc. Unfortunately, the dataset does not have any information on the guest side, and we make reasonable assumptions to construct the relevant primitives for the guests.
For each host type $i \in S$, we aggregate the yearly pre-rental nights over all hosts of this type. We divide this quantity by 365 to obtain the (average daily) supply. Rounding this number to the closest integer, we define the host population $s_i$ for the relevant type.

The dataset also includes a single price for each listing. This is the host’s default price, which the host may change for specific dates (such as holidays). Some hosts also require the guests to pay a cleaning fee (also included in the dataset), which is paid per stay (regardless of the duration). We divide this quantity by the average stay length (which for Chicago is 3.3 days; see Airbnb (2017a)) and add it to the listing’s price. Airbnb keeps 3% of this sum as its commission from the hosts (see Airbnb (2017b)), thus we define the remainder as the net transfer to the host for a stay of one night. We assume that the reservation value of a host is equal to this net transfer. Furthermore, we assume that for each host of type $i \in S$ the reservation values are drawn from an underlying Weibull distribution, $F_{s_i}(v) = 1 - \exp\left(-\left(\frac{v}{\lambda_i}\right)^{k_i}\right)$. For $i \in S$, using the aforementioned net transfers as our observations, we estimate the parameters $(\lambda_i, k_i)$ of this distribution using a maximum likelihood estimator. Columns 3-5 of Table 1 summarize the host distribution and population parameters obtained.

![Figure 3 Chicago Compatibility Network](image)

### Table 1 Hosts and Guest Information

<table>
<thead>
<tr>
<th>Location</th>
<th>Room#</th>
<th>$s$</th>
<th>$\lambda^*$</th>
<th>$k^*$</th>
<th>$b$</th>
<th>$\lambda^b$</th>
<th>$k^b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>North</td>
<td>1</td>
<td>147</td>
<td>147</td>
<td>5.73</td>
<td>315</td>
<td>140</td>
<td>25</td>
</tr>
<tr>
<td>North</td>
<td>2</td>
<td>146</td>
<td>190</td>
<td>5.67</td>
<td>57</td>
<td>207</td>
<td>25</td>
</tr>
<tr>
<td></td>
<td>≥ 3</td>
<td>124</td>
<td>324</td>
<td>4.73</td>
<td>35</td>
<td>535</td>
<td>25</td>
</tr>
<tr>
<td>Central</td>
<td>1</td>
<td>121</td>
<td>211</td>
<td>5.31</td>
<td>55</td>
<td>200</td>
<td>25</td>
</tr>
<tr>
<td></td>
<td>≥ 3</td>
<td>16</td>
<td>582</td>
<td>4.51</td>
<td>4</td>
<td>712</td>
<td>25</td>
</tr>
<tr>
<td>West</td>
<td>1</td>
<td>104</td>
<td>132</td>
<td>6.57</td>
<td>724</td>
<td>127</td>
<td>25</td>
</tr>
<tr>
<td></td>
<td>≥ 3</td>
<td>86</td>
<td>342</td>
<td>4.87</td>
<td>32</td>
<td>419</td>
<td>25</td>
</tr>
</tbody>
</table>

On the guest side, we similarly assume that the values come from a Weibull distribution $F_{b_j}(v) = 1 - \exp\left(-\left(\frac{v}{\lambda_j}\right)^{k_j}\right)$ for $j \in B$. Due to limited availability of information on guests, we make two simplifying assumptions on the parameters of these distributions. First, we assume that the coefficient of variation for value distribution of guests is 5% across all types, which is equivalent to setting $k^b = 25$ for all $j \in B$. Second, we assume that the expected value of a guest of type $X^*_i$ is 0%, 15%, and 30% higher than the expected value of a host of type $X^*_i$ respectively for $i \in \{1,2,3\}$, where $X \in \{N,C,W\}$. That is, the expected value of a guest of type $N^*_i$ who needs two bedrooms and

---

7 There were 1161 listings satisfying our criteria (involving location, rental duration requirements, etc.), and on average 84% of the nights were available on the hosts’ calendar, yielding a total supply of roughly 972 units.
prefers the North Side is 15% higher than the expected (reservation) value of hosts who have 2-bedroom listings in this area. Note that while for listings with 1 or 2 bedrooms the hotels may be a good substitute, for 3-bedroom listings this is less likely to be the case. Hence, we assume that the expected value of the guests relative to the reservation values their hosts is higher for such listing (30%), than for other types of listings. Taken together, assumptions pin down \((\lambda^b_j, k^b_j)\) for all \(j \in \mathcal{B}\).

The only unspecified part of the model is the guest population \(b_j\) for \(j \in \mathcal{B}\). We solve for the guest populations such that in the induced equilibrium the number of stays in each region is consistent with the data. We do not have data on the exact number of stays in each region. However, the Airbnb data includes the number of reviews (made in a calendar year) for each of the listings. Following the model discussed in [InsideAirbnb (2017)], we assume that 50% of the bookings result in a review and each property has at most 70% occupancy (in a year). Using these assumptions, we obtain an estimate of the total number of stays in each region per day. Our exploration on the Airbnb platform suggests that in Chicago, Airbnb charges about 14% of the price and cleaning fee to the guests as the platform’s commission. Using this information, together with the estimated value distributions, we solve for the host population vector that matches our estimated total number of stays in each region. This defines our guest population vector \(b\). The guest parameters are reported in columns 6-8.

Given the primitives of the model, we analyze the performance of different commission/subscription schemes. Since Airbnb relies exclusively on commissions, we set all subscription fees to zero. We consider three different settings: (i) baseline model with homogeneous \(\gamma^s = 3\%\) commission for the hosts and \(\gamma^b = 14\%\) commission for the guests (which is consistent with the observed data), (ii) optimal homogeneous commissions for both sides, and (iii) optimal heterogeneous commissions for both sides. As Airbnb sets the guest commissions to be between 0% and 20% (Airbnb (2017b)), in analyzing the second case, for both the host and guest sides, we search the commissions over a grid over \([0\%, 20\%]\) in a brute force fashion and obtain the commission pair that induces the largest revenues for the platform. In the third case, we compute the optimal commissions provided in Theorem 1 and whenever the commission rate for a type is above 20%, we set it equal to 20%.

In the first case, we compute the (daily) revenue of the platform to be \(V_{curr} = 9.96 \times 10^3\). In the second case, we obtain the homogeneous commission rates \((\gamma^s, \gamma^b) = (19\%, 13\%)\), yielding a revenue of \(V_{h(com)} = 13.9 \times 10^3\). In the third case, we obtain the commission vectors \(\gamma^s = (16\%, 17\%, 20\%, 17\%, 15\%, 20\%, 15\%, 17\%, 19\%)\) and \(\gamma^b = (2\%, 9\%, 20\%, 5\%, 11\%, 20\%, 1\%, 9\%, 17\%)\) that achieve a revenue of \(V_{opt} = 15.0 \times 10^3\). We point out that relative to the first scenario, charging optimal homogeneous commission rates improves the revenues by \(\frac{V_{h(com)}}{V_{curr}} - 1 = 39.5\%\). Similarly, targeting different types with different commissions improves the revenues by \(\frac{V_{opt}}{V_{curr}} - 1 = 50.5\%\).

In summary, in this example, using the optimal homogeneous commissions may lead to roughly 40% revenue improvement (relative to \(V_{curr}\)), while an additional 10% (relative to \(V_{curr}\)) can be
obtained using heterogeneous commissions. We also point out that in the last two scenarios we analyzed, the commissions for the hosts are higher than those for the guests, unlike what we see in the current practice. We emphasize that there may be other considerations (e.g., competition with other short-term rental platforms) that incentivize platforms to offer lower commissions to hosts (despite inducing suboptimal revenues), which are beyond the scope of our model. Nevertheless, our observation suggests that from a purely revenue point of view, it may be worthwhile to considering charging higher commissions to the hosts and, at least in some regions, lower ones to the guests.

5. Welfare and Revenue Maximization

We next analyze the welfare implications of the platform’s choice of revenue-maximizing commissions/subscriptions. We start by formally defining the aggregate welfare (that is, the sum of the surpluses of the buyers, sellers, and the revenues of the platform) associated with a pair of commissions and subscriptions in terms of the values of the marginal agents.

**Definition 3.** Under commission-subscription vector \((\gamma, \mu) \in \Gamma \times U\), the associated welfare is given by:

\[
W(\gamma, \mu) := \sum_{j \in B} b_j \int_{v_{m, b_j}}^{\bar{v}_{b_j}} v dF_{b_j}(v) - \sum_{i \in S} s_i \int_{0}^{v_{m, s_i}} v dF_{s_i}(v),
\]

where \(v_{m, b_j}\) and \(v_{m, s_i}\) are as given in (4) and (5), respectively.

To see that (13) captures the welfare associated with \((\gamma, \mu)\), first note that for given \((\gamma, \mu)\), all equilibria in \(X(\gamma, \mu)\) share the same supply-demand vector \((q^s, q^h)\) (by Proposition 1). Hence, in all equilibria, the marginal agents have the same values \(\{v_{b_j}, v_{s_i}\}\). Moreover, all buyers of type \(j\) (sellers of type \(i\)) who have values higher (lower) than those of the marginal agent of their type trade. Thus, the aggregate value that buyers of type \(j\) derive from the received goods/services is given by \(b_j \int_{v_{m, b_j}}^{\bar{v}_{b_j}} v dF_{b_j}(v)\). Similarly, the aggregate reservation value of the sellers of type \(i\) who sell goods is given by \(s_i \int_{0}^{v_{m, s_i}} v dF_{s_i}(v)\). Summing these quantities over all buyers/sellers (after negating the reservation values) and observing that any monetary transfers between buyers, sellers, and the platform cancel out, we obtain the welfare function in (13). Finally, since all equilibria in \(X(\gamma, \mu)\) share the same marginal agents, they induce the same welfare, and \(W(\gamma, \mu)\) is well defined.

We first establish that welfare is maximized when the platform does not charge any payments to buyers/sellers. This is because, in this case, there are no trade frictions due to commissions/subscriptions, and in a competitive equilibrium goods are allocated efficiently.

**Proposition 6.** The maximum welfare is achieved by the commission-subscription vector \((\gamma, \mu) = (0, 0)\), i.e., \(W_{opt} := W(0, 0) \geq W(\gamma, \mu)\) for any \((\gamma, \mu) \in \Gamma \times U\).

Recall that, for any given commissions/subscriptions, a corresponding equilibrium can be obtained by solving a convex optimization program (see Appendix A). Thus in light of Proposition 6 we conclude
that the maximum welfare can be computed by solving this optimization problem for \((\gamma, \mu) = (0, 0)\), obtaining the valuations of the marginal agents in the equilibrium, and evaluating \([13]\).

Since the platform maximizes its revenues by employing nonzero commissions/subscriptions, loss of social welfare can be expected under the optimal commissions and subscriptions. Formally, let \((\gamma', \mu')\) denote the revenue-maximizing commissions-subscriptions chosen by the platform, and let \(W_r := \frac{W(\gamma', \mu')}{W_{opt}}\) be the ratio of the welfare associated with the revenue-maximizing and socially optimal solutions. Our next result provides bounds on this welfare ratio.

**Proposition 7.**

(i) For any \(\epsilon > 0\), there exists a problem instance such that \(W_r < \epsilon\).

(ii) Suppose that the value distributions are uniform, i.e., \(F_{si}(v) = \frac{v}{\bar{v}_{si}}\) for all \(v \in [0, \bar{v}_{si}]\) and \(F_{bj}(v) = \frac{v}{\bar{v}_{bj}}\) for all \(v \in [0, \bar{v}_{bj}]\). Then, for any network \(G(S \cup B, E)\), we have \(W_r \geq \frac{3}{4}\). Moreover, this lower bound is tight.

(iii) Suppose that the value distributions are homogeneous, i.e., \(F_{bj}(v) = F_b(v)\) for all \(j \in B\) and \(F_{si}(v) = F_s(v)\) for all \(i \in S\), with \(F_s(v)\) concave in \(v \in [0, \bar{v}_{s}]\) and \(F_b(v)\) convex in \(v \in [0, \bar{v}_{b}]\). Then for any network \(G(S \cup B, E)\), we have \(W_r \geq \frac{2}{3}\). Moreover, this lower bound is tight.

The first part of this result implies that in the worst case the welfare under revenue-maximizing commissions/subscriptions can be arbitrarily low (relative to the socially optimal level). The result is obtained by considering settings where different types have very different value distributions, and the platform maximizes its revenue by imposing high fees. These fees allow for adequately extracting revenue from the high-value buyers (or low-value sellers) but, at the same time, they deter the majority of buyers/sellers from trading, thereby inducing a low welfare. However, under additional assumptions on the value distributions, Proposition 7 establishes that the welfare achieved under revenue-maximizing commissions/subscriptions is not arbitrarily small. In particular, under uniform valuations, the revenue-maximizing commissions/subscriptions still yield at least 75% of the maximum achievable welfare. If instead we have homogeneous value distributions that are convex for buyers and concave for sellers, the welfare ratio is at least 66.7%.

6. Conclusion

In this paper, we consider a platform that facilitates trade between buyers and sellers. Different types of buyers/sellers have different value distributions, and a bipartite network captures which buyer/seller types are compatible. The platform charges commissions rates and subscription fees to the trading agents, but does not dictate the prices at which trades need to occur. That is, the prices are determined endogenously and are only indirectly influenced by the platform’s choice of commissions-subscriptions. In this setting, we study how the platform should choose its commissions/subscriptions.

\(^8\) Although, we are not aware of any formal connection, this ratio is reminiscent of the well-known price of anarchy ratios in congestion games with linear edge costs (see, e.g., Roughgarden and Tardos (2002)).
to maximize its revenues. We show that, in general, the platform finds it optimal to target different buyer/seller types with different commissions/subscriptions that depend on their value distributions and network positions. Furthermore, charging agents only on one side of the market, or using identical commissions/subscriptions for all agents on the same side (both common practices in real-world platforms) may result in substantial revenue loss. We characterize how the network structure impacts the surplus of different types and the revenues of the platform. We also provide bounds on the revenue loss due to using simpler commission/subscription schemes, as well as on the welfare loss due to employing nonzero commissions/subscriptions. Our results highlight the suboptimality of some commonly used payment schemes, and showcase the importance of understanding the compatibility between different user types present on the platform.

This paper opens up a number of interesting future directions. First, in our baseline model, we assume that each buyer has the same value for trading with a compatible seller. This assumption enables us to focus on the impact of the compatibility network, while abstracting away any preference heterogeneity buyers may have. It is of interest to allow for such preference heterogeneity and understand how it impacts the platform’s choice of commissions/subscriptions. Second, we assume that the population sizes are deterministic and known to the platform. If there is only noisy information available on the population sizes, the platform’s problem of choosing commissions/subscriptions becomes a very challenging, yet important problem. Finally, many platforms impact the trades that take place not only by choosing commissions/subscriptions, but also by facilitating a search protocol that allows buyers to find compatible sellers. The design of such search protocols is likely have to a first-order impact on the trading outcome, and promises to be an exciting future research direction.

References


Online Appendix

A. Auxiliary Results

In this section, we first establish that given commissions/subscriptions the corresponding competitive equilibrium can be obtained through the optimal primal/dual solutions of a convex optimization problem (see (14)). Then, we show that optimal solutions of both this problem and the revenue optimization problem (7) can be characterized by maximizing a concave function over a polymatroid. The solutions of such problems admit an interesting structure, which we leverage in the subsequent sections to establish our key findings.

We start by presenting our equilibrium problem. Given $\gamma, \mu \in \Gamma \times U$ such that $\gamma_i < 1$ for all $i \in S$, define functions $F_i^{-1} : [0, 1] \rightarrow \left[ \frac{-\mu_i}{1-\gamma_i}, \frac{-\mu_i}{1+\gamma_i} \right]$ and $F_b^{-1} : [0, 1] \rightarrow \left[ \frac{-\mu_b}{1-\gamma_b}, \frac{-\mu_b}{1+\gamma_b} \right]$ such that $F_i^{-1}(x) = \frac{x - \mu_i}{1-\gamma_i}$ and $F_b^{-1}(x) = \frac{x - \mu_b}{1+\gamma_b}$. Consider the following problem:

\[
\begin{align*}
\max_{x, q^i, q^b} & \sum_{j \in S} \int_0^{q^i_j} \tilde{F}_{b_j}^{-1} \left( 1 - \frac{x}{b_j} \right) dx - \sum_{i \in S} \int_0^{q^i_i} \tilde{F}_{s_i}^{-1} \left( \frac{x}{s_i} \right) dx \\
\text{s.t.} & \sum_{j : (i, j) \in E} x_{ij} = q^i_i, \quad \forall i \in S, \\
& \sum_{i : (i, j) \in E} x_{ij} = q^b_j, \quad \forall j \in B, \\
& q^i_i \leq s_i, \quad \forall i \in S, \\
& q^b_j \leq b_j, \quad \forall j \in B, \\
& x_{ij} \geq 0, \quad \forall (i, j) \in E.
\end{align*}
\] (14a)

We refer to this problem as the equilibrium problem, since its solutions correspond to competitive equilibria:

**Proposition 8.** Given $(\gamma, \mu)$, the tuple $(p, x, q^i, q^b)$ constitutes a competitive equilibrium if and only if

(i) $(x, q^i, q^b)$ is an optimal solution to optimization problem (14);

(ii) the price vector $p$ satisfies $p_i = \theta_i^*$ for all $i$ such that $q^i_i > 0$ and $\max_{j : (i, j) \in E} \theta_j^* \leq p_i \leq \theta_i^*$ for all $i$ such that $q^i_i = 0$, where $(\theta^*, \theta^b, \eta^*, \eta^b)$ are the unique optimal dual multipliers associated with constraints (14b) - (14c).

Next we introduce a class of optimization problems involving polymatroids. Let $g, h : [0, 1] \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ be functions satisfying the following assumptions:

**[AL] $g(r)$ is differentiable, strictly concave in $r \in (0, 1)$, continuous at $r = 0$, and continuous at $r = 1$ if $g(1) = -\infty$.**

**[A2] $h(r)$ is differentiable, strictly convex in $r \in (0, 1)$, continuous at $r = 0$, and continuous at $r = 1$ if $h(1) < \infty$.**

**[A3] $ch(\frac{r}{c})$ is strictly decreasing in $c$ and jointly convex in $(r, c)$.**

**[A4] $g'(0) > h'(0) \geq 0$ and $g'(1) \leq 0$.**

Define a function $f : (0, \infty) \rightarrow \mathbb{R}$, and correspondence $r : (0, \infty) \rightarrow \mathbb{R}$ which (for $c > 0$) are given by:

\[
f(c) = \max_{r \in (0, \min(1, c)]} g(r) - ch \left( \frac{r}{c} \right),
\]

\[
r(c) = \arg \max_{r \in (0, \min(1, c)]} g(r) - ch \left( \frac{r}{c} \right).
\] (15)

We have the following properties for $f(\cdot), r(\cdot)$:

**Lemma 1.** Suppose Assumptions [AL][A2][A3][A4] hold. Then:

(i) $f(c)$ is continuous, strictly increasing, strictly concave in $(0, \infty)$. Furthermore, $\lim_{c \rightarrow 0} f(c) = g(0)$.

With some abuse of notation we let $g'(0) = \lim_{r \rightarrow 0} g'(r)$, $g'(1) = \lim_{r \rightarrow 1} g'(r)$, $h'(0) = \lim_{r \rightarrow 0} h'(r)$, and $h'(1) = \lim_{r \rightarrow 1} h'(r)$. 
(ii) \( r(c) \) is a singleton for \( c > 0 \), and hence \( r(\cdot) \) is a function. Moreover, \( r(\cdot) \) is strictly increasing in \( c \).

Define \( c_0 = [g']^{-1}(h'(1)) \) with the convention \([g']^{-1}(x) = 0 \) for \( x \geq \sup_{x' \in (0,1)} g'(x') \) and \([g']^{-1}(x) = 1 \) for \( x \leq \inf_{x' \in (0,1)} g'(x') \). We have \( r(c)/c = 1 \) for \( c \leq c_0 \) and \( r(c)/c \) is strictly decreasing in \( c \) for \( c > c_0 \).

Using the first part of this lemma, we extend the domain of \( f(\cdot) \) to include 0, and in particular we let \( f(0) = \lim_{c \to 0} f(c) \). We consider the following optimization problem:

\[
\begin{align*}
\max_y & \quad \sum_{j \in \mathcal{B}} b_j f \left( \frac{y_j}{b_j} \right) \\
\text{s.t.} & \quad \sum_{j \in \mathcal{B}} y_j \leq \sum_{i \in \mathcal{N}(\mathcal{B})} s_i, \quad \forall \mathcal{B} \subseteq \mathcal{B}, \\
& \quad y_j \geq 0, \quad \forall j \in \mathcal{B}.
\end{align*}
\]

(iii) The optimal objective value of \( (17) \) can be characterized in terms of the sets \((A_1)\) and \((A_2)\).

**Proposition 10.** Suppose Assumptions \( (A_1) \) holds, and \( F_0(\cdot) = F_0(\cdot), \ F_s(\cdot) = F_s(\cdot), \ \gamma^b = \gamma^b, \ \gamma^s = \gamma^s, \ \mu^b = \mu^b, \ \mu^s = \mu^s, \) for \( j \in \mathcal{B}, \ i \in \mathcal{S} \). Let \( f(\cdot) \) be given as in \( (15) \), where \( g(r) = F_{-1}^{-1}(1-r) r \) and \( h(r) = F_{s}^{-1}(r) r \) for \( r \in [0,1] \). Then:

(i) Assumptions \( (A_1) \) holds.

(ii) Let \( y^* \) be an optimal solution to \( (17) \), and \( r_* = \{r_*^j\}_{j \in \mathcal{B}} \) be such that \( r_*^j = r(y_*^j/b_j) \). Any optimal solution \( (x, q^*, q^*) \) of the revenue optimization problem \( (7) \) is such that (i) \( q_*^j = r_*^j b_j \) for all \( j \in \mathcal{B}, \) (ii) \( q_*^i = r_*^i b_s \) for all \( i \in \mathcal{S} \) and \( j \in \mathcal{B} \) such that \( x_{ij} > 0 \).

(iii) The optimal objective value of \( (17) \) is the optimal revenue in \( (7) \), i.e., \( V_{opt} = \sum_{j \in \mathcal{B}} b_j f \left( \frac{y_*^j}{b_j} \right) \).

**Proof of Theorem 2** Since the seller/buyer value distributions are homogeneous, Proposition \( (A) \) implies that the solution of the revenue optimization problem \( (7) \) can be characterized in terms of the solution of \( (17) \), where \( f(\cdot) \) is as in \( (15) \), and \( g(r) = F_{-1}^{-1}(1-r) r \) and \( h(r) = F_{s}^{-1}(r) r \) for \( r \in [0,1] \). Let \( y^* \) be an optimal solution to this problem and \( \mathbf{r}^* = \{r_*^j\}_{j \in \mathcal{B}} \) be such that \( r_*^j = r(y_*^j/b_j) \) (where \( r(\cdot) \) is as in \( (16) \)). By Lemma \( (2) \) it follows that \( y^* \) is the lexicographically optimal base of the feasible set of \( (17) \) (which we denote by \( \mathcal{P} \)) with respect to weight vector \( \mathbf{b} \).

To establish the claim, we proceed in two steps. First, we show that the lexicographically optimal base \( y^* \) can be characterized in terms of the sets \( \{\mathcal{S}, \mathcal{B}\} \) given in the theorem statement. Then, we relate this characterization to the the ranking of the marginal agents.

**Step 1:** Characterization of the lexicographically optimal base. The construction in the theorem statement identifies a nonempty set \( \mathcal{B} \) at each step and removes the elements of this set from \( \mathcal{B}^{(t-1)} \). Since \( \mathcal{B} \) is a finite set, it follows that in \( t \leq |\mathcal{B}| \) iterations sets \( \{\mathcal{B}_t\}_{t=1}^T \) and \( \{\mathcal{S}_t\}_{t=1}^T \) are constructed. Let vector \( \mathbf{c}' = \{c'_j\}_{j \in \mathcal{B}} \)

\[ 10 \text{ Note that } \mu^b + \frac{1-r}{1-r} \mu^s \geq F_{-1}^{-1}(1) \text{ yields the trivial equilibrium where no one trades in the system.} \]
be such that \( c'_j = \frac{\sum_{i \in N_E(t-1)_{(t)}} s_i}{\sum_{k \in B_t} b_k} \) for \( j \in B_t \) and \( t = 1, \ldots, \ell \). We claim that the vector \( y' = \{ c'_j b_j \}_{j \in B} \) is the unique lexicographical optimal basis of polymatroid \( P \) under weight vector \( b \), i.e., \( y = y' \).

In order to establish this claim, we employ Theorem 3.1 of Fujishige (1980). To this end, define the vector \( v^a = (ab_j)_{j \in B} \) for any \( a \geq 0 \). Second, let the vector \( u^a \) be a vector such that: (i) \( u^a_j = ab_j \) if \( 0 \leq a \leq c'_j \), and (ii) \( u^a_j = c'_j b_j \) if \( a > c'_j \). It can be readily seen that \( u^a \leq u^a' \) (where the inequality is entrywise) for \( 0 \leq a \leq a' \). We claim that for any \( a \geq 0 \), \( u^a \) is a base of the polymatroid \( P_{\alpha} = \{ z \in P | z \leq v^a \} \) (where once again the inequality is entrywise). Note that if this claim holds, then the above observations imply that the weight vector \( b \) and vector \( c' \) satisfy the conditions (3.1) - (3.5) of Fujishige (1980). Hence, Theorem 3.1 of Fujishige (1980) applies, and it implies that \( y' = y' \).

Fix some \( a \geq 0 \). We complete the proof of Step 1, by establishing that \( u^a \) is a base of \( P_{\alpha} \). Note that for any \( B \subset B \) we have

\[
\sum_{j \in B} u^a_j \leq \sum_{j \in B} c'_j b_j \leq \sum_{i \in N_E(t)} \sum_{j \in B_t} s_i \leq \sum_{i \in N_E(t)} \frac{\sum_{j \in B \setminus B_t} b_j}{\sum_{j \in B_t} b_j}
\]

Here, (1) uses the definition of \( u^a \), (2) follows since \( \sum_{j \in B_t} b_j = B_t \), and (3) uses the definition of \( c'_j \). Finally, (4) follows since the definition of \( B_t \) implies that \( \sum_{i \in N_E(t)} s_i \leq \sum_{j \in B_t} b_j \). The inequality (18) together with the fact that \( u^a \leq u^a' \) (which holds by construction of \( u^a \)), implies that \( u^a \in P_{\alpha} \).

Let \( B_1 = \{ j | a \leq c'_j \} \) and \( B_2 = B \setminus B_1 \). Note that by the definition of \( \{ B_t \}_{t=1}^{\ell} \) it follows that \( c'_j = c'_j \) for \( j_1 \in B_1 \) and \( j_2 \in B_2 \) and \( t_1, t_2 \in \{ 1, \ldots, \ell \} \) such that \( t_1 < t_2 \). Moreover, \( c'_j = c'_j \) for \( j_1, j_2 \in B_2 \) for \( t \in \{ 1, \ldots, \ell \} \). These observations imply that \( B_1 = \sum_{t=1}^{\ell} B_1 = \sum_{t=1}^{\ell} B_2 \) for some \( t' \in \{ 1, \ldots, \ell \} \). By the definition of \( u^a \), we have \( u^a_j = u^a_{j'} \) for \( j \in B_1 \). Moreover, for \( t \leq \ell' \) we have \( B_t \subset B_{t'} \), and hence the definition of \( u^a \) and \( c'_j \) imply that \( u^a_k = \sum_{k \in B_{t'}} c'_k b_k = \sum_{i \in N_E(t-1)} s_i \). Hence, we conclude

\[
\sum_{k \in B} u^a_k = \sum_{k \in B_1} u^a_k + \sum_{k \in B_2} u^a_k + \sum_{t=1}^{\ell'} \sum_{k \in N_E(t)} s_k.
\]

Consider an arbitrary \( z \in P_{\alpha} \), and observe that

\[
\sum_{k \in B} z_k = \sum_{k \in B_1} z_k + \sum_{t=1}^{\ell'} \sum_{k \in B_2} z_k \leq \sum_{k \in B_1} u^a_k + \sum_{t=1}^{\ell'} \sum_{k \in N_E(t)} u^a_k \]

where we use \( z_k \leq v^a_k \) for \( k \in B_1 \) and the fact that \( \sum_{k \in B_2} z_k \leq \sum_{k \in N_E(t)} s_k \) for any \( z \in P_{\alpha} \). Together with (19), this implies that \( u^a \) is a base of \( P_{\alpha} \). Hence, we conclude \( y' = y' \).

Step 2: Characterization of marginal agents. By Proposition 9, we have \( \frac{\partial v^a_j}{\partial y_j} = r_j \frac{y_j}{b_j} \) for all \( j \in B \), which implies that the value of marginal agent of type \( j \in B \) is given by \( v^a_j = F_b^{-1}(1 - \frac{\partial v^a_j}{\partial y_j}) = F_b^{-1}(1 - r_j y_j b_j) \). By Lemma 9 \( r(\cdot) \) is a strictly increasing function, and hence the entries of the vector \( \{ r_j(y_j^* b_j) \}_{j \in B} \) admit the same ranking as the entries of the vector \( \{ v_j^* \}_{j \in B} \). Since \( y^* = y' \) recalling the definition of \( y' \), these observations imply that

\[
\frac{\partial v^a_j}{\partial y_j} = \frac{\partial v^a_{k}}{\partial y_k} \quad \text{for} \quad j, k \in B_1 \quad \text{with} \quad t \in \{ 1, \ldots, \ell \},
\]

\[
\frac{\partial v^a_{j}}{\partial y_k} > \frac{\partial v^a_{k}}{\partial y_k} \quad \text{for} \quad j \in B_1, k \in B_2, \quad t_1, t_2 \in \{ 1, \ldots, \ell \} \quad \text{and} \quad t_1 < t_2.
\]

Thus, we have established that \( j \in B \) if and only if type \( j \)'s marginal agent has the 3rd highest among the values of marginal agents of all buyer types.

We next establish the ranking result for the marginal agents of seller types. By Theorem 1 under any optimal commission/subscription pair \( (\gamma, \mu) \) the mass of agents who trade \( (q^b, q^a) \), and therefore the marginal
agents, are the same. Furthermore, by Proposition 5(ii), and the fact that buyers (sellers) share identical distributions under the assumptions of the theorem, it follows that there exists an optimal commission/subscription pair where $\gamma^b = \mu^b = 0$.

Consider such an optimal commission/subscription pair, and note that since $\gamma^b = \mu^b = 0$, at this solution the price trading buyers of type $j \in B$ pay is equal to $v^m_{s_i} = F_{s_i}^{-1}(1 - \gamma^b_j)$. We claim that buyers in $B_1$ only trade with the sellers in $S_1$, and vice versa. To see this first focus on $t = 1$ and observe that buyers in $B_1$ can only trade with sellers in $S_1 = N_{S/E}(B_1)$, as they are not adjacent to any other seller type. Since the price the trading buyers in $B_1$ pay is given by $v^m_{s_i}$ for any $j \in B_1$ (by (21)), it follows by (21) that type $B_1$ buyers trade at the highest prices. Thus, if sellers of type $S_1$ trade, then they trade with buyers whose types belong to $B_1$. On the other hand, by Proposition 5(ii) of the e-companion, it follows that all seller types involve in some trade at the revenue maximizing solution. Hence, it follows that all seller types in $S_1$ have nonzero trade with some buyer type in $B_1$, and buyers in $B_1$ only trade with sellers in $S_1$ and vice versa. By induction, the same argument implies that buyers in $B_1$ trade positive quantities with the sellers in $S_1$. Thus, Proposition 9 implies that for any $i \in S_1$, we have $v^m_{s_i} = F_{s_i}^{-1}(\frac{R}{c_{s_i}}) = F_{s_i}^{-1}(\frac{r(c_{s_i})/c_{s_i}}{b_{s_i}})$ for some buyer type $j \in B_1$. Moreover, since $y^b_j = y^b_j = c'_{s_i}b_j$, it follows that $v_{s_i}^m = F_{s_i}^{-1}(\frac{r(c_{s_i})}{c_{s_i}})$. By Lemma 1 it follows that $r(c)/c = 1$ for $c \leq c_0$, where $c_0$ is a constant that depends on the value distributions. Let $T = \{t|c_j < c_0 \text{ for } j \in B\}$, and observe that by construction of $\{c'_j\}$ we have $T = \{1, \ldots, \bar{t} \}$ for some $\bar{t} \in \mathbb{Z}$. Consider some $t \leq \bar{t}$ and $i \in S_1$. It follows from the observations above that $v^m_{s_i} = F_{s_i}^{-1}(r(c_{s_i}'')) = F_{s_i}^{-1}(1) = \bar{v}_{s_i}$. Next, consider $t > \bar{t}$ and $i \in S_1$. Observe that Lemma 1 also implies that $r(c)/c$ is strictly decreasing for $c > c_0$. Hence, for such $t$ we have $v^m_{s_i} = F_{s_i}^{-1}(r(c_{s_i}'')) < \bar{v}_{s_i}$, where $j \in B_1$. Moreover, $v_{s_i}^m$ is strictly decreasing in $t$. These observations imply that

$$v^m_{s_i} = \bar{v}_{s_i} \quad \text{for } i \in S_1, \quad t \leq \bar{t},$$

$$v^m_{s_{i_1} s_{i_2}} = v^m_{s_{i_1} s_{i_2}} \quad \text{for } i_1, i_2 \in S_1 \text{ with } t \in \{1, \ldots, \ell\},$$

$$v^m_{s_{i_1} s_{i_2}} > v^m_{s_{i_1} s_{i_2}} \quad \text{for } i_1, i_2 \in S_1, i_2 \in S_2, \quad \text{with } t_1, t_2 \in \{1, \ldots, \ell\}, \quad t_2 > \bar{t} \quad \text{and} \quad t_1 < t_2.$$  

Hence the claim follows. Q.E.D.

**Proof of Theorem 3.**

Proof of part (i). Let

$$f(c) = \max_{r \in [0, \min(1, c)]} \left[ F_{s_i}^{-1}(1 - r) - F_{s_i}^{-1}\left(\frac{r}{c}\right) \right]$$  

for $c > 0$ and let $\hat{V}(s, b)$ be defined as $\hat{V}(s, b) := b_0 f\left(\frac{a_{s_i}}{b_0}\right)$ where $a_0$ and $b_0$ are as defined in the statement of the theorem. Therefore, we need to show that $V_{max}(s, b) = \hat{V}(s, b)$.

As a first step, we first prove that $V_{max}(s, b) \leq \hat{V}(s, b)$. Fix any arbitrary network $G(S \cup B, E)$, and let $y^*$ be an optimal solution to Problem (17) when $f$ is as defined by (25).

Then,

$$V_{opt}(E, s, b) := \left(\sum_{j \in B} b_j \sum_{j \in B} \frac{b_j}{\sum_{j \in B} b_j} f\left(\frac{y^*_j}{b_j}\right)\right) \leq \left(\sum_{j \in B} \frac{b_j}{\sum_{j \in B} b_j} f\left(\sum_{j \in B} \frac{y^*_j}{b_j}\right)\right) \leq \hat{V}(s, b),$$  

where the equality in (a) follows from Proposition 9 part (iii) (note that the assumptions in the statement of the theorem imply that the assumptions in Proposition 9 hold). The inequality in (b) follows from the fact that $f$ is strictly concave (Lemma 1). Inequality (c) follows from the fact that $f$ is strictly increasing and that $y^*$ is feasible and thus must satisfy $\sum_{j \in B} y^*_j \leq \sum_{i \in S(E)} s_i$. Finally, equality (d) follows from the fact that, by definition, $\hat{V}(s, b) = b_0 f\left(\frac{a_{s_i}}{b_0}\right)$. Therefore, we have established that $V_{opt}(E, s, b) \leq \hat{V}(s, b)$ for any network $G(S \cup B, E)$ and, thus, we must have that $V_{max}(s, b) = \max_{E \in B \times S} V_{opt}(E, s, b) \leq \hat{V}(s, b)$.

Next, let $y$ be defined such that $y_j = b_0 \sum_{i \in S(E)} s_i$ for all $j \in B$. As the function $f$ is strictly concave (Lemma 1), the inequality (b) in (26) holds by equality if and only if $y^* = y$. Note that, in fact, such a $y$ is feasible for Problem (17) with $f$ given by (25) (i.e. it satisfies constraints (17b) and (17c)) if and only if the network
satisfies the Hall’s marriage condition; this follows straightforward from Definition 2. Moreover, by the definition of \( y \), we have that \( \sum_{j \in B} b_j = \sum_{i \in S} s_i \), and thus inequality (c) in Equation (26) holds by equality as well. Therefore, we have established that \( V_{opt}(E, s, b) = V(s, b) \) if and only if network \( G(S \cup B, E) \) satisfies weighted Hall’s marriage condition. Because a complete bipartite network satisfies Hall’s marriage condition, we obtain that \( V_{max}(s, b) = \max_{E \subseteq B \times S} V_{opt}(E, s, b) \geq V(s, b) \). This completes the proof of part (i).

Proof of part (ii): By claim (ii) in Proposition 4 we have that, if a network \( G(S \cup B, E) \) satisfies the \( \varepsilon \)-marriage condition, then \( V_h(E, s, b) \geq (1 - \varepsilon)V_{max}(s, b) \) where \( V_h \) is the revenue induced by the optimal homogeneous commissions/subscriptions. Hence, \( V_{opt}(E, s, b) \geq V_h(E, s, b) \geq (1 - \varepsilon)V_{max}(s, b) \), as desired. Q.E.D.

Proof of Proposition 2. Let \( y_1^*, y_2^* \) be optimal solutions to problem (17) associated with networks \( G(S \cup B, E_1) \) and \( G(S \cup B, E_2) \), respectively, when \( f \) is as defined in the statement of Proposition 9. As \( \sum_{i \in N_{E_1}(B)} s_i = \sum_{i \in N_{E_2}(B)} s_i \), for all \( B \subseteq \bar{B} \), we have that \( y_2^* \) is a feasible solution for the problem associated with network \( G(S \cup \bar{B}, E_1) \). Therefore, by Proposition 9:

\[
V_{opt}(E_1, s, b) = \sum_{j \in \bar{B}} b_j f \left( \frac{(y_1^* - y_2^*)_j}{b_j} \right) \geq \sum_{j \in \bar{B}} b_j f \left( \frac{(y_2^*)_j}{b_j} \right) = V_{opt}(E_2, s, b),
\]

which completes the proof. Q.E.D.

C. Proof of Results in Section 4

Proof of Proposition 4. Fix a network \( G(S \cup B, E) \). To ease exposition, throughout the rest of the proof we omit the dependence on the network in the notation.

With a slight abuse of notation, let \( V_h(\mu^b) \) denote the revenue under homogeneous commissions/subscriptions as a function of the buyers’ subscription fee \( \mu^b \), when \( \mu^*, \gamma^b \), and \( \gamma^s \) are all set to zero, and define \( \bar{V}_h = \max_{\mu^b \geq 0} V_h(\mu^b) \). Note that \( \bar{V}_h \geq V_h \), and therefore it suffices to establish the results in parts (i) and (ii) of Proposition 4.

Before proceeding to the proofs of parts (i) and (ii) of this proposition, we establish a couple of intermediate results that will be exploited later on.

Claim 1: \( V_h(\mu^b) = \sum_{j \in B} \mu^b b_j r_j^*(\mu^b) \). As a first step in our proof, we start by calculating an expression for \( V_h(\mu^b) \). By convexity of value distribution function \( F_b(\cdot) \), we have \( \bar{v}_b < \infty \). It suffices to consider \( \mu^b \in [0, \bar{v}_b] \) as otherwise no trade will take place. We start by finding the optimal equilibrium demand as a function of the buyers’ subscription fee \( \mu^b \). Hence, \( \bar{V}_h = \max_{\mu^b \geq 0} V_h(\mu^b) \).

Let \( \mu^b \in [0, \bar{v}_b] \). By Proposition 10 (i), Assumptions (A1) – (A4) hold. Using Proposition 10 (ii), we know that the equilibrium demands \( y^*_j(\mu^b) = b_j r_j^*(\mu^b) \), where

\[
r_j^*(\mu^b) = \arg \max_{x \in [0, \min\{1, y^*_j(\mu^b)/b_j\}]} \int_0^r F_b^{-1}(1 - x) - \mu^b - F_s^{-1} \left( \frac{x}{F_b^{-1}(p_j^*(\mu^b)/b_j)} \right) dx,
\]

and \( y^*(\mu^b) \) is an optimal solution to (17). By Lemma 2, \( y^* \) is the lexicographical optimal base of polymatroid \( P = \{y \geq 0 : \sum_{j \in B} y_j \leq \sum_{i \in N_B} s_i, \forall B \subseteq \bar{B}\} \), which is independent from \( \mu^b \). Therefore, we drop the dependency on \( \mu^b \) in \( r_j^*(\mu^b) \), and define \( c_j := \frac{y_j^*}{b_j} \) for all \( j \in B \). From expression (27), \( r_j^*(\mu^b) = 0 \) (corresponding to no trade in equilibrium). We can apply expression (27) for all \( \mu^b \in [0, \bar{v}_b] \).

Notice that the problem to solve \( r_j^*(\mu^b) \) is a concave maximization problem. Using the first order optimality condition, we can express \( r_j^*(\mu^b) \) as

\[
r_j^*(\mu^b) = \max \left\{ r : F_b^{-1}(1 - r) - F_s^{-1} \left( \frac{r}{c_j} \right) \geq \mu^b, \ 0 \leq r \leq \min\{1, c_j\} \right\}, \quad \forall \mu^b \in [0, \bar{v}_b] \tag{28}
\]

Finally, the revenue can be expressed as

\[
V_h(\mu^b) = \sum_{j \in B} \mu^b b_j r_j^*(\mu^b) = \sum_{j \in B} \mu^b b_j r_j^*(\mu^b) \quad \text{for } \mu^b \in [0, \bar{v}_b].
\]

Claim 2: The functions \( \mu^b r_j^*(\mu^b) \) are continuous, concave, \( 0 \leq \mu^b r_j^*(\mu^b) \leq f(c_j) \) for all \( \mu^b \in [0, \bar{v}_b] \). Moreover, these bounds are tight. We now derive some properties on the functions \( \mu^b r_j^*(\mu^b) \) that we will later use to
Define a finite set of continuous and concave functions. Moreover, as each $r_j^\ast(\mu^b)$ is weakly decreasing in $\mu^b$, by the maximum theorem in page 116 of [Berger 1963], the value function $r_j^\ast(\mu^b)$ is thus continuous in $\mu^b \in [0, \bar{\nu}]$. Moreover, as the function $F^{-1}_b(1-r) - F^{-1}_s(\frac{r}{c})$ is continuous and strictly decreasing in $r$, for any $\mu_1^b \geq \mu_2^b$ we have $\{ r : F^{-1}_b(1-r) - F^{-1}_s(\frac{r}{c}) \geq \mu_1^b, 0 \leq r \leq \min\{1, c_j\} \} \subset \{ r : F^{-1}_b(1-r) - F^{-1}_s(\frac{r}{c}) \geq \mu_2^b, 0 \leq r \leq \min\{1, c_j\} \}$, which implies that $r_j^\ast(\mu^b)$ is weakly decreasing in $\mu^b$. Finally, for any $\mu_1^b, \mu_2^b \in [0, \bar{\nu}]$, let $\bar{r} := r_j^\ast(\frac{1}{2} \mu_1^b + \frac{1}{2} \mu_2^b), r_1 := r_j^\ast(\mu_1^b)$ and $r_2 := r_j^\ast(\mu_2^b)$. Given that functions $F^{-1}_b(1-r)$ and $- F^{-1}_s(1-r)$ are concave, we have $F^{-1}_b(1-r) - F^{-1}_s(1-r) = F^{-1}_b(1-r) - F^{-1}_s(1-r) \leq \frac{1}{2} \mu_1^b + \frac{1}{2} \mu_2^b$. Moreover, by definition of $r_1$ and $r_2$, we have $\frac{r_1 + r_2}{2} \leq c_j$, and $\frac{r_1 + r_2}{2} \leq 1$ because both $r_1, r_2 \leq \min\{1, c_j\}$. This implies that $\frac{r_1 + r_2}{2} \in \{ r : F^{-1}_b(1-r) - F^{-1}_s(\frac{r}{c}) \geq \frac{1}{2} \mu_1^b + \frac{1}{2} \mu_2^b, 0 \leq r \leq \min\{1, c_j\} \}$. Thus, we have $\bar{r} \geq \frac{r_1 + r_2}{2} c_j$, and hence $r_j^\ast(\mu^b)$ is a concave function in $\mu^b \in [0, \bar{\nu}]$. Therefore, we have established that $r_j^\ast(\mu^b)$ is continuous, weakly decreasing and concave, which implies that the function $\mu^b r_j^\ast(\mu^b)$ is continuous and concave in $\mu^b \in [0, \bar{\nu}]$.

Next, we establish tight lower and upper bounds on $\mu^b r_j^\ast(\mu^b)$. To that end, we start by showing that $\max_{\mu \in [0, \bar{\nu}]} \mu r_j^\ast(\mu^b) = f(c_j)$ for all $j \in \mathcal{B}$, where $f$ as defined in (30) and recall that we have defined $c_j = y_j^\ast / b_j$. Let $\tilde{\mu}^b := \max_{\mu \in [0, \bar{\nu}]} \mu r_j^\ast(\mu^b)$. From (28), we have that $\mu^b \leq F^{-1}_b(1-r) - F^{-1}_s(\frac{r_1 + r_2}{2})$. Therefore, $\mu^b r_j^\ast(\mu^b) \leq [F^{-1}_b(1-r) - F^{-1}_s(\frac{r_1 + r_2}{2})] r_j^\ast(\mu^b) \leq \max_{\mu \in [0, \min\{1, c_j\}]} [F^{-1}_b(1-r) - F^{-1}_s(\frac{r}{c})] r = f(c_j)$. To show that this bound is tight, let $\bar{r} := \max_{\mu \in [0, \min\{1, c_j\}]} [F^{-1}_b(1-r) - F^{-1}_s(\frac{r}{c})] r$, and consider $\mu^b = F^{-1}_b(1-r) - F^{-1}_s(\frac{r}{c})$. Notice that $\mu^b \in [0, \bar{\nu}]$ and, from (28), we have $r_j^\ast(\mu^b) = \bar{r}$. Therefore, $\mu^b r_j^\ast(\mu^b) = \mu^b \bar{r} = [F^{-1}_b(1-r) - F^{-1}_s(\frac{r}{c})] \bar{r} = f(c_j)$. Finally, from (28), it is immediate to see that $r_j^\ast(\mu^b) \mu^b \geq 0$ and it is equal to zero if $\mu^b \in [0, \bar{\nu}]$. Therefore, we conclude that $0 \leq \mu^b r_j^\ast(\mu^b) \leq f(c_j)$, and both these bounds are tight.

Proof of part (i): $\frac{V_h}{V_{opt}} \geq \frac{1}{2}$. Define $\tilde{b}_j := b_j f(c_j)$, and define $\bar{r}_j^\ast(\mu^b) := b_j r_j^\ast(\mu^b)$ for $\mu^b \in [0, \bar{\nu}]$. Note that the function $r_j(\mu^b) \mu^b$ is still concave in $\mu^b \in [0, \bar{\nu}]$, as $\tilde{b}_j$ is a positive constant. Moreover, by using the tight upper bound on $r_j^\ast$, we conclude that $\max_{\mu \in [0, \bar{\nu}]} \mu \bar{r}_j^\ast(\mu^b) = 1$ for every $j \in \mathcal{B}$. Let the weight vector $\bm{w}$ be defined as $w_j := \sum_{j' \in \mathcal{B}} \tilde{b}_{j'}$ for all $j \in \mathcal{B}$.

$$
\frac{V_h}{V_{opt}} \geq \max_{\mu \in [0, \bar{\nu}]} \sum_{j' \in \mathcal{B}} b_j r_j^\ast(\mu^b) \mu^b \geq \max_{\mu \in [0, \bar{\nu}]} \sum_{j' \in \mathcal{B}} b_j \bar{r}_j^\ast(\mu^b) \mu^b \geq \max_{\mu \in [0, \bar{\nu}]} \sum_{j' \in \mathcal{B}} w_j \bar{r}_j^\ast(\mu^b) \mu^b \geq \max_{\mu \in [0, \bar{\nu}]} \min_{j' \in \mathcal{B}} \bar{r}_j^\ast(\mu^b) \mu^b.
$$

where the inequality in (a) follows from the fact that the numerator fact satisfies $V_h \geq \max_{\mu \in [0, \bar{\nu}]} \sum_{j' \in \mathcal{B}} b_j r_j^\ast(\mu^b) \mu^b$, and in the denominator we used Proposition (iii) to derive $V_{opt} = \sum_{j' \in \mathcal{B}} b_j f(\frac{\tilde{b}_{j'}}{c_j})$. The equality in (b) follows from the definition of $\tilde{b}$ and, similarly, the equality in (c) follows from using the definition of $\tilde{w}$. Finally, the inequality in (d), follows from the fact that $\sum_{j' \in \mathcal{B}} w_j \tilde{r}_j^\ast(\mu^b) \mu^b \geq \min_{j' \in \mathcal{B}} \tilde{r}_j^\ast(\mu^b) \mu^b \mu^b$ for all $\mu^b \in [0, \bar{\nu}]$.

Next, we provide a lower bound on $\max_{\mu \in [0, \bar{\nu}]} \min_{j' \in \mathcal{B}} \tilde{r}_j^\ast(\mu^b) \mu^b$. To that end, note $H_j(\mu^b) := \mu^b \tilde{r}_j^\ast(\mu^b)$ for all $\mu^b \in [0, \bar{\nu}]$ and define two functions $H, G : [0, \bar{\nu}] \rightarrow [0, 1]$ where $H(\mu^b) := \min_{j' \in \mathcal{B}} H_j(\mu^b)$ and $G(\mu^b) := \min_{\mu^b \in [0, \bar{\nu}]} \left( \frac{\mu^b}{b_j} - 1 \right)$. By Claim 2 above, we have that $H_j(\mu^b)$ is continuous and concave for every $\mu^b \in [0, \bar{\nu}]$. Therefore, we have that $H(\mu^b)$ is continuous and concave in $\mu^b \in [0, \bar{\nu}]$ because it is the minimum of a finite set of continuous and concave functions. Moreover, as each $H_j$ is continuous and concave, it is differentiable almost everywhere, and thus $H$ is differentiable almost everywhere. By Claim 2 above, we also
have $H(0) = H(\bar{v}_b) = 0$. In addition, note that $G(\mu^b)$ is a piecewise linear concave function, consisting of two pieces, and it is symmetric around $\mu^b = \frac{1}{2} \bar{v}_b$, where it achieves its peak value of $\frac{1}{2}$.

We want to show that $H\left(\frac{1}{2} \bar{v}_b\right) \geq G\left(\frac{1}{2} \bar{v}_b\right) = 1/2$; this will imply the desired bound as $\max_{\mu^b \in [0,\bar{v}_b]} \min_{v \in \mathcal{E}} r_j^b(\mu^b) \geq \min_{v \in \mathcal{E}} r_j^b \left(\frac{1}{2} \bar{v}_b\right) \geq 1/2 \bar{v}_b = H\left(\frac{1}{2} \bar{v}_b\right)$. In fact, we show a stronger result: we show that $H(\mu^b) \geq G(\mu^b)$ for all $\mu^b \in [0,\bar{v}_b]$.

Suppose towards contradiction that there exists $\mu \in [0,\bar{v}_b]$ such that $H(\mu) < G(\mu)$, and suppose that $H$ is differentiable at $\mu$. The latter is without loss of generality; as both $H$ and $G$ are continuous in $[0,\bar{v}_b]$ and $H$ is differentiable almost everywhere, next, we derive contradictions all possible values of $H'(\mu)$:

(1) If $H'(\mu) > \frac{1}{v}$, then by the concavity of $H$, we have $H(0) \leq H(\mu) + H'(\mu)(0 - \mu)$. As $H'(\mu)(0 - \mu) \leq -\frac{1}{v}$, we have $H(\mu) + H'(\mu)(0 - \mu) < G(\mu) - \bar{v}_b = 0$, and thus $H(0) < 0$, which is a contradiction.

(2) If $0 \leq H'(\mu) \leq \frac{1}{v}$; as we assumed that $H$ is differentiable at $\mu$, then there exist a $\bar{j} \in \mathcal{B}$ and an error $\epsilon > 0$ such that $H(\mu) = H_{\bar{j}}(\mu)$ for every $\mu \in (\mu - \epsilon, \mu + \epsilon)$, and thus we must have $0 \leq H'_{\bar{j}}(\mu) \leq \frac{1}{v}$. We argue that this will contradict $\max_{x \in [0,\bar{v}_b]} H_{\bar{j}}(x) = 1$, which was established in Claim 2 above. For that, we use the fact that the concavity of $H_{\bar{j}}(\mu)$ implies that $H_{\bar{j}}(\mu) \leq H_{\bar{j}}(\mu) + H'_{\bar{j}}(\mu)(\mu - \mu)$, for any $\mu \in [0,\bar{v}_b]$. Then, for $\mu \in [0,\bar{v}_b]$, we have $H_{\bar{j}}(\mu) \leq H_{\bar{j}}(\mu) < G(\mu)$ 2/2. Moreover, for any $\mu \in [\bar{\mu},\bar{v}_b]$, we have $H'_{\bar{j}}(\mu)(\mu - \mu) \leq \frac{1}{\bar{v}_b}(\mu - \mu)$, which implies that $H_{\bar{j}}(\mu) < G(\mu) + \frac{1}{\bar{v}_b}(\mu - \mu) \leq \frac{2}{\bar{v}_b}$. Therefore, we have that $H_{\bar{j}}(\mu) < 1$ for every $\mu \in [0,\bar{v}_b]$, and thus $\max_{x \in [0,\bar{v}_b]} H_{\bar{j}}(x) < 1$, which is a contradiction.

(3) if $-\frac{1}{v} \leq H'(\mu) < 0$, we follow an argument along the lines of the one in case (2). Let $\bar{j}_0 \in \mathcal{B}$ be as defined in (2), and we prove a contradiction to $\max_{x \in [0,\bar{v}_b]} H_{\bar{j}_0}(\mu) = 1$. For any $\mu \in [0,\bar{v}_b]$, we have $H'_{\bar{j}_0}(\mu)(\mu - \mu) \leq -\frac{1}{v}(\mu - \mu)$. As $H_{\bar{j}_0}(\mu) < G(\mu) \leq 1 - \frac{1}{v}$, we have $H_{\bar{j}_0}(\mu) < (1 - \frac{1}{v})(\mu - \mu) \leq 1$. For any $\mu \in [\bar{\mu},\bar{v}_b]$, we have $H'_{\bar{j}_0}(\mu)(\mu - \mu) \leq 0$, which implies that $H_{\bar{j}_0}(\mu) < G(\mu) < 1$. Therefore, we have that $H_{\bar{j}_0}(\mu) < 1$ for every $\mu \in [0,\bar{v}_b]$, and thus $\max_{x \in [0,\bar{v}_b]} H_{\bar{j}_0}(x) < 1$, which is a contradiction.

(4) If $H'(\mu) < -\frac{1}{v}$, the argument is similar to that in case (1). By the concavity of $H$, we have $H(\bar{v}_b) \leq H(\mu) + H'(\mu)(\bar{v}_b - \mu)$. Since $H'(\mu)(\bar{v}_b - \mu) \leq -\frac{1}{v}(\bar{v}_b - \bar{v}_b) \leq -1 + \frac{1}{v}$, this leads to a contradiction as $0 = H_{\bar{j}_0}(\bar{v}_b) < G(\bar{v}_b) - 1 + \frac{1}{v} = -1 + 2 \frac{1}{v} < 0$.

Therefore, we established that $H(\mu^b) \geq G(\mu^b)$ for all $\mu^b \in [0,\bar{v}_b]$. To conclude the proof, note that

$$ V_{\mu} \geq \max_{\mu^b \in [0,\bar{v}_b]} \min_{v \in \mathcal{E}} r_j^b(\mu^b) \geq \max_{\mu^b \in [0,\bar{v}_b]} \frac{1}{2} \bar{v}_b \geq G\left(\frac{1}{2} \bar{v}_b\right) = \frac{1}{2}. $$

where the first inequality follows from (31). Thus, we obtained $V_{\mu} \geq \frac{1}{2} V_{opt}$ as desired.

Proof of Part (ii). Suppose that the network satisfies the $\epsilon$-marriage condition for $\epsilon \in [0,1)$. (The claim holds trivially when $\epsilon = 1$ because $V_{\mu} \geq 0$.) From part (i) in Theorem 39 we have $V_{\max} = \sum_{j \in \mathcal{B}} \ell_j(f,\hat{c})$, where $\hat{c} := \sum_{i \in \mathcal{E}} \ell_i \tilde{s}_i$ and $f$ is as defined in (30). Define $\hat{r} := \arg \max_{x \in [0,\min(1,\tilde{c})]} \left[ F_{\nu}^{-1}(1 - r) - F_{\nu}^{-1}(1 - \tilde{c})\right] r$, and define $\hat{\nu} := \nu f \tilde{c}$. $\nu$ and $\hat{\nu}$ are defined in (28) for all $j \in \mathcal{B}$. The claim (ii)-1: $V_{\mu}^j \geq (1 - \epsilon) \hat{c}$ for all $j \in \mathcal{B}$. Let vector $y^*$ be the optimal solution to problem (17) and, as before, let $c_j = y_j^* / b_j$. We further let the distinct values of $c_j$’s to be given by $c_1 < c_2 \cdots < c_l$ and let $\mathcal{B}_1 := \{j \in \mathcal{B} : c_j = c_1\}$. This implies that

$$ y_j^* / b_j \geq c_1 = \sum_{j \in \mathcal{B}_1} y_j^* / b_j = \sum_{j \in \mathcal{B}_1} s_j \gamma_j = \sum_{j \in \mathcal{B}_1} \sum_{i \in \mathcal{E}} \ell_i \tilde{s}_i = (1 - \epsilon) \sum_{j \in \mathcal{B}_1} \sum_{i \in \mathcal{E}} \ell_i \tilde{s}_i = (1 - \epsilon) \hat{c} \quad \text{for every } j \in \mathcal{B}. $$

where the equality in (e) follows from the fact that, by Lemma 2, vector $y^*$ is a lexicographical optimal base in polymatroid $P := \{y \geq 0 : \sum_{j \in \mathcal{B}} y_j \leq \sum_{j \in \mathcal{B}} b_j, \mathcal{B} \in \mathcal{B} \}$ and, by the equivalence of item (i) and item (ii) of Theorem 3 in Fujishige [1980], we have $\sum_{j \in \mathcal{B}_1} y_j^* = \sum_{i \in \mathcal{E}} \tilde{s}_i$. The equality in (f) follows directly from the definition of the $\epsilon$-marriage condition.

Claim (ii)-2: $r_j^*(\hat{\nu}) \leq (1 - \epsilon) \hat{r}$ for all $j \in \mathcal{B}$. Let $r_j := r_j^*(\hat{\nu})$ where $r_j^*(\hat{\nu})$ is as defined in (28). We want to show $r_j^* \geq (1 - \epsilon) \hat{r}$ for all $j \in \mathcal{B}$. As the $F_{\nu}^{-1}(1 - r) - F_{\nu}^{-1}(1 - \tilde{c})$ is decreasing in $r$, then we have that one of the constraints in (29) must bind, that is, either $\hat{\nu} = F_{\nu}^{-1}(1 - r_j^*) - F_{\nu}^{-1}\left(\frac{r_j^*}{b_j}\right)$ or $r_j^* = \frac{y_j^*}{b_j}$. If $r_j^* = \frac{y_j^*}{b_j}$,
then \( r_j^* = \frac{y_j^*}{b_j^*} \geq (1-\varepsilon)\hat{c} \) follows from (33), and thus \( \frac{1}{1-\varepsilon} \frac{r_j^*}{\hat{c}} \geq 1 \geq \frac{\hat{c}}{c} \), where the last inequality follows from the definition of \( \hat{r} \). Hence, if \( r_j^* = \frac{y_j^*}{b_j^*} \), we have \( r_j^* \geq (1-\varepsilon)\hat{r} \).

Suppose that \( \bar{\mu}^b = F_b^{-1}(1-r_j^*) - F_s^{-1}(\frac{r_j^*}{y_j^*/b_j^*}) \). Recall that, by the definition of \( \bar{\mu}^b \), we also have \( \bar{\mu}^b = F_b^{-1}(1-\bar{r}) - F_s^{-1}(\frac{\bar{r}}{c}) \), and therefore

\[
F_b^{-1}(1-r_j^*) - F_s^{-1}\left(\frac{r_j^*}{y_j^*/b_j^*}\right) = F_b^{-1}(1-\bar{r}) - F_s^{-1}\left(\frac{\bar{r}}{c}\right).
\] (34)

We show that \( r_j^* \geq (1-\varepsilon)\hat{r} \) by discussing the two subcases depending on whether \( \frac{y_j^*}{b_j^*} \geq \hat{c} \) or \( \frac{y_j^*}{b_j^*} < \hat{c} \). If \( \frac{y_j^*}{b_j^*} \geq \hat{c} \), then (34) implies that \( r_j^* \geq \hat{r} \geq (1-\varepsilon)\hat{r} \). On the other hand, if \( \frac{y_j^*}{b_j^*} < \hat{c} \), then (34) implies that \( r_j^* < \hat{r} \). In this case, to show that \( \frac{r_j^*}{y_j^*/b_j^*} \geq \frac{\hat{c}}{c} \), suppose towards contradiction that \( \frac{r_j^*}{y_j^*/b_j^*} < \frac{\hat{c}}{c} \). Then, we must have \( F_s^{-1}(\frac{r_j^*}{y_j^*/b_j^*}) < F_s^{-1}(\frac{\hat{c}}{c}) \). Given \( r_j^* < \hat{r} \), we have \( F_b^{-1}(1-r_j^*) > F_b^{-1}(1-\hat{r}) \). This leads to contradtions as \( \bar{\mu}^b = F_b^{-1}(1-r_j^*) - F_s^{-1}(\frac{r_j^*}{y_j^*/b_j^*}) \). Hence, we have must have \( \frac{r_j^*}{y_j^*/b_j^*} \geq \frac{\hat{c}}{c} \) or equivalently \( r_j^* \geq (1-\varepsilon)\hat{r} \).

Finally, note that

\[
V_h(q) = \sum_{j \in B} \bar{\mu}^b_j r_j^*(\bar{\mu}^b)
\]

\[
\geq \sum_{j \in B} -\bar{\mu}^b_j (1-\varepsilon)\hat{r} \geq (1-\varepsilon) \sum_{j \in B} \left[ F_b^{-1}(1-\bar{r}) - F_s^{-1}\left(\frac{\bar{r}}{c}\right) \right] \hat{r} \geq (1-\varepsilon) V_{\max}. \tag{35}
\]

where inequality (g) follows from the definition of \( \bar{V}_h(\bar{\mu}^b) \), and equality (h) follows from (33). Inequality (i) follows from the fact \( r_j \geq (1-\varepsilon)\hat{r} \) for all \( j \in B \) (Claim (ii)-1). Inequality (j) follows from the fact that \( \bar{\mu}^b = F_b^{-1}(1-\hat{r}) - F_s^{-1}(\frac{\hat{r}}{c}) \). Equality (k) follows from the definition of \( V_{\max} \). This completes the proof that \( V_h \geq (1-\varepsilon) V_{\max} \).

Q.E.D.

**Proof of Proposition 5** We prove part (i) of the proposition, and defer the proof of part (ii) to the online appendix. By Proposition 1 for any \((0, \mu) \in \Gamma \times \mathcal{U} \), the induced equilibrium supply-demand vector \((\bar{q}^*, \bar{q}'')\) is unique. Thus, we can define function \( q_j^* : \mathcal{U} \rightarrow \mathbb{R}^+ \) for all \( i \in \mathcal{S} \) and \( \bar{q}_j^* : \mathcal{U} \rightarrow \mathbb{R}^+ \) for all \( j \in \mathcal{B} \) such that \((p, x, q^* (\mu), q^b (\mu)) \in \mathcal{X}(0, \mu) \). We consider the following lemma on the parametrized equilibrium supply-demand vectors, the proof of which can be found in Section (EC.6) of the electronic companion.

**Lemma 3.** For any \( \mu \in \mathcal{U} \), we have
(i) for all \( j_0 \in B \), function \( q_j^* (\mu) \) is decreasing in \( \mu_j \) for all \( i \in \mathcal{S} \);
(ii) for all \( i_0 \in \mathcal{S} \), function \( q_j^* (\mu) \) is decreasing in \( \mu_j \) for all \( j \in B \).

Let \( \bar{\mu} = (\bar{\mu}^*, \bar{\mu}^b) \) denote the optimal subscription vector obtained from part (ii) in Theorem 1. Then

\[
\max \{ V_0, V_1 \} \geq \frac{1}{2} \left( \sum_{i \in S} \mu_i^* g_i^*(\bar{\mu}^*, 0) + \sum_{j \in B} \mu_j^b g_j^b(0, \bar{\mu}) \right) \geq \frac{1}{2} \left( \sum_{i \in S} \mu_i^* g_i^*(\bar{\mu}^*, 0) + \sum_{j \in B} \mu_j^b g_j^b(0, \bar{\mu}) \right) \geq \frac{1}{2} V_{opt}. \tag{36}
\]

where inequality (a) follows directly from Lemma 3, which proves that \( q_j^* (\bar{\mu}^*, 0) \geq q_j^* (\bar{\mu}^*, \bar{\mu}^b) \) and \( q_j^* (0, \bar{\mu}^b) \geq q_j^* (0, \bar{\mu}^b) \). Equality (b) follows from the optimality of subscription vector \( \bar{\mu} \). Q.E.D.
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EC.1. Auxiliary Results for Appendix A

In this section, we consider an optimization framework that unifies all of the optimization problems covered in this paper. Define functions $g_j : [0, b_j] \to \mathbb{R} \cup \{-\infty, \infty\}$ with $b_j \in \mathbb{R}_+$ for all $j \in \mathcal{B}$ and $h_i : [0, s_i] \to \mathbb{R} \cup \{-\infty, \infty\}$ with $s_i \in \mathbb{R}_+$ for all $i \in \mathcal{S}$. We consider the following optimization problem

$$\max \sum_{j \in \mathcal{B}} g_j(q_j^0) - \sum_{i \in \mathcal{S}} h_i(q_i^0) \quad \text{(EC.1a)}$$

subject to

$$x_{ij} = q_j^b \quad \forall j \in \mathcal{B}, \quad \text{(EC.1b)}$$

$$x_{ij} = q_i^s \quad \forall i \in \mathcal{S}, \quad \text{(EC.1c)}$$

$$q_j^b \leq b_j \quad \forall j \in \mathcal{B}, \quad \text{(EC.1d)}$$

$$q_i^s \leq s_i \quad \forall i \in \mathcal{S}, \quad \text{(EC.1e)}$$

$$x_{ij} \geq 0 \quad \forall (i, j) \in \mathcal{E}. \quad \text{(EC.1f)}$$

In this paper, we characterize $g_j(q)$ and $h_i(q)$ under the following set of properties:

- (EC.1.1) for all $j \in \mathcal{B}$, function $g_j(q)$ is differentiable and strictly concave in $q \in (0, b_j)$, continuous at $q = 0$, and continuous at $q = b_j$ if $g_j(b_j) > -\infty$;
- (EC.1.2) for all $i \in \mathcal{S}$, function $h_i(q)$ is differentiable and strictly convex in $q \in (0, s_i)$, continuous at $q = 0$, and continuous at $q = s_i$ if $h_i(s_i) < \infty$;
- (EC.1.3) for all $i \in \mathcal{S}, j \in \mathcal{B}$, $g_j(b_j) \leq 0$ and $h_i(0) \geq 0$;
- (EC.1.4) for all $j \in \mathcal{B}$, $g_j(q) = b_j g\left(\frac{q}{b_j}\right)$;
- (EC.1.5) for all $i \in \mathcal{S}$, $h_i(q) = s_i h\left(\frac{q}{s_i}\right)$;
- (EC.1.6) for all $i \in \mathcal{S}$, $h_i(0) = 0$.

In the next result, we prove a list of properties related to optimization problem [EC.1].

**Proposition EC.1.** Under Assumptions [EC.1.1] - [EC.1.2], optimization problem [EC.1] has the following properties:

(i) there exists an optimal solution $(\mathbf{x}, \mathbf{q}^0, \mathbf{q}^b)$ to problem [EC.1];

(ii) in problem [EC.1], Slater’s condition holds.

Moreover, let $(\mathbf{x}, \mathbf{q}^0, \mathbf{q}^b)$ be an optimal solution to problem [EC.1] and $(\mathbf{\theta}^b, \mathbf{\theta}^s, \mathbf{\eta}^b, \mathbf{\eta}^s, \mathbf{\pi})$ be a dual optimal solution corresponding to constraints [EC.1b] - [EC.1f]. Mangasarian-Fromovitz constraint qualification (MFCQ) holds at $(\mathbf{x}, \mathbf{q}^0, \mathbf{q}^b)$, and the optimal primal-dual solution vector satisfies the KKT conditions

$$g_j'(q_j^b) - \theta_j^b - \eta_j^b = 0 \quad \forall j \in \mathcal{B}, \quad \text{(EC.2a)}$$

$$h_i'(q_i^s) - \theta_i^s + \pi_{ij} = 0 \quad \forall (i, j) \in \mathcal{E}, \quad \text{(EC.2b)}$$

$$\theta_j^b - \theta_i^s + \pi_{ij} = 0 \quad \forall (i, j) \in \mathcal{E}, \quad \text{(EC.2c)}$$

$$q_j^b \leq b_j \quad \perp \eta_j^b \geq 0 \quad \forall j \in \mathcal{B}, \quad \text{(EC.2d)}$$

$$q_i^s \leq s_i \quad \perp \eta_i^s \geq 0 \quad \forall i \in \mathcal{S}, \quad \text{(EC.2e)}$$

$$x_{ij} \geq 0 \quad \perp \pi_{ij} \geq 0 \quad \forall (i, j) \in \mathcal{E}; \quad \text{(EC.2f)}$$

11 Abusing some notation, we let $g_j'(0) = \lim_{q \to 0} g_j'(q)$ and $g_j'(b_j) = \lim_{q \to b_j} g_j'(q)$ for all $j \in \mathcal{B}$. We also let $h_i'(0) = \lim_{q \to 0} h_i'(q)$ and $h_i'(s_i) = \lim_{q \to s_i} h_i'(q)$ for all $i \in \mathcal{S}$. 
(iii) there is a unique optimal supply-demand vector \((q^*, q^b)\);
(iv) under Assumption \((EC.1-3)\), \(q_j^b < b_j \) for all \(j \in B\);
(v) under Assumption \((EC.1-3)\), there is a unique dual optimal vector \((\theta^*, \theta^b, \eta^*, \eta^b)\);
(vi) under Assumption \((EC.1-3)\) and \((EC.1-6)\), \(q_i^d > 0\) for all \(i \in S\);
(vii) under Assumption \((EC.1-4)\), optimal solution \((x, q^*, q^b)\) satisfies that if \(x_{ij}, x_{ij'} > 0\), then \(\frac{q_i^b}{b_j} = \frac{q_{ij}}{\frac{q_i}{b_j}}\); if \(x_{ij} > 0\) and \(x_{ij'} = 0\), then \(\frac{q_i^b}{b_j} \leq \frac{q_{ij}}{\frac{q_i}{b_j}}\);
(viii) under Assumption \((EC.1-5)\), optimal solution \((x, q^*, q^b)\) satisfies that if \(x_{ij}, x_{ij'} > 0\), then \(\frac{q_i^b}{s_i} = \frac{\partial x}{\partial s_i} \); if \(x_{ij} > 0\) and \(x_{ij'} = 0\), then \(\frac{q_i^b}{s_i} \leq \frac{\partial x}{\partial s_i} \).

In this paper, problem framework \((EC.1)\) summarizes the following three optimization problems:
(1) equilibrium problem \((14)\);
(2) revenue optimization problem \((7)\);
(3) welfare optimization problem \((EC.3)\) which is formulated as

\[
\begin{align*}
\max_{x, q^*, q^b} & \quad \sum_{j \in B} \int_0^{q_j^b} F_{b_j}^{-1} \left(1 - \frac{x}{b_j}\right) dx - \sum_{i \in S} \int_0^{q_i^d} F_{s_i}^{-1} \left(\frac{x}{s_i}\right) dx \\
\text{s.t.} & \quad \sum_{j : (i, j) \in E} x_{ij} = q_i^* \quad \forall i \in S, \\
& \quad \sum_{i : (i, j) \in E} x_{ij} = q_j^b \quad \forall j \in B, \\
& \quad q_i^* \leq s_i \quad \forall i \in S, \\
& \quad q_j^b \leq b_j \quad \forall j \in B, \\
& \quad x_{ij} \geq 0, \quad \forall (i, j) \in E.
\end{align*}
\]

\((EC.3a)\) \(\text{(EC.3b)}\) \(\text{(EC.3c)}\) \(\text{(EC.3d)}\) \(\text{(EC.3e)}\) \(\text{(EC.3f)}\)

In the following lemma, we formally show that all three problems fit into framework problem \((EC.1)\).

**Lemma EC.1.** (i) In problem \((14)\), \(g_j(q) = \int_0^q \tilde{F}_{b_j}^{-1} \left(1 - \frac{x}{b_j}\right) dx\) for all \(j \in B\) and \(h_i(q) = \int_0^q \tilde{F}_{s_i}^{-1} \left(\frac{x}{s_i}\right) dx\) where Assumptions \((EC.1-1)\) - \((EC.1-3)\) hold;
(ii) In problem \((7)\), \(g_j(q) = F_{b_j}^{-1} \left(1 - \frac{q}{b_j}\right) q\) for all \(j \in B\) and \(h_i(q) = F_{s_i}^{-1} \left(\frac{q}{s_i}\right) q\) where Assumptions \((EC.1-1)\) - \((EC.1-3)\) and \((EC.1-6)\) hold;
(iii) In problem \((EC.3)\), \(g_j(q) = \int_0^q \tilde{F}_{b_j}^{-1} \left(1 - \frac{x}{b_j}\right) dx\) for all \(j \in B\) and \(h_i(q) = \int_0^q \tilde{F}_{s_i}^{-1} \left(\frac{x}{s_i}\right) dx\) where Assumptions \((EC.1-1)\) - \((EC.1-3)\) and \((EC.1-6)\) hold;
(iv) In all three problems, if we denote the value function of problem \((EC.1)\) as \(V(s, b)\), then \(\frac{\partial V}{\partial s_i} V(s, b)\) is well-defined at \(s_i > 0\) for any \(i \in S\) and \(\frac{\partial V}{\partial b_j} V(s, b)\) is well-defined at \(b_j > 0\) for any \(j \in B\).

We also discuss a special case for problem framework \((EC.1)\). Under Assumptions \((EC.1-1)\) - \((EC.1-5)\) and Assumptions \((A1)\) - \((A4)\) we show that problem \((EC.1)\) can be reformulated into the polymatroid problem in \((17)\).

**Proposition EC.2.** Under Assumptions \((EC.1-1)\) - \((EC.1-5)\) and Assumption \((A4)\) \(\text{12}\) problem \((EC.1)\) and problem \((17)\) are equivalent.

\(\text{12}\) Assumptions \((A1)\) - \((A3)\) can be derived directly from Assumptions \((EC.1-1)\) - \((EC.1-5)\).
EC.1.1. Proof of Results in Section EC.1

Proof of Proposition EC.1. Proof of claim (i). In problem \((\text{EC.1})\), we denote \(Z\) as the compact set characterized by constraints \((\text{EC.1b}) - (\text{EC.1i})\). It is easy to verify that \(Z \neq \emptyset\) because \(0 \in Z\). To prove the claim in this proposition statement, we discuss two cases:

Case (i)-1: if function \(g_j(q)\) is continuous in \([0, b_j]\) for all \(j \in B\) and \(h_i(q)\) is continuous in \([0, s_i]\) for all \(i \in S\), then the objective function \(\sum_{j \in B} g_j(q_j^b) - \sum_{i \in S} h_i(q_i^s)\) is continuous in compact set \(Z\). By the extreme value theorem, an optimal solution \((\tilde{x}, \tilde{q}^s, \tilde{q}^b)\) exists.

Case (i)-2: in the more general case, under Assumptions \((\text{EC.1.1}) - (\text{EC.1.2})\) denote index set

\[
\begin{align*}
I &= \{i \in S : h_i(s_i) < \infty\}, \quad I^c = \{i \in S : i \notin I\}, \\
J &= \{j \in B : g_j(b_j) > -\infty\}, \quad J^c = \{j \in B : j \notin J\}.
\end{align*}
\]

We can compactly express the “sup” version of problem \((\text{EC.1})\) as

\[
\hat{Y} = \sup_{(\bar{x}, \bar{q}^s, \bar{q}^b) \in Z} \left\{ \sum_{j \in J} g_j \left( \frac{1}{2} b_j \right) + \sum_{j \in J^c} g_j \left( \frac{1}{2} b_j \right) - \sum_{i \in I} h_i(\bar{q}_i^s) - \sum_{i \in I^c} h_i(\bar{q}_i^s) \right\}. \tag{EC.5}
\]

By Assumption \((\text{EC.1.1})\), function \(g_j(\cdot)\) is concave and differentiable in \(q \in (0, b_j)\) and continuous at \(q = 0\). Thus, there exists

\[
K_j^b = \max \left\{ \max_{q \in [0, \frac{1}{2} b_j]} g_j(q), g_j \left( \frac{1}{2} b_j \right) \right\} < \infty, \tag{EC.6}
\]

such that

\[
g_j(q) \leq K_j^b < \infty \quad \text{for all} \quad q \in [0, b_j], \quad j \in B. \tag{EC.7}
\]

Similarly, by Assumption \((\text{EC.1.2})\) function \(h(\cdot)\) is convex and differentiable in \(q \in (0, s_i)\) and continuous at \(q = 0\). Thus, there exists

\[
K_i^s = \min \left\{ \min_{q \in [0, \frac{1}{2} s_i]} h_i(q), h_i \left( \frac{1}{2} s_i \right) \right\} > -\infty, \tag{EC.8}
\]

such that

\[
h_i(q) \geq K_i^s > -\infty \quad \text{for all} \quad q \in [0, s_i], \quad i \in S. \tag{EC.9}
\]

By \((\text{EC.7})\) and \((\text{EC.9})\), the objective function of problem \((\text{EC.5})\) satisfies

\[
\hat{Y} \leq \sum_{j \in J} K_j^b - \sum_{i \in I^c} K_i^s < \infty. \tag{EC.10}
\]

For any \(j \in J^c\), function \(g_j(q)\) is continuous in \([0, b_j]\) and \(\lim_{q \to b_j} g_j(q) = -\infty\), so there exists \(\tilde{b}_j < b_j\) such that for all \(q \geq \tilde{b}_j\), \(g_j(q) \leq -3|\hat{Y}|\). Similarly, for any \(i \in I^c\), function \(h_i(q)\) is continuous in \(q \in [0, s_i]\) with \(\lim_{q \to s_i} h_i(q) = \infty\), so there exists \(\tilde{s}_i < s_i\) such that for all \(q \geq \tilde{s}_i\), \(h_i(q) \geq 3|\hat{Y}|\). As a result, any feasible solution \((\tilde{x}, \tilde{q}^s, \tilde{q}^b) \in Z\) where there exists \(i \in I^c\) with \(\tilde{q}_i^s > \tilde{s}_i\) or \(j \in J^c\) with \(\tilde{q}_j^b > \tilde{b}_j\) is not optimal, otherwise, the objective function value would lead to a contradiction

\[
\sum_{j \in J} g_j(q_j^b) + \sum_{j \in J^c} g_j(q_j^b) - \sum_{i \in I} h_i(\tilde{q}_i^s) - \sum_{i \in I^c} h_i(\tilde{q}_i^s) \leq -3|\hat{Y}| + \hat{Y} \leq -2|\hat{Y}| < \hat{Y}.
\]

Thus, we can construct a compact subset

\[
Z_0 = Z \cap \left\{ (\tilde{x}, \tilde{q}^s, \tilde{q}^b) : \tilde{q}_i^s \leq \bar{s}_i, \forall i \in I^c, \tilde{q}_j^b \leq \bar{b}_j, \forall j \in J^c \right\}. \tag{EC.11}
\]
where optimization problem \( \text{(EC.5)} \) is simplified to

\[
\sup_{(x, q^b, q^h) \in Z_0} \left\{ \sum_{j \in J} g_j^b (q^b_j) + \sum_{j \in J^c} g_j (q^b_j) - \sum_{i \in I} h_i (q^h_i) - \sum_{i \in I^c} h_i (q^h_i) \right\}. \tag{EC.12}
\]

In problem \( \text{(EC.12)} \), we have \( Z_0 \neq \emptyset \) because \( 0 \in Z_0 \). Moreover, the objective function is continuous in \( Z_0 \). Thus, by the extreme value theorem, problem \( \text{(EC.12)} \) has an optimal solution \((x, q^b, q^h)\). This completes the proof that an optimal solution exists in optimization problem \( \text{(EC.1)} \).

Proof of claim (iii). To prove this claim, we show that Slater’s condition is satisfied. Consider a solution vector \((x, q^b, q^h)\) that satisfies

\[
x_{ij} = \frac{1}{2|E|} \min \{s, b\} \text{ for all } (i, j) \in E, \tag{EC.13a}
\]

\[
q^h_i = \sum_{j: (i,j) \in E} x_{ij} \text{ for all } i \in S, \tag{EC.13b}
\]

\[
q^b_j = \sum_{i: (i,j) \in E} x_{ij} \text{ for all } j \in B. \tag{EC.13c}
\]

In this construction, \( \text{(EC.13b)} \) and \( \text{(EC.13c)} \) ensure that equality constraints \( \text{(EC.1b)} - \text{(EC.1c)} \) are satisfied. Thus, Slater’s condition holds. In a convex optimization problem with finite dimensional space, by Lemma 2.3 of Li (1997), the Slater’s condition implies MFCQ at the optimal solution.

Based on MFCQ, KKT conditions follow.

Proof of claim (iv). To ease notation, denote the objective function for problem \( \text{(EC.1)} \) as

\[
f(q^b, q^h) = \sum_{j \in B} g_j(q^b_j) - \sum_{i \in S} h_i(q^h_i),
\]

which is strictly concave by Assumption \( \text{(EC.1-1)} \). Suppose towards contradiction that there exists two optimal solution vectors \((x_0, q^b_0, q^h_0)\) and \((x_1, q^b_1, q^h_1)\) with \((q^b_0, q^h_0) \neq (q^b_1, q^h_1)\). Pick any \( \lambda \in (0,1) \) and let \((\lambda x_0 + (1 - \lambda) x_1, \lambda q^b_0 + (1 - \lambda) q^b_1, \lambda q^h_0 + (1 - \lambda) q^h_1)\). Since the feasible region for problem \( \text{(EC.1)} \) is convex, \((\lambda x_0 + (1 - \lambda) x_1, \lambda q^b_0 + (1 - \lambda) q^b_1, \lambda q^h_0 + (1 - \lambda) q^h_1)\) is feasible. By Jensen’s inequality, the objective function satisfies

\[
f(q^b, q^h) > \lambda f(q^b_1, q^h_1) + (1 - \lambda) f(q^b_0, q^h_0),
\]

which is a contradiction that \((x_0, q^b_0, q^h_0)\) and \((x_1, q^b_1, q^h_1)\) are optimal. As a result, for any optimal solution \((x, q^b, q^h)\), vector \((q^b, q^h)\) is unique.

Proof of claim (v). Let \((x, q^b, q^h)\) and \((\theta^b, \theta^h, \eta^b, \eta^h, \pi)\) respectively be a primal optimal solution vector and a dual optimal solution vector to problem \( \text{(EC.1)} \). Suppose towards contradiction that \(q^b_j = b_j\). This allows us to find \( i : x_{ij} > 0 \) or correspondingly \( q^h_i > 0 \) such that

\[
0 \overset{(a)}{\geq} g'_j(b_j) \overset{(b)}{=} \theta^b_j + \eta^b_j \overset{(c)}{\geq} \theta^h_j \overset{(d)}{=} h'_i(q^h_i) + \eta^h_i \overset{(f)}{\geq} h'(0) \overset{(g)}{=} 0,
\]

where step (a) follows directly from Assumption \( \text{(EC.13)} \). Step (b) follows directly from condition \( \text{(EC.2a)} \). In step (c), the inequality is satisfied under \( \eta^h_i \geq 0 \) by condition \( \text{(EC.2d)} \). In step (d), given \( x_{ij} > 0 \), we have \( \pi_{ij} = 0 \) by condition \( \text{(EC.2f)} \). From \( \theta^b_j + \theta^h_j + \pi_{ij} = 0 \) in condition \( \text{(EC.2e)} \), we obtain \( \theta^h_j = \theta^b_j \). Step (e) follows directly from condition \( \text{(EC.2b)} \). In step (f), we have \( \eta^h_i \geq 0 \) by condition \( \text{(EC.2e)} \) and \( h'(q^h_i) \geq h'(0) \) by the strict convexity of \( h_i(\cdot) \) in Assumption \( \text{(EC.1-2)} \). Step (g) follows directly from Assumption \( \text{(EC.1-3)} \). All together, this implies that \( h'(q^h_i) = h'(0) \) with \( q^h_i > 0 \), which is a contradiction to strict concavity of \( h_i(\cdot) \) in Assumption \( \text{(EC.1-2)} \). Thus, we have \( q^h_j < b_j \) for all \( j \in B \).

Proof of claim (vi). Let \((x, q^b, q^h)\) and \((\theta^b, \theta^h, \eta^b, \eta^h, \pi)\) respectively be a primal optimal solution and dual optimal solution to problem \( \text{(EC.1)} \).

We start from showing the uniqueness of \( \eta^b \). For all \( j \in B \), by item (iv) we have \( q^h_j < b_j \), which implies \( \eta^h_j = 0 \) by condition \( \text{(EC.2d)} \).

To see the uniqueness of \( \theta^h \), for all \( j \in B \), given \( \eta^h_j = 0 \), condition \( \text{(EC.2a)} \) suggests that \( \theta^b_j = g'_j(q^b_j) \). The uniqueness of \( \theta^h_j \) follows from the uniqueness of \( q^h_j \) in item (iii).
For any $i \in S$, we discuss two cases: (vi) 1 if $q_i^* = 0$, then by condition (EC.2e), we have $\eta_i^* = 0$, which further implies from condition (EC.2b) that $\theta_i^* = h_i'(q_i^*)$. The uniqueness of $(\eta_i^*, \theta_i^*)$ follows from the uniqueness of $q_i^*$ in item (iii) (vi) 2 if $q_i^* > 0$, then we can find $j : x_{ij} > 0$ such that $\pi_{ij} = 0$ by condition (EC.2f). This further implies $\theta_i^* = \theta_j^*$ by condition (EC.2e). The uniqueness of $\theta_j^*$ follows from the uniqueness of $\theta_i^*$. With $\eta_i^* = h_i'(q_i^*) - \theta_i^*$ by condition (EC.2b), the uniqueness of $\eta_i^*$ follows from the uniqueness of $q_i^*$ and $\theta_i^*$.

Proof of claim (vi). Let $(x, q^*, \theta^*, \eta^*, \pi)$ respectively be a primal optimal solution and a dual optimal solution to problem (EC.1). We start by showing $g_j'(q_j^*) > 0$ for all $j \in B$. Suppose towards contradiction that $g_j'(q_j^*) \leq 0$ for some $j \in B$. From $g_j'(0) > 0$ in Assumption (EC.1-3) and the strictly concavity of $g_j(q)$ in $(0, b_j)$ in Assumption (EC.1-1), we obtain $q_j^* > 0$. Then, we can find $i : x_{ij} > 0$, or correspondingly $q_i^* > 0$. Given $h_i'(0) \geq 0$ in Assumption (EC.1-3) and the strict convexity of $h_i(q)$ over $(0, s_i)$ in Assumption (EC.1-2), we obtain $h_i'(q_i^*) = h_i'(0) > 0$. This would lead to the following contradiction

$$0 < h_i'(q_i^*) + \eta_i^* = \theta_i^* - \theta_i^* = g_j'(q_j^*) - \eta_j^* = g_j'(q_j^*) \leq 0,$$

(E.C.15)

where step (h) follows from inequality $h_i'(q_i^*) > 0$ and $\eta_i^* \geq 0$ by condition (EC.2e). Step (i) follows directly from condition (EC.2b). In step (j), we have $\pi_{ij} = 0$ by condition (EC.2b) given $x_{ij} > 0$, which further implies $\theta_i^* = \theta_j^*$ by condition (EC.2e). Step (k) follows directly from condition (EC.2a). In step (l), we have $\eta_j^* = 0$ by condition (EC.2d) given $q_j^* < b_j$ shown in item (iv). Step (m) comes from Assumption (EC.1-3). By the contradiction in (E.C.15), we obtain

$$g_j'(q_j^*) > 0 \quad \forall j \in B.$$

(E.C.16)

Next, suppose towards contradiction that there exists $i \in S$ with $q_i^* = 0$, we can derive the following contradiction

$$0 = (n) h_i'(0) = \theta_i^* - \theta_i^* = g_j'(q_j^*) - \eta_j^* > 0,$$

(E.C.17)

where step (n) follows directly from Assumption (EC.1-6). In step (o), given that $q_i^* = 0$, we first have $\eta_i^* = 0$ by condition (EC.2e). With $\eta_i^* = 0$, step (n) directly follows from $h_i'(q_i^*) - \theta_i^* + \eta_i^* = 0$ in condition (EC.2b). The inequality in step (p) follows from $\theta_i^* - \theta_j^* = \pi_{ij}$ by condition (EC.2e) where we have $\pi_{ij} \geq 0$ by condition (EC.2b). Step (q) follows directly from condition (EC.2a). In step (r), we obtain $\eta_j^* = 0$ by condition (EC.2d) given $q_j^* < b_j$ in item (iv). We also obtain $g_j'(q_j^*) > 0$ from (E.C.16). The contradiction in (E.C.17) implies that $q_i^* > 0$ for all $i \in S$.

Proof of claim (vii). We prove this claim by discussing the following two cases.

Case (vii)-1: If $x_{ij}, x_{ij'} > 0$, then we have $\pi_{ij} = \pi_{ij'} = 0$ from condition (EC.2f). By condition (EC.2e), we obtain

$$\theta_j^* = \theta_i^* = \theta_j^*,$$

(E.C.18)

With $g_j(q) = b_j g\left(\frac{q}{b_j}\right)$ in Assumption (EC.1-4), we have $g_j'(q) = g'\left(\frac{q}{b_j}\right)$. Suppose towards contradiction that $\frac{q}{b_j} \neq \frac{q}{b_j'}$. Without loss of generality, we assume $\frac{q}{b_j} > \frac{q}{b_j'}$. Condition (EC.2d) implies

$$\eta_j^* \geq 0 = \eta_j^*.$$

(E.C.19)

From (E.C.18) and (E.C.19), we obtain $\theta_j^* = \eta_j^* \geq \theta_j^* = \eta_j^*$. Condition (EC.2a) suggests that $\theta_j^* + \eta_j^* = g'\left(\frac{q}{b_j}\right)$ and $\theta_j^* + \eta_j^* = g'\left(\frac{q}{b_j'}\right)$. From (E.C.18) and (E.C.19), we further obtain

$$g'\left(\frac{q}{b_j}\right) \geq g'\left(\frac{q}{b_j'}\right).$$

(E.C.20)
However, from \( \frac{q^i_j}{b_j} > \frac{q^i_{j'}}{b_j} \) and the concavity of function \( g(\cdot) \) by Assumption \((A1)\), we obtain

\[
g'\left(\frac{q^i_j}{b_j}\right) < g'\left(\frac{q^i_{j'}{b_j}}{b_j}\right). \tag{EC.21}
\]

By the contradiction in \(\text{(EC.20)}\) and \(\text{(EC.21)}\), we obtain \( \frac{q^i_j}{b_j} = \frac{q^i_{j'}}{b_j} \).

Case \((\text{vii})2:\)
If \( x_{ij} > 0 \) and \( x_{ij'} = 0 \), by condition \(\text{(EC.2f)}\), we have \( \pi_{ij} = 0 \leq \pi_{ij'} \). Condition \(\text{(EC.2c)}\) implies \( \theta^i_j = \theta^i_{j'} \). To prove the claim, we assume towards contradiction that \( \theta^i_j > \theta^i_{j'} \), then repeat the same argument as in case \((\text{vii})1\) to achieve a contradiction between \( g'\left(\frac{q^i_j}{b_j}\right) \geq g'\left(\frac{q^i_{j'}{b_j}}{b_j}\right) \) and \( g'\left(\frac{q^i_j}{b_j}\right) > g'\left(\frac{q^i_{j'}{b_j}}{b_j}\right) \).

Proof of claim \((\text{viii})\) We prove this claim by discussing the following two cases.

Case \((\text{viii})1:\)
If \( x_{ij}, x_{ij'} > 0 \), then by condition \(\text{(EC.2f)}\), we have \( \pi_{ij} = \pi_{ij'} = 0 \). From condition \(\text{(EC.2c)}\), we further deduce

\[
\theta^i_j = \theta^i_{j'} = \theta^i_j. \tag{EC.22}
\]

Suppose towards contradiction that \( \frac{q^i_j}{s_i} \neq \frac{q^i_{j'}}{s_i} \). Without loss, we assume \( \frac{q^i_j}{s_i} > \frac{q^i_{j'}}{s_i} \). Condition \(\text{(EC.2c)}\) implies

\[
\eta^i_j \geq 0 = \eta^i_{j'}. \tag{EC.23}
\]

From \(\text{(EC.22)}\) and \(\text{(EC.23)}\), we obtain \( \theta^i_j - \eta^i_j \leq \theta^i_{j'} - \eta^i_{j'} \). By condition \(\text{(EC.2b)}\), we have \( \theta^i_j - \eta^i_j = h'\left(\frac{q^i_j}{s_i}\right) \) and \( \theta^i_{j'} - \eta^i_{j'} = h'\left(\frac{q^i_{j'}}{s_i}\right) \). By \(\text{(EC.22)}\) and \(\text{(EC.23)}\), we have

\[
h'\left(\frac{q^i_j}{s_i}\right) \leq h'\left(\frac{q^i_{j'}}{s_i}\right). \tag{EC.24}
\]

However, with \( \frac{q^i_j}{s_i} > \frac{q^i_{j'}}{s_i} \) and the strict convexity of function \( h(\cdot) \) from Assumption \((A2)\), we have

\[
h'\left(\frac{q^i_j}{s_i}\right) > h'\left(\frac{q^i_{j'}}{s_i}\right). \tag{EC.25}
\]

Thus, from the contradiction in \(\text{(EC.24)}\) and \(\text{(EC.25)}\), we have \( \frac{q^i_j}{s_i} = \frac{q^i_{j'}}{s_i} \).

Case \((\text{viii})2:\)
If \( x_{ij} > 0 \) and \( x_{ij'} = 0 \), then we have \( \pi_{ij} = 0 \leq \pi_{ij'} \) by condition \(\text{(EC.2f)}\), which further implies \( \theta^i_j = \theta^i_{j'} \) by condition \(\text{(EC.2c)}\). To prove the claim, we assume towards contradiction that \( \frac{q^i_j}{s_i} > \frac{q^i_{j'}}{s_i} \) and repeat the same proof-by-contradiction argument as in case \((\text{viii})1\) to achieve a contradiction between \( h'\left(\frac{q^i_j}{s_i}\right) \leq h'\left(\frac{q^i_{j'}}{s_i}\right) \) and \( h'\left(\frac{q^i_j}{s_i}\right) > h'\left(\frac{q^i_{j'}}{s_i}\right) \). This contradiction suggests that \( \frac{q^i_j}{s_i} \leq \frac{q^i_{j'}}{s_i} \). Q.E.D.

Proof of Lemma \((\text{EC.1})\)
Proof of Claim \((i)\) In problem \((14)\), we have

\[
g_j(q) = \int_0^q \hat{F}_{b_j}^{-1}\left(1 - \frac{x}{b_j}\right) \, dx \quad \text{where} \quad \hat{F}_{b_j}^{-1}(q) = \frac{F_{b_j}^{-1}(q) - \mu_j}{1 + \gamma_j} \quad \forall j \in \mathcal{B}, \tag{EC.26a}
\]

\[
h_i(q) = \int_0^q \hat{F}_s^{-1}\left(\frac{x}{s_i}\right) \, dx \quad \text{where} \quad \hat{F}_s^{-1}(q) = \frac{F_{s_i}^{-1}(q) + \mu_i}{1 - \gamma_i} \quad \forall i \in \mathcal{S}. \tag{EC.26b}
\]
To verify Assumption [EC.1-1] in problem [14], we start from showing that \( g_j(q) \) is differentiable in \((0, b_j)\). For any \( q \in (0, b_j) \), we pick \( c \in (0, q) \) and \( d \in (q, b_j) \) such that

\[
g_j(q) = \int_c^q \bar{F}_{b_j}^{-1}(1 - \frac{x}{b_j})dx + \int_c^q \bar{F}_{b_j}^{-1}(1 - \frac{x}{b_j})dx. \tag{EC.27}
\]

By [EC.26a], the first component in [EC.27] satisfies \( \int_c^c \bar{F}_{b_j}^{-1}(1 - \frac{x}{b_j})dx = \frac{1}{1 + \gamma_j} \int_0^c F_{b_j}^{-1}(1 - \frac{x}{b_j}) - \mu^b dx \).

By Assumption 1, it is finite constant term. In the second component of [EC.27], function \( \bar{F}_{b_j}^{-1}(1 - \frac{x}{b_j}) \) is differentiable in compact set \([c, d]\). By the Fundamental Theorem of Calculus, function \( g_j(q) \) is differentiable at \( q \in [c, d] \subset (0, b_j) \). Since \( q \) is picked arbitrarily within \((0, b_j)\), we obtain that \( g_j(q) \) is differentiable in \((0, b_j)\).

To see strict concavity, we calculate the derivative function \( g_j'(q) = \bar{F}_{b_j}^{-1}(1 - \frac{q}{b_j}) \). Since \( g_j'(q) \) is strictly decreasing in \((0, b_j)\), function \( g_j(q) \) is strictly concave in \((0, b_j)\).

To see continuity of \( g_j(q) \) at \( q = 0 \), it is sufficient to show that the limiting point is well-defined. We discuss two cases:

1. when \( \bar{v}_{b_j} < \infty \), \( \lim_{q \to 0} g_j(q) = 0 \) follows directly from its expression in [EC.26a].
2. when \( \bar{v}_{b_j} = \infty \), if \( (v, q) \) is such that \( F_{b_j}(v) = 1 - \frac{q}{b_j} \), then

\[
\int_0^q F_{b_j}^{-1}(1 - \frac{x}{b_j})dx = \frac{v}{b_j} (1 - F_{b_j}(v)) + \frac{1}{b_j} \int_v^\infty \left(1 - F_{b_j}(x)\right)dx. \tag{EC.28}
\]

Moreover, we have

\[
v \left(1 - F_{b_j}(v)\right) = v \int_v^\infty F_{b_j}'(x)dx \leq \int_0^\infty xF_{b_j}'(x)dx - \int_v^\infty xF_{b_j}'(x)dx \tag{EC.29}
\]

Thus,

\[
\lim_{q \to 0} \int_0^q \bar{F}_{b_j}^{-1}(1 - \frac{x}{b_j})dx \equiv (a) \lim_{v \to \infty} \frac{1}{b_j(1 + \gamma_j)} v \left(1 - F_{b_j}(v)\right) + \lim_{v \to \infty} \frac{1}{b_j(1 + \gamma_j)} \int_v^\infty \left(1 - F_{b_j}(x)\right)dx - \frac{1}{1 + \gamma_j} \lim_{q \to 0} \int_0^q \mu^b dx \\
\leq (b) \frac{1}{b_j(1 + \gamma_j)} \left[ \int_0^\infty xF_{b_j}'(x)dx - \lim_{v \to \infty} \int_v^\infty xF_{b_j}'(x)dx \right] + \frac{1}{1 + \gamma_j b_j} \int_0^\infty 1 - F_{b_j}(x)dx - \lim_{q \to \infty} \int_0^q \mu^b dx \leq (c) 0. \tag{EC.30}
\]

where step (a) follows from [EC.26a] and [EC.28]. In step (b), the first term follows directly from [EC.29]. The second term follows from a reorganization of integral form. In step (c), both the first term and the second term are zero because of the finite expectation assumption of \( F_{b_j}(-) \) in Assumption 1. The third term is clearly zero.

To show \( \lim_{q \to 0} \int_0^q \bar{F}_{b_j}^{-1}(1 - \frac{x}{b_j})dx \geq 0 \), we start from the expression in step (a). Since first two terms are nonnegative and the third term is zero, we have

\[
\lim_{q \to 0} \int_0^q \bar{F}_{b_j}^{-1}(1 - \frac{x}{b_j})dx \geq 0. \tag{EC.31}
\]

Thus, from [EC.26a], [EC.30] and [EC.31], we conclude that \( g_j(q) \) is continuous at \( q = 0 \) with \( g_j(0) = 0 \).
To show the continuity of \( g_j(q) \) at \( q = b_j \), we deduce that
\[
g_j(b_j) = \frac{1}{1 + \gamma_j} \int_0^{b_j} F_{b_j}^{-1} \left( 1 - \frac{x}{b_j} \right) dx - \frac{\mu_j^b}{1 + \gamma_j} b_j
\]
\[= \frac{1}{b_j(1 + \gamma_j)} \int_0^{\gamma_j b_j} \left( 1 - F_{b_j}(x) \right) dx - \frac{\mu_j^b}{1 + \gamma_j} b_j \in \left[ -\frac{\mu_j^b}{1 + \gamma_j} b_j, \infty \right), \tag{EC.32}
\]
where step (d) follows from (EC.26a). Step (e) follows from an alternative form of integration. Step (f) follows from the finite expectation in Assumption 1. Thus, the limiting point \( g_j(b_j) \) is well-defined.

To verify Assumption [EC.1-2] in problem [14], we start from showing that \( h_i(q) \) is differentiable in \((0,s_i]\). Similar to verifying Assumption [EC.1-1], for any \( q \in (0,s_i) \), we can pick a compact set \([c,d] \subset (0,s_i)\) such that for all \( q \in [c,d] \), we express \( h_i(q) \) as
\[
h_i(q) = \int_0^q \tilde{F}_{s_i}^{-1} \left( \frac{x}{s_i} \right) dx = \int_0^c \tilde{F}_{s_i}^{-1} \left( \frac{x}{s_i} \right) dx + \int_c^q \tilde{F}_{s_i}^{-1} \left( \frac{x}{s_i} \right) dx.	ag{EC.33}
\]
By [EC.26a], the first term in [EC.33] satisfies \( \int_0^c \tilde{F}_{s_i}^{-1} \left( \frac{x}{s_i} \right) dx = \frac{1}{1 - \gamma_i^s} \int_0^c F_{s_i}^{-1} \left( \frac{x}{s_i} \right) + \mu_i^s dx \). By Assumption 1, it is a finite constant. In the second term of [EC.33], \( \tilde{F}_{s_i}^{-1} \left( \frac{x}{s_i} \right) \) is differentiable in \( q \in [c,d] \). By Fundamental Theorem of Calculus, function \( h_i(q) \) is differentiable in \( q \in (0,s_i) \).

To see the convexity of function \( h_i(q) \), we obtain the derivative function \( h_i'(q) = \tilde{F}_{s_i}^{-1} \left( \frac{q}{s_i} \right) \), which is strictly increasing in \( q \in (0,s_i) \). Thus, function \( h_i(q) \) is strictly convex in \( (0,s_i) \).

The see the continuity of \( h_i(q) \) at \( q = 0 \), from \( \lim_{q \to 0} \int_0^q \tilde{F}_{s_i}^{-1} \left( \frac{x}{s_i} \right) dx = 0 \) and \( \lim_{q \to 0} \int_0^q \mu_i^s dx = 0 \), we obtain \( \lim_{q \to 0} \int_0^q \tilde{F}_{s_i}^{-1} \left( \frac{x}{s_i} \right) dx = 0 \).

To see the conditional continuity at \( q = s_i \), we show that
\[
h_i(s_i) = \int_0^{s_i} \tilde{F}_{s_i}^{-1} \left( \frac{x}{s_i} \right) dx = \frac{1}{1 - \gamma_i^s} \int_0^{s_i} F_{s_i}^{-1} \left( \frac{x}{s_i} \right) dx + \frac{\mu_i^s}{1 - \gamma_i^s} s_i
\]
\[= \frac{1}{s_i(1 - \gamma_i^s)} \int_0^s_i \left( 1 - F_{s_i}(x) \right) dx + \frac{\mu_i^s}{1 - \gamma_i^s} s_i \in (0,\infty), \tag{EC.34}
\]
where step (f) and (g) follow directly from (EC.26b). Step (h) follows from Assumption 1.

To verify Assumption [EC.1-3] in problem [14], from [EC.26a], we have
\[
\lim_{q \to b_j} g_j(q) = \lim_{q \to b_j} \tilde{F}_{b_j}^{-1} \left( 1 - \frac{q}{b_j} \right) \leq 0 \quad \forall j \in \mathcal{B},
\]
\[
\lim_{q \to 0} h_i'(q) = \lim_{q \to s_i} \tilde{F}_{s_i}^{-1} \left( \frac{q}{s_i} \right) \geq 0 \quad \forall i \in \mathcal{S}. \tag{EC.35}
\]

**Proof of Claim (ii)** In problem [7], we let
\[
g_j(q) = F_{b_j}^{-1} \left( 1 - \frac{q}{b_j} \right) q \quad \forall q \in [0,b_j], j \in \mathcal{B}, \tag{EC.36a}
\]
\[
h_i(q) = F_{s_i}^{-1} \left( \frac{q}{s_i} \right) q \quad \forall q \in [0,s_i], i \in \mathcal{S}. \tag{EC.36b}
\]

To verify Assumption [EC.1-1] in problem [7], with [EC.36a], the differentiability of \( g_j(q) \) follows directly from the differentiability of \( F_{b_j}^{-1}(\cdot) \) in \((0,b_j)\) by Assumption 1. The concavity follows directly from Assumption 2.

To see the continuity of \( g_j(q) \) at \( q = 0 \), we show that
\[
0 \leq \lim_{q \to 0} g_j(q) = \lim_{q \to 0} F_{b_j}^{-1} \left( 1 - \frac{q}{b_j} \right) q \leq \lim_{q \to 0} \int_0^q F_{b_j}^{-1} \left( 1 - \frac{x}{b_j} \right) dx \equiv 0, \tag{EC.37}
\]
where step (i) follows from nonnegativity of $F_{s_j}(\cdot)$ in Assumption 1. Step (j) follows from the increasing property of $F_{s_j}(\cdot)$ by Assumption 1. Step (k) follows from the continuity discussion from (EC.30) and (EC.31). Thus, $g_j(q)$ is continuous at $q = 0$.

Regarding the continuity of $g_j(q)$ at $q = b_j$, we have $\lim_{q \to b_j} g_j(q) = 0$ by (EC.36).

To verify Assumption (EC.1-2) in problem (7), by (EC.36b), the differentiability of $h_i(q)$ follows directly from the differentiability of $F_{s_i}^{-1}(\cdot)$ in $(0, s_i)$ by Assumption 1. The convexity condition follows directly from Assumption 2.

For the continuity of $h_i(q)$ at $q = 0$, we have $\lim_{q \to 0} h_i(q) = 0$ by (EC.36). For the continuity of $h_i(q)$ at $q = s_i$, $h_i(s_i) < \infty$ corresponds to the case when $\tilde{v}_{s_i} < \infty$ such that $\lim_{q \to s_i} h_i(q) = \lim_{q \to s_i} F_{s_i}^{-1}(\frac{q}{s_i}) < \infty$.

To verify Assumption (EC.1-3) in problem (7), from (EC.36), we first calculate that

$$g_j(q) = F_{b_j}^{-1}\left(1 - \frac{q}{b_j}\right) - [F_{b_j}^{-1}]' \left(1 - \frac{q}{b_j}\right) \frac{q}{b_j} \quad \forall j \in B,$$

(E38a)

$$h_i(q) = F_{s_i}^{-1}\left(\frac{q}{s_i}\right) + [F_{s_i}^{-1}]' \left(\frac{q}{s_i}\right) \frac{q}{s_i} \quad \forall i \in S.$$

(E38b)

In (EC.38a), by Assumption 1 we have

$$\lim_{q \to b_j} F_{b_j}^{-1} \left(1 - \frac{q}{b_j}\right) = 0,$$

$$\lim_{q \to b_j} [F_{b_j}^{-1}]' (q) > 0.$$  

(EC.39)

From (EC.38a) and (EC.39), we obtain $\lim_{q \to b_j} g_j(q) \leq 0$.

Regarding $h_i'(0)$, we show that

$$0 \leq (i) \lim_{q \to 0} h_i'(q) \leq (m) \lim_{q \to 0} \left[2F_{s_i}^{-1}\left(\frac{2q}{s_i}\right) - F_{s_i}^{-1}\left(\frac{q}{s_i}\right)\right] = 0,$$

(EC.40)

where in step (l), based on (EC.38b), we obtain the inequality from the nonnegativity of $F_{s_i}^{-1}(\cdot)$ and $[F_{s_i}^{-1}]'(\cdot)$ by Assumption 1. In step (m), by convexity of $h_i(q)$ in Assumption 2 we have $h_i'(q) \leq (h_i(2q) - h_i(q))/q$.

Thus, Assumption (EC.1-3) is satisfied.

To verify Assumption (EC.1-6) in problem (7), we directly apply the derivation in (EC.40) to obtain $h_i'(0) = 0$.

Proof of Claim (iii) In problem (EC.3), as a special case of the equilibrium problem (14) with $(\gamma, \mu) = 0$, Assumptions (EC.1-1) - (EC.1-3) are satisfied. To see Assumption (EC.1-6) for all $i \in S$, we have

$$\lim_{q \to 0} h_i'(q) = \lim_{q \to 0} F_{s_i}^{-1}\left(\frac{q}{s_i}\right) = 0.$$ 

(EC.41)

Thus, Assumption (EC.1-6) is satisfied.

Proof of claim (iv) To prove the claim, we plan to use Proposition 6 of Morand et al. (2015). Abusing some notation, we denote by $g_j(q_j, b_j)$ for all $j \in B$ and $h_i(q_i^*, s_i)$ for all $i \in S$ be the parameterized objective function to problem (EC.1).

We first verify Assumption 1 of Morand et al. (2015). By continuous differentiability of $(F_{b_j})_{j \in B}$ and $(F_{s_i})_{i \in S}$ in Assumption 1, the objective function $\sum_{j \in B} g_j(q_j^*, b_j) - \sum_{i \in S} h_i(q_i^*, s_i)$ is continuously differentiable in $(x, q^*, q^b, s, b)$ in all three related problems: equilibrium problem (14), revenue optimization problem (7) and welfare optimization problem (EC.3). Thus, it is locally Lipschitz in $(x, q^*, q^b, s, b)$. The constraint functions in (EC.1b) - (EC.11) are all affine, thus also continuously differentiable in $(x, q^*, q^b, s, b)$. Regarding the total number of equality constraints, we have $|S| + |B| + |E| \geq |S| + |B|$. Thus, Assumption 1 of Morand et al. (2015) is satisfied.
Next, we verify the uniform compactness condition. Denote by $K$ a neighborhood of $(s, b)$ that satisfies $K = \times_{i \in S} \left( \frac{1}{2}s_i, 2s_i \right) \times_{j \in B} \left( \frac{1}{2}b_j, 2b_j \right)$. The closure of the union of feasible region (EC.1) - (EC.11) regarding $(s', b') \in K$ is equivalent to the feasible region under parameter $(2s, 2b)$, which is compact. Thus, by the definition uniform compactness before Proposition 6 in page 159 of Morand et al. (2015), the feasible region characterized by (EC.1b) - (EC.1f) near $(s, b)$ is uniformly compact by the definition in page.

Lastly, by Proposition (EC.1)(ii) any optimal solution $x(s, b), q^*(s, b), q^b(s, b)$ to problem (EC.1) satisfies MFCQ.

We let $\nabla_{(s, b)} (q^*, q^b, s, b)$ be the gradient of the objective function $\sum_{j \in B} g_j(q^b, b_j) - \sum_{i \in S} h_i(q^i, s_i)$ in terms of $(s, b)$ evaluated at $(q^*, q^b, s, b)$. Letting “$\circ$” be the Hadamard product. For any $\delta^* \in [0, 1]$ and $\delta^b \in [0, 1]$, we deduce that

$$\lim_{t \to 0^+} \frac{V(s + t\delta^*, b + t\delta^b) - V(s, b)}{t} = \frac{\nabla_{(s, b)} (q^*(s, b), q^b(s, b), s, b) + \left( \eta^* \right) \left( \eta^b \right)}{\delta^* \delta^b},$$

which follows from Proposition 6 of Morand et al. (2015), continuous differentiability of $\sum_{j \in B} g_j(q^b, b_j) - \sum_{i \in S} h_i(q^i, s_i)$, the uniqueness of optimal supply-demand vector $(q^*(s, b), q^b(s, b))$, and the uniqueness of dual optimal solution $(\eta^*(s, b), \eta^b(s, b))$. If we let $(\delta^*, \delta^b)$ to take the value of a positive unit vector and a negative unit vector, we obtain that the gradient $\nabla_{(s, b)} V(s, b)$ is well-defined.

**Proof of Proposition EC.2.** To prove the result, we establish a bridging auxiliary problem (EC.43) and establish its equivalence to problem (EC.1) under Assumptions [EC.1] - [EC.15]. Consider the following auxiliary Proposition.

**PROPOSITION EC.3.** Let $f(\cdot)$ be defined in (15). Consider problem

$$\max_{(y, z)} \sum_{j \in B} b_j f \left( \frac{y_j}{b_j} \right),$$

s.t.

$$\sum_{i : (i, j) \in E} z_{ij} = y_j, \quad \forall j \in B.$$ (EC.43b)

$$\sum_{j : (i, j) \in E} z_{ij} \leq s_i, \quad \forall i \in S.$$ (EC.43c)

$$z_{ij} \geq 0, \quad \forall (i, j) \in E.$$ (EC.43d)

Given Assumptions [EC.1] - [EC.15], optimization problem (EC.43) is equivalent to optimization problem (EC.1) under the following solution mappings:

(i) let $(x, q^*, q^b)$ be an optimal solution to problem (EC.1), we construct a solution $(z, y, r)$ that satisfies (1) $z_{ij} = \frac{z_{ij}^*}{y_j}$ for all $(i, j) \in E$; (2) $r_j = \frac{q^b_j}{y_j}$ for all $j \in B$; (3) $y_j = \frac{z_{ij}^*}{q^i_j}$ for all $i \in S$ and $z_{ij} > 0$. Solution $(z, y)$ is optimal to problem (EC.43), and solution $r$ is optimal to problem (15).

(ii) let $(z, y)$ be an optimal solution to problem (EC.43) and $r$ be an optimal solution to problem (15). We construct solution $(x, q^*, q^b)$ that satisfies (1) $q^i_j = r_j b_j$ for all $j \in B$; (2) $q^b_j = r_j y_j b_j$ for all $i \in S$ and $j : z_{ij} > 0$; (3) $x_{ij} = \frac{z_{ij}^* q^i_j}{y_j}$ for all $(i, j) \in E$. Solution $(x, q^*, q^b)$ is optimal to problem (EC.1).

We denote the optimal objective value for optimization problem (EC.1) as $Y^f_{opt}$ and the optimal objective value for optimization problem (EC.43) as $Y^a_{opt}$. We show $Y^f_{opt} \leq Y^a_{opt}$ in step (1) and $Y^f_{opt} \geq Y^a_{opt}$ in step (2) to establish $Y^f_{opt} = Y^a_{opt}$ under the solution mappings.
Step 1: $Y^f_{opt} \leq Y^a_{opt}$. Let $(\mathbf{x}, \mathbf{q}^*, \mathbf{q}^b)$ be an optimal solution to optimization formulation (EC.1) and let $(\mathbf{z}, \mathbf{y}, \mathbf{r})$ be the solution in item (i) of this proposition statement. In particular, $y_j = \frac{s_j}{q_j^b}$ for all $j \in \mathcal{B}$ and for all $i : x_{ij} > 0$ is a valid construction by Proposition EC.1(viii).

To establish that the constructed vector $(\mathbf{z}, \mathbf{y})$ is feasible to optimization problem (EC.43), we verify constraints (EC.43b) - (EC.43d);

1. to see constraint (EC.43b), construction $z_{ij} = \frac{x_{ij}s_j}{q_j^b}$ implies that $\sum_{i:(i,j)\in E} z_{ij} = \sum_{i:(i,j)\in E} \frac{x_{ij}s_j}{q_j^b} = x_{ij}$. Construction $y_j = \frac{s_j}{q_j^b}$ for $x_{ij} > 0$ implies that $\sum_{i:(i,j)\in E} y_j = \sum_{i:(i,j)\in E} \frac{s_j}{q_j^b} = s_j$. Thus, constraint $\sum_{i:(i,j)\in E} z_{ij} = y_j$ in (EC.43b) holds;

2. to establish constraint (EC.43c), from $z_{ij} = \frac{x_{ij}s_j}{q_j^b}$, we can derive that $\sum_{i:(i,j)\in E} z_{ij} = \frac{s_j}{q_j^b} = s_j$ where the second equality follows from constraint (EC.1d);

3. constraint (EC.43d) follows directly from $x_{ij} \geq 0$ in constraint (EC.1f).

Next, we establish that $r_j = \frac{q_j^*}{b_j}$ is a feasible solution to problem (15). We verify the constraints $r_j \in [0, \min\{1, \frac{q_j^*}{b_j}\}]$.

1. to see $r_j \leq 1$, we have $r_j = \frac{q_j^*}{b_j} \leq 1$ by constraint (EC.1d);

2. $r_j = \frac{q_j^*}{b_j} = \frac{q_j^*}{b_j} \leq \frac{y_j}{y_j}$ where the second equality follows from $y_j = \frac{s_j}{q_j^b}$, and the inequality follows from constraint $q_j^* \leq s_j$ in (EC.1c).

We conclude $Y^f_{opt} \leq Y^a_{opt}$ by deriving that

$$Y^f_{opt} = \begin{array}{c}
(a) \sum_{j \in \mathcal{B}} b_j g \left( \frac{q_j^b}{b_j} \right) - \sum_{i \in \mathcal{S}} s_i h \left( \frac{q_i^s}{s_i} \right) \\
(b) \sum_{j \in \mathcal{B}} b_j g \left( \frac{q_j^b}{b_j} \right) - \sum_{j \in \mathcal{B}} \sum_{i:(i,j)\in E} h \left( \frac{q_j^b}{s_j} \right) s_i x_{ij} \\
(c) \sum_{j \in \mathcal{B}} b_j g \left( \frac{q_j^b}{b_j} \right) - \sum_{j \in \mathcal{B}} \sum_{d_j \geq 0} h \left( \frac{q_j^b}{y_j} \right) \frac{y_j}{q_j^b} x_{ij} \\
(d) \sum_{j \in \mathcal{B}} \sum_{d_j \geq 0} b_j h \left( \frac{q_j^b}{b_j} \right) \frac{y_j}{b_j} x_{ij} \\
(e) \sum_{j \in \mathcal{B}} b_j f \left( \frac{y_j}{b_j} \right) \leq Y^a_{opt},
\end{array}$$

(44)

where step (a) follows from the definition of $Y^f_{opt}$ as the optimal objective value where we implement $g_j(q_j^b) = b_j g \left( \frac{q_j^b}{b_j} \right)$ and $h_i(q_j^s) = s_i h \left( \frac{q_j^s}{s_i} \right)$ from Assumption (EC.14) - (EC.15). In step (b), we implement equation $\sum_{j:(i,j)\in E} x_{ij} = q_j^s$ from constraint (EC.1d). In step (c), by Proposition EC.1(vii) ratio $\frac{q_j^s}{s_i}$ is homogeneous for all $i : x_{ij} > 0$ and $j \in \mathcal{B}$. With $\frac{q_j^s}{s_i} = \frac{q_j^b}{y_j}$, we replace $\frac{q_j^s}{s_i}$ with $\frac{q_j^b}{y_j}$ to obtain step (c). In step (d), for each $j \in \mathcal{B}$, we aggregate $x_{ij}$ to obtain $\sum_{i:(i,j)\in E} x_{ij} = q_j^b$. Step (e) follows from the optimality of value function $f(\cdot)$ in problem (15). Inequality (f) follows from the optimality of $Y^a_{opt}$ in problem (EC.43). Thus, we have $Y^f_{opt} \leq Y^a_{opt}$.

Step 2: $Y^a_{opt} \leq Y^f_{opt}$. Let $(\mathbf{z}, \mathbf{y})$ an optimal solution to problem (EC.43) and $\mathbf{r}$ be an optimal solution to problem (15) given vector $\mathbf{y}$. We construct a solution vector $(\mathbf{x}, \mathbf{q}^*, \mathbf{q}^b)$ by item (ii) in the proposition statement.

To establish the feasibility of $(\mathbf{x}, \mathbf{q}^*, \mathbf{q}^b)$ in problem (EC.1), we verify constraints (EC.1b) - (EC.1f):
(1) to see constraint (EC.1b), we first show that in problem (EC.1), constraint $\sum_{j(i,j) \in E} z_{ij} \leq s_i$ is tight because objective function $f(\cdot)$ is increasing. Given $x_{ij} = z_{ij} \frac{q_i}{s_i}$, we have $\sum_{j(i,j) \in E} x_{ij} = \sum_{j(i,j) \in E} \frac{z_{ij} q_i}{s_i} = q_i^*.$

(2) to see constraint (EC.1c), from construction $x_{ij} = z_{ij} \frac{q_i}{s_i}$, we first obtain $\sum_{j(i,j) \in E} x_{ij} = \sum_{i(j,i) \in E} z_{ij} \frac{q_i}{s_i}$. From construction $q_i^* = s_i \frac{r_j b_j}{y_j}$ for any $i : z_{ij} > 0$, we next establish that $\sum_{j(i,j) \in E} z_{ij} \frac{q_i}{s_i} = (\sum_{j(i,j) \in E} z_{ij}) \frac{r_j b_j}{y_j}$. Following this, from constraint $\sum_{i(j,i) \in E} z_{ij} = y_j$ in (EC.43b), we can further derive that $(\sum_{j(i,j) \in E} z_{ij}) \frac{r_j b_j}{y_j} = r_j b_j = q_j^*$.

(3) to see constraint (EC.1d), from construction $q_j^* = r_j b_j$ and from constraint $r_j \leq 1$ in problem (15), we obtain inequality $\frac{q_j^*}{b_j} = r_j \leq 1$;

(4) to see constraint (EC.1e), from construction $q_i^* = s_i \frac{r_j b_j}{y_j}$ for all $j : z_{ij} > 0$ and from constraint $r_j \leq \frac{y_j}{b_j}$ in problem (15), we obtain $\frac{q_i^*}{s_i} = \frac{r_j b_j}{y_j} \leq 1$;

(5) constraint (EC.1f) follows directly from $z_{ij} \geq 0$.

We conclude $Y_{\alpha} \leq Y_{\alpha}^f$ by showing that

$$Y_{\alpha}^f = \sum_{j \in B} b_j f\left(\frac{y_j}{b_j}\right) \leq \sum_{j \in B} b_j \left[ g\left(\frac{q_j^*}{b_j}\right) - \frac{y_j}{b_j} h\left(\frac{q_j^*}{b_j}\right) \right]$$

$(g)$ follows from the definition of value function $f(\cdot)$ in problem (15), where $r_j = \frac{q_j^*}{b_j}$ is optimal in problem (15). In step (i), we add $\sum_{j(i,j) \in E} x_{ij} = 1$. In step (j), from construction $q_j^* = r_j b_j$, we first have $\frac{r_j b_j}{y_j} = \frac{y_j}{b_j}$.

From construction $q_i^* = s_i \frac{r_j b_j}{y_j}$ for any $i : x_{ij} > 0$, we further obtain $\frac{r_j b_j}{y_j} = \frac{q_i^*}{s_i}$. Thus, we can replace $\frac{r_j b_j}{y_j}$ with $\frac{q_i^*}{s_i}$ in step (j). Step (k) follows from $\sum_{j(i,j) \in E} x_{ij} = 1$. The inequality in step (l) follows by optimality of $Y_{\alpha}^f$ in problem (EC.1).

In summary the arguments in step 1 and step 2, we obtain $Y_{\alpha}^f = Y_{\alpha}^\prime$. This implies that the inequalities steps (e), (f) and (l) are tight, which further suggests that the solution mappings in this proposition statement are optimal. Q.E.D.

After establishing the equivalence of formulation (EC.1) and (EC.43), by Lemma 4.1 of Megiddo (1974), solution $(z, y)$ is feasible to problem (EC.43) if and only if $y$ is feasible to problem (17). Given that the objective functions of both problems are the same, the equivalence of problem (EC.43) and problem (17) follows directly. Q.E.D.

### EC.2. Proof of Results in Appendix A

#### Proof of Proposition 8

We prove the necessity argument in step 1 and the sufficiency argument in step 2.
Step 1: necessity. Let \((p, x, q^*, q^b)\) be a competitive equilibrium satisfying expressions (EC.2a) - (EC.2d).

The competitive equilibrium has the following two properties:

(8.1) by equilibrium expression (EC.2d), if \((1 - \gamma^*_i)p_i - \mu^*_i \leq 0\), then \(\frac{q^*_i}{s^*_i} = 0\) and \(\bar{F}_{s^*_i}^{-1}(\frac{q^*_i}{s^*_i}) \geq p_i\); if \(0 < (1 - \gamma^*_i)p_i - \mu^*_i < \bar{v}_{s_i}\), then \(0 < \frac{q^*_i}{s^*_i} < 1\) and \(\bar{F}_{s^*_i}^{-1}(\frac{q^*_i}{s^*_i}) = p_i\); if \((1 - \gamma^*_i)p_i - \mu^*_i \geq \bar{v}_{s_i}\), then \(\frac{q^*_i}{s^*_i} = 1\) and \(\bar{F}_{s^*_i}^{-1}(\frac{q^*_i}{s^*_i}) \leq p_i\);

(8.2) by equilibrium expression (EC.2b), if \((1 + \gamma^*_j)\min_{(i,j) \in E}\{p_i\} + \mu^*_j < \bar{v}_{b_j}\), then \(\frac{q^*_j}{b_j} > 0\) and \(\bar{F}_{b_j}^{-1}(1 - \frac{q^*_j}{b_j}) = \min_{(i,j) \in E}\{p_i\}\); if \((1 + \gamma^*_j)\min_{(i,j) \in E}\{p_i\} + \mu^*_j \geq \bar{v}_{b_j}\), then \(\frac{q^*_j}{b_j} = 0\) and \(\bar{F}_{b_j}^{-1}(1 - \frac{q^*_j}{b_j}) \leq \min_{(i,j) \in E}\{p_i\}\).

We verify that \((p, x, q^*, q^b)\) satisfies conditions (i) and (ii) in the proposition statement.

Step 1-1: verification of condition (i). We first argue that \((x, q^*, q^b)\) is a feasible solution to problem (EC.1). Supply expression (2a) and demand expression (2b) guarantee that vector \((x, q^*, q^b)\) satisfies constraints (EC.1e) and (EC.1f) in problem (EC.1). Flow expression (2c) guarantees constraints (EC.1b), (EC.1c), and (EC.1h). Next, we construct vector \((\theta^*, \eta^*, \theta^b, \eta^b, \pi)\) as follows:

• for all \(i \in S\), if \((1 - \gamma^*_i)p_i - \mu^*_i \leq 0\), let \(\theta^*_i = \bar{F}_{s^*_i}^{-1}(\frac{q^*_i}{s^*_i})\); if \((1 - \gamma^*_i)p_i - \mu^*_i > 0\), then let \(\theta^*_i = p_i\);

• for all \(i \in S\), if \((1 - \gamma^*_i)p_i - \mu^*_i \leq \bar{v}_{s_i}\), let \(\eta^*_i = 0\); conversely if \((1 - \gamma^*_i)p_i - \mu^*_i > \bar{v}_{s_i}\), let \(\eta^*_i = p_i - \bar{F}_{s^*_i}^{-1}(\frac{q^*_i}{s^*_i})\);

• for each \(j \in B\), let \(\theta^*_j = \bar{F}_{b_j}^{-1}(1 - \frac{q^*_j}{b_j})\);

• for all \(j \in B\), let \(\eta^*_j = 0\);

• for all \((i, j) \in E\): if \(x_{ij} > 0\), then let \(\pi_{ij} = 0\); otherwise, let \(\pi_{ij} = \theta^*_i - \theta^*_j\).

With a strictly concave objective function and linear constraints in problem (EC.1), it is sufficient to establish that vector \((x, q^*, q^b)\) and vector \((\theta^*, \theta^b, \eta^*, \eta^b, \pi)\) satisfy KKT optimality conditions. In framework problem (EC.1), this is equivalent to checking conditions (EC.2a) - (EC.2b).

(1) To see condition (EC.2a), given that \(g_j(q) = \int_0^q \bar{F}_{b_j}^{-1}(1 - \frac{x}{b_j})dx\), we have \(g_j'(q^*_j) = \bar{F}_{b_j}^{-1}(1 - \frac{q^*_j}{b_j})\).

By construction, we have \(\theta^*_j = \bar{F}_{b_j}^{-1}(1 - \frac{q^*_j}{b_j})\) and \(\eta^*_j = 0\), which satisfies condition (EC.2a).

(2) To see condition (EC.2b), given that \(h_i(q) = \int_0^q \bar{F}_{s_i}^{-1}(\frac{x}{s_i})dx\), we start from \(h'_i(q^*_i) = \bar{F}_{s_i}^{-1}(\frac{q^*_i}{s_i})\). We further consider the following two possibilities: (2-i) if \((1 - \gamma^*_i)p_i - \mu^*_i \leq 0\), condition (EC.2b) holds given our construction \(\theta^*_i = \bar{F}_{s^*_i}^{-1}(\frac{q^*_i}{s^*_i})\) and \(\eta^*_i = 0\); (2-ii) if \((1 - \gamma^*_i)p_i - \mu^*_i > \bar{v}_{s_i}\), then condition (EC.2b) holds because we have \(\theta^*_i = p_i\) by construction, \(p_i = \bar{F}_{s^*_i}^{-1}(\frac{q^*_i}{s^*_i})\) by equilibrium property (8.1) and \(\eta^*_i = 0\) by construction; (2-iii) if \((1 - \gamma^*_i)p_i - \mu^*_i \geq \bar{v}_{s_i}\), then condition (EC.2b) holds because we have \(\bar{F}_{s_i}^{-1}(\frac{q^*_i}{s^*_i}) = p_i - \eta^*_i\) and \(p_i - \eta^*_i = \theta^*_i - \eta^*_i\) by construction of \((\eta^*_i, \theta^*_i)\);

(3) To see equation (EC.2c), we discuss two cases: (3-i) if \(x_{ij} > 0\), then we have \(q^*_j > 0\), which implies \((1 + \gamma^*_j)\min_{(i,j) \in E}\{p_i\} + \mu^*_j < \bar{v}_{b_j}\) and \(\bar{F}_{b_j}^{-1}(1 - \frac{q^*_j}{b_j}) = \min_{(i',j') \in E}\{p_{i'}\}\) by equilibrium property (8.2). By construction of \(\theta^*_j\), we obtain \(\theta^*_j = \bar{F}_{b_j}^{-1}(1 - \frac{q^*_j}{b_j}) = \min_{(i',j') \in E}\{p_{i'}\}\). Moreover, we have \(p_i = \theta^*_i\) by construction. By expression (2d), \(x_{ij} > 0\) implies that \(\min_{(i',j') \in E}\{p_{i'}\} = p_i\). This allows us to establish \(\theta^*_j = \theta^*_i\). When \(x_{ij} > 0\), we have \(\pi_{ij} = 0\) by construction. Thus, \(\theta^*_j - \theta^*_i + \pi_{ij} = 0\) in condition (EC.2c) follows; (3-ii) if \(x_{ij} = 0\), then condition (EC.2c) holds because we let \(\pi_{ij} = \theta^*_i - \theta^*_j\) in our construction;

(4) To see condition (EC.2d), we first note that \(x_{ij} \geq 0\) follows from equilibrium expression (2c). Under the construction of \(\pi_{ij}\), we discuss two cases: (4-ii) if \(x_{ij} > 0\), we have \(\pi_{ij} = 0\) and condition (EC.2d) follows because \(\pi_{ij}x_{ij} = 0\); (4-ii) if \(x_{ij} = 0\), by construction of \(\theta^*_j\), we have \(\theta^*_j = \bar{F}_{b_j}^{-1}(1 - \frac{q^*_j}{b_j})\). Since \(\bar{F}_{b_j}^{-1}(1 - \frac{q^*_j}{b_j}) \leq \min_{(i',j') \in E}\{p_{i'}\}\) by equilibrium property
this implies that \( \pi_{ij} = \theta^b - \theta^a \geq p_i - \min_{i', (i', j) \in E} \{ \rho_{i'} \} \geq 0 \). Since \( x_{ij} = 0 \), we have \( x_{ij} \pi_{ij} = 0 \), which suggests that condition (EC.2d) holds;

(5) To see condition (EC.2e), \( q^*_j \leq 1 \) follows from \( F_{s_i}(v) \leq 1 \) in expression (2a). There are two cases to discuss: (5-i) if \( (1 - \gamma^b_j) p_i - \mu^*_i \leq \tilde{v}_{s_i} \), then we have \( \eta^*_i = 0 \) by construction, which directly implies condition (EC.2e); (5-ii) if \( (1 - \gamma^b_j) p_i - \mu^*_i > \tilde{v}_{s_i} \), then we have \( q^*_i = s_i \) and \( \tilde{F}_{s_i}^{-1}(q^*_i) \leq p_i \) by equilibrium property (8-1). The construction of \( \eta^*_i \) further suggests that \( \eta^*_i = p_i - \tilde{F}_{s_i}^{-1}(q^*_i) \geq 0 \). As a result, we obtain equation \( \eta^*_i (q^*_i - s_i) = 0 \), which further suggests that condition (EC.2e) holds.

(6) To see condition (EC.2f), \( q^*_j / b_j \leq 1 \) follows from \( F_{b_j}(v) \leq 1 \) in expression (2b). In addition, our construction \( \eta^*_j = 0 \) directly suggests that condition (EC.2f) holds.

In summary, vector \((x, q^*, q^b)\) and \((\theta^b, \theta^a, \eta^a, \eta^b, \pi)\) satisfy the KKT optimality condition (EC.2a) - (EC.2f). Thus, \((x, q^*, q^b)\) is an optimal solution to problem (EC.1).

Step 1-2: verification of condition (ii). To prove this claim, we start from the construction \((\theta^b, \theta^a, \eta^a, \eta^b, \pi)\), which has proven to be the optimal dual solution corresponding to constraints (EC.1b) - (EC.1c) given that \((x, q^*, q^b)\) is an optimal solution to problem (EC.1).

For all \( s \), we discuss three cases:

1. If \( \theta^b < s_i \), we have \( p_i = \tilde{F}_{s_i}^{-1}(\frac{q^*_i}{s_i}) \) by equilibrium property (8-1). Furthermore, we have \( \eta^*_i = 0 \) by condition (EC.2e), which implies \( \tilde{F}_{s_i}^{-1}(\frac{q^*_i}{s_i}) = \theta^*_i \) by condition (EC.2b). Thus, we have \( p_i = \theta^*_i \);

2. If \( q^*_i = s_i \), then we can pick any \( j : x_{ij} > 0 \) or correspondingly \( q^*_j > 0 \), which implies \( 1 + \gamma^b_j \min_{i (i, j) \in E} \{ p_i \} + \mu^*_j < \tilde{v}_{b_j} \) and \( \min_{i' (i', j) \in E} \{ \rho_{i'} \} \geq \tilde{F}_{b_j}^{-1}(1 - \frac{q^*_j}{b_j}) \) by equilibrium property (8-2). By Proposition EC.1 (iv), we have \( q^*_j < b_j \), which implies \( \eta^*_j = 0 \) by condition (EC.2b) and \( \tilde{F}_{b_j}^{-1}(1 - \frac{q^*_j}{b_j}) = \theta^*_j \) by condition (EC.2a). Furthermore, we have \( \pi_{ij} = 0 \) by condition (EC.2b) and \( \theta^*_i = \theta^*_j \) by condition (EC.2c). Given \( x_{ij} > 0 \), equilibrium expression (2d) implies \( p_i = \min_{i' (i', j) \in E} \{ \rho_{i'} \} \). Thus, we obtain equation \( p_i = \min_{i' (i', j) \in E} \{ \rho_{i'} \} = \tilde{F}_{b_j}^{-1}(1 - \frac{q^*_j}{b_j}) = \theta^*_j = \theta^*_i \);

3. If \( q^*_i > 0 \), then by condition (EC.2e), we have \( \eta^*_i = 0 \). To argue \( p_i \leq \theta^*_i \), suppose towards contradiction that \( p_i > \theta^*_i \). By condition (EC.2b), we have \( \tilde{F}_{s_i}^{-1}(\frac{q^*_i}{s_i}) = \tilde{F}_{s_i}^{-1}(0) = 0 \). However, we have \( p_i \leq F_{s_i}^{-1}(0) \) from equilibrium property (8-1). This results in a contradiction \( F_{s_i}^{-1}(0) = \theta^*_i < p_i \leq F_{s_i}^{-1}(0) \). Thus, we have \( p_i \leq \theta^*_i \). Moreover, since \( q^*_i = 0 \), for all \( j : (i, j) \in E \), we obtain \( \min_{i' (i', j) \in E} \{ \rho_{i'} \} \geq \tilde{F}_{b_j}^{-1}(1 - \frac{q^*_j}{b_j}) \) from equilibrium property (8-2). Given \( \tilde{F}_{b_j}^{-1}(1 - \frac{q^*_j}{b_j}) = \theta^*_j \) for all \( j : (i, j) \in E \) by condition (EC.2a), we have \( p_i \geq \min_{i' (i', j) \in E} \{ \rho_{i'} \} \geq \theta^*_j \) for all \( j : (i, j) \in E \).

Step 2: sufficiency. Let vector \((p, x, q^*, q^b)\) satisfy conditions (i) and (ii). Let \((\theta^b, \theta^a, \eta^a, \eta^b, \pi)\) be the corresponding optimal dual solution corresponding to constraints (EC.1b) - (EC.1f). We argue that \((p, x, q^*, q^b) \in X(\gamma, \mu)\).

We define \( \tilde{F}_{s_i} : \mathbb{R} \to [0, 1] \) by \( \tilde{F}_{s_i}(v) = F_{s_i}(v) = (1 - \gamma^a) v - \mu^*_i \) for all \( i \in S \) and \( \tilde{F}_{b_j} : \mathbb{R} \to [0, 1] \) by \( \tilde{F}_{b_j}(v) = F_{b_j}(v = (1 + \gamma^b_j) v + \mu^*_j) \). Note that this definition is consistent with its inverse definition \( \tilde{F}_{s_i}^{-1}(\cdot) \) and \( \tilde{F}_{b_j}^{-1}(\cdot) \) used throughout the proof.

First of all, we establish expression (2a) by discussing two cases:

1. If \( q^*_i = 0 \), then we have \( \eta^*_i = 0 \) by condition (EC.2e). By condition (EC.2b), we have \( \tilde{F}_{s_i}^{-1}(\frac{q^*_i}{s_i}) = \theta^*_i \). Given \( \theta^*_i \geq p_i \) in condition (ii) of this proposition statement, we obtain that \( 0 \leq \tilde{F}_{s_i}(p_i) \leq \tilde{F}_{s_i}(\theta^*_i) = \tilde{F}_{s_i}(\tilde{F}_{s_i}^{-1}(0)) = 0 \). Thus, we have \( \tilde{F}_{s_i}(p_i) = 0 = \frac{q^*_i}{s_i} \);

2. If \( q^*_i > 0 \), then we have \( \theta^*_i \geq \tilde{F}_{s_i}^{-1}(\frac{q^*_i}{s_i}) \) by condition (EC.2b) and \( p_i = \theta^*_i \) by condition (ii). This in turn implies that \( \frac{q^*_i}{s_i} \geq \tilde{F}_{s_i}(p_i) = \tilde{F}_{s_i}(\theta^*_i) \geq \tilde{F}_{s_i}(\tilde{F}_{s_i}^{-1}(\frac{q^*_i}{s_i})) = \frac{q^*_i}{s_i} \), where the first inequality follows from \( F_{s_i}(v) \leq 1 \), the second equality follows from \( p_i = \theta^*_i \), the third inequality follows from \( \theta^*_i \geq \tilde{F}_{s_i}^{-1}(\frac{q^*_i}{s_i}) \).

As a result, we have \( \tilde{F}_{s_i}(p_i) = \frac{q^*_i}{s_i} \).

In summary, expression (2a) holds.
Next, we establish expression (2b) by discussing the following two cases:

(1) if \( q_i^b = 0 \), we have \( \eta_i^b = 0 \) by condition (EC.2f) and \( \tilde{F}_b^{-1}(1 - \frac{q_i^b}{b_i}) = \theta_i^b \) by condition (EC.2a). Since \( p_i \geq \theta_i^b \) for all \( i \), we know \( \min_{i \in (i,j) \in E} \{ p_i \} \geq \theta_i^b \). Thus, we have \( \theta_i^b = \tilde{F}_b^{-1}(1 - \frac{q_i^b}{b_i}) = \tilde{F}_b^{-1}(1) \), which allows us to further derive that \( 1 \geq \tilde{F}_b \min_{i \in (i,j) \in E} \{ p_i \} \geq \tilde{F}_b(\theta_i^b) = \tilde{F}_b(\tilde{F}_b^{-1}(1)) \), where the first inequality follows from \( F_b(v) \leq 1 \), the second inequality follows from \( \min_{i \in (i,j) \in E} \{ p_i \} \geq \theta_i^b \), and the third equality follows from \( \theta_i^b = \tilde{F}_b^{-1}(1) \). Equivalently, we have
\[
\tilde{F}_b(\min_{i \in (i,j) \in E} \{ p_i \}) = 1 - 1 - \frac{q_i^b}{b_i} \;
\]

(2) if \( q_i^b > 0 \), we have \( q_i^b < b_j \) by Proposition EC.1(iv), \( \eta_i^b = 0 \) by condition (EC.2f), and \( \tilde{F}_b^{-1}(1 - \frac{q_i^b}{b_j}) = \theta_i^b \) by condition (EC.2a), which further implies equation \( \tilde{F}_b(\theta_i^b) = \tilde{F}_b(\tilde{F}_b^{-1}(1 - \frac{q_i^b}{b_j})) = 1 - \frac{q_i^b}{b_j} \).

Pick any \( i \in x_i > 0 \), and we obtain \( \pi_i = 0 \) by condition (EC.2c) and \( \theta_i^s = \theta_i^b \) by condition (EC.2e). From condition (EC.2c), we know \( \theta_i^s \geq \theta_i^b \) for all \( i \in \{ i', j \} \in E \), from which we further obtain that \( i \in \arg \min_{i \in (i', j) \in E} \{ \theta_i^b \} \). Given \( p_i = \theta_i^b \) by condition (EC.2a), we have \( \tilde{F}_b(\min_{i \in (i', j) \in E} \{ p_i \}) = \tilde{F}_b(\theta_i^b) = \tilde{F}_b(1 - \frac{q_i^b}{b_j}) \), or in short \( \tilde{F}_b(\min_{i \in (i,j) \in E} \{ p_i \}) = 1 - \frac{q_i^b}{b_j} \).

Thus, expression (2b) holds.

To establish expression (2c), we observe that constraints (EC.1b), (EC.1c) and (EC.1e) directly guarantee the flow conservation expression in (2c).

To establish expression (2d), we have \( \theta_i^s - \theta_i^b = \pi_i \geq 0 \) by condition (EC.2c) and (EC.2d). By condition (EC.2f), if \( x_i > 0 \), then \( \theta_i^s = \theta_i^b \), which further implies that \( p_i = \theta_i^s = \min_{i \in (i', j) \in E} \{ \theta_i^b \} = \min_{i \in (i', j) \in E} \{ \theta_i^b \} \). This establishes the second part of expression (2d).

Having verified expressions (2a) - (2d), we conclude that \( (p, x, q^i, q^j) \in \mathcal{X}(\gamma, \mu) \). Q.E.D.

**Proof of Lemma 1.** In step 1, we show that the optimizer \( r(c) \) in problem \((15)\) exists and is unique. In step 2, we further show that \( r(c) \) strictly increases in \( c > 0 \). In step 3, we show that \( \frac{r(c)}{c} \) strictly decreases in \( c \geq c_0 \) and \( \frac{r(c)}{c} = 1 \) for \( c \leq c_0 \). In step 4, we show that the value function \( f(c) \) is differentiable. We prove \( f(c) \) is strictly increasing in step 5 and strictly concave for \( c > 0 \) in step 6. In step 7, we prove the limiting point.

Step 1: \( r(c) \) is well-defined. For any \( c > 0 \), we discuss two cases:

1. \( g(r) - ch \left( \frac{r}{c} \right) \) is continuous in compact set \([0, \min\{1, c\}]\), we directly apply the extreme value theorem to show that an optimal solution \( r(c) \) exists;

2. if \( \lim_{r \to 1} g(r) \to -\infty \) or \( \lim_{r \to 1} h(r) = \infty \), we first show that there shows \( M > 0 \) such that
\[
\sup_{r \in [0, \min\{1, c\}]} g(r) - ch \left( \frac{r}{c} \right) \leq M. \tag{EC.46}
\]

We let \( r_0 = \frac{1}{2} \min\{1, c\} \). By Assumptions (A1) - (AP) functions \( g(r) \) and \( -h(r) \) are concave and differentiable in \((0, 1)\). Thus, for all \( r \in [r_0, \min\{1, c\}] \), we have
\[
g(r) - ch \left( \frac{r}{c} \right) \leq g(r_0) - ch \left( \frac{r_0}{c} \right) + g'(r_0) - h' \left( \frac{r_0}{c} \right) (r - r_0). \tag{EC.47}
\]

The upper bound expression is continuous in \([r_0, \min\{1, c\}]\). Thus, there exists
\[
M = \max \left\{ \max_{r \in [r_0, r_0]} \left\{ g(r) - ch \left( \frac{r}{c} \right) \right\}, \right. \\
\left. g(r_0) - ch \left( \frac{r_0}{c} \right) + \max_{r \in [r_0, \min\{1, c\}]} \left\{ g'(r_0) - h' \left( \frac{r_0}{c} \right) (r - r_0) \right\} \right\} < \infty \tag{EC.48}
\]
such that [EC.46] is satisfied. As a result, there exists \( \bar{r} > 0 \) such that \( g(r) - ch\left(\frac{r}{c}\right) \) is continuous in \( r \in [0, \min\{1, c, \bar{r}\}] \) and \( g(r) - ch\left(\frac{r}{c}\right) < -2M \) for all \( r > \min\{1, c, \bar{r}\} \). By [EC.46], \( r(c) \leq \bar{r} \). Thus, problem (15) becomes

\[
\max_{r \in [0, \min\{1, c, \bar{r}\}]} g(r) - ch\left(\frac{r}{c}\right),
\]

(EC.49)

whose objective function is continuous in compact set \( [0, \min\{1, c, \bar{r}\}] \). Thus, we apply the extreme value theorem to obtain that an optimal solution \( r(c) \) exists.

To show \( r(c) \) is well-defined, for any \( c > 0 \), from condition [A4], we have \( g'(0) - h'(0) > 0 \). By Assumptions [A1], [A2], \( g(r) \) and \( -h(r) \) are strictly concave in \((0,1)\), which implies that function \( g'(r) - h'(r) \) is strictly decreasing in \( r \). Thus, we can derive from the first order optimality condition that

\[
r(c) = \max \left\{ r \in [0, \min\{1, c\}] : g'(r) - h'(\frac{r}{c}) \geq 0 \right\},
\]

(EC.50)

which is uniquely determined. Thus, optimizer \( r(c) \) is well-defined for \( c > 0 \).

Step 2: \( r(c) \) is strictly increasing in \( c > 0 \). When \( c > 0 \), we first show \( r(c) > 0 \). Suppose towards contradiction that \( r(c) = 0 \). From \( g'(0) - h'(0) > 0 \) in Assumption [A1], we can increase \( r(c) \) to strictly increase the objective value, which contradicts the optimality of \( r(c) \). Thus,

\[
r(c) > 0.
\]

(EC.51)

Next, we show that \( r(c) < 1 \). When \( c < 1 \), constraint \( r \in [0, \min\{1, c\}] \) directly implies that \( r(c) < 1 \). When \( c \geq 1 \), to show \( r(c) < 1 \), suppose towards contradiction that \( r(c) = 1 \). This implies

\[
g'(1) - h'(1) = (a) < g'(1) - h'(0) \leq 0,
\]

(EC.52)

where step (a) follows from the strict convexity of \( h(r) \) in \( r \in (0,1) \) from Assumption [A2], and step (b) follows from Assumption [A1]. By [EC.52], decreasing \( r(c) \) would strictly increase the objective value, which contradicts the optimality of \( r(c) \). As a result,

\[
r(c) < 1.
\]

(EC.53)

By [EC.51] and [EC.53], expression (EC.50) can be further simplified to

\[
r(c) = \max \left\{ r : g'(r) - h'(\frac{r}{c}) \geq 0, \ r \leq c \right\}.
\]

(EC.54)

For any \( 0 < c_1 < c_2 \), suppose towards contradiction that \( r(c_1) \geq r(c_2) \). By strict convexity of \( h(r) \) in \( r \in (0,1) \) from Assumption [A2], function \( h'(\cdot) \) is strictly increasing. Thus, given \( c_2 > c_1 \), we have

\[
g'(r(c_1)) - h'(\frac{r(c_1)}{c_2}) > g'(r(c_2)) - h'(\frac{r(c_1)}{c_1})
\]

(EC.55)

Moreover, by strict concavity of \( g(r) \) and \( -h(r) \) in \( r \in (0,1) \) from Assumption [A1], [A2], function \( g'(r) - h'(r) \) is strictly decreasing in \( r \). Given \( r(c_2) \leq r(c_1) \), we have

\[
g'(r(c_2)) - h'(\frac{r(c_2)}{c_2}) \geq g'(r(c_1)) - h'(\frac{r(c_1)}{c_2}).
\]

(EC.56)

As a summary of [EC.54], [EC.55] and [EC.56], we obtain

\[
g'(r(c_2)) - h'(\frac{r(c_2)}{c_2}) > g'(r(c_1)) - h'(\frac{r(c_1)}{c_1}) \geq 0.
\]

(EC.57)
With \( r(c_2) \leq r(c_1) \leq c_1 < c_2 \) and (EC.57), \( r(c_2) \) does not satisfy (EC.54), which is a contradiction. Thus, \( r(c_1) < r(c_2) \) for all \( 0 < c_1 < c_2 \).

Step 3: \( r(c) \) is strictly decreasing for \( c \geq c_0 \). Defined in the lemma statement, we have \( c_0 = [g']^{-1}(h(1)) \). By Assumption (A4), we have \( g'(1) \leq 0 \leq h'(0) < g'(0) \). By strict convexity of function \( h(\cdot) \) over \((0,1)\) in Assumption (A4), we have \( h'(1) > h'(0) \).

Value \( c_0 \) has two possibilities: if \( h'(1) \in (0, g'(0)) \), then \( c_0 = [g']^{-1}(h(1)) \in (0, 1) \); if \( h'(1) \geq g'(0) \), then \( c_0 = 0 \).

When \( c \leq c_0 \), it is sufficient to discuss the case when \( c_0 \in (0, 1) \) because \( r(c) \) is defined on \((0,1)\). By strict concavity of \( g(\cdot) \) over \((0,1)\) in Assumption (A4), \( g'(\cdot) \) is a strictly decreasing function in \((0,1)\), this implies that \( g'(c) \geq g'(c_0) = h'(1) \). By Assumption (A4), function \( g'(r) - h'(\xi) \) is strictly decreasing in \( r \), which implies \( g'(r) - h'(\xi) > 0 \) for all \( r \in [0, c) \). From (EC.54), this implies that \( r(c) = c \).

When \( c \geq c_0 \), by the strict convexity of \( g(\cdot) \) from Assumptions (A4) and \( g'(0) - h'(1) \leq 0 \) in Assumption (A4), we know that
\[
g'(c) - h'(1) < g'(0) - h'(1) \leq 0.
\]
From (EC.58) and (EC.54), we obtain that \( r(c) = c \) is not the optimal solution to problem (15).

Thus,
\[
r(c) < c.
\]

To show the strict decreasing property of \( r(c) \), for any \( c_0 \leq c < c_2 \), assume towards contradiction that \( \frac{r(c_1)}{c_1} \leq \frac{r(c_2)}{c_2} \). From the previous step, we have shown that \( r(c_1) < r(c_2) \). By strict concavity of \( g(\cdot) \) and \( -h(\cdot) \) over \((0,1)\) in Assumption (A4), we have \( g'(r(c_1)) - h'(\frac{r(c_1)}{c_1}) > g'(r(c_2)) - h'(\frac{r(c_2)}{c_2}) \). Moreover, we also have \( r(c_1) \leq \frac{c_2}{c_2} r(c_2) < c_1 \) where the second inequality follows from \( \frac{r(c_2)}{c_2} < 1 \) in (EC.59). Then \( g'(r(c_1)) - h'(\frac{r(c_1)}{c_1}) > 0 \) and \( r(c_1) < c_1 \) is a contradiction to the optimality of \( r(c_1) \) by (EC.54). As a result, we have \( \frac{r(c_1)}{c_1} > \frac{r(c_2)}{c_2} \).

Step 4: value function \( f(c) \) is differentiable in \( c > 0 \). We show that \( f(c) \) is differentiable in \( c > 0 \). Denote by \((\theta(c), \eta(c), \pi(c))\) a dual optimal solution vector corresponding to constraints \( r \leq 1, r \leq c, r \geq 0 \). By KKT condition, we have
\[
g'(r(c)) - h'(\frac{r(c)}{c}) - \theta(c) - \eta(c) + \pi(c) = 0, \tag{EC.60a}
\]
\[
r(c) \leq 1 \quad \perp \quad \theta(c) \geq 0, \tag{EC.60b}
\]
\[
r(c) \leq c \quad \perp \quad \eta(c) \geq 0, \tag{EC.60c}
\]
\[
r(c) \geq 0 \quad \perp \quad \pi(c) \geq 0. \tag{EC.60d}
\]

From (EC.51) and (EC.53), we have shown that \( 0 < r(c) < 1 \). By (EC.60a) and (EC.60d), we have \( \theta(c) = \pi(c) = 0 \). Given that the optimizer \( r(c) \) is unique, by Assumptions (A4) - (A5), function \( g'(r) - h'(\xi) \) is strictly decreasing in \( r \in [0, \min\{1, c\}] \) for \( c > 0 \). Condition (EC.60a) suggests that \( \eta(c) \) is uniquely determined by \( c \).

To use Proposition 6 of Morand et al. (2015), we verify the sufficient conditions. Firstly, for any \( c > 0 \), function \( g'(r) - c h'(\xi) \) is jointly concave, and thus locally Lipschitz. Constraint functions \( r - c, r - 1, -r \) are continuously differentiable. Thus, Assumption 1 in Morand et al. (2015) is satisfied. Secondly, feasible region \([0, \min\{1, c\}]\) is uniformly compact because we can consider a neighborhood \((\frac{1}{2} c, 2c)\) and closed set of \( \bigcup_{\xi \in (\frac{1}{2} c, 2c)} [0, \min\{1, \xi\}] \) is precisely the compact set \([0, \min\{1, 2c\}]\). Lastly, to see that the optimizer \( r(c) \) satisfies the MFCQ, from (EC.51) and (EC.53), we have \( 0 < r(c) < 1 \). Note that there is no equality constraint in problem (15). The gradient of active constraint functions \( r - c, r - 1, -r \) is \([-1, 1, 1]\) (or \([-1, 1]\) if \( r(c) = c \), which implies that there exists \( z = [-1, 1, 1] \) (or \( z = [-1, 1] \)
If \( r(c) = c \) such that item (ii) of Definition 2 in Morand et al. (2015) is satisfied. As a result, the optimizer \( r(c) \) satisfies the MFCQ. By Proposition 6 of Morand et al. (2015), the uniqueness of primal optimal \( r(c) \), and the uniqueness of the dual optimal \((\theta(c), \eta(c), \pi(c))\), for any \( \delta \in \mathbb{R} \), we obtain

\[
\lim_{t \to 0} \frac{f(c + t\delta) - f(c)}{t} = \lim_{t \to 0} \frac{f(c + t\delta) - f(c)}{t} = \left[ \frac{r(c)}{c^2} h' \left( \frac{r(c)}{c} \right) + \eta(c) \right] \delta. \tag{EC.61}
\]

Thus, value function \( f(c) \) is differentiable in \( c > 0 \).

Step 5: value function \( f(c) \) is strictly increasing. To establish the strict increasing property, for any \( 0 < c_1 < c_2 \), we firstly have \( 0 < r(c_1) < r(c_2) \) from step (2). We further derive that

\[
f(c_1) = g(r(c_1)) - c_1 h \left( \frac{r(c_1)}{c_1} \right) \leq g(r(c_1)) - c_2 h \left( \frac{r(c_1)}{c_2} \right) \leq g(r(c_2)) - c_2 h \left( \frac{r(c_2)}{c_2} \right) = f(c_2),
\]

where inequality (c) follows from the strict decreasing property of function \( ch(\frac{r}{c}) \) in \( c \) from Assumption [A5]. Inequality (d) follows from the optimality of \( r(c_2) \) given \( c_2 \).

Step 6: value function \( f(c) \) is strictly concave in \( c > 0 \). To establish strict concavity of function \( f(c) \), for any \( c_0, c_1 > 0 \) with \( c_0 \neq c_1 \), let \( r(c_0), r(c_1) \) respectively be the optimal solution to problem (15). Since \( c_0 \neq c_1 \), by strict increasing property of \( r(c) \) in step (2), we have \( r(c_0) \neq r(c_1) \). For any \( \theta \in (0, 1) \), denote \( c_0 = \theta c_1 + (1 - \theta) c_0, r_0 = \theta r(c_1) + (1 - \theta) r(c_0) \), and \( r(c_0) = \arg \max_{r \in [0, \min(1, c_0)]} g(r) - c_0 h \left( \frac{r}{c_0} \right) \).

We deduce that

\[
f(c_0) = g(r(c_0)) + c_0 h \left( \frac{r(c_0)}{c_0} \right) \geq g(r_0) + c_0 h \left( \frac{r_0}{c_0} \right) \geq \theta g(r(c_1)) + (1 - \theta) g(r(c_0)) + \theta c_1 h \left( \frac{r(c_1)}{c_1} \right) + (1 - \theta) c_0 h \left( \frac{r(c_0)}{c_0} \right) \equiv \theta f(c_1) + (1 - \theta) f(c_0),
\]

where in step (e), the inequality holds because \( r(c_0) \) is the optimal solution whereas \( r_0 \) is only a feasible solution to problem (15). The strict inequality in (f) follows from Jensen’s inequality. When \( g(r) \) is strictly concave and \( -ch(\frac{r}{c}) \) is jointly concave, Jensen’s inequality is strict when \( r(c_0) \neq r(c_1) \). Step (g) follows from the optimality of \( f(c_1) \) and \( f(c_0) \) in (15). This completes the proof that \( f(c) \) is a strictly concave function for \( c > 0 \).

Step 7: show \( \lim_{c \to 0} f(c) = g(0) \). To show \( \lim_{c \to 0} f(c) = g(0) \), we first derive from \( 0 \leq r(c) \leq c \) that

\[
\lim_{c \to 0} r(c) = 0. \tag{EC.64}
\]

We pick constant \( c_h = h \left( \frac{1}{2} \right) + \min_{r \in [0, 1]} h' \left( \frac{1}{2} \right)(r - \frac{1}{2}) \). By strictly convexity of function \( h(r) \) in \( r \in (0, 1) \) from Assumption [A2] and we have \( h(r) \geq h \left( \frac{1}{2} \right) + h' \left( \frac{1}{2} \right)(r - \frac{1}{2}) \geq c_h \) for all \( r \in [0, 1] \). This allows us to establish that

\[
f(c) \leq g(r(c)) - cc_h, \quad \forall c \in (0, 1]. \tag{EC.65}
\]

When \( c \leq 1 \), \( r = \frac{1}{2} c \) is feasible solution to problem (15). Thus,

\[
f(c) \geq g \left( \frac{c}{2} \right) - c h \left( \frac{1}{2} \right), \quad \forall c \in (0, 1]. \tag{EC.66}
\]

By (EC.64), \( \lim_{c \to 0} g(r(c)) - cc_h = g(0) \) and \( \lim_{c \to 0} g(\frac{r}{c}) - ch(\frac{r}{c}) = g(0) \). Thus, by (EC.65) and (EC.66), we have \( \lim_{c \to 0} f(c) = g(0) \). Q.E.D.
Proof of Lemma 2. We prove the claim under the framework provided by Fujishige (1980).

Under Assumptions [A1] - [A4] by item (i) of Lemma 1, function \( f(c) \) is differentiable and strictly concave for \( c > 0 \). We define \( f'(c) \) as the derivative of \( f(c) \) evaluated at \( c > 0 \).

In the optimal solution \( y^* \), we let the distinct values of \( \{ \frac{y_j^*}{b_j} \}_{j \in B} \) be given by \( c_1 < c_2 \cdots < c_l \). Given that \( y^* \) is a feasible solution, it is an independent vector of \( \mathcal{P} \). By Lemma 2.3 of Fujishige (1980),

we can define \( \text{dep}(y^*, j) = \bigcap \{ B : j \in B \subset \mathcal{B}, \sum_{i \in B} y_i^* = \sum_{i \in N_k(B)} s_i \} \). If \( \text{dep}(y^*, j) = \emptyset \), by strict increasing property of function \( f(\cdot) \), we can always increase \( y_j^* \) to improve the objective function, which is a contradiction. Thus, we have

\[
\text{dep}(y^*, j) \neq \emptyset, \quad \forall j \in \mathcal{B}.
\]

We also define a collection of sets as \( \mathcal{J} = \{ B \subset \mathcal{B} : \sum_{j \in B} y_j^* = \sum_{i \in N_k(B)} s_i \} \).

Next, we want to show that if \( \frac{y_{j_1}^*}{b_{j_1}} < \frac{y_{j_2}^*}{b_{j_2}} \), then \( j_2 \notin \text{dep}(y^*, j_1) \). Suppose towards contradiction that we have \( j_2 \in \text{dep}(y^*, j_1) \). By strict concavity and strict increasing property of \( f(c) \) in \( c > 0 \) from Lemma 1(i), we have \( f'(\frac{y_{j_1}^*}{b_{j_1}}) > f'(\frac{y_{j_2}^*}{b_{j_2}}) > 0 \). For any \( \delta > 0 \) small enough, we can obtain the following two conditions:

1. \( \delta < \min_B \{ \sum_{i \in N_k(B)} s_i - \sum_{j \in B} y_j \} \);
2. \( f'(\frac{y_{j_1}^* + \delta}{b_{j_1}}) > f'(\frac{y_{j_2}^*}{b_{j_2}}) > 0 \).

We construct a new vector \( \bar{y} \) such that \( \bar{y}_{j_1} = y_{j_1}^* + \delta, \bar{y}_{j_2} = y_{j_2}^* - \delta \), and \( \bar{y}_j = y_j^* \) for all \( j \neq j_1, j_2 \). Given that \( j_2 \in \text{dep}(y^*, j_1) \) and \( \delta \) is small, the constructed new vector \( \bar{y} \) satisfies that \( \bar{y} \in \mathcal{P} \). Define function \( H(y) = \sum_{j \in B} b_j f \left( \frac{y_j}{b_j} \right) \). By strict concavity of function \( f(\cdot) \), \( H(y) \) is strictly concave, which implies that \( H(y^*) < H(\bar{y}) + z^T(\bar{y} - y) \) for any \( z \) satisfying \( z_j = f'(\frac{\bar{y}_j}{b_j}) \) for \( j \in \mathcal{B} \). We can further show that \( z^T(\bar{y} - y) = -\delta_3 z_{j_1} + \delta z_{j_2} = -\delta f'(\frac{y_{j_1}^*}{b_{j_1}}) + \delta f'(\frac{y_{j_2}^*}{b_{j_2}}) < 0 \). This leads to \( H(y^*) < H(\bar{y}) \), which is a contradiction to the optimality of \( y^* \).

Thus,

\[
\text{if } \frac{y_{j_1}^*}{b_{j_1}} < \frac{y_{j_2}^*}{b_{j_2}}, \text{ then } j_2 \notin \text{dep}(y^*, j_1).
\]

For each \( k = 1, \ldots, l \), we define \( \mathcal{B}^{(k)} = \{ j \in \mathcal{B} : \frac{y_j^*}{b_j} \leq c_k \} \). With (EC.67) and (EC.68), we conclude that

\[
\emptyset \neq \text{dep}(y^*, j) \subset \mathcal{B}^{(k)}, \forall j \in \mathcal{B}^{(k)}, k = 1, \ldots, l
\]

Using the equivalence of item (i) and (iii) in Theorem 3.2 of Fujishige (1980), we conclude that \( y^* \) is a lexicographical optimal base of \( \mathcal{P} \). Q.E.D.

Proof of Proposition 9. Proof of claim (i). With \( g(r) = F_{b_r}^{-1}(1-r)r \) and \( h(r) = F_{b_r}^{-1}(r(1-r)) \), when we verify Assumptions [A1] and [A2], we can replicate exactly the same verification procedures for Assumptions [EC.1-1] - [EC.1-2] in Lemma EC.1(ii) (by letting \( r = \frac{g_r}{b_r} \) in \( g_r(q) = F_{b_r}^{-1}(1-\frac{q}{b_r}) \) and \( r = \frac{h_r(q)}{b_r} \) in \( h_r(q) = F_{b_r}^{-1}(\frac{q}{b_r}) \)). As a result, Assumption [A1] - [A2] directly follow from Assumptions [EC.1-1] - [EC.1-2] in Lemma EC.1(ii).

To see Assumption [A4] by Lemma EC.1(ii) (Assumptions [EC.1-5] and [EC.1-6]), we have

\[
h'(0) = 0.
\]

Given that \( g(r) \) is a strictly concave differentiable function in \((0,1)\), we have

\[
g'(0) = \lim_{r \to 0} g'(r) = \lim_{r \to 0} \frac{g(r) - g(0)}{r} = \lim_{r \to 0} F_{b_r}^{-1}(1-r) > 0,
\]

which implies that \( g'(0) > h'(0) \) in Assumption [A4].
To see Assumption [A3], we have \( \frac{ch(\xi)}{c} = F_s^{-1}(\frac{\xi}{c})r \), which is strictly decreasing in \( c > 0 \) because function \( F_s^{-1}(\cdot) \) is a strictly increasing function. To see the joint convexity property, we calculate the Hessian matrix for function \( ch(\xi) \) as

\[
H = \begin{pmatrix}
[F_s^{-1}''(\frac{\xi}{c})]_{12} + 2[F_s^{-1}''(\frac{\xi}{c})]_{12} - [F_s^{-1}''(\frac{\xi}{c})]_{12}^2 - 2[F_s^{-1}''(\frac{\xi}{c})]_{12}^2 & \frac{1}{c} [F_s^{-1}''(\frac{\xi}{c})]_{12}^2 - 2[F_s^{-1}''(\frac{\xi}{c})]_{12}^2 \\
\frac{1}{c} [F_s^{-1}''(\frac{\xi}{c})]_{12}^2 - 2[F_s^{-1}''(\frac{\xi}{c})]_{12}^2 & \left[ F_s^{-1}''(\frac{\xi}{c}) \right]_{12}^2 + 2[F_s^{-1}''(\frac{\xi}{c})]_{12}^2
\end{pmatrix}
\]

\[= \begin{pmatrix}
[F_s^{-1}''(\frac{\xi}{c})]_{12} + 2[F_s^{-1}''(\frac{\xi}{c})]_{12}^2 & \frac{1}{c} [F_s^{-1}''(\frac{\xi}{c})]_{12}^2 - 2[F_s^{-1}''(\frac{\xi}{c})]_{12}^2 \\
\frac{1}{c} [F_s^{-1}''(\frac{\xi}{c})]_{12}^2 - 2[F_s^{-1}''(\frac{\xi}{c})]_{12}^2 & \left[ F_s^{-1}''(\frac{\xi}{c}) \right]_{12}^2 + 2[F_s^{-1}''(\frac{\xi}{c})]_{12}^2
\end{pmatrix}.
\]

Matrix \( H \) satisfies the following properties: (1) \( H(1,1) > 0 \) and \( H(2,2) > 0 \), because \( [F_s^{-1}''(\frac{\xi}{c})]_{12} + 2[F_s^{-1}''(\frac{\xi}{c})]_{12}^2 > 0 \) follows from the positivity of the second derivative for strictly convex function \( F_s^{-1}(\frac{\xi}{c})r \) in \( r \) by Assumption [A2]; (2) its determinant satisfies \( |H| = 0 \); (3) \( H \) is symmetric. Thus, function \( ch(\xi) \) is jointly convex in \((r,c)\).

Proof of claim (ii). By Lemma EC.1, the equilibrium problem [14] can be characterized by formulation (EC.1). In the proof of Proposition EC.2, we established an auxiliary Proposition EC.3 where we show that there is an equivalent formulation (EC.43) to problem (EC.1). Proposition EC.2 establishes that formulation (EC.43) is also equivalent to formulation (17).

As a result, claim (ii) directly follows from the solution mappings proposed in Proposition EC.3(ii).

Proof of claim (iii). Claim (iii) follows from the equivalence of formulation (17) and (EC.1) established by Proposition EC.2 which further suggests that the objective value is the same i.e., \( V_{opt} = \sum_{j \in \mathbb{N}} b_j f \left( \frac{\gamma^*}{b_j} \right) \). Q.E.D.

Proof of Proposition 10. Proof of claim(i). With \( g(r) = \int_0^r \frac{1}{1+\gamma} F_b^{-1}(1-x) - \frac{\mu^b}{1+\gamma^b} dx \) and \( h(r) = \int_0^r \frac{1}{1+\gamma} F_s^{-1}(x) + \frac{\mu^s}{1+\gamma^s} dx \), the verification of Assumptions (A1) and (A2) is the same as the verification of Assumptions (EC.1) and (EC.1.2) in Lemma EC.1(ii) (by letting \( r = \frac{\xi}{b_j} \) in \( g_j(q) = \int_0^\xi \frac{1}{1+\gamma} F_b^{-1}(1-x) - \frac{\mu^b}{1+\gamma^b} dx \) and \( r = \frac{\xi^2}{s_{ij}} \) in \( h_i(q) = \int_0^\xi \frac{1}{1+\gamma} F_s^{-1}(x) + \frac{\mu^s}{1+\gamma^s} dx \)).

If \( \mu^b + \frac{1+\gamma^b}{1+\gamma^b} \mu^s < F_b^{-1}(1) \), this implies that \( g(0) - h(0) = \frac{1}{1+\gamma^b} F_b^{-1}(1) - \frac{\mu^b}{1+\gamma^b} \mu^s > 0 \), which implies Assumption (A1).

To see Assumption (A3), we firstly have \( ch(\xi) = c \int_0^{r/c} \frac{1}{1+\gamma} F_s^{-1}(x) + \frac{\mu^s}{1+\gamma^s} dx = \int_0^{r/c} \frac{1}{1+\gamma} F_s^{-1}(\frac{\xi}{c}) + \frac{\mu^s}{1+\gamma^s} dx \). Because function \( F_s^{-1}(x) \) is strictly increasing in \( x \in (0,1) \), \( ch(\xi) \) is strictly decreasing in \( c \) for \( c > 0 \). To show \( ch(\xi) \) is jointly convex in \((r,c)\), the Hessian matrix for function \( ch(\xi) \) can be expressed as

\[
H = \begin{pmatrix}
[F_s^{-1}''(\frac{\xi}{c})]_{12} + 2[F_s^{-1}''(\frac{\xi}{c})]_{12}^2 & \frac{1}{c} [F_s^{-1}''(\frac{\xi}{c})]_{12}^2 - 2[F_s^{-1}''(\frac{\xi}{c})]_{12}^2 \\
\frac{1}{c} [F_s^{-1}''(\frac{\xi}{c})]_{12}^2 - 2[F_s^{-1}''(\frac{\xi}{c})]_{12}^2 & \left[ F_s^{-1}''(\frac{\xi}{c}) \right]_{12}^2 + 2[F_s^{-1}''(\frac{\xi}{c})]_{12}^2
\end{pmatrix}.
\]

The Hessian matrix satisfies the following properties: (1) \( H(1,1) > 0 \) and \( H(2,2) > 0 \), which follows from \( [F_s^{-1}''(\frac{\xi}{c})]_{12} + 2[F_s^{-1}''(\frac{\xi}{c})]_{12}^2 > 0 \) by the strict convexity of \( \int_0^r F_s^{-1}(\frac{\xi}{c}) dx \) in \( r \) in Lemma EC.1(ii) (Assumption [AE]); (2) the determinant satisfies \( |H| = 0 \); (3) matrix \( H \) is symmetric. Thus, function \( ch(\xi) \) is jointly convex in \((r,c)\). In summary, Assumption [A3] is satisfied.

Proof of claim (ii). The proof is exactly the same as the proof of claim (ii) in Proposition 7 Q.E.D.

EC.3. Flexibility of Equilibrium Implementation

As an auxiliary result, we prove the following lemma on the flexibility of implementing a competitive equilibrium.
**Lemma EC.2.** For any competitive equilibrium \((\bm{p}, \bm{x}, \bm{q}^*, \bm{q}^b) \in \mathcal{X}(\gamma^*, \mu^*, \gamma^b, \mu^b)\) induced by \((\gamma, \mu) \in \Gamma \times \mathcal{U}\), there exists

- (i) \(\gamma_i^* \in \Gamma^i\) with \(\overline{\gamma_i}^* = \min\{\gamma_i^* + \frac{\mu_i^b}{p_i}, 1\}\) for all \(i \in \mathcal{S}\) such that \((\bm{p}, \bm{x}, \bm{q}^*, \bm{q}^b) \in \mathcal{X}(\gamma^*, 0, \gamma^b, \mu^b)\);
- (ii) \(\mu_i^* \in \mathcal{U}^i\) with \(\overline{\mu_i}^* = \gamma_i^* p_i + \mu_i^b\) for all \(i \in \mathcal{S}\) such that \((\bm{p}, \bm{x}, \bm{q}^*, \bm{q}^b) \in \mathcal{X}(0, \mu^*, \gamma^b, \mu^b)\);
- (iii) \(\gamma^b_j \in \Gamma^b\) with \(\overline{\gamma^b_j} = \gamma^b_j + \min_{b_{i,j} \in \mathcal{E}\{p_i\}} \mu_j^b\) for all \(j \in \mathcal{B}\) such that \((\bm{p}, \bm{x}, \bm{q}^*, \bm{q}^b) \in \mathcal{X}(\gamma^*, \mu^*, \gamma^b_j, 0)\);
- (iv) \(\mu_j^b \in \mathcal{U}^b\) with \(\overline{\mu_j^b} = \min_{b_{i,j} \in \mathcal{E}\{p_i\}} \mu_j^b\) for all \(j \in \mathcal{B}\) such that \((\bm{p}, \bm{x}, \bm{q}^*, \bm{q}^b) \in \mathcal{X}(\gamma^*, \mu^*, 0, \overline{\mu_j^b})\).

**Proof of Lemma EC.2.** For any \((\bm{p}, \bm{x}, \bm{q}^*, \bm{q}^b) \in \mathcal{X}(\gamma, \mu),\) we have (1) \(s_i F_i((1 - \gamma_i^*) p_i - \mu_i^*) = q_i^*\) for all \(i \in \mathcal{S}\) by expression (2a); (2) \(b_j [1 - F_{b_j}((1 + \gamma_j^b) \min_{b_{i,j} \in \mathcal{E}\{p_i\}} \mu_j^b)] = q_j^b\) for all \(j \in \mathcal{B}\) by expression (2b).

Proof of claim (i). To implement \((\gamma_i^*, 0, \gamma^b_j, 0),\) we define \(\tilde{\gamma}_i^* = \min\{\gamma_i^* + \frac{\mu_i^b}{p_i}, 1\}\) for all \(i \in \mathcal{S}\). To show \((\bm{p}, \bm{x}, \bm{q}^*, \bm{q}^b) \in \mathcal{X}(\gamma^*, 0, \gamma^b_j, 0),\) it is sufficient to verify expression (2a). We discuss two cases: (1) if \((1 - \gamma_i^*) p_i - \mu_i^* < 0,\) then we have \(\tilde{\gamma}_i^* = 1 \in [0, 1]\) by construction, and the claim holds because \(s_i F_i((1 - \gamma_i^*) p_i) = s_i F_i((1 - \gamma_i^*) p_i - \mu_i^*) = 0;\) (2) if \((1 - \gamma_i^*) p_i - \mu_i^* \geq 0,\) then \(\tilde{\gamma}_i^* = \gamma_i^* + \frac{\mu_i^b}{p_i} \leq 1,\) which further implies that \(\tilde{\gamma}_i^* \in [0, 1]\) and \(s_i F_i((1 - \gamma_i^*) p_i) = s_i F_i((1 - \gamma_i^*) p_i - \mu_i^*) = q_i^*\).

Proof of claim (ii). To implement \((\bm{0}, \mu^*, \gamma^b_j, \mu^b)\), we define \(\tilde{\mu}_i^* = \gamma_i^* p_i + \mu_i^b\) for all \(i \in \mathcal{S}\). Again, it is sufficient to verify expression (2b). By construction, we have \(\tilde{\mu}_i^* \geq 0,\) Expression (2b) follows because \(s_i F_i(p_i - \tilde{\mu}_i^*) = s_i F_i((1 - \gamma_i^*) p_i - \mu_i^b) = q_i^b\).

Proof of claim (iii). To implement \((\gamma_i^*, \mu_i^*, \gamma^b_j, 0),\) we define \(\tilde{\gamma}_j^b = \gamma_j^b + \frac{\mu_j^b}{\min_{b_{i,j} \in \mathcal{E}\{p_i\}} \mu_j^b}\) for all \(j \in \mathcal{B}\). To show \((\bm{p}, \bm{x}, \bm{q}^*, \bm{q}^b) \in \mathcal{X}(\gamma^*, \mu^*, \gamma^b_j, 0),\) it is sufficient to verify expression (2b). Note that we have \(\tilde{\gamma}_j^b \geq 0\) by construction, and expression (2b) follows from \(b_j [1 - F_{b_j}((1 + \gamma_j^b) \min_{b_{i,j} \in \mathcal{E}\{p_i\}} \mu_j^b)] = b_j [1 - F_{b_j}((1 + \gamma_j^b) \min_{b_{i,j} \in \mathcal{E}\{p_i\}} \mu_j^b)] = q_j^b\).

Proof of claim (iv). To implement \((\gamma^*, \mu^*, 0, \overline{\mu_j^b})\), we define \(\tilde{\mu}_j^b = \min_{b_{i,j} \in \mathcal{E}\{p_i\}} \mu_j^b\) for all \(j \in \mathcal{B}\). Again, it is sufficient to verify expression (2b). By construction, we have \(\tilde{\mu}_j^b \geq 0\) and \(b_j [1 - F_{b_j}((1 + \gamma_j^b) \min_{b_{i,j} \in \mathcal{E}\{p_i\}} \mu_j^b)] = b_j [1 - F_{b_j}((\min_{b_{i,j} \in \mathcal{E}\{p_i\}} \mu_j^b)] = q_j^b\). Q.E.D.

**EC.4. Proof of Results in Section 2**

**EC.4.1. Major Results for Section 2**

**Proof of Proposition 1.** We prove the result in the following two steps.

Step 1: existence of competitive equilibrium. By Lemma EC.1, we can apply Proposition EC.1(i) to establish that there exists an optimal solution to problem (14). By Proposition 8 an optimal solution to problem (14) is a competitive equilibrium. This completes the proof for the existence of a competitive equilibrium.

Step 2: uniqueness. By Proposition EC.1(iii) we establish that optimal supply-demand vector \((\bm{q}^*, \bm{q}^b)\) is unique. By Proposition EC.1(v) we established that optimal dual vector \((\bm{\theta}^*, \bm{\theta}^b)\) is unique. Given that \(p_i = \theta_i^* \forall i: q_i^* > 0,\) we establish that price vector \((p_i)_{i:q_i^*>0}\) is unique. Q.E.D.

**Proof of Corollary 1.** For any competitive equilibrium \((\bm{p}, \bm{x}, \bm{q}^*, \bm{q}^b) \in \mathcal{X}(\gamma, \mu)\) induced by \((\gamma, \mu) \in \Gamma \times \mathcal{U}\), we first derive an equivalent revenue expression

\[
V(\gamma, \mu) = \sum_{i,j:(i,j) \in \mathcal{E}} (\gamma_i^* + \gamma_j^b) p_{ij} x_{ij} + \sum_{i,j:(i,j) \in \mathcal{E}} (\mu_i^* + \mu_j^b) x_{ij} \tag{EC.74}
\]

(a) \(= \sum_{j \in \mathcal{B}, q_j > 0} (\gamma_j^b \min_{b_{i,j} \in \mathcal{E}\{p_i\}} \mu_j^b) q_j^b + \sum_{i \in \mathcal{S}, q_i > 0} (\gamma_i^* p_i + \mu_i^b) q_i^b\),

where in step (a), we first aggregate all \(\sum_{j:(i,j) \in \mathcal{E}} (\gamma_i^* p_i + \mu_i^b) x_{ij}\) together and replace \(\sum_{j:(i,j) \in \mathcal{E}} x_{ij}\) with \(q_j^b\). Next, if \(x_{ij} > 0\), we have \(p_{ij} = \min_{b_{i,j} \in \mathcal{E}\{p_i\}} \mu_j^b\) by equilibrium expression (2a), which allows us to group all \(\sum_{i:x_{ij} > 0} (\gamma_i^* p_i + \mu_i^b) x_{ij}\) together and replace \(\sum_{i:(i,j) \in \mathcal{E}} x_{ij}\) with \(q_j^b\). By Proposition 1, ...
vectors \((q^s, q^b)\) and \((p_i)_{i,q^s>0}\) are unique. Thus, the revenue function \(V(\gamma, \mu)\) is well-defined over the platform’s decision space \(\Gamma \times \mathcal{U}\). Q.E.D.

**EC.4.2. Nonconvexity of Revenue Function over Commissions/Subscriptions**

We use problem (14) to illustrate non-convexity of the platform’s revenue function \(V(\gamma, \mu)\).

**EXAMPLE EC.1.** Consider a network with one seller type and one buyer type. Let the value distribution be \(F_s(v) = F_b(v) = 1 - \exp(-v)\) for \(v \geq 0\) and the population be \(s = b = 1\). Consider the following three commission-subscription vectors \((\gamma^s, \gamma^b, \mu^s, \mu^b)\): \((0, 0, 1, 1)\), \((0, 0, 2, 2)\) and \((0, 0, 3, 3)\). An equilibrium for these commissions-subscriptions can be obtained by solving (14). The corresponding revenue can be computed using (3a) and is given by \(V(0, 0, 1, 1) = 0.24, V(0, 0, 2, 2) = 0.07\) and \(V(0, 0, 3, 3) = 0.01\). Note that since \(\bar{V}(0, 0, 2, 2) < \frac{1}{2}V(0, 0, 1, 1) + \frac{1}{2}V(0, 0, 3, 3)\), it follows that \(V(\gamma^s, \gamma^b, \mu^s, \mu^b)\) is not concave in \((\gamma^s, \gamma^b, \mu^s, \mu^b)\). Q.E.D.

**EC.5. Proof of Results in Section 3**

**Proof of Theorem 1**

Let \((\bar{x}, \bar{q}^s, \bar{q}^b)\) be optimal solution to problem (7). We prove the claim (i) and (ii) of this theorem in the following steps. In step 1, we establish that \(V_{opt} \leq \bar{V}_{opt}\) i.e., problem (7) is an upper bound problem for (3). In step 2, we show \(\bar{V}_{opt}\) can be achieved by the constructions in claim (ii), which also allows us to conclude that \(V_{opt} = \bar{V}_{opt}\) in claim (i). In step 3, we show that any optimal solution to problem (3) must have supply/demand vectors equal to \((\bar{q}^s, \bar{q}^b)\).

Step 1: upper bound problem. First consider the optimization problem obtained after replacing the objective of (3) with that of (7). Problem (3) is a relaxation of (3).

Next, consider a feasible solution of (3). Note that the constraint \((p, x, q^s, q^b) \in \mathcal{X}(\gamma, \mu)\) of (3) is equivalent to requiring that \((p, x, q^s, q^b)\) satisfies (2a), (2b), and (2c). In order to obtain the feasible set of (7), we need to (1) drop constraints \(x_{ij} = 0\) for all \(i \notin \mathcal{E}(i; j) \in \mathcal{E}(p)\) in (2c), (2) replace equilibrium expression (2a) with \(q^s_i \leq s_i\), and (3) replace equilibrium expression (2b) with \(q^b_j \leq b_j\). Note that since for distribution functions \(F_{x_i}(x), F_{s_i}(x) \in [0, 1]\) for all \(x\), it follows that \(q^s_i \leq s_i\) and \(q^b_j \leq b_j\). These observations jointly imply that any \((x, q^s, q^b)\) feasible in (3) (together with some \((\gamma, \mu)\)) is feasible in (7). Thus, it follows that (7) is a relaxation of (3). Since (3) is a relaxation of (3), it follows that (7) is a relaxation of (3), and we conclude that \(V_{opt} \leq \bar{V}_{opt}\).

Step 2: optimal commissions/subscriptions. We consider the framework problem (EC.1) with \(g_j(q) = F_{q_j}^{-1}(1 - \frac{b_j}{a_j})q\) and \(h_i(q) = F_{s_i}^{-1}(\frac{q_i}{b_i})\). With primal optimal solution \((\bar{x}, \bar{q}^s, \bar{q}^b)\), we let \((\theta^s, \theta^b, \eta^b, \eta^s, \pi)\) be a dual optimal solution vector corresponding to constraints (EC.1b) - (EC.1f). We set the price vector \(p\) be such that

\[
p_i = \theta^s_i, \quad \forall i \in \mathcal{S}.
\]  

We discuss the optimality of the constructions in claim (ii).

Step 2 - 1 optimal commissions: In the commission-only model, we set the subscription fee vector as \(\mu = 0\) and prove the optimality of the following commissions

\[
\gamma^s_i = 1 - \frac{1}{\theta^s_i}F_{s_i}^{-1}\left(\frac{q^s_i}{s_i}\right) \quad \forall i \in \mathcal{S},
\]  

\[
\gamma^b_j = \frac{1}{\theta^b_j}F_{b_j}^{-1}\left(1 - \frac{q^b_j}{b_j}\right) - 1 \quad \forall j \in \mathcal{B}.
\]  

We start from establishing that \((\gamma^s, \gamma^b) \in \Gamma\). By Lemma EC.1(b) Assumptions EC.1(b) - EC.1(d) hold. For all \(i \in \mathcal{S}\), by condition (EC.2b) in Proposition EC.1(b) (ii) we have

\[
\theta^s_i = \eta^s_i + \left[F_{s_i}^{-1}\left(\frac{q^s_i}{s_i}\right)\right] \frac{q^s_i}{s_i} + F_{s_i}^{-1}\left(\frac{q^s_i}{s_i}\right).
\]
Note that we have \( \eta_i^* \geq 0 \) by condition (EC.2e) and \( F_{s_i}^{-1}(\eta_i^*) > 0 \) by nonnegativity function \( F_{s_i}(\cdot) \) from Assumption 1. To show \( [F_{s_i}^{-1}]'(\eta_i^*) > 0 \), by Proposition (EC.1iv) we have \( q_i^b > 0 \). By strict increasing property of function \( F_{s_i}^{-1} \) in \((0,1)\) we have \([F_{s_i}^{-1}]'(\eta_i^*) > 0\). Thus, we obtain that \( \theta_i^b > F_{s_i}^{-1}(\eta_i^*) > 0 \) and \( 1 - \gamma_i^b = \frac{1}{\theta_i^b} F_{s_i}^{-1}(\eta_i^*) \in (0,1) \).

For all \( j \in B \), we have \( q_j^b < b_j \) by Proposition (EC.1iv). By condition (EC.2d) in Proposition (EC.1ii) we have \( \eta_j^b = 0 \). Condition (EC.2a) implies that

\[
\theta_j^b = - [F_{b_j}^{-1}'] \left( 1 - \frac{q_j^b}{b_j} \right) \frac{q_j^b}{b_j} + F_{b_j}^{-1} \left( 1 - \frac{q_j^b}{b_j} \right).
\]  

(EC.78)

With \([F_{b_j}^{-1}'] \left( 1 - \frac{q_j^b}{b_j} \right) \frac{q_j^b}{b_j} \geq 0 \) by the increasing property of \( F_{b_j}(\cdot) \) in Assumption 1 we have

\[
\theta_j^b \leq F_{b_j}^{-1} \left( 1 - \frac{q_j^b}{b_j} \right), \quad \forall j \in B.
\]  

(EC.79)

To show that \( \theta_j^b \geq 0 \), we suppose towards contradiction that \( \theta_j^b < 0 \), which further suggests \( g_j'(q_j^b) < 0 \) from condition (EC.2e). This also suggests \( q_j^b > 0 \) by the strict concavity of function \( g_j(\cdot) \) by Lemma (EC.1ii)(Assumption (EC.1-1)) and \( g_j'(0) \geq 0 \). Furthermore, we can find \( i : \bar{x}_{i,j} > 0 \) such that \( h_i'(\bar{x}_{i,j}) > 0 \) because \( h_i'(0) \geq 0 \) and function \( h_i(\cdot) \) is strictly convex by Lemma (EC.1ii)(Assumption (EC.1-2)) - (EC.13). This suggests that we can decrease \( q_i^b, q_j^b \) and \( \bar{x}_{i,j} \) simultaneously to strictly increase the objective value, contradicting to the optimality of \((\bar{x}, \bar{q}^*, \bar{q}^b)\). Thus, we have

\[
\theta_j^b \geq 0, \quad \forall j \in B.
\]  

(EO.80)

By (EC.79) and (EC.80), we obtain that \( 1 + \gamma_j^b = \frac{1}{\theta_j^b} F_{b_j}^{-1} \left( 1 - \frac{q_j^b}{b_j} \right) \geq 1 \), or correspondingly \( \gamma_j^b \geq 0 \). This allows us to conclude that \( (\gamma^*, \gamma^b) \in \Gamma \).

Next, we verify that \((p, \bar{x}, \bar{q}^*, \bar{q}^b) \in \mathcal{X}(\gamma, 0)\) by checking equilibrium expressions (2a) - (2c).

To see expression (2a), for each \( i \in S \), we can derive that

\[
s_i F_{s_i} \left( (1 - \gamma_i^b)p_i \right) \overset{(a)}{=} s_i F_{s_i} \left( (1 - \gamma_i^b)\theta_i^b \right) \overset{(b)}{=} s_i F_{s_i} \left( F_{s_i}^{-1} \left( \frac{q_i^b}{\theta_i^b} \right) \right) = \bar{q}_i^s,
\]  

(EO.81)

where step (a) follows from \( p_i = \theta_i^b \) in (EC.75). Step (b) follows from the construction of \( \gamma_i^b \) in (EC.76a). Thus, expression (2a) holds.

To see expression (2b), we start from \( \theta_i^b \geq \theta_j^b \), which is obtained from \( \theta_i^b - \theta_j^b + \pi_{ij} = 0 \) by condition (EC.2d) and \( \pi_{ij} \geq 0 \) by condition (EC.2d) in Proposition (EC.1ii). For each \( j \in B \), we discuss two subcases.

When \( q_j^b > 0 \), then for all \( i : \bar{x}_{i,j} > 0 \), we have \( \pi_{ij} = 0 \) by condition (EC.2d) and then \( \theta_j^b = \theta_i^b \). Thus, if \( i : \bar{x}_{i,j} > 0 \), we have

\[
\theta_j^b = \theta_i^b = \min_{i':(i',j) \in E} \{ \theta_{i'}^b \}.
\]  

(EO.82)

This allows us to derive that

\[
b_j \left[ 1 - F_{b_j} \left( (1 + \gamma_j^b) \min_{i':(i',j) \in E} \{ p_{i'} \} \right) \right] \overset{(d)}{=} b_j \left[ 1 - F_{b_j} \left( (1 + \gamma_j^b) \min_{i':(i',j) \in E} \{ \theta_{i'}^b \} \right) \right] \overset{(e)}{=} b_j \left[ 1 - F_{b_j} \left( F_{b_j}^{-1} \left( 1 - \frac{q_j^b}{b_j} \right) \right) \right] = \bar{d}_j^b,
\]  

(EO.83)
where step (c) follows from price expression (EC.75). Step (d) follows from expression (EC.82). Step (e) follows from the construction of $\gamma_{ij}^b$ in (EC.76b).

When $q_{ij}^b = 0$, we have $\theta_i^b \leq \min_{i,(i,j) \in E} \{ \theta_{ij}^b \} = \min_{i,(i,j) \in E} \{ p_i \}$, which is obtained from $\theta_i^s \geq \theta_i^b$ for all $(i, j) \in E$ by condition (EC.2c). This in turn implies that $(1 + \gamma_{ij}^b) \min_{i,(i,j) \in E} \{ p_i \} \geq (1 + \gamma_{ij}^b) \theta_j^b = F_{b_j}^{-1} (1 - \frac{\gamma_{ij}^b}{\theta_j^b}) = F_{b_j}^{-1} (1)$, from which we obtain

$$b_j \left[ 1 - F_{b_j} \left( \left( 1 + \gamma_{ij}^b \right) \min_{i,(i,j) \in E} \{ p_i \} \right) \right] = b_j \left[ 1 - F_{b_j} \left( F_{b_j}^{-1} (1) \right) \right] = 0 = q_{ij}^b. \quad \text{(EC.84)}$$

To see expression (2c), constraints (EC.1b), (EC.1c) and (EC.1i) guarantee expression (2c). Moreover, if $\bar{x}_{ij} > 0$, then we have $i \in \arg \min_{i,(i',j) \in E} \{ \theta_{ij}^s \} = \arg \min_{i,(i',j) \in E} \{ p_{i'} \}$, which guarantees expression (2d).

In summary, we have verified $(p, \bar{x}, \bar{q}^s, \bar{q}^b) \in X(\gamma, 0)$.

To establish optimality of our construction, we verify that the revenue induced by $\gamma \in \Gamma$ satisfies

$$V(\gamma, 0) = \sum_{i,j : \bar{x}_{ij} > 0} \left( 1 + \gamma_{ij}^b \right) p_i \bar{x}_{ij} - \sum_{i,j : \bar{x}_{ij} > 0} \left( 1 - \gamma_{ij}^s \right) p_i \bar{x}_{ij}$$

$$\overset{(f)}{=} \left[ \sum_{i \in S : \bar{q}_{ij}^s > 0} \left( 1 + \gamma_{ij}^b \right) \theta_i^b \bar{q}_{ij}^b \right] - \left[ \sum_{i \in S : \bar{q}_{ij}^s > 0} \left( 1 - \gamma_{ij}^s \right) \theta_i^s \bar{q}_{ij}^s \right]$$

$$= \sum_{j \in B} F_{b_j}^{-1} (1 - \bar{q}_{ij}^b) \bar{q}_{ij}^b - \sum_{i \in S} F_{s_i}^{-1} \left( \frac{\bar{q}_{ij}^s}{s_i} \right) \bar{q}_{ij}^s = \tilde{V}_{opt}, \quad \text{(EC.85)}$$

where step (f) follows from aggregating $\bar{x}_{ij}$ by $\sum_{i : \bar{x}_{ij} > 0} \bar{x}_{ij} = \bar{q}_{ij}^b$ and $\sum_{j : \bar{x}_{ij} > 0} \bar{x}_{ij} = \bar{q}_{ij}^s$. Note that when we aggregate $\bar{x}_{ij}$ regarding $i$ given $\bar{x}_{ij} > 0$, we have $p_i = \theta_i^s$ from (EC.75) and $\theta_j^b = \theta_j^s$ from (EC.82).

Step 2-2 optimal subscriptions: given $(p, \bar{x}, \bar{q}^s, \bar{q}^b) \in X(\gamma, 0)$, we consider the construction of $\mu$ given by

$$\mu_i^s = \theta_i^s - F_{s_i}^{-1} \left( \frac{\bar{q}_{ij}^s}{s_i} \right) \quad \forall i \in S, \quad \text{(EC.86a)}$$

$$\mu_j^b = F_{b_j}^{-1} \left( 1 - \frac{\bar{q}_{ij}^b}{b_j} \right) - \theta_j^b \quad \forall j \in B. \quad \text{(EC.86b)}$$

Note that this construction follows directly from the construction in Lemma [EC.2][iv] and Lemma [EC.2][v]. By Lemma EC.2, we have $(p, \bar{x}, \bar{q}^s, \bar{q}^b) \in X(0, \mu)$.

Given the feasibility $(\gamma, 0) \in \Gamma \times U$ and $(0, \mu) \in \Gamma \times U$, we have $V_{opt} \geq \tilde{V}_{opt}$. With $V_{opt} \leq \tilde{V}_{opt}$, this concludes the proof that $V_{opt} = \tilde{V}_{opt} = V(\gamma, 0)$.

Step 3: any optimal equilibrium supply-demand is $(\bar{q}^*, \bar{q}^b)$. By Lemma EC.1[i], we can use problem framework [EC.1] and apply Proposition EC.1[iii] to conclude that $(\bar{q}^*, \bar{q}^b)$ is unique. By Proposition [EC.1] for any equilibrium $(p, x, q^s, q^b)$ induced by optimal $(\gamma, \mu)$, equilibrium supply-demand vector $(\bar{q}^*, \bar{q}^b)$ is unique. We have established that a commission-subscription profile $(\gamma, \mu)$ is optimal if and only if $(\bar{q}^*, \bar{q}^b)$ is supported in the induced equilibrium. This implies that $(q^*, q^b) = (\bar{q}^*, \bar{q}^b)$ for any optimal commission-subscription profile. Q.E.D.

**Proof of Proposition 3.** Consider the following class of networks characterized by problem size $n$.

In problem index by $n$, we let the network be a collection of network components $G(S_i \cup B_i, E_i)$ for $i = 1, 2, \ldots, n$ where each component has one seller type and one buyer type (see the network below).
For each \( i \in \{1, \ldots, n\} \), we first let the value distributions be

\[
F_{s_i}(v) = \min \left\{ \frac{v}{\bar{v}_{s_i}}, 1 \right\}, \quad \text{for any } v \geq 0 \text{ with } \bar{v}_{s_i} = 4^n
\]

\[
F_{b_i}(v) = \min \left\{ \frac{v}{\bar{v}_{b_i}}, 1 \right\}, \quad \text{for any } v \geq 0 \text{ with } \bar{v}_{b_i} = 4^i
\]  

(EC.87)

For all \( i \in \{1, \ldots, n\} \), we also let the population size be

\[
s_i = \bar{v}_{s_i} S^{-i}
\]

\[
b_i = \bar{v}_{b_i} 16^{-i}
\]  

(EC.88)

For any feasible commission-subscription pair \((\gamma, \mu) \in \Gamma \times \mathcal{U}\) and for each \( i \in \{1, \ldots, n\} \), it is without loss of optimality to consider \( \gamma_i^* < 1 \), because setting \( \gamma_i^* = 1 \) would result in no trade in component \( i \). By Proposition \[1\] there exists a price vector \((p_i)_{i=1}^n\) such that the type-\( i \) supply is

\[
q_i^* = s_i \min \left\{ \frac{1}{\bar{v}_{s_i}} \left( (1 - \gamma_i^*) p_i - \mu_i^* \right) , 1 \right\}
\]

by expression (2a), the type-\( i \) demand is

\[
q_i^* = b_i [1 - \min \left\{ \frac{1}{\bar{v}_{b_i}} \left( (1 + \gamma_i^*) p_i + \mu_i^* \right) , 1 \right\}]
\]

by (2b), and the flow conservation condition implies \( q_i^* = x_{ii} = q_i^* \) by (2c).

In our problem instance, we simplify the equilibrium expression by considering the following two properties:

1. We first show that \((1 - \gamma_i^*) p_i - \mu_i^* \geq 0\) for all \( i \in \{1, \ldots, n\} \) is without loss of generality. Suppose there exists \( i \in \{1, \ldots, n\} \) such that \((1 - \gamma_i^*) p_i - \mu_i^* < 0\), then we have \( q_i^* = 0 = q_i^* \) and \((1 + \gamma_i^*) p_i + \mu_i^* \geq b_i\). Given that \( \gamma_i^* < 1 \), we can always increase \( p_i \) to the level where \((1 - \gamma_i^*) p_i - \mu_i^* = 0\). This does not change the equilibrium supply \( q_i^* = 0 \) and thus, it is without loss of generality.

2. We also show that \((1 - \gamma_i^*) p_i - \mu_i^* \leq \bar{v}_{s_i} \) for all \( i \in \{1, \ldots, n\} \). Suppose towards contradiction that there exists \( i \) such that \((1 - \gamma_i^*) p_i - \mu_i^* \geq \bar{v}_{s_i} \). On the supply side, the assumption implies that \( x_{ii} = q_i^* = s_i > 0 \). However, on the demand side, given that \( \frac{1 + \gamma_i^*}{1 - \gamma_i^*} \geq 1 \), \( \mu_i^* \geq 0 \) and \( 4^n = \bar{v}_{s_i} \geq \bar{v}_{b_i} = 4^i \), the assumption implies that \((1 + \gamma_i^*) p_i + \mu_i^* \geq (1 + \gamma_i^*) \bar{v}_{s_i} + \mu_i^* \geq \bar{v}_{b_i}\), which further suggests \( x_{ii} = q_i^* = 0 \). We achieve a contradiction from \( 0 < x_{ii} = 0 \). Thus, we have \((1 - \gamma_i^*) p_i - \mu_i^* \leq \bar{v}_{s_i}\).

With these two simplifications, we end up with the following supply-demand balance equation

\[
s_i \frac{1}{\bar{v}_{s_i}} [(1 - \gamma_i^*) p_i - \mu_i^*] = x_{ii} = b_i - b_i \min \left\{ \frac{1}{\bar{v}_{b_i}} \left[ (1 + \gamma_i^*) p_i + \mu_i^* \right] , 1 \right\}.
\]  

(EC.89)

We reorganize expression (EC.89) to obtain the following expressions for \((p_i, x_{ii})\) for \( i \in \{1, \ldots, n\}\)

\[
p_i = \begin{cases} 
\frac{b_i - (b_i/\bar{v}_{s_i}) \mu_i^* + (s_i/\bar{v}_{s_i}) \mu_i^*}{(s_i/\bar{v}_{s_i})(1 - \gamma_i^*) + (b_i/\bar{v}_{b_i})(1 + \gamma_i^*)} & \text{if } \frac{1 + \gamma_i^*}{1 - \gamma_i^*} \mu_i^* + \mu_i^* \leq \bar{v}_{b_i} \\
\frac{\mu_i^*}{1 - \gamma_i^*} & \text{if } \frac{1 + \gamma_i^*}{1 - \gamma_i^*} \mu_i^* + \mu_i^* > \bar{v}_{b_i}
\end{cases}
\]

\[
x_{ii} = \begin{cases} 
\frac{(s_i/\bar{v}_{s_i})(1 - \gamma_i^*) b_i - (b_i/\bar{v}_{b_i})(1 - \gamma_i^*) \mu_i^* + (1 + \gamma_i^*) \mu_i^*}{(s_i/\bar{v}_{s_i})(1 - \gamma_i^*) + (b_i/\bar{v}_{b_i})(1 + \gamma_i^*)} & \text{if } \frac{1 + \gamma_i^*}{1 - \gamma_i^*} \mu_i^* + \mu_i^* \leq \bar{v}_{b_i} \\
0 & \text{if } \frac{1 + \gamma_i^*}{1 - \gamma_i^*} \mu_i^* + \mu_i^* > \bar{v}_{b_i}
\end{cases}
\]  

(EC.90)
For simplicity of notation, set \( r_i = \frac{1+\gamma_i^b}{1-\gamma_i} - 1 \) and \( \xi_i = (r_i + 1)\mu_i^s + \mu_i^b \). Let \( V_i \) be the amount of revenue from network component \( G(S_i \cup B_i, E_i) \) in a competitive equilibrium. We can show that the expression for \( V_i \) satisfies

\[
V_i = \left[(\gamma_i^s + \gamma_i^b) p_i + (\mu_i^s + \mu_i^b)\right] x_{ii}
\]

\[
\leq \begin{cases} 
\left[1 + \gamma_i - (1 - \gamma_i)\right] \frac{b_{i} - (b_i / \bar{v}_{b_i})}{(s_i / \bar{v}_{s_i})} + (\mu_i^s + \mu_i^b) & \text{if } \frac{1 + \gamma_i^b}{1 - \gamma_i} \mu_i^s + \mu_i^b \leq \bar{v}_{b_i} \\
0 & \text{if } \frac{1 + \gamma_i^b}{1 - \gamma_i} \mu_i^s + \mu_i^b \geq \bar{v}_{b_i} 
\end{cases}
\]

Step (c) follows from writing the expression in step (b) in a compact way as (EC.90). Step (d) follows from reorganization of the expression in step (a) with changes of variables \( r_i = \frac{1+\gamma_i^b}{1-\gamma_i} - 1 \) and \( \xi_i = (r_i + 1)\mu_i^s + \mu_i^b \). Step (c) follows from writing the expression in step (b) in a compact way as \( (\bar{v}_i^b - \xi_i)^+ = \max\{\bar{v}_i^b - \xi_i, 0\} \). Step (d) follows from plugging in the values of \( \bar{v}_{s_i} = 4^n, \bar{v}_{b_i} = 4^i, s_i / \bar{v}_{s_i} = 8^{-i} \) and \( b_i / \bar{v}_{b_i} = 16^{-i} \). Abusing some notation, we can parameterize the equilibrium revenue expression as a function of \( (r_i, \xi_i) \) and denote \( V_i(r_i, \xi_i) \) as the revenue from network component \( G(S_i \cup B_i, E_i) \) for \( i \in \{1, \ldots, n\} \). 

Step 1: heterogeneous commission-subscription pairs. Consider a lower bound for the optimal revenue under the heterogeneous commission-subscription pair. Consider a feasible commission-subscription pair \( (\gamma, \mu) \) where \( \gamma_i^s = \gamma_i^b = 0, \mu_i^s = 0, \) and \( \mu_i^b = 2^{2i-1} \), for each \( i \in \{1, \ldots, n\} \). This is equivalent to setting \( \xi_i = \frac{4^{2i+1}}{2(2^i+1)} \) for all \( i \in \{1, \ldots, n\} \). We further deduce that

\[
V_{opt}(n) = \sum_{i=1}^{n} V_i(r_i, \xi_i)
\]

\[
= \sum_{i=1}^{n} \frac{8^{-i}}{[2^i + (1+r_i)]} [4^i r_i + (2^i + 1)\xi_i]
\]

\[
= \sum_{i=1}^{n} \frac{8^{-i}}{2^i + 1} \cdot \frac{16^i (2^i + 2)^2}{4^i}
\]

\[
= \frac{1}{4} \sum_{i=1}^{n} \frac{2^i}{1+2^i} \geq \frac{1}{8} n, \tag{EC.92}
\]

where the inequality in step (e) holds because the constructed \( (r, \xi) \) is a feasible solution. Inequality in step (f) follows from \( \frac{2^i}{1+2^i} \geq \frac{1}{2} \) for all \( i \in \{1, \ldots, n\} \).
Step 2: homogeneous commission-subscription pairs. In the homogeneous case, we can only use one vector of commission-subscription pair \((\gamma^s, \gamma^b, \mu^s, \mu^b)\), which corresponds to homogeneous \((r, \xi)\). Using the revenue expression in \((EC.91)\), we derive the following upper bound for the optimal revenue

\[
V_h(n) \leq \max_{r, \xi \geq 0} \sum_{i=1}^{n} \frac{8^{-i}}{[2^i + (1 + r)]^2} (4^i - \xi)^+ [4^i r + (2^i + 1) \xi], \tag{EC.93}
\]

where in inequality \((g)\), given the change of variables \(r = \frac{1}{1 + x^2} - 1\) and \(\xi = (r + 1) \mu^s + \mu^b\), we see that \(\gamma^s, \gamma^b, \mu^s, \mu^b \geq 0\) implies \(r, \xi \geq 0\) in the. We denote \((r, \xi)\) as the optimal solution for the upper bound optimization problem in \((EC.93)\). It is easy to see that the upper bound expression \((EC.93)\) is at least 0. Moreover, if \(\xi \geq 4^n\), then the upper bound expression \((EC.93)\) is 0. Thus, it is without loss of optimality to say that there exists \(s_i \in [0, \ldots, n]\) and \(t_i \in [-1, 3 \cdot 4^i\] such that \(\xi = 4^i + t_i\). For any \(\xi \geq 0\), the expression in each component \(i \in \{1, \ldots, n\}\) is decreasing in \(r \geq (1 + 2^i) - 2\xi^2 + 1\), which implies that the upper bound expression \((EC.93)\) is decreasing in \(r \geq 2^{n+1}\). Thus, without loss of optimality, there exists \(s_r \in [0, \ldots, n]\) and \(t_r \in [-1, 2^{r}]\) such that \(r = 2^r + t_r\). Following the expression of the optimal solution \(r = 2^r + t_r\) and \(\xi = 4^i + t_i\), we can further derive that

\[
\frac{V_h(n)}{V_{opt}(n)} \leq \frac{8}{n} \sum_{i=1}^{n} \frac{8^{-i}}{(2^i + 1 + r)^2} (4^i - \xi)^+ [4^i r + (2^i + 1) \xi] 
\]

\[
= \frac{8}{n} \sum_{i=1}^{n} \frac{8^{-i}}{(2^i + 1 + r)^2} [- (2^i + 1) \xi^2 + 4^i (2^i + 1 - r) \xi + 16^i r]^+ 
\]

\[
\leq \frac{8}{n} \sum_{i=1}^{n} \frac{8^{-i}}{(2^i + 1 + r)^2} 16^i r + \frac{8}{n} \sum_{i=1}^{n} \frac{8^{-i}}{(2^i + 1 + r)^2} [- (2^i + 1) \xi^2 + 4^i (2^i + 1 - r) \xi]^+ 
\]

\[
\leq \frac{8}{n} \sum_{i=1}^{n} \frac{2^i r}{(2^i + 1)^2} + \frac{8}{n} \sum_{i=1}^{n} \frac{8^{-i}}{(2^i + 1)^2} [- (2^i + 1) \xi^2 + 4^i (2^i + 1) \xi]^+ 
\]

\[
= \frac{8}{n} \sum_{i=1}^{n} \frac{2^i r}{2^i + 2} + \frac{8}{n} \sum_{i=1}^{n} \frac{8^{-i}}{2^i + 1} [4^i - \xi]^+ 
\]

\[
= \frac{8}{n} \sum_{i=1}^{n} \frac{2}{2^i + 2} + \frac{8}{n} \sum_{i=1}^{n} \frac{8^{-i}}{2^i + 1} [4^i - \xi]^+ 
\]

\[
\leq \frac{8}{n} \sum_{i=1}^{n} \frac{1}{2^i - 4} + \frac{8}{n} \sum_{i=1}^{n} \frac{1}{2^{r-i} - 1} + \frac{8}{n} \sum_{i=1}^{n} \frac{1}{4^i - 1} 
\]

\[
\leq \frac{272}{n}, \tag{EC.94}
\]

where step \((h)\) follows from implementing the lower bound of \(V_{opt}(n)\) in \((EC.92)\) and the upper bound of \(V_h(n)\) in \((EC.93)\). In step \((i)\), we implement the following upper bound \([- (2^i + 1) \xi^2 + 4^i (2^i + 1) \xi + 16^i r]^+ \leq [- (2^i + 1) \xi^2 + 4^i (2^i + 1) \xi]^+ + 16^i r\). In step \((j)\), we implement \(2^i + 1 + r \geq 2^i + r\) in the first expression, and we also implement \(2^i + 1 + r \geq 2^i + 1 + 2^i + 1 - r \leq 2^i + 1\) in the second expression. Step \((k)\) follows from plugging in the expression \(r = 2^r + t_r\). In step \((l)\), we first implement inequalities \(\frac{2^r + t_r}{2} \geq \frac{1}{2} \cdot 2^r - \frac{1}{2} \) and \(\frac{2^r + t_r}{2} \geq \frac{1}{2} \cdot 2^r - \frac{1}{2}\) in the first expression. In the second expression, since \(4^i - \xi \leq 0\) for all \(i \leq s_r - 1\), we have \(- \xi^2 + 4^i \xi \leq 4^i 4^i \xi + 1\). As a result of this derivation, we end up with the upper bound expression in step \((l)\). The inequality in step \((m)\) follows from \(\sum_{i=1}^{n} \frac{1}{2^i - 1} \leq 2, \sum_{i=s_r+1}^{n} \frac{1}{2^i - 1} \leq 1,\) and \(\sum_{i=1}^{n} \frac{1}{4^i - 1} \leq \frac{16}{3}\). This allows us to derive that \(\frac{V_h(n)}{V_{opt}(n)} \rightarrow 0\) as \(n \rightarrow \infty\). Q.E.D.
Proof of Corollary 2: By Proposition 9, we can apply framework problem (17) to revenue optimization problem (7) with the value function

\[ f(c) = \max_{r \in [0, \min\{1, c\}]} g(r) - rh \left( \frac{r}{c} \right), \tag{EC.95} \]

where \( g(r) = F^{-1}_c(1-r)c \) and \( h(r) = F^{-1}_c(r) \).

By Lemma 1(i), value function \( f(c) \) is differentiable. We let \((\theta, \pi) = ((\theta_B)_{B \subseteq E}, (\pi_j)_{j \in B})\) be the dual solution corresponding to constraints (17b) and (17c). The Langrangian for problem (17) can be written as

\[ \mathcal{L}(y, \theta, s, b) = \sum_{j \in B} b_j f \left( \frac{y_j}{b_j} \right) - \sum_{B \subseteq E} \theta_B \left( \sum_{j \in B} y_j - \sum_{i \in N_E(B)} s_i \right) + \sum_{j \in B} \pi_j y_j \tag{EC.96} \]

We let \( y^* \) be the optimal solution to problem (17) and \((\theta^*, \pi^*)\) be the corresponding dual optimal solution. We further let \( 0 < c_1 < c_2 < \ldots < c_l \) be the distinct values of \( \frac{y_j}{b_j} \). Define \( B^{(k)} = \{ j \in B : \frac{y_j}{b_j} \leq c_k \} \) and \( B_k = B^{(k-1)} \setminus B^{(k)} \). With the definition of \( B_k \), we obtain that

\[ \frac{y_{j_1}^*}{b_{j_1}} = \frac{y_{j_2}^*}{b_{j_2}}, \quad \forall j_1, j_2 \in B_k \]
\[ \frac{y_{j_1}^*}{b_{j_1}} < \frac{y_{j_2}^*}{b_{j_2}}, \quad \forall j_1 \in B_{k_1}, j_2 \in B_{k_2}, k_1 < k_2 \tag{EC.97} \]

By Lemma 2, \( y^* \) is a lexicographical optimal base of polymatroid \( P \) with respect to weight vector \( b \). By equivalence of item (i) and (ii) Theorem 3.2 in Fujishige (1980), we have \( \sum_{j \in B_k} y_j = \sum_{i \in N_E(B_k)} s_i \). Define \( X_k = \{ B_k, \ldots, B_l \} \) for all \( k \in \{1, \ldots, l\} \). For all \( i \in S \), we can find minimum index \( k \in \{1, \ldots, l\} \) with \( i \in N_E(B_k) \) such that for any \( j \in B_k \), we have

\[ \frac{\partial}{\partial s_i} \mathcal{V}_{opt}(s, b) \xlongequal{(a)} \frac{\partial}{\partial s_i} \mathcal{L}(y^*, \theta^*, s, b) \xlongequal{(b)} \sum_{B \subseteq E, i \in N_E(B)} \theta_B^* \]
\[ \xlongequal{(c)} \sum_{B \subseteq X_k} \theta_B^* \xlongequal{(d)} \sum_{B \subseteq E, j \in B} \theta_B^* \xlongequal{(e)} f' \left( \frac{y_j^*}{b_j} \right), \tag{EC.98} \]

where step (a) follows from applying the envelope theorem to formulation (17). Step (b) follows from taking the partial derivative of \( \mathcal{L}(\cdot) \) in terms of \( s_i \) given expression (EC.96). In step (c), the KKT condition for problem (17) says that

\[ f' \left( \frac{y_j^*}{b_j} \right) - \sum_{B \subseteq E, j \in B} \theta_B^* + \pi_j^* = 0 \quad \forall j \in B, \tag{EC.99a} \]
\[ y_j^* \geq 0 \quad \perp \quad \pi_j^* \geq 0 \quad \forall j \in B, \tag{EC.99b} \]
\[ \sum_{j \in B} y_j^* - \sum_{i \in N_E(B)} s_i \leq 0 \quad \perp \quad \theta_B^* \geq 0 \quad \forall B \subseteq E, \tag{EC.99c} \]

Since \( y_j^* > 0 \), we have \( \pi_j^* = 0 \) by (EC.99b). Moreover, for all \( B \not\subseteq X_k \) where \( X_k = \{ B_k, B_{k+1}, \ldots, B_l \} \), we have \( \sum_{j \in B} y_j^* < \sum_{i \in N_E(B)} s_i \), which implies \( \theta_B^* = 0 \) by condition (EC.99c). Since \( i \in N_E(B_k) \), we obtain that \( \sum_{B \subseteq E, j \in N_E(B)} \theta_B^* = \sum_{B \subseteq X_k} \theta_B^* \). In step (d), since \( j \in B_k \), we can again apply \( \theta_B^* = 0 \) for all \( B \not\subseteq X_k \) to obtain that \( \sum_{B \subseteq X_k} \theta_B^* = \sum_{B \subseteq E, j \in B} \theta_B^* \). In step (e), because \( \pi_B^* = 0 \), we obtain the equation from (EC.99a).

By strict concavity of \( f(c) \) for \( c > 0 \) in Lemma 1(i), \( f'(\cdot) \) is strictly decreasing. Thus, \( \frac{\partial}{\partial s_i} \mathcal{V}_{opt}(s, b) \) is the \( k^{th} \) highest distinct value if and only if \( i \in N_E(B_k) \).
Similarly, for any $j \in \mathcal{B}$, we can derive that
\[
\frac{\partial}{\partial b_j} V_{opt}(s, b) \overset{(f)}{=} \frac{\partial}{\partial b_j} \mathcal{L}(y^*, \theta^*, s, b) \overset{(g)}{=} f \left( \frac{y_{j}}{b_j} \right) - y_{j} f' \left( \frac{y_{j}}{b_j} \right)
\]
where step (f) follows from the envelope theorem. We consider function $F(c) = f(c) - cf'(c)$ for $c > 0$. For any $c_1 > c_2$, we want to show that $F(c_1) > F(c_2)$. By differentiability and strict concavity of $f(c)$ and by mean value theorem, there exists $z_0 \in (f'(c_1), f'(c_2))$ such that $f(c_1) - f(c_2) = z_0(c_1 - c_2)$. This implies that $F(c_1) - F(c_2) = f(c_1) - f(c_2) - c_1 f'(c_1) + c_2 f'(c_2) = c_1 (z_0 - f'(c_1)) + c_2 (f'(c_2) - z_0) > 0$, where the second equality follows from replacing $f(c_1) - f(c_2)$ with $z_0(c_1 - c_2)$ and the last inequality follows from $z_0 > f'(c_1)$ or $f'(c_2) > z_0$. Thus, the ranking of $\frac{\partial}{\partial b_j} V_{opt}(s, b)$ matches the ranking of vector $\frac{y_j}{b_j}$.

As a result, $\frac{\partial}{\partial b_j} V_{opt}(s, b)$ is the $k^{th}$ lowest distinct value if and only if $j \in \mathcal{B}_k$. Q.E.D.

**EC.6. Proof of Results in Section 4**

**Proof of Proposition 3.** The proof for claim (ii) is in Appendix [C]. We prove claim (i) here. Let $(x, \bar{q}^*, \bar{q}^h)$ be an optimal solution to problem (EC.1) and $(\theta^0, \theta^*, \eta^0, \eta^*, \pi)$ be the dual optimal solution corresponding to constraints (EC.1b) - (EC.1h). In step (1), we show $V_s = V_{opt}$ when the buyer value distributions are homogeneous. In step (2), we show that $V_b = V_{opt}$ when the seller value distributions are homogeneous.

Step 1: homogeneous buyer value distribution. To prove the claim, it is sufficient to show that there exists $\gamma^* \in \Gamma^*$ and $\mu^* \in \Upsilon^*$ such that $V(\gamma^*, 0, 0, 0) = V(0, 0, \mu^*, 0) = V_{opt}$.

We start from the construction of seller commission rates $\gamma^*$. By Lemma [EC.3 (ii)], Assumption [EC.1(iii)] and [EC.1(iv)] hold. Thus, for any $i \in S$, we obtain $\bar{q}^*_i > 0$ by Proposition [EC.1(iv)]. This allows us to pick any $j : x_{ij} > 0$, and let $p_j = F_b^{-1}(1 - \frac{\bar{q}^*_i}{b_j})$ and $\gamma^*_i = 1 - \frac{F_b^{-1}(\bar{q}^*/b_j)}{F_b^{-1}(1 - \frac{\bar{q}^*_i}{b_j})}$. Note that when buyer value distributions are homogeneous, we have $g_j(q) = F_b^{-1}(1 - \frac{\bar{q}^*_i}{b_j})$ which can be further expressed as $g_j(q^*_i) = b_j g \left( \frac{q^*_i}{b_j} \right)$ where $g(r) = F_b^{-1}(1 - r)r$. Thus, by Proposition [EC.1(vii)] it does not matter which $j : x_{ij} > 0$ is picked because $\frac{q^*_i}{b_j}$ is homogeneous for all $j : x_{ij} > 0$.

To see the feasibility of $\gamma^*$, for any $i \in S$, we fix $j : x_{ij} > 0$. We first calculate that $h_j' \left( \bar{q}^*_i \right) = -\frac{\bar{q}^*_i}{F_b^{-1} \left( \bar{q}^*/b_j \right)} - F_b^{-1} \left( \frac{\bar{q}^*_i}{b_j} \right)$ and $g_j' \left( \bar{q}^*_i \right) = -\frac{\bar{q}^*_i}{F_b^{-1} \left( 1 - \frac{\bar{q}^*_i}{b_j} \right)} + F_b^{-1} \left( 1 - \frac{\bar{q}^*_i}{b_j} \right)$. By the increasing property of $F_b^{-1}$ and $F_{s_i}^{-1}$, we have $\left[ F_b^{-1} \right]' \left( 1 - \frac{\bar{q}^*_i}{b_j} \right) \geq 0$ and $\left[ F_{s_i}^{-1} \right]' \left( \frac{\bar{q}^*_i}{b_j} \right) \geq 0$, which implies that $F_b^{-1} \left( 1 - \frac{\bar{q}^*_i}{b_j} \right) \geq g_j' \left( \bar{q}^*_i \right)$ and $-h_j' \left( \bar{q}^*_i \right) \geq F_{s_i}^{-1} \left( \frac{\bar{q}^*_i}{s_i} \right)$. By Proposition [EC.1(iv)] we have $\bar{q}^*_j < b_j$, which further implies that $s_i \geq 0$ by condition (EC.2d). This in turn implies $g_j \left( \bar{q}^*/b_j \right) = \theta^i_j$ by condition (EC.2a). Moreover, we have $\eta^i_j \geq 0$ by condition (EC.2e), which further implies $\theta^i_j \geq h^i \left( \bar{q}^*_i \right)$ by condition (EC.2f). Lastly, since $x_{ij} > 0$, we have $\pi_j > 0$ by condition (EC.2f) and $\theta^i_j = \theta^i_j$ by condition (EC.2e). Together, these inequalities imply that $F_b^{-1} \left( 1 - \frac{\bar{q}^*_i}{b_j} \right) \geq F_{s_i}^{-1} \left( \frac{\bar{q}^*_i}{s_i} \right)$. By Proposition [EC.1(iv)] we have $\bar{q}^*_i > 0$, which further implies that $F_{s_i}^{-1} \left( \frac{\bar{q}^*_i}{s_i} \right) > 0$. Thus, we have
\[
F_b^{-1} \left( 1 - \frac{\bar{q}^*_i}{b_j} \right) \geq F_{s_i}^{-1} \left( \frac{\bar{q}^*_i}{s_i} \right) > 0.
\]
By construction, $\gamma^*_i = 1 - \frac{F_{s_i}^{-1} \left( \frac{\bar{q}^*_i}{s_i} \right)}{F_b^{-1} \left( 1 - \frac{\bar{q}^*_i}{b_j} \right)} \in [0, 1]$, which implies that $\gamma^* \in \Gamma^*$.

To validate this construction, we also need to verify $(p, x, q^*, q^h) \in \mathcal{K}(\gamma^*, 0, 0, 0)$ by checking equilibrium expressions (2a), (2c):

(1) to see expression (2a), by construction of $p$ and $\gamma^*$, we have $s_i F_{s_i} \left( (1 - s_i) p_i \right) = s_i F_{s_i} \left( F_b^{-1} \left( \frac{\bar{q}^*_i}{s_i} \right) \right) = \bar{q}^*_i$.
(2) to see expression (2b), we first show that for all \( j \in B \), if \( \bar{x}_{ij} > 0 \) and \( \bar{x}_{i2j} = 0 \), then \( p_i \leq p_{i2} \). By Proposition EC.1(vii), we have \( \tilde{q}_{ij} \leq \tilde{q}_{i2j} \). By construction of price vector \( \mathbf{p} \), this leads to \( p_i = F_b^{-1}(1 - \frac{\tilde{q}_{ij}}{b_j}) \leq F_b^{-1}(1 - \frac{\tilde{q}_{i2j}}{b_{ij}}) = p_{i2} \). Furthermore, if \( \bar{x}_{ij} > 0 \) and \( \bar{x}_{i2j} > 0 \), then by construction, we have \( p_i = F_b^{-1}(1 - \frac{\tilde{q}_{ij}}{b_j}) \leq F_b^{-1}(1 - \frac{\tilde{q}_{i2j}}{b_{ij}}) = p_{i2} \). Thus,

if \( \bar{x}_{ij} > 0 \), then \( p_i = \min_{i' : v'(i',j) \in E} \{ p_{i'} \} \). (EC.102)

Thus, we have \( b_j |1 - F_b(\min_{i : i(i) \in E} \{ p_i \})| = b_j |1 - F_b(F_b^{-1}(1 - \frac{\tilde{q}_{ij}}{b_j}))| = \tilde{q}_{ij}^b \);

(3) to see expression (2c), feasibility constraints (EC.1b, EC.1c), and (EC.1i) says that \((\bar{x}, \tilde{q}^s, \tilde{q}^b)\) satisfies the first part of expression (2c);

(4) to see expression (2d), from (EC.102), if \( \bar{x}_{ij} > 0 \), then \( i \in \arg \min_{v'(i',j) \in E} \{ p_{i'} \} \). Thus, expression (2d) holds.

In the end, we can verify the optimality of \( \gamma^o \) by deriving that

\[
V(\gamma^o, 0, 0, 0) = \sum_{(i,j) \in E} \gamma_i^o p_{i} \bar{x}_{ij} = \sum_{j \in B} \sum_{i : \bar{x}_{ij} > 0} p_i \bar{x}_{ij} - \sum_{i \in S} \sum_{j : \bar{x}_{ij} > 0} (1 - \gamma_i^o) p_i \bar{x}_{ij}
\]

\[
= \sum_{j \in B} \sum_{i : \bar{x}_{ij} > 0} F_b^{-1}(1 - \frac{\tilde{q}_{ij}^b}{b_j}) \bar{x}_{ij} - \sum_{i \in S} \sum_{j : \bar{x}_{ij} > 0} F_s^{-1}(\frac{\tilde{q}_{ij}^s}{s_i}) \bar{x}_{ij}
\]

\[
= \sum_{j \in B} F_b^{-1}(1 - \frac{\tilde{q}_{ij}^b}{b_j}) \tilde{q}_{ij}^b - \sum_{i \in S} F_s^{-1}(\frac{\tilde{q}_{ij}^s}{s_i}) \tilde{q}_{ij}^s = V_{opt},
\]

(ECC.103)

where step (a) follows from reorganizing of expression \( \sum_{(i,j) \in E} \gamma_i^o p_{i} \bar{x}_{ij} \). Step (b) follows directly from the construction of price vector \( \mathbf{p} \) and commission vector \( \gamma^o \). Step (c) follows from aggregating \( \bar{x}_{ij} \) by \( \tilde{q}_{ij}^b = \sum_{i : \bar{x}_{ij} > 0} \bar{x}_{ij} \) and \( \tilde{q}_{ij}^s = \sum_{j : \bar{x}_{ij} > 0} \bar{x}_{ij} \). Step (d) follows directly from Theorem 1(i) given the optimality of solution vector \((\bar{x}, \tilde{q}^s, \tilde{q}^b)\).

To establish that there exists \( \mu^o \in \mathcal{U}^b \) such that \((p, \bar{x}, \tilde{q}^s, \tilde{q}^b) \in \mathcal{X}(0, 0, \mu^o, 0) \), we start from \((p, \bar{x}, \tilde{q}^s, \tilde{q}^b) \in \mathcal{X}(\gamma^o, 0, 0, 0) \) and apply Lemma EC.2(ii) to establish the optimal seller subscription vector \( \mu^o \). By Theorem 1(i), we have \( V(0, 0, \mu^o, 0) = V_{opt} \).

Step 1-2: homogeneous seller value function. To prove the claim, it is sufficient to show that there exists \( \gamma^b \in \Gamma^b \) and \( \mu^o \in \mathcal{U}^b \) such that \( V(0, \gamma^b, \mu^o, 0) = V(0, 0, \mu^o, 0) = V_{opt} \).

We construct a buyer commission vector \( \gamma^b \). Let price vector \( \mathbf{p} \) be \( p_i = F_b^{-1}(\frac{s_{i}}{s_{ij}}) \) for all \( i \in S \). For each \( j \in B \) with \( \tilde{q}_{ij}^b > 0 \), pick one \( i : \bar{x}_{ij} > 0 \) and let \( \gamma_j^b = \frac{F_b^{-1}(1 - \frac{\tilde{q}_{ij}^b}{b_j})}{F_s^{-1}(\frac{\tilde{q}_{ij}^s}{s_i})} - 1 \). Note that it does not matter which \( i : \bar{x}_{ij} > 0 \) is picked because of the homogeneity of \( \frac{s_i}{s_{ij}} \) for \( i : \bar{x}_{ij} > 0 \) by Proposition EC.1(viii).

For the \( j \in B \) with \( \tilde{q}_{ij}^b = 0 \), we set \( \gamma_j^b = \max \left\{ \frac{F_b^{-1}(1 - \frac{\tilde{q}_{ij}^b}{b_j})}{F_s^{-1}(\frac{\tilde{q}_{ij}^s}{s_i})} - 1, 0 \right\} \).

To verify the feasibility of \( \gamma^b \), for any \( j \in B \) with \( \tilde{q}_{ij}^b > 0 \), we fix \( i : \bar{x}_{ij} > 0 \). From (EC.101), we have \( \gamma_j^b = \frac{F_b^{-1}(1 - \frac{\tilde{q}_{ij}^b}{b_j})}{F_s^{-1}(\frac{\tilde{q}_{ij}^s}{s_i})} - 1 \geq 0 \). For all \( j \in B \) with \( \tilde{q}_{ij}^b = 0 \), we have \( \gamma_j^b \geq 0 \) by construction. Thus, we have \( \gamma^b \in \Gamma^b \).

Next, we verify \((p, \bar{x}, \tilde{q}^s, \tilde{q}^b) \in \mathcal{X}(0, \gamma^b, \mu^o, 0) \) by checking expressions (2a)-(2d):

(1) to see expression (2a), by construction, we have \( s_i F_s(p_i) = s_i F_s(F_s^{-1}(\frac{\tilde{q}_{ij}^s}{s_i})) = \tilde{q}_{ij}^s \);

(2) to see expression (2b), for all \( j \in B \), we first show that if \( \bar{x}_{ij} > 0 \), then \( i \in \arg \min_{i' : v'(i',j) \in E} \{ p_{i'} \} \). By Proposition EC.1(viii), if \( \bar{x}_{ij} > 0 \) and \( \bar{x}_{i2j} = 0 \), we obtain \( \frac{s_i}{s_{ij}} \leq \frac{s_i}{s_{i2j}} \). By construction of price vector \( \mathbf{p} \) and the strict increasing property of \( F_s^{-1}(\cdot) \) over \([0, s_i]\) from Assumption 1, we have \( p_i =
By construction of commission vector $\gamma^b$, we obtain $b_j \left[ 1 - F_{b_j}\left((1 + \gamma^b_j) \min_{i:j(i,j) \in E} \{p_i\}\right) \right] = b_j \left[ 1 - F_{b_j}\left(1 - \frac{q^b_j}{b_j}\right)\right] = q^b_j$.

(3) to see expression (2c), feasibility constraints (EC.1b), (EC.1c), and (EC.1f) suggest that $(\bar{x}, \bar{q}^s, \bar{q}^b)$ satisfies expression (2c);

(4) to see expression (2d), from [EC.104], if $\bar{x}_{ij} > 0$, then $\bar{x} \in \arg \min_{i:j(i',j) \in E} \{p_{ij}\}$. This completes the verification of expression (2d).

In the end, we verify the optimality of $\gamma^b$ with the following derivation

$$V(0,0,\gamma^b,0) = \sum_{i,j:(i,j) \in E} \gamma^b_{ij} \bar{x}_{ij} = \sum_{j \in \mathcal{B} : \bar{x}_{ij} > 0} \sum_{(1 + \gamma^b_j) \min_{i:j(i,j) \in E} \{p_{ij}\}} \bar{x}_{ij} = \sum_{i \in \mathcal{S}} \sum_{j \in \mathcal{B} : \bar{x}_{ij} > 0} p_i \bar{x}_{ij}$$

Step (e) follows from (EC.104) and then a reorganization of $\sum_{i,j:(i,j) \in E} \gamma^b_{ij} p_{ij} \bar{x}_{ij}$. Step (f) follows from aggregating $\bar{x}_{ij}$ by constraints (EC.1b) and (EC.1c) to obtain $\bar{q}^b_{ij}$ and $\bar{q}^{s}_{ij}$. Step (g) follows from Theorem (1) given the optimality of solution vector $(\bar{x}, \bar{q}^s, \bar{q}^b)$.

Finally, to establish the existence of $\mu^b \in \mathcal{U}^b$ where $(\bar{p}, \bar{x}, \bar{q}^s, \bar{q}^b) \in \mathcal{X}(0,0,\mu^b)$, we start from $(\bar{p}, \bar{x}, \bar{q}^s, \bar{q}^b) \in \mathcal{X}(0,\gamma^b,0,0)$ and then apply Lemma (EC.2iv) Optimality of $\mu^b$ follows from Theorem (1) given that $\mu^b$ induces the implementation of optimal equilibrium flow $(\bar{x}, \bar{q}^s, \bar{q}^b)$. Q.E.D.

**Proof of Lemma 3** Let $\mu^b_{i,j}$ be a subvector of $\mu^b$ excluding component $\mu^b_i$ and $\mu^b_{-i}$ be a subvector of $\mu^*$ excluding component $\mu^*_i$. For any $\mu^* \in \mathcal{U}$, let $(p, x, q^s(\mu), q^b(\mu)) \in \mathcal{X}(0, \mu)$.

By Lemma (EC.1iii), Assumptions (EC.1a), (EC.1i), and (EC.1ii) are satisfied. In framework problem (EC.1), we let $(\theta^*, \theta^0, \bar{x}^*, \bar{q}^s, \bar{q}^b, \pi)$ be a dual optimal solution vector corresponding to constraints (EC.1b) - (EC.1f). From the KKT conditions in Proposition (EC.1ii) we can cluster sellers and buyers $(S, B)$ into disjoint components $\{ \mathcal{S}_k, \mathcal{B}_k \}_{k=1}^l$ where (1) $\mathcal{S}_k \subseteq \mathcal{S}$ is such that the dual solution vector satisfies $\theta^*_i = \theta^b_i$ if $i, i' \in \mathcal{S}_k$, and (2) $\mathcal{B}_k \subseteq \mathcal{B}$ is such that the dual solution vector satisfies $\theta^*_j = \theta^b_j$ for all $j, j' \in \mathcal{B}_k$. If $\bar{x}_{ij} > 0$, then we have $\pi_{ij} = 0$ by condition (EC.2a) and $\theta^*_i = \theta^b_i$ by condition (EC.2c). We can match $\mathcal{S}_k$ with $\mathcal{B}_k$ such that $\theta^*_i = \theta^b_i$ for $i \in \mathcal{S}_k$ and $j \in \mathcal{B}_k$. Note that there can be two possible degenerate cases in this clustering approach: (1) $\mathcal{S}_k \neq \emptyset$ and $\mathcal{B}_k = \emptyset$, which corresponds to $q^b_i(\mu) = 0$ for all $i \in \mathcal{S}_k$; (2) $\mathcal{S}_k = \emptyset$ and $\mathcal{B}_k \neq \emptyset$, which corresponds to $q^b_i(\mu) = 0$ for all $j \in \mathcal{B}_k$.

For simplicity of notation, we let $c_k = \theta^b_j$ for $j \in \mathcal{B}_k$ (or $c_k = \theta^b_i$ for $i \in \mathcal{S}_k$). By Proposition (8), $c_k$ is the equilibrium price used in cluster $(\mathcal{S}_k, \mathcal{B}_k)$. Without loss of generality, we can sort components $(\mathcal{S}_k, \mathcal{B}_k)$ by increasing order of $(c_k)_{k=1}^l$, i.e., $c_1 < c_2 \cdots < c_l$ (for completion, let $c_0 = 0$ and $c_{l+1} = \infty$). Note that no edge connects $\mathcal{S}_{k_1}$ with $\mathcal{B}_{k_2}$ where $k_1 < k_2$, otherwise, buyers in $\mathcal{B}_{k_2}$ will work with $\mathcal{S}_{k_1}$ given $c_{k_1} < c_{k_2}$.

To prove claim (i), for any $\mu \in \mathcal{U}$, $(p, x, q^s(\mu), q^b(\mu)) \in \mathcal{X}(0, \mu)$ and $j_0 \in \mathcal{B}$, we want to show that there exists $\tilde{d} > 0$ such that for all $d \in [0, \tilde{d}]$ with $\bar{\mu} = (\mu^*, \mu_{j_0} + \delta, \mu^*_{-j_0})$ and its induced equilibrium $(\bar{p}, x, q^s(\bar{\mu}), q^b(\bar{\mu})) \in \mathcal{X}(0, \bar{\mu})$, and we have $q^b_i(\bar{\mu}) \leq q^b_i(\mu)$ for all $i \in \mathcal{S}$.

For any $j_0 \in \mathcal{B}$, we find the index $k_0 \in \{1, \ldots, l\}$ such that $j_0 \in \mathcal{B}_{k_0}$. Then, we discuss two possibilities:

1. if $q^b_i(\bar{\mu}) > 0$, we can always pick any $i : x_{ij} > 0$ and find the matching seller cluster $\mathcal{S}_{k_0}$. Under subscription profile $\bar{\mu}$, we have supply-demand balance in cluster $(\mathcal{S}_{k_0}, \mathcal{B}_{k_0})$ by construction.
suggests that \( \sum_{j \in B_{k_0}} q_{j}^b(\mu) - \sum_{i \in S_{k_0}} q_{i}^*(\mu) = 0 \). Expressing \( q_{i}^*(\mu) \) and \( q_{j}^b(\mu) \) by (2a) and (2b), we have

\[
\sum_{j \in B_{k_0}} b_j \left[ 1 - F_{b_j}(c_{k_0} + \mu_{j}^b) \right] - \sum_{i \in S_{k_0}} s_i F_{s_i}(c_{k_0} - \mu_{i}^*) = 0. \tag{EC.106}
\]

Since \( c_{k_0-1} < c_{k_0} \), we have

\[
\sum_{j \in B_{k_0}} b_j \left[ 1 - F_{b_j}(c_{k_0-1} + \mu_{j}^b) \right] - \sum_{i \in S_{k_0}} s_i F_{s_i}(c_{k_0-1} - \mu_{i}^*) > 0. \tag{EC.107}
\]

By the increasing property of function \( F_{b_j}(\cdot) \) over \([0, \tilde{v}_{b_j}]\) and \( F_{s_i}(\cdot) \) over \([0, \bar{v}_{s_i}]\) from Assumption [1], function \( \sum_{j \in B_{k_0}} b_j \left( 1 - F_{b_j}(c + \mu_{j}^b) \right) - \sum_{i \in S_{k_0}} s_i F_{s_i}(c - \mu_{i}^*) \) is continuous and decreasing in \( c \).

We can pick \( \delta > 0 \) small enough such that for any \( \delta \in (0, \tilde{\delta}) \), under the new subscription profile \( \tilde{\mu} = (\mu_k^*, \mu_{k_0}^b + \delta, \mu_{k_0}^b) \), we have

\[
\begin{align*}
(1-1) & \sum_{j \in B_{k_0}} b_j \left[ 1 - F_{b_j}(c_{k_0} + \tilde{\mu}_{j}^b) \right] - \sum_{i \in S_{k_0}} s_i F_{s_i}(c_{k_0} - \mu_{i}^*) < 0, \\
(1-2) & \sum_{j \in B_{k_0}} b_j \left[ 1 - F_{b_j}(c_{k_0-1} + \tilde{\mu}_{j}^b) \right] - \sum_{i \in S_{k_0}} s_i F_{s_i}(c_{k_0-1} - \mu_{i}^*) > 0, \\
(1-3) & \text{there exists } \tilde{c}_{k_0} \in [c_{k_0-1}, c_{k_0}] \text{ such that } \sum_{j \in B_{k_0}} b_j \left[ 1 - F_{b_j}(c_{k_0} + \tilde{\mu}_{j}^b) \right] - \sum_{i \in S_{k_0}} s_i F_{s_i}(c_{k_0} - \mu_{i}^*) = 0, \\
(1-4) & \sum_{i \in S_{k_0}} |q_{i}^b(\tilde{\mu}) - q_{i}^*(\mu)| + \sum_{j \in B_{k_0}} |q_{j}^b(\tilde{\mu}) - q_{j}^b(\mu)| < \min\{x_{ij} > 0 : i \in S_{k_0}, j \in B_{k_0}\},
\end{align*}
\]

Conditions (1-1) - (1-2) guarantee that the price adjustment will preserve the cluster structure \( \{S_{k_0}, B_{k_0}\} \). Condition (1-3) says that price is reduced to \( \tilde{c}_{k_0} \) in component \( (S_{k_0}, B_{k_0}) \). Condition (1-4) guarantees that it is sufficient to reduce the current flow vector \( \bar{x} \) on each edge to achieve new balance.

Under this construction, for small enough \( \tilde{\delta} > 0 \), we can make adjustment to achieve a new equilibrium via the following procedures:

\[
\begin{align*}
(1) & \text{1 adjust the supply to } q_{i}^*(\tilde{\mu}) \text{ such that } q_{i}^*(\tilde{\mu}) = s_i F_{s_i}(1 - \gamma_{i}^* \tilde{c}_{k_0} - \mu_{i}^*) \text{ for all } i \in S_{k_0}, \\
(2) & \text{2 adjust the demand to } q_{j}^b(\tilde{\mu}) \text{ such that } q_{j}^b(\tilde{\mu}) = b_j \left[ 1 - F_{b_j}(c_{k_0} + \tilde{\mu}_{j}^b) \right] \text{ for } j \in B_{k_0}, \\
(3) & \text{3 adjust the price } \mu \text{ where } \tilde{\mu}_i = \bar{\mu}_i \text{ for all } i \in S_{k_0} \text{ and } \tilde{\mu}_j = \bar{\mu}_j \text{ for all } j \notin S_{k_0}, \\
(4) & \text{4 adjust the flow to } \bar{x} \text{ such that it satisfies } \sum_{i \in S_{k_0}} \bar{x}_{ij} = q_{i}^*(\tilde{\mu}) \text{ and } \sum_{j \in B_{k_0}} \bar{x}_{ij} = q_{j}^b(\tilde{\mu}) \text{ in the following iterative updates} \tag{13} \\
& \text{1) create a node set } N = \{ i \in S_{k_0}, j \in B_{k_0} \} \text{ and a set of target adjustment } Q = \{|q_{i}^*(\tilde{\mu}) - q_{i}^*(\mu)|, \forall i \in S_{k_0}, |q_{j}^b(\tilde{\mu}) - q_{j}^b(\mu)|, \forall j \in B_{k_0}\}; \\
& \text{2) pick } \delta_s = \min Q \text{ with } s \in N \text{ being the corresponding node index, and pick any } t \in N \text{ such that } (s, t) \in E; \\
& \text{3) adjust the flow on } (s, t) \text{ to } \bar{x}_{st} = x_{st} - \delta_s, \text{ eliminate } s \text{ from node set } N \text{ i.e., } N = N / \{s\}, \text{ and update } \tilde{\delta}_i := \tilde{\delta}_i - \delta_s; \\
& \text{4) repeat step } \text{2) until } N = \emptyset.
\end{align*}
\]

In this construction, expressions (2a) - (2b) are preserved by step (1-1) and step (1-2). Expression (2c) and (2d) are preserved by step (1-4). Given the adjustment in (1-3), we end up with another equilibrium \( (\tilde{\mu}, \bar{x}, q^*(\tilde{\mu}), q^b(\tilde{\mu})) \in X(0, \bar{\mu}) \) where \( q^*(\tilde{\mu}) \leq q_{j_0}^b(\mu) \).

(2) If \( q_{j_0}^b(\mu) = 0 \), then increasing \( \mu_{j_0}^b \) to \( \tilde{\mu}_{j_0}^b = \mu_{j_0}^b + \delta \) for any \( \delta > 0 \) still results in \( q_{j_0}^b(\tilde{\mu}) = 0 \). This does not change the equilibrium flow, so we have \( q^*(\tilde{\mu}) = q^*(\mu) \) for all \( i \in S \).

This completes the proof of claim (i).

The proof for claim (ii) follows the same sensitivity arguments as that in claim (i) Q.E.D.

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13 This is a simple greedy algorithm to achieve flow conservation condition.
EC.7. Proof of Results in Section 5

EC.7.1. Auxiliary Results for Welfare Maximization

In this section, we consider the following auxiliary result that will be used throughout the proof of results in Section 5.

**Lemma EC.3.** For any \((p, x, q^s, q^b) \in \mathcal{X}(\gamma, \mu)\), the induced welfare \(W(\gamma, \mu)\) satisfies

\[
W(\gamma, \mu) = \sum_{j \in \mathcal{B}} \int_0^{q^b} F_{b_j}^{-1} \left(1 - \frac{q}{b_j}\right) dq - \sum_{i \in \mathcal{S}} \int_0^{q^s} F_{s_i}^{-1} \left(\frac{q}{s_i}\right) dq.
\]

(EC.108)

**Proof of Lemma EC.3** Let \((p, x, q^s, q^b) \in \mathcal{X}(\gamma, \mu)\) be a competitive equilibrium induced by \((\gamma, \mu) \in \Gamma \times \mathcal{U}\). Given \(\bar{v}^q_b = F_{b_j}^{-1} \left(1 - \frac{q^b}{b_j}\right)\) and \(\bar{v}^m_s = F_{s_i}^{-1} \left(\frac{q^s}{s_i}\right)\), for simplicity of notation, we let

\[
g_j(q) = b_j \int_{F_{b_j}^{-1}(1-q/b_j)}^{\bar{v}^q_b} v dF_{b_j}(v) \quad \forall j \in \mathcal{B},
\]

(EC.109a)

\[
h_i(q) = s_i \int_0^{F_{s_i}^{-1}(q/s_i)} v dF_{s_i}(v) \quad \forall i \in \mathcal{S}.
\]

(EC.109b)

From (EC.109), we deduce that

\[
g_j'(q) = \frac{d}{dq} \left[ b_j \int_{F_{b_j}^{-1}(1-q/b_j)}^{\bar{v}^q_b} v dF_{b_j}(v) \right]
\]

\[
= \left(F_{b_j}^{-1}\right)' \left(1 - \frac{q}{b_j}\right) F_{b_j}^{-1} \left(1 - \frac{q}{b_j}\right) F_{b_j}' \left(F_{b_j}^{-1} \left(1 - \frac{q}{b_j}\right)\right)
\]

\[
= F_{b_j}^{-1} \left(1 - \frac{q}{b_j}\right) \quad \forall j \in \mathcal{B},
\]

(EC.110a)

\[
h_i'(q) = \frac{d}{dq} \left[ s_i \int_0^{F_{s_i}^{-1}(q/s_i)} v dF_{s_i}(v) \right]
\]

\[
= \left(F_{s_i}^{-1}\right)' \left(\frac{q}{s_i}\right) F_{s_i}^{-1} \left(\frac{q}{s_i}\right) F_{s_i}' \left(F_{s_i}^{-1} \left(\frac{q}{s_i}\right)\right)
\]

\[
= F_{s_i}^{-1} \left(\frac{q}{s_i}\right) \quad \forall i \in \mathcal{S},
\]

(EC.110b)

where step (a) and step (c) follow from Leibniz’s rule. Step (b) and step (d) follow from the derivative rule on inverse function i.e., \([F^{-1}]’(x) = \frac{1}{F’(F^{-1}(x))}\).

From (EC.109) and (EC.110), we obtain

\[
g_j(q) = \int_0^{q^b} F_{b_j}^{-1} \left(1 - \frac{x}{b_j}\right) dx \quad \forall j \in \mathcal{B},
\]

(EC.111a)

\[
h_i(q) = \int_0^{q^s} F_{s_i}^{-1} \left(\frac{x}{s_i}\right) dx \quad \forall i \in \mathcal{S}.
\]

(EC.111b)

Thus, the induced welfare satisfies

\[
W(\gamma, \mu) \overset{(a)}{=} \sum_{j \in \mathcal{B}} g_j(q^b_j) - \sum_{i \in \mathcal{S}} h_i(q^s_i) \overset{(b)}{=} \int_0^{q^b} F_{b_j}^{-1} \left(1 - \frac{x}{b_j}\right) dx - \int_0^{q^s} F_{s_i}^{-1} \left(\frac{x}{s_i}\right) dx,
\]

(EC.112)

where step (a) follows from (13) and step (b) follows from (EC.111). Q.E.D.
EC.7.2. Proof of Major Results.

Proof of Proposition 6. By Lemma EC.3, the welfare maximization problem can be formulated as

\[
W_{\text{opt}} = \max_{(p,x,q^*,q^b,\gamma,\mu)} \sum_{j \in S} \int_0^{q^b_j} F_{b_j}^{-1}(1 - \frac{q}{b_j}) \, dq - \sum_{i \in S} \int_0^{q^i_i} F_{s_i}^{-1}(\frac{q}{s_i}) \, dq
\]  

s.t. \( (p,x,q^*,q^b) \in \mathcal{X}(\gamma,\mu) \), \( (\gamma,\mu) \in \Gamma \times \mathcal{U} \).  

(EC.113)

Problem (EC.113) suggests that

\[
W(\gamma,\mu) \leq W_{\text{opt}} \text{ for all } (\gamma,\mu) \in \Gamma \times \mathcal{U}.
\]  

(EC.114)

Note that problem (EC.3) is precisely the equilibrium problem (14) with \( (\gamma,\mu) = 0 \). By Proposition 8, the optimal solution to problem (EC.3) is a competitive equilibrium given \( (\gamma,\mu) = 0 \). By Lemma EC.3, the optimal objective of problem (EC.3) is denoted by \( W(0,0) \).

By eliminating (EC.113c) and expression (2c)-(2d) in (EC.113b), we obtain that problem (EC.3) is a relaxation of problem (EC.13). Thus,

\[
W_{\text{opt}} \leq W(0,0).
\]  

(EC.115)

By (EC.113) and (EC.115), we obtain \( W(0,0) = W_{\text{opt}} \). Q.E.D.

Proof of Proposition 7. Proof of claim (i). To prove the claim, we consider a class of problems indexed by \( n \in \{1,2,\ldots\} \). Let problem instance \( n \) be a network with one seller type and one buyer type where the population vector is \( (s_n,b_n) = (n,2n) \). For simplicity of notation, we denote by \( q \) such that \( q = q^s = q^b = x \). We let the value distribution functions in problem instance \( n \) be defined as

\[
F^n_b(v) = \begin{cases} 
\frac{n}{2} v, & \text{for } 0 \leq v \leq \frac{1}{2} \\
\frac{1}{2} - \frac{1}{2n} v, & \text{for } \frac{1}{2} \leq v \leq 1 \\
1 - \frac{1}{2n} + \frac{1}{2n} v, & \text{for } 1 \leq v \leq 2 \\
1, & \text{for } v \geq 2
\end{cases}
\]  

(EC.116a)

\[
F^n_s(v) = \begin{cases} 
 nv, & \text{for } 0 \leq v \leq \frac{1}{n} \\
1, & \text{for } v \geq \frac{1}{n}
\end{cases}
\]  

(EC.116b)

As a quick verification, the construction in (EC.116) satisfies Assumption 1.

With \( (s_n,b_n) = (n,2n) \), it is sufficient to consider \( q \in [0,n] \). By (EC.116), we obtain

\[
[F^n_b]^{-1}(1 - \frac{q}{b_n}) - [F^n_s]^{-1}(\frac{q}{s_n}) = \begin{cases} 
2 - q - \frac{1}{n} q, & \text{for } 0 \leq q \leq 1, \\
\frac{1}{q} - \frac{1}{n} q, & \text{for } 1 \leq q \leq n.
\end{cases}
\]  

(EC.117)

In problem instance \( n \in \{1,2,\ldots\} \), welfare maximization problem (EC.3) becomes

\[
\max_{0 \in [0,n]} \int_0^q [F^n_b]^{-1}(1 - \frac{x}{b_n}) - [F^n_s]^{-1}(\frac{x}{s_n}) \, dx.
\]  

(EC.118)

We denote by \( \bar{q} \) the optimal solution to problem (EC.118). By calculating the first order optimality condition \( [F^n_b]^{-1}(1 - \frac{\bar{q}}{b_n}) = [F^n_s]^{-1}(\frac{\bar{q}}{s_n}) \) based on (EC.117), we obtain that \( \bar{q} = n \). Thus, the induced maximum welfare in problem instance \( n \in \{1,2,\ldots\} \) satisfies

\[
W^n_{\text{opt}} = \int_0^n [F^n_b]^{-1}(1 - \frac{q}{b_n}) - [F^n_s]^{-1}(\frac{q}{s_n}) \, dq = \ln n + 1.
\]  

(EC.119)
By solving the first order optimality condition based on (EC.117), we obtain
\[ [F^n_b]^{-1}(1 - \frac{q}{b_n}) - [F^n_s]^{-1}(\frac{q}{s_n})q. \]  
(EC.120)

We let \( \bar{q} \) be the optimal solution to revenue maximization problem (EC.120) for \( n \in \{1, 2, \ldots, \} \). By solving the first order optimality condition based on (EC.117), we obtain \( 2 - 2\bar{q} - \frac{2}{n^2}\bar{q} = 0 \), which implies that \( \bar{q} = \frac{n^2}{n^2 + 1} \). Let \((\gamma', \mu')\) be the induced optimal commission-subscription vector that supports optimal flow \( \bar{q} \) in equilibrium. By Lemma EC.3, the induced welfare in problem instance \( n \in \{1, 2, \ldots, \} \) satisfies
\[
W^n(\gamma', \mu') = \int_0^\bar{q} F_b^{-1}(1 - \frac{x}{b_n}) - F_s^{-1}(\frac{x}{s_n})dx \\
\leq (2\bar{q} - \frac{\bar{q}^2}{2} - \frac{\bar{q}^2}{2n^2})^{\frac{1}{2}} = \frac{3}{2} - \frac{1}{2n^2} \leq \frac{3}{2}.
\]  
(EC.121)

As \( n \to \infty \), the welfare efficiency satisfies
\[
\frac{W^n(\gamma', \mu')}{W^n_{opt}} \to 0.
\]  
(EC.122)

Proof of claim (ii). Given that \( F_{b_j}^{-1}(u) = \bar{v}_{b_j}u \) and \( F_{s_j}^{-1}(u) = \bar{v}_{s_j}u \) for \( u \in [0, 1] \), the matrix representation for constraint \( \sum_{i,j \in E} x_{ij} = q^b_j \) can be expressed as
\[
A^b x = q^b, 
\]  
(EC.123)
where matrix \( A^b \) is a \(|B| \times |E| \) matrix where \( A^b_{ij} = 1 \) if \( e = (i, j) \in E \) and \( j \in B \). Similarly, we can represent constraint \( \sum_{j:(i,j) \in E} x_{ij} = q^s_i \) compactly as
\[
A^s x = q^s, 
\]  
(EC.124)
where matrix \( A^s \) is a \(|S| \times |E| \) matrix satisfying \( A^s_{ie} = 1 \) if \( e = (i, j) \in E \) and \( i \in S \). To ease notation, we construct the following diagonal matrices
\[
\Lambda^s = diag(s), \Lambda^b = diag(b), \Lambda^v^s = diag(v^s), \Lambda^v^b = diag(v^b).
\]  
(EC.125)
Based on (EC.123), (EC.124) and (EC.125), constraints \( q^b_i \leq s_i \) and \( q^b_j \leq b_j \) can be compactly expressed as
\[
(\Lambda^s)^{-1} A^s x \leq 1,
\]
\[
(\Lambda^b)^{-1} A^b x \leq 1.
\]  
(EC.126)

To further ease notation, we let
\[
r = (A^b)^\top \bar{o}^b,
\]
\[
H = (A^s)^\top \Lambda^b (A^b)^{-1} A^b + (A^s)^\top \Lambda^s (A^s)^{-1} A^s,
\]
\[
A = \left(\begin{array}{c}
(\Lambda^b)^{-1} A^b \\
(\Lambda^s)^{-1} A^s
\end{array}\right).
\]  
(EC.127)

By construction, matrix \( H \) is symmetric. Moreover, \( \Lambda^b (A^b)^{-1} > 0 \) and \( \Lambda^s (A^s)^{-1} > 0 \), because they are diagonal matrices with positive diagonals. For all \( z \in \mathbb{R}^{|E|}/0 \), we have
\[
z^\top H z = (A^b z)^\top \Lambda^b (A^b)^{-1} (A^b z) + (A^s z)^\top \Lambda^s (A^s)^{-1} (A^s z) > 0,
\]  
(EC.128)
which implies that matrix $H$ is positive definite.

In revenue optimization problem (EC.3), the objective function can be expressed as

$$
\sum_{j \in S} F_{b_j}^{-1} \left(1 - \frac{q^b_j}{b_j}\right)q^b - \sum_{i \in S} F_{s_i}^{-1} \left(\frac{q^s_i}{s_i}\right)q^s
= \sum_{j \in S} \tilde{v}_{b_j} \left(1 - \frac{q^b_j}{b_j}\right)q^b - \sum_{i \in S} \tilde{v}_{s_i} \left(\frac{q^s_i}{s_i}\right)q^s
= (\tilde{v}^b)^\top q^b - (q^b)^\top [A^v b (A^b)^{-1}]q^b - (q^b)^\top [A^v s (A^s)^{-1}]q^s
= (\tilde{v}^b)^\top A^b x - x^\top \left[(A^b)^\top A^v b (A^b)^{-1} A^b + (A^s)^\top A^v s (A^s)^{-1} A^s\right] x
= x^\top (r - H x),
$$

where step (a) follows from $F_{b_j}^{-1}(u) = \tilde{v}_{b_j} u$ and $F_{s_i}^{-1}(u) = \tilde{v}_{s_i} u$. Step (b) follows from implementation of matrix notations based on (EC.125). Step (c) follows from (EC.123) and (EC.124). Step (d) follows directly from (EC.127).

Based on (EC.123), (EC.124), (EC.125) and (EC.129), we can rewrite revenue optimization problem (EC.3) compactly as the following convex quadratic program

$$
V_{\text{opt}} = \max_x x^\top (r - H x)
\text{s.t. } A x \leq 1,
\quad x \geq 0.
$$

Let $\bar{x}$ be the optimal primal solution to (EC.130) and $(\bar{\pi}, \bar{\theta})$ be the optimal dual solution to constraints (EC.130b) - (EC.130c). By KKT optimality conditions, we have

$$
r - 2H \bar{x} - A^\top \bar{\pi} + \bar{\theta} = 0,
\pi \geq 0 \perp A \bar{x} - 1 \leq 0,
\bar{\theta} \geq 0 \perp \bar{x} \geq 0.
$$

Similarly, the objective function of problem (EC.3) can be compactly written as

$$
\sum_{j \in S} \int_0^{q^b_j} F_{b_j}^{-1} \left(1 - \frac{q^b_j}{b_j}\right) dq - \sum_{i \in S} \int_0^{q^s_i} F_{s_i}^{-1} \left(\frac{q^s_i}{s_i}\right) dq
= \sum_{j \in S} \tilde{v}_{b_j} \left(1 - \frac{q^b_j}{2b_j}\right)q^b - \sum_{i \in S} \tilde{v}_{s_i} \left(\frac{q^s_i}{2s_i}\right)q^s
= (\tilde{v}^b)^\top q^b - \frac{1}{2} (q^b)^\top [A^v b (A^b)^{-1}]q^b - \frac{1}{2} (q^b)^\top [A^v s (A^s)^{-1}]q^s
= (\tilde{v}^b)^\top A^b x - \frac{1}{2} x^\top \left[(A^b)^\top A^v b (A^b)^{-1} A^b + (A^s)^\top A^v s (A^s)^{-1} A^s\right] x
= x^\top (r - \frac{1}{2} H x),
$$

where step (e) follows from $F_{b_j}^{-1}(u) = \tilde{v}_{b_j} u$ and $F_{s_i}^{-1}(u) = \tilde{v}_{s_i} u$. Step (f) follows from implementation of the matrix notations based on (EC.125). Step (g) follows from (EC.123) and (EC.124). Step (h) follows from (EC.127).

Based on (EC.123), (EC.124), (EC.125) and (EC.132), we can express welfare optimization problem (EC.3) compactly as the following convex quadratic program

$$
W_{\text{opt}} = \max_x x^\top (r - \frac{1}{2} H x)
$$
Given the dual optimal solution \( \bar{\pi} \) from problem (EC.130), we consider the following Lagrangian relaxation problem for welfare optimization problem (EC.133):

\[
\tilde{W}_{opt} = \max_x x^T \left( r - \frac{1}{2} H x \right) - \tilde{\pi}^T (A x - 1)
\]

s.t. \( x \geq 0 \).

We let \( (\tilde{x}, \tilde{\theta}) \) be such that \( \tilde{x} = 2x \) and \( \tilde{\theta} = \theta \). Next, we show that \( (\tilde{x}, \tilde{\theta}) \) is an optimal solution to optimization problem (EC.134). In a concave maximization problem with linear constraints and quadratic objective function, it is sufficient to verify the KKT conditions under problem (EC.134):

\[
\begin{align*}
r - H \tilde{x} - A^\top \tilde{\pi} + \tilde{\theta} & \overset{(i)}{=} 0, \\
\tilde{\theta} & \overset{(j)}{=} 0 \perp \tilde{x} \geq 0,
\end{align*}
\]

where in step (i), the construction \( (\tilde{x}, \tilde{\theta}) = (2x, \theta) \) suggests that \( r - H \tilde{x} - A^\top \tilde{\pi} + \tilde{\theta} = r - 2H \bar{x} - A^\top \bar{\pi} + \theta \). Then, \( r - 2H \bar{x} - A^\top \bar{\pi} + \theta = 0 \) follows directly from (EC.131a). Step (j) follows directly from \( \tilde{\theta} \geq 0 \perp \tilde{x} \geq 0 \) in (EC.131c). Thus, \( (\tilde{x}, \tilde{\theta}) \) is an optimal solution to optimization problem (EC.134).

Note that since vector \( \bar{x} \) is an optimal solution to revenue optimization problem (EC.130), by Lemma EC.3 and expression (EC.132), the revenue-maximum welfare induced by the supporting optimal commission-subscription vector \( (\gamma', \mu') \) can be compactly expressed as

\[
W(\gamma', \mu') = \bar{x}^\top (r - \frac{1}{2} H \bar{x}).
\]

Thus,

\[
\frac{W(\gamma', \mu')}{W_{opt}} \overset{(k)}{\geq} \frac{\bar{W}(\gamma', \mu')}{\bar{W}_{opt}} \overset{(l)}{=} \frac{\bar{x}^\top (r - \frac{1}{2} H \bar{x})}{\bar{x}^\top (r - \frac{1}{2} H \bar{x}) - \bar{\pi}^\top (A \bar{x} - 1)}
\]

\[
\overset{(m)}{=} \frac{\bar{x}^\top (r - \frac{1}{2} H \bar{x})}{\bar{x}^\top (r - \frac{1}{2} H \bar{x}) - \bar{\pi}^\top (A \bar{x} - 1)}
\]

\[
= \frac{\bar{x}^\top (r - 2H \bar{x} - A^\top \bar{\pi} + \bar{\theta}) + \frac{3}{2} \bar{x}^\top H \bar{x} + \bar{\pi}^\top 1}{\bar{x}^\top (r - H \bar{x} - A^\top \bar{\pi} + \bar{\theta}) + \frac{3}{2} \bar{x}^\top H \bar{x} + \bar{\pi}^\top 1}
\]

\[
\overset{(n)}{=} \frac{\bar{x}^\top (r - 2H \bar{x} - A^\top \bar{\pi} + \bar{\theta}) + \frac{3}{2} \bar{x}^\top H \bar{x} + \bar{\pi}^\top 1}{\bar{x}^\top (r - H \bar{x} - A^\top \bar{\pi} + \bar{\theta}) + \frac{3}{2} \bar{x}^\top H \bar{x} + \bar{\pi}^\top 1}
\]

\[
\overset{(o)}{=} \frac{\frac{3}{2} \bar{x}^\top H \bar{x} + \bar{\pi}^\top 1}{\frac{3}{2} \bar{x}^\top H \bar{x} + \bar{\pi}^\top 1} \overset{(p)}{\geq} \frac{\frac{3}{2} \bar{x}^\top H \bar{x}}{\frac{3}{2} \bar{x}^\top H \bar{x}} \geq \frac{3}{4},
\]

where step (k) follows directly from (EC.135). Step (l) follows directly from (EC.137) and (EC.134). Step (m) follows from \( \bar{\pi}^\top (A \bar{x} - 1) = 0 \) in (EC.131b). In step (n), we add \( \bar{x}^\top \bar{\theta} \) to the numerator and \( \bar{x}^\top \bar{\theta} \) to the denominator. Note that we have \( \bar{x}^\top \bar{\theta} = \bar{x}^\top \bar{\theta} = 0 \) by (EC.131c) and (EC.136b). Step (o) follows from reorganizing the previous step. In step (p), we have \( r - 2H \bar{x} - A^\top \bar{\pi} + \bar{\theta} = 0 \) from (EC.131a) and \( r - H \bar{x} - A^\top \bar{\pi} + \bar{\theta} = 0 \) from (EC.136a). Moreover, we also have \( \bar{x} = 2 \bar{x} \) by construction. In
step (q), we have $\pi^T 1 \geq 0$ by (EC.131b) and $\tilde{\alpha}^T H \tilde{x} > 0$ by positive definiteness of matrix $H$. This completes the proof that
\[
\frac{W(\gamma', \mu')}{W_{opt}} \geq \frac{3}{4}.
\]
(139)

To show the tightness, we consider an example of a network with one seller type and one buyer type. Let the value distributions be $F_i(v) = F_i(v) = \min\{1, v\}$ for $v \geq 0$ and the population profile be $s = b = 1$. In this example, welfare maximization problem (EC.133) becomes
\[
\max_{0 \leq x \leq 1} x - \frac{1}{2} x^2 - \frac{1}{2} x^2.
\]
(140)

Let $\tilde{x}$ be the optimal solution to (EC.140). By the first order optimality condition, we obtain $\tilde{x} = \frac{1}{2}$ and $W_{opt} = \frac{1}{2} - \frac{1}{8} - \frac{1}{8} = \frac{1}{4}$.

Similarly, revenue optimization problem (EC.130) becomes
\[
\max_{0 \leq x \leq 1} x - x^2 - x^2
\]
(141)

Let $\tilde{x}$ be the optimal solution to (EC.141). By the first order optimality condition, we obtain $\tilde{x} = \frac{1}{4}$. By (EC.137) and $W(\gamma', \mu') = \frac{1}{4} - \frac{1}{32} - \frac{1}{32} = \frac{1}{16}$ where the revenue-maximum equilibrium can be implemented by commission-subscription profile $(\gamma'^r, \gamma'^b, \mu'^r, \mu'^b) = (0, 0, 0, \frac{1}{2})$.

This construction satisfies $\frac{W(\gamma', \mu')}{W_{opt}} = \frac{3}{4}$.

Proof of claim (iii). We first focus on welfare maximization problem (EC.3) (or equivalently equilibrium problem (14) given $(\gamma, \mu) = 0$). Let $(p, x, q^r, q^b) \in \mathcal{X}(0, 0)$ be a welfare-maximum equilibrium.

Given the homogeneity of seller/buyer value distributions, by Proposition 10.1, Assumption (A1) - (A4) hold when $(\gamma, \mu) = 0$. Let $y^*$ be the optimal solution to problem (17) where $f(c) = \max_{r \in [0, \min\{1, c\}]} \tilde{g}(r) - c \theta(\xi)$ with $\tilde{g}(r) = \int_0^r F_b^{-1}(1 - x)dx$ and $\tilde{h}(r) = \int_0^r F_s^{-1}(x)dx$. Let $\tilde{r}_j$ be the optimal solution to problem (15) given $y^*_j$ for all $j \in \mathcal{B}$. Using the first order optimality condition, we obtain
\[
\tilde{r}_j = \max \left\{ r : \tilde{g}'(r) - \tilde{h}' \left( \frac{r}{y^*_j/b_j} \right) \geq 0, \ 0 \leq r \leq \min \left\{ 1, \frac{y^*_j}{b_j} \right\} \right\}.
\]
(142)

By Assumption (A4) we have $\tilde{g}'(0) - \tilde{h}'(0) > 0$, which implies that $\tilde{r}_j > 0$. Moreover, by Assumption (A3) and Assumption (A4) we also have $\tilde{h}'(r) > \tilde{h}'(0) \geq 0$ for $r > 0$ and $\tilde{g}'(1) \leq 0$, which further implies that $\tilde{r}_j < 1$. Thus, optimality expression (EC.142) can be further simplified to
\[
\tilde{r}_j = \max \left\{ r : F_b^{-1}(1 - r) - F_s^{-1} \left( \frac{r}{y^*_j/b_j} \right) \geq 0, \ r \leq \frac{y^*_j}{b_j} \right\}.
\]
(143)

Regarding the maximum welfare, we deduce that
\[
W_{opt} = \sum_{j \in \mathcal{B}} \int_0^{\tilde{r}_j} F_b^{-1} \left( 1 - \frac{x}{b_j} \right) dx - \sum_{i \in \mathcal{S}} \int_0^{\tilde{r}_i} F_s^{-1} \left( \frac{x}{s_i} \right) dx
\]
\[
= \sum_{j \in \mathcal{B}} b_j \left[ \int_0^{\tilde{r}_j} F_b^{-1}(1 - x) - F_s^{-1} \left( \frac{x}{y^*_j/b_j} \right) dx \right],
\]
(144)

where step (r) follows from Lemma EC.3 In step (s), Lemma EC.4(iii) suggests that Assumptions (EC.11) - (EC.1-3) hold. Together with Assumptions (EC.1-4) - (EC.1-5) from the homogeneity of value distributions, we have Assumptions (A1) - (A4) and step (s) follows directly from Proposition EC.2.
Next, we consider revenue optimization problem (7). By Proposition 9(i), Assumption [AI] hold. By Lemma 2, $y^*$ is the lexicographical optimal base for polymatroid $P = \{ y \geq 0 : \sum_{j \in B} y_j \leq \sum_{i \in N \in \mathcal{B}} s_i, \forall B \subseteq \mathcal{B} \}$. Thus, $y^*$ is optimal in problem (17) where $f(c) = \max_{r \in [0, \min(1, c)]} \tilde{g}(r) - ch(\frac{x}{r})$ with $\tilde{g}(r) = F_b^{-1}(1-r)F_s^{-1}(r)$. Let $r_j$ be the optimal solution to problem (15) given $y^*_j$. Applying the first order optimality condition, we obtain

$$
\tilde{r}_j = \max \left\{ r : \tilde{g}'(r) - \bar{h}'(r) \left( \frac{r}{y^*_j/b_j} \right) \geq 0, 0 \leq r \leq \min \left\{ 1, \frac{y^*_j}{b_j} \right\} \right\}. \quad (EC.145)
$$

By Assumption [AI], we have $g'(0) - h'(0) > 0$, which implies that $\tilde{r}_j > 0$. Moreover, by Assumption [AE] and Assumption [AI], we also have $h'(r) - h'(0) \geq 0$ for $r > 0$ and $g'(1) \leq 0$, which also implies that $\tilde{r}_j < 1$. Thus, optimality expression (EC.145) can be simplified to

$$
\tilde{r}_j = \max \left\{ r : F_b^{-1}(1-r) - F_s^{-1}(r) \left( \frac{r}{y^*_j/b_j} \right) - r [F_b^{-1}(1-r) - F_s^{-1}(r) \left( \frac{r}{y^*_j/b_j} \right)] \geq 0, 0 \leq r \leq \frac{y^*_j}{b_j} \right\}. \quad (EC.146)
$$

Let $(\gamma', \mu')$ be the revenue-optimal commission-subscription profile that induces equilibrium $(\tilde{p}, \bar{x}, q^*, \bar{q}^*) \in X(\gamma', \mu')$. We deduce that

$$
W(\gamma', \mu') \overset{(t)}{=} \sum_{j \in \mathcal{B}} \int_0^{\tilde{r}_j} F_b^{-1} \left( 1 - \frac{x}{b_j} \right) dx - \sum_{i \in \mathcal{S}} \int_0^{\tilde{r}_i} F_s^{-1} \left( \frac{x}{s_i} \right) dx
$$

$$
\overset{(u)}{=} \sum_{j \in \mathcal{B}} b_j \left[ \int_0^{\tilde{r}_j} F_b^{-1}(1-x) - F_s^{-1} \left( \frac{x}{y^*_j/b_j} \right) dx \right], \quad (EC.147)
$$

where step (t) follows from Lemma EC.3 In step (u), Lemma EC.1(ii) suggests that Assumptions [EC.1(ii)] hold. Together with Assumptions [EC.1-4] and [EC.1-5], from the homogeneity of value distributions, we have Assumptions [AI][AI] and step (u) follows directly from Proposition 9(ii).

To complete the proof, we first show that $\tilde{r}_j \leq \bar{r}_j$ for all $j \in \mathcal{B}$. By the strict increasing property of $F_b(\cdot)$ and $F_s(\cdot)$ from Assumption 4, we obtain

$$
-r [F_b^{-1}(1-r) - r F_s^{-1} \left( \frac{r}{y^*_j/b_j} \right)] < 0 \text{ for all } r > 0,
$$

$$
F_b^{-1}(1-r) - F_s^{-1} \left( \frac{r}{y^*_j/b_j} \right) \text{ is strictly decreasing in } r > 0. \quad (EC.148)
$$

From (EC.148), by comparing expression $\tilde{r}$ in (EC.146) and $\bar{r}$ in (EC.143), we obtain

$$
\tilde{r}_j \leq \bar{r}_j \text{ for all } j \in \mathcal{B}. \quad (EC.149)
$$

Next, we discuss the following two possibilities:

1. If $\tilde{r}_j = \frac{y^*_j}{b_j}$, we have $\bar{r}_j = \frac{y^*_j}{b_j}$ by (EC.149) and (EC.143), which further suggests that

$$
\frac{b_j \left[ \int_0^{\tilde{r}_j} F_b^{-1}(1-x) - F_s^{-1} \left( \frac{x}{y^*_j/b_j} \right) dx \right]}{b_j \left[ \int_0^{\tilde{r}_i} F_b^{-1}(1-x) - F_s^{-1} \left( \frac{x}{y^*_j/b_j} \right) dx \right]} = 1. \quad (EC.150)
$$
(2) if $\bar{r}_j < \frac{y_j^*}{b_j}$, we derive that

$$
\bar{r}_j \left[ F_b^{-1}'(1 - \bar{r}_j) + \frac{1}{y_j^*/b_j} \left( \frac{\bar{r}_j}{y_j^*/b_j} \right) \right] 
\equiv F_b^{-1}(1 - \bar{r}_j) - F_s^{-1} \left( \frac{\bar{r}_j}{y_j^*/b_j} \right)
\geq \left[ F_b^{-1}(1 - \bar{r}_j) - F_s^{-1} \left( \frac{\bar{r}_j}{y_j^*/b_j} \right) \right] - \left[ F_b^{-1}(1 - \bar{r}_j) - F_s^{-1} \left( \frac{\bar{r}_j}{y_j^*/b_j} \right) \right]
\geq (\bar{r}_j - \bar{r}_j) \left[ F_b^{-1}'(1 - \bar{r}_j) + \frac{1}{y_j^*/b_j} \left( \frac{\bar{r}_j}{y_j^*/b_j} \right) \right],
$$

(EC.151)

where the equation in step (v) follows from (EC.146) given that $\bar{r}_j < \frac{y_j^*}{b_j}$. Step (w) follows from $F_b^{-1}(1 - \bar{r}_j) - F_s^{-1} \left( \frac{\bar{r}_j}{y_j^*/b_j} \right) \geq 0$ in (EC.143). In step (x), given that functions $F_b(v)$ and $-F_s(v)$ are convex respectively in $v \in (0, \bar{v}_b)$ and $v \in (0, \bar{v}_s)$, function $F_b^{-1}(1 - r) - F_s^{-1}(\bar{r})$ is concave in $r$ for all $c > 0$. As a result, step (x) follows from the concavity of function $F_b^{-1}(1 - r) - F_s^{-1}(\bar{r})$.

Since $[F_b^{-1}]'(1 - \bar{r}_j) + [F_s^{-1}]' \left( \frac{\bar{r}_j}{y_j^*/b_j} \right) > 0$, the inequality in (EC.151) further implies that $\bar{r}_j \geq \bar{r}_j - \bar{r}_j$ or equivalently

$$
\bar{r}_j \geq \frac{1}{2} \bar{r}_j \text{ for all } j \in \mathcal{B}.
$$

(EC.152)

Since $F_b^{-1}(1 - r) - F_s^{-1} \left( \frac{r}{y_j^*/b_j} \right)$ is strictly decreasing and concave in $r$, it implies that

$$
F_b^{-1}(1 - \bar{r}_j) - F_s^{-1} \left( \frac{\bar{r}_j}{y_j^*/b_j} \right) \geq F_b^{-1}(1 - r) - F_s^{-1} \left( \frac{r}{y_j^*/b_j} \right) \text{ for all } r \geq \bar{r}_j
$$

$$
F_b^{-1}(1 - \bar{r}_j) - F_s^{-1} \left( \frac{\bar{r}_j}{y_j^*/b_j} \right) \leq F_b^{-1}(1 - r) - F_s^{-1} \left( \frac{r}{y_j^*/b_j} \right) \text{ for all } r \leq \bar{r}_j.
$$

(EC.153)

By the concavity of $F_b^{-1}(1 - x) - F_s^{-1} \left( \frac{x}{y_j^*/b_j} \right)$, from (EC.143) and (EC.149), we deduce that

$$
0 \leq \int_0^{\bar{r}_j} F_b^{-1}(1 - x) - F_s^{-1} \left( \frac{x}{y_j^*/b_j} \right) dx \leq \int_0^{\bar{r}_j} F_b^{-1}(1 - x) - F_s^{-1} \left( \frac{x}{y_j^*/b_j} \right) dx,
$$

$$
0 \leq \int_{\bar{r}_j}^{\bar{r}_j} F_b^{-1}(1 - x) - F_s^{-1} \left( \frac{x}{y_j^*/b_j} \right) dx \leq \frac{1}{2} \left[ F_b^{-1}(1 - \bar{r}_j) - F_s^{-1} \left( \frac{\bar{r}_j}{y_j^*/b_j} \right) \right] (\bar{r}_j - \bar{r}_j)
$$

$$
\leq \frac{1}{2} \left[ F_b^{-1}(1 - \bar{r}_j) - F_s^{-1} \left( \frac{\bar{r}_j}{y_j^*/b_j} \right) \right] \bar{r}_j.
$$

(EC.154)

Thus,

$$
\frac{b_j \int_0^{\bar{r}_j} F_b^{-1}(1 - x) - F_s^{-1} \left( \frac{x}{y_j^*/b_j} \right) dx}{b_j \int_0^{\bar{r}_j} F_b^{-1}(1 - x) - F_s^{-1} \left( \frac{x}{y_j^*/b_j} \right) dx}
\equiv \frac{\int_0^{\bar{r}_j} F_b^{-1}(1 - x) - F_s^{-1} \left( \frac{x}{y_j^*/b_j} \right) dx}{\int_0^{\bar{r}_j} F_b^{-1}(1 - x) - F_s^{-1} \left( \frac{x}{y_j^*/b_j} \right) dx}
$$
\[
\frac{\left(\frac{1}{2} + \frac{1}{3}\right)}{1} = \frac{2}{3}
\]

where step (y) follows from a decomposition based on (EC.153). Step (z) follows from (EC.153) and (EC.154).

In summary of (EC.150) and (EC.155), for \( W_r = \frac{W(\gamma', \mu')}{W_{opt}} \), we complete the proof that

\[
W_r = \frac{W(\gamma', \mu')}{W_{opt}} = \frac{\sum_{j \in B} b_j \left[ \int_0^{q_j} F_b^{-1}(1 - x) - F_s^{-1}\left(\frac{x}{y_j/b_j}\right) dx \right]}{\sum_{j \in B} b_j \left[ \int_0^{q_j} F_b^{-1}(1 - x) - F_s^{-1}\left(\frac{x}{y_j/b_j}\right) dx \right]} \geq \frac{2}{3}
\]

(EC.156)

To show its tightness, consider a class of examples characterized by index \( n = 1, 2, \ldots \). In example instance \( n \), let there be one seller type and one buyer type with the population vector \((s_n, b_n) = (2n, \frac{4n}{2n-1})\) and value distribution functions

\[
F_b^n(v) = \begin{cases} 
\frac{1}{2} v, & \text{for } 0 \leq v \leq \frac{2n}{2n+1} \\
\frac{2n-1}{2n} v - \frac{(2n+1)(n-1)}{2n}, & \text{for } \frac{2n}{2n+1} < v \leq \frac{n+1}{n} \\
1, & \text{for } v \geq \frac{n+1}{n}
\end{cases}
\]

\[
F_s^n(v) = \min\{v, 1\}, \quad \text{for } v \geq 0.
\]

(EC.157)

For any \( n \geq 2 \), we have

\[
[F_b^n]^{-1}\left(1 - \frac{q}{b_n}\right) - [F_s^n]^{-1}\left(\frac{q}{s_n}\right) = \begin{cases} 
1 + \frac{1}{n} q, & \text{for } 0 \leq q \leq 1, \\
2 - q, & \text{for } 1 \leq q \leq \frac{4n}{2n-1}.
\end{cases}
\]

(EC.158)

In this example, we can calculate by first order condition that the welfare-maximum flow is \( \tilde{q} = \arg\max_{0 \leq q \leq \frac{4n}{2n-1}} \left\{ \int_0^q [F_b^n]^{-1}(1 - \frac{x}{b_n}) - [F_s^n]^{-1}\left(\frac{x}{s_n}\right) dx \right\} = 2 \) and the revenue-maximum flow is \( \tilde{q} = \arg\max_{0 \leq q \leq \frac{4n}{2n-1}} \left\{ [F_b^n]^{-1}(1 - \frac{q}{b_n}) q - [F_s^n]^{-1}\left(\frac{q}{s_n}\right) q \right\} = 1 \). Thus, we can derive that the welfare ratio satisfies

\[
W^n_r = \frac{W^n(\gamma', \mu')}{W^n_{opt}} = \frac{\int_0^{\tilde{q}} [F_b^n]^{-1}\left(1 - \frac{q}{b_n}\right) - [F_s^n]^{-1}\left(\frac{q}{s_n}\right) dq}{\int_0^{\tilde{q}} [F_b^n]^{-1}\left(1 - \frac{q}{b_n}\right) - [F_s^n]^{-1}\left(\frac{q}{s_n}\right) dq} = \frac{1 + \frac{1}{2n}}{\frac{3}{2} + \frac{1}{2n}}.
\]

(EC.159)

For this class of example, the welfare efficiency ratio satisfies \( \lim_{n \to \infty} W^n_r = \frac{2}{3} \). Thus, the ratio of \( W_r \geq \frac{2}{3} \) is tight. Q.E.D.
EC.8. Neighborhood Map

Figure EC.1   Chicago Community Map