On How Well Generative Adversarial Networks Learn Densities: Nonparametric and Parametric Results

Tengyuan Liang*[1]

[1]University of Chicago, Booth School of Business

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Abstract

We study in this paper the rate of convergence for learning distributions with the Generative Adversarial Networks (GAN) framework, which subsumes Wasserstein, Sobolev and MMD GANs as special cases. We study a wide range of parametric and nonparametric target distributions, under a collection of objective evaluation metrics. On the nonparametric end, we investigate the minimax optimal rates and fundamental difficulty of the density estimation under the adversarial framework. On the parametric end, we establish theory for neural network classes, that characterizes the interplay between the choice of generator and discriminator. We investigate how to improve the GAN framework with better theoretical guarantee through the lens of regularization. We discover and isolate a new notion of regularization, called the *generator/discriminator pair regularization*, that sheds light on the advantage of GAN compared to classic parametric and nonparametric approaches for density estimation.

Keywords: Generative adversarial networks, density estimation, oracle inequality, neural network learning, pair regularization, leaky ReLU, nonparametric statistics.

1 Introduction

Generative Adversarial Networks (GANs) (Goodfellow et al., 2014; Li et al., 2015; Arjovsky et al., 2017; Dziugaite et al., 2015) have stood out as an important unsupervised method for learning and efficient sampling from a complicated, multi-modal target data distribution. Despite its celebrated empirical success in image tasks, there are many theoretical questions yet to be answered (Liu et al., 2017; Arora and Zhang, 2017; Arora et al., 2017; Liang, 2017; Daskalakis et al., 2017; Liang and Stokes, 2018; Singh et al., 2018; Bai et al., 2018; Liu and Chaudhuri, 2018).

One formulation of the GAN framework (Arjovsky et al., 2017; Li et al., 2015; Dziugaite et al., 2015; Liu et al., 2017; Mroueh et al., 2017) solves the following minimax problem, at the population level,

$$\min_{\mu \sim \mathcal{D}_G} \max_{f \in \mathcal{F}_D} \mathop{\mathbb{E}}_{Y \sim \mu} f(Y) - \mathop{\mathbb{E}}_{X \sim \nu} f(X).$$

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In plain language, given a target distribution ν, one seeks for a distribution μ from a probability distribution *generator class* $\mathcal{D}_G$, such that it minimizes the loss incurred by the best critic function inside the *discriminator class* $\mathcal{F}_D$. In practice, both the *generator class* and the *discriminator class* are represented by deep neural networks. $\mathcal{D}_G$ quantifies the transformed distributions realized by a neural network with random inputs (either isotropic Gaussian or uniform distribution), and $\mathcal{F}_D$ represents the functions that are realizable by a certain neural network architecture. We refer the readers to Liu et al. (2017) for other more general formulations of GANs.

In practice, one only has access to finite samples of the real data (target) distribution ν. Denote $\widehat{\nu}_n$ to be a measure based on n i.i.d. samples from ν, where the empirical density is typically used. Given the samples, the GAN framework solves the following problem

$$\widehat{\mu}_n = \arg\min_{\mu \sim \mathcal{D}_G} \max_{f \in \mathcal{F}_D} \mathop{\mathbb{E}}_{Y \sim \mu} f(Y) - \mathop{\mathbb{E}}_{X \sim \widehat{\nu}^n} f(X). \tag{1.1}$$

In the statistics literature, density estimation has been a central topic in nonparametric statistics (Nemirovski, 2000; Tsybakov, 2009; Wassermann, 2006), as well as in parametric estimation (Brown, 1986). In the nonparametric case, the minimax optimal rate of convergence has been understood fairly well, for a wide range of density function classes quantified by its smoothness (Stone, 1982). We would like to point out a simple yet important connection between two fields: in nonparametric statistics, the model grows in size to accommodate the complexity of the data, which is reminiscent of the complexity (such as depth/width, or norms of weights) of the deep neural networks in GANs.

Note the GAN framework mentioned above is flexible. Define the integral probability metric (IPM) for a symmetric function class $\mathcal{F}$,

$$d_{\mathcal{F}}(\mu, \nu) := \sup_{f \in \mathcal{F}} \mathop{\mathbb{E}}_{Y \sim \mu} f(Y) - \mathop{\mathbb{E}}_{X \sim \nu} f(X) = \sup_{f \in \mathcal{F}} \int_{x \in \Omega} f(x) \left(\mu(x) - \nu(x)\right) dx.$$

If $\mathcal{F}$ contains all Lipschitz-1 functions, the IPM is the Wasserstein-1 metric (Wasserstein-GAN (Arjovsky, Chintala, and Bottou, 2017)) $d_W(\cdot,\cdot)$. When $\mathcal{F}$ represents all functions bounded by 1, the IPM is the total variation metric (Radon metric) $d_{TV}(\cdot,\cdot)$. Let $\mathcal{H}$ be a reproducing kernel Hilbert space (RKHS) and $K(\cdot,\cdot)$ be its kernel. When $\mathcal{F}$ consists of functions with bounded RKHS norm $\mathcal{F} = \{f \in \mathcal{H} \mid \|f\|_{\mathcal{H}} \le 1\}$, it is referred to as the maximum mean discrepancy (MMD) GAN (Dziugaite, Roy, and Ghahramani, 2015; Li, Swersky, and Zemel, 2015; Arbel, Sutherland, Bińkowski, and Gretton, 2018). It is called Sobolev GAN (Mroueh, Li, Sercu, Raj, and Cheng, 2017) when $\mathcal{F}$ is the Sobolev space with certain smoothness.

The current paper studies the GAN framework for learning densities from a statistical viewpoint. The focus of the current paper is *not* on the optimization side of how to solve for $\widehat{\mu}_n$ efficiently, rather on the statistical front: how well GAN estimates a wide range of target distributions (both nonparametric and parametric) under a collection of objective evaluation metrics, and how to improve the GAN framework to achieve better theoretical guarantee through the lens of regularization. We discover and isolate a new notion of regularization, which we call *generator/discriminator pair regularization*. Remark that several curious features of this pair regularization appear to be new to the literature. Throughout the nonparametric and parametric theory, we rely on several simple yet powerful oracle inequalities for analyzing GAN, which could be of independent interest.

1.1 Summary and Organization

The paper is organized as follows. Section 2 consists of main nonparametric results for GAN, and more generally the adversarial framework. Section 3 contains the main parametric results for GAN with neural network generator and discriminator classes, where we introduce the new notion of pair regularization. Further discussions on the generator/discriminator pair regularization is deferred to Section 4. The main proofs are collected in Section 5, with supporting lemmas in Appendix A.

In Section 2, we start with studying the minimax optimal rates for the Wasserstein, Sobolev, and MMD GAN framework, when the target distributions belong to nonparametric classes.

Optimal Nonparametric Rates and Oracle Results Let $\widetilde{\nu}_n$ be any estimator for the target density ν based on n i.i.d. drawn samples. Consider the target density $\nu(x) \in \mathcal{G} = W^\alpha$ lies in the Sobolev space with smoothness $\alpha > 0$, and evaluation metric $\mathcal{F} = W^\beta$ with smoothness $\beta > 0$, the minimax optimal rate for Sobolev GAN (with Wasserstein GAN $\beta = 1$ as special case) is

$$\inf_{\widetilde{\nu}_n} \sup_{\nu \in \mathcal{G}} \mathbb{E}\, d_{\mathcal{F}}\left(\nu, \widetilde{\nu}_n\right) \asymp n^{-\frac{\alpha+\beta}{2\alpha+d}} \vee n^{-\frac{1}{2}}.$$

To make the theory better suited for distributions on image manifolds, we extend the above results to MMD GAN with explicit dependence on the "effective dimension" (denoted by κ). For MMD GAN, the optimal rate is

$$\sup_{\nu \in \mathcal{G}} \mathbb{E}\, d_{\mathcal{F}}\left(\nu, \widetilde{\nu}_n\right) \asymp n^{-\frac{(\alpha+1)\kappa}{2\alpha\kappa+2}} \vee n^{-\frac{1}{2}}.$$

Here we consider a reproducing kernel Hilbert space $\mathcal{H}$ whose eigenvalues of the integral operator $\mathcal{T}$ decay as $t_i \asymp i^{-\kappa}$. The evaluation metric $\mathcal{F} = \{f \mid \|f\|_{\mathcal{H}} \leq 1\}$, and the target density $\nu(x)$ lies in a subset of RKHS with smoothness parameter α.

By means of a simple oracle inequality, we establish performance guarantee on GAN estimator (1.1) for learning the target density $\nu(x) \in W^\alpha$. Let $\mathcal{D}_G$ be *any generator* class that can be *mis-specified*, and $\mathcal{F}_D = W^\beta$ be the discriminator metric. With the empirical density as plug-in for (1.1), we show a sub-optimal rate

$$\mathbb{E}\, d_{\mathcal{F}_D}(\widehat{\mu}_n, \nu) \leq \min_{\mu \in \mathcal{D}_G} d_{\mathcal{F}_D}(\mu, \nu) + n^{-\frac{\beta}{d}} \vee \frac{\log n}{\sqrt{n}}.$$

In contrast, with a smoothed/regularized empirical density as plug-in, a faster rate is attainable

$$\mathbb{E}\, d_{\mathcal{F}_D}(\widetilde{\mu}_n, \nu) \leq \min_{\mu \in \mathcal{D}_G} d_{\mathcal{F}_D}(\mu, \nu) + n^{-\frac{\alpha+\beta}{2\alpha+d}} \vee \frac{1}{\sqrt{n}}.$$

In Section 3, we move on to studying the parametric rates for GANs when both the generator and discriminator are neural networks. Conceptually, this can be thought as the nonparametric case when $\alpha \gg d$.

Generator/Discriminator Pair Regularization Consider the parametrized GAN estimator with generator network $\mathcal{G}$ (parametrized by θ) and discriminator network $\mathcal{F}$ (parametrized by ω)

$$\widehat{\theta}_{m,n} \in \arg\min_{\theta: g_\theta \in \mathcal{G}} \max_{\omega: f_\omega \in \mathcal{F}} \left\{ \widehat{\mathbb{E}}_m f_\omega(g_\theta(Z)) - \widehat{\mathbb{E}}_n f_\omega(X) \right\},$$

where m and n denote the number of the generator samples and real samples. We generalize the oracle inequality to establish the following upper bound

$$\begin{aligned}&\mathbb{E}\, d_{TV}^2\left(\nu, \mu_{\widehat{\theta}_{m,n}}\right), \mathbb{E}\, d_W^2\left(\nu, \mu_{\widehat{\theta}_{m,n}}\right), \mathbb{E}\, d_{KL}\left(\nu || \mu_{\widehat{\theta}_{m,n}}\right) + \mathbb{E}\, d_{KL}\left(\mu_{\widehat{\theta}_{m,n}} || \nu\right)\\ &\leq A_1(\mathcal{F}, \mathcal{G}, \nu) + A_2(\mathcal{G}, \nu) + S_{n,m}(\mathcal{F}, \mathcal{G}),\end{aligned}$$

where μ_θ and ν are absolute continuous, and

$$\begin{aligned}&\text{approx. err.} \quad A_1(\mathcal{F}, \mathcal{G}, \nu) := \sup_\theta \inf_\omega \left\| \log \frac{\nu}{\mu_\theta} - f_\omega \right\|_\infty, \quad A_2(\mathcal{G}, \nu) := \inf_\theta \left\| \log \frac{\mu_\theta}{\nu} \right\|_\infty^{1/2},\\ &\text{sto. err.} \quad S_{n,m}(\mathcal{F}, \mathcal{G}) := \sqrt{\text{Pdim}(\mathcal{F}) \left(\frac{\log m}{m} \vee \frac{\log n}{n} \right)} \vee \sqrt{\text{Pdim}(\mathcal{F} \circ \mathcal{G}) \frac{\log m}{m}},\end{aligned}$$

with $\text{Pdim}(\cdot)$ denotes the pseudo-dimension of the function computed by the neural networks.

We emphasize on the interplay between $(\mathcal{G}, \mathcal{F})$ as a pair of tuning parameters for regularization, for instance, one simple form of the interplay is:

$$\begin{aligned}&\text{fix } \mathcal{G}, \text{ as } \mathcal{F} \text{ increase}: \quad A_1(\mathcal{F}, \mathcal{G}, \nu) \text{ decrease, } A_2(\mathcal{G}, \nu) \text{ constant, } S_{n,m}(\mathcal{F}, \mathcal{G}) \text{ increase,}\\ &\text{fix } \mathcal{F}, \text{ as } \mathcal{G} \text{ increase}: \quad A_1(\mathcal{F}, \mathcal{G}, \nu) \text{ increase, } A_2(\mathcal{G}, \nu) \text{ decrease, } S_{n,m}(\mathcal{F}, \mathcal{G}) \text{ increase.}\end{aligned}$$

We call this the *generator/discriminator pair regularization*. In Section 3.2, we elaborate on the intricacies of this phenomenon using diagram in Fig. 1, and on implications of the above result on generator/discriminator design for pair regularization.

Parametric Rates for Leaky ReLU Networks As a direct application of the pair regularization aforementioned, we establish the following parametric rate when the generator $\mathcal{G}$ and discriminator $\mathcal{F}$ are both leaky ReLU networks with depth scale with L (width properly chosen depends on dimension). When the target density is realizable by the generator, we have

$$\mathbb{E}\, d_{TV}^2\left(\nu, \mu_{\widehat{\theta}_{m,n}}\right) \precsim \sqrt{d^2 L^2 \log(dL) \left(\frac{\log m}{m} \vee \frac{\log n}{n} \right)}.$$

In addition, we present the analytic form of the parametric densities that are realizable by leaky ReLU generator networks. Remark that the results hold for very deep networks with depth $L = o(\sqrt{n/\log n})$.

Finally, as a sanity check, GANs enjoy near optimal sampling complexity (w.r.t. dimension d) for estimating multivariate Gaussians, with proper choices of the architecture and activation, i.e.,

$$\mathbb{E}\, d_{TV}^2\left(\nu, \mu_{\widehat{\theta}_{m,n}}\right) \precsim \sqrt{\frac{d^2 \log d}{n \wedge m}}.$$

1.2 Preliminaries

Let's introduce the notations used in this paper. During the discussion, we restrict the input space to be $\Omega = [0,1]^d$ when the input measure is uniform, and $\Omega = \mathbb{R}^d$ when the input measure is isotropic Gaussian. We use μ, ν to denote the distribution/measure, and the Radon-Nikodym derivatives $\mu(x), \nu(x)$ to denote the corresponding density functions, with slight abuse of notation. $\|f\|_q := \left(\int_\Omega |f(x)|^q dx\right)^{1/q}$ denotes the ℓ_q-norm under the Lebesgue measure, for $1 \leq q \leq \infty$. For a vector w, $\|w\|_q$ denotes the vector ℓ_q-norm. Denote $A(n) \precsim n^\alpha$, if $\overline{\lim}_{n\to\infty} \frac{\log A(n)}{\log n} \leq \alpha$, holding other parameters fixed, similarly $A(n) \succsim n^\alpha$ if $\underline{\lim}_{n\to\infty} \frac{\log A(n)}{\log n} \geq \alpha$. Denote $A(n) \asymp n^\alpha$ if $A(n) \precsim n^\alpha$ and $A(n) \succsim n^\alpha$. $[K] := \{0, 1, \ldots, K\}$ denotes the index set, for any $K \in \mathbb{N}$.

The Sobolev space is defined as follows.

Definition 1 (Sobolev space: $k \in \mathbb{N}$). *For an integer k, define the Sobolev space $W^{k,q}(r)$ ($1 \leq q \leq \infty$)*

$$W^{k,q}(r) := \left\{ f \in \Omega \to \mathbb{R} : (\sum_{|\alpha| \leq k} \|D^{(\alpha)} f\|_q^q)^{1/q} \leq r \right\},$$

where α is a multi-index and $D^{(\alpha)}$ denotes the α-weak derivative.

The definition when $q = 2$ naturally extends to fractional $\alpha \in \mathbb{R}$ through the Bessel potential, with $\mathbf{F}f(\xi) : \mathbb{R} \to \mathbb{C}$ denotes the Fourier transform of f, and $\mathbf{F}^{-1}$ as its inverse.

Definition 2 (Sobolev space: $\alpha \in \mathbb{R}$). *For $\alpha \in \mathbb{R}$, the Sobolev space $W^{\alpha,2}(r)$ definition extends to non-integer α,*

$$W^\alpha(r) := \left\{ f \in \Omega \to \mathbb{R} : \left\| \mathbf{F}^{-1} \left[(1 + |\xi|^2)^{\frac{\alpha}{2}} \mathbf{F} f(\xi) \right] \right\|_2 \leq r \right\}.$$

Consider reproducing kernel Hilbert space (RKHS) $\mathcal{H}$ endowed with norm $\|\cdot\|_{\mathcal{H}}$, and the corresponding positive semidefinite kernel $K(\cdot, \cdot) : \Omega \times \Omega \to \mathbb{R}$. Let us introduce the integral operator $\mathcal{T}_\pi : L^2_\pi \to \mathcal{H}$ with respect to the base measure $\pi(x)$:

Definition 3 (Integral operator of RKHS). *Define the integral operator $\mathcal{T}_\pi : L^2_\pi \to \mathcal{H}$,*

$$\mathcal{T}_\pi f(t) = \int K(t, x) f(x) \pi(x) dx,$$

and denote the eigenfunctions of this operator by $\psi_i(x)$, and eigenvalues by $t_i, i \in \mathbb{N}$, with

$$\mathcal{T}_\pi \psi_i = t_i \psi_i, \text{ and } \int \psi_i(x) \psi_j(x) \pi(x) dx = \delta_{ij}.$$

The following notion of combinatorial dimension for real-valued function is credited to Pollard (1990), which we will employ in deriving parametric rates for leaky ReLU GANs.

Definition 4 (Pseudo-dimension). *Let $\mathcal{F} : \Omega \to \mathbb{R}$ be a class of functions. The pseudo-dimension of $\mathcal{F}$, denoted by $\mathrm{Pdim}(\mathcal{F})$, is the largest integer m such that there exists $(X_i, y_i) \in \Omega \times \mathbb{R}, 1 \leq i \leq m$ such that for any $(b_1, \ldots, b_m) \in \{-1, 1\}^m$ there exists $f \in \mathcal{F}$ such that $\mathrm{sign}(f(X_i) - y_i) = b_i, \forall 1 \leq i \leq m$. For a class $\mathcal{F}$ of real-valued functions, we can also define its Vapnik-Chervonenkis dimension $\mathrm{VCdim}(\mathcal{F}) := \mathrm{VCdim}(\mathrm{sign}(\mathcal{F}))$.*

Finally, for two functions $f : \mathbb{R}^d \to \mathbb{R}, g : \mathbb{R}^p \in \mathbb{R}^d$, we denote $f \circ g$ to be the composition $f(g(x))$. We use the following notion for composition of function classes

$$\mathcal{F} \circ \mathcal{G} := \{f \circ g \mid \forall f \in \mathcal{F}, g \in \mathcal{G}\}. \tag{1.2}$$

2 Nonparametric Results of GANs: Framework

We start by investigating the optimal statistical rates of the Wasserstein, Sobolev, and MMD GAN framework, for density estimation. The goal of this section is to answer the fundamental difficulty of learning a wide range of nonparametric densities for different evaluation/discriminator metric.

2.1 Minimax Optimal Rates

Theorem 1 (Minimax optimal rates, Sobolev space)**.** *Consider the target density $\nu(x) \in \mathcal{G} = W^{\alpha}(r)$ lies in the Sobolev space with smoothness $\alpha > 0$ for some constant $r > 0$, and the evaluation metric $\mathcal{F} = W^{\beta}(1)$ with smoothness $\beta > 0$. The minimax optimal rate is*

$$\inf_{\widetilde{\nu}_n} \sup_{\nu \in \mathcal{G}} \mathbb{E}\, d_{\mathcal{F}}\left(\nu, \widetilde{\nu}_n\right) \asymp n^{-\frac{\alpha+\beta}{2\alpha+d}} \vee n^{-\frac{1}{2}},$$

where $\widetilde{\nu}_n$ is any estimator for the target density ν based on n i.i.d. drawn samples.

Remark 1 (Relations to the literature)**.** The above theorem is an improvement to an earlier version (Liang, 2017) of this paper. Admittedly, the improvement is simply in one line of the original argument for the upper bound, specifically Eqn. (5.1). The minimax lower bound was first proved in the earlier version of this paper (Liang (2017), P. 18-19, for density estimation). The improvement in the upper bound was also obtained in (Singh, Uppal, Li, Li, Zaheer, and Póczos, 2018) in a more general form. We remark that the optimal upper bound can also be attained using similar approach as in Mair and Ruymgaart (1996).

One can generalize the above theorem to the reproducing kernel Hilbert space (RKHS) to accommodate the fact that distribution of images are typically supported on manifolds, with image similarity measured by kernels. It will be useful to spell out the explicit dependence on the intrinsic dimension of the manifold and the kernel, rather than the ambient dimension. The Sobolev space considered in Theorem 1 is a special case of RKHS. In addition, this will enable us to provide theoretical rates for MMD GAN (Dziugaite et al., 2015; Li et al., 2015; Arbel et al., 2018).

In the next theorem, we assume that for all target density $\nu \in \mathcal{G}$ of interest and all $i \in \mathbb{N}$, there exists a universal constant on the variance of eigenfunctions,

$$\mathop{\mathbb{E}}_{X\sim\nu} \psi_i(X)^2 \leq C. \tag{2.1}$$

This is a very mild condition that is satisfied in many cases.

Corollary 1 (MMD rates, RKHS)**.** *Consider a RKHS $\mathcal{H}$ whose eigenvalues of the integral operator $\mathcal{T}_\pi$ decay as $t_i \asymp i^{-\kappa}$, $0 < \kappa < \infty$. Consider the evaluation metric $\mathcal{F} = \{f \mid \|f\|_{\mathcal{H}} \leq 1\}$, and the target density $\nu(x)$ lies in a subspace $\mathcal{G} = \{\nu \mid \|\mathcal{T}_\pi^{-(\alpha-1)/2}\nu\|_{\mathcal{H}} \leq r\}$ of smooth densities with smoothness parameter $\alpha > 0$, for some fixed $r > 0$. Under the assumption (2.1), we have*

$$\sup_{\nu \in \mathcal{G}} \mathbb{E}\, d_{\mathcal{F}}\left(\nu, \widetilde{\nu}_n\right) \precsim n^{-\frac{(\alpha+1)\kappa}{2\alpha\kappa+2}} \vee n^{-\frac{1}{2}}.$$

Remark 2. Here the target density class considers a subset of the RKHS, with α quantifies its smoothness w.r.t. the kernel: the high frequency component decays sufficiently fast. This is a standard formulation studied in the RKHS literature, see Caponnetto and De Vito (2007). The parameter κ describes the intrinsic dimension of the integral operator. When $\kappa > 1$, the intrinsic dimension (trace of $\mathcal{T}_\pi$) is bounded as $\sum_{i\geq 1} t_i = \sum_{i\geq 1} i^{-k} \leq C$, therefore the upper bound reads the parametric rate $n^{-\frac{(\alpha+1)\kappa}{2\alpha\kappa+2}} \vee n^{-\frac{1}{2}} = n^{-1/2}$. When $\kappa < 1$, to obtain $\mathbb{E}\, d_{\mathcal{F}}(\nu, \widetilde{\nu}_n) \leq \epsilon$, the sample complexity scales

$$n = \epsilon^{2+\frac{2}{\alpha+1}\left(\frac{1}{\kappa}-1\right)}.$$

Therefore we see that the curse of dimensionality (exponential dependence of ϵ on the sample size n) only reflects in the "effective dimension", described by $1/\kappa - 1$.

The Sobolev space $W^1(r)$ can be regarded as a RKHS with $\kappa = \frac{2}{d}$, therefore the lower bound in Theorem 1 suggests that the rate for MMD GAN is also tight, for a particular subclass.

2.2 Oracle Inequality, Sub-optimality and Regularization

In this section, we use a simple oracle inequality to show that when the generator class $\mathcal{D}_G$ — typically represented by neural networks — is mis-specified, one can still derive oracle results based on the theoretical rates obtained so far. In addition, as a byproduct of the theory, we show that when learning hard nonparametric densities, the direct GAN formulation may end up with suboptimal rates. However, a technique in training GANs called instance noise (Sønderby, Caballero, Theis, Shi, and Huszár, 2016; Arjovsky and Bottou, 2017; Mescheder, Geiger, and Nowozin, 2018), can potentially improve the sub-optimal rates to optimal nonparametric rates. In practice, this is known to alleviate the mode collapse problem, and enables better convergence during training.

Recall the GAN formulation, denote $\mathcal{D}_G$ to be class of distributions represented by the generator, and $\mathcal{F}_D$ to be the class of functions realized by the discriminator

$$\mu_n = \arg\min_{\mu\sim\mathcal{D}_G} \max_{f\in\mathcal{F}_D} \left\{ \mathop{\mathbb{E}}_{Y\sim\mu} f(Y) - \mathop{\mathbb{E}}_{X\sim\nu_n} f(X) \right\}. \tag{2.2}$$

where ν_n is an empirical version of the density based on n i.i.d. drawn samples from the target distribution ν.

In this section, we consider the generator class $\mathcal{D}_G$ to be *any* class of distributions, and discriminator class $\mathcal{F}_D$ to be certain nonparametric function class, say Lipschitz functions as in Wasserstein GAN, Sobolev class in Sobolev GAN, and functions with bounded RKHS norm in MMD GAN. Remark that $\mathcal{D}_G$ may not contain the target density ν (which we call mis-specified), as in reality, it might be questionable to assume that the target density is realizable by a neural network transformation of simple densities (say, uniform or Gaussian). In Section 3, we will consider the further case when both the generator and discriminator are ReLU neural networks.

Theorem 2 (GANs framework: nonparametric)**.** *Let $\mathcal{D}_G$ be any generator class. Consider the discriminator metric $\mathcal{F}_D = W^\beta(1)$, and the target density $\nu(x) \in W^\alpha(r)$. With the empirical density $\widehat{\nu}^n(x) := \frac{1}{n}\sum_{i=1}^n \delta_{X_i}(x)$ as the plug-in, the GAN estimator*

$$\widehat{\mu}_n \in \arg\min_{\mu\in\mathcal{D}_G} \max_{f\in\mathcal{F}_D} \left\{ \int f(x)\mu(x)dx - \int f(x)\widehat{\nu}^n(x)dx \right\},$$

learns the target density with a sub-optimal rate

$$\mathbb{E}\, d_{\mathcal{F}_D}(\widehat{\mu}_n, \nu) \leq \min_{\mu\in\mathcal{D}_G} d_{\mathcal{F}_D}(\mu, \nu) + n^{-\frac{\beta}{d}} \vee \frac{\log n}{\sqrt{n}}.$$

In contrast, there exists a smoothed/regularized empirical density $\widetilde{\nu}^n(x)$ *as plug-in*

$$\widetilde{\mu}_n \in \arg\min_{\mu \in \mathcal{D}_G} \max_{f \in \mathcal{F}_D} \left\{ \int f(x)\mu(x)dx - \int f(x)\widetilde{\nu}^n(x)dx \right\},$$

where a faster rate is attainable

$$\mathbb{E}\, d_{\mathcal{F}_D}(\widetilde{\mu}_n, \nu) \le \min_{\mu \in \mathcal{D}_G} d_{\mathcal{F}_D}(\mu, \nu) + n^{-\frac{\alpha+\beta}{2\alpha+d}} \vee \frac{1}{\sqrt{n}}.$$

The proof of the above theorem is based on the following simple oracle inequality. Later, we will generalize the oracle inequality (see Lemma 2) in Section 3 to establish rates when both the generator and discriminator are ReLU networks. In addition, a generalization of the oracle inequality gives rise to a curious notion of pair regularization, which we will study in Section 3.

Lemma 1 (Simple oracle inequality)**.** *Under the condition that* $\mathcal{F}_D$ *is symmetric class, i.e.,* $\mathcal{F}_D = -\mathcal{F}_D$*, the GAN estimator in* (2.2) *satisfies*

$$d_{\mathcal{F}_D}(\nu, \mu_n) \le \min_{\mu \in D_G} d_{\mathcal{F}_D}(\mu, \nu) + 2 d_{\mathcal{F}_D}(\nu, \nu_n),$$

where we refer the first term as the approximation error, and second as the stochastic error.

Remark 3 (Regularization)**.** We would also like to remark that to obtain a implementable algorithm for the smoothed/regularized empirical density $\widetilde{\nu}^n(x)$ in Theorem 2, one may use the kernel methods

$$\widetilde{\nu}^n(x) = \frac{1}{nh_n} K\left(\frac{x - x_i}{h_n}\right),$$

with specific choices of the kernel K and bandwidth h_n. When using the Gaussian kernel, this so-called "instance noise" technique (Sønderby et al., 2016; Arjovsky and Bottou, 2017; Mescheder et al., 2018) is used in GAN training: each time when evaluating the stochastic gradients for generator/discriminator, sample a mini-batch of data and then perturb them by a Gaussian. One may view this data augmentation (or stability to data perturbation) as a form of *regularization* (Yu, 2013) statistically, to prevent the generator from memorizing the empirical data and learning a too complex model, besides the computational advantages in training. We will show in Section 3 that, specific choice of generator and discriminator pair can also serve the goal of regularization in the parametric regime, in a curious way.

3 Parametric Results of GANs: Networks

From the above nonparametric results, the optimal rates should be $n^{-\frac{\alpha+1}{2\alpha+d}}$ for nonparametric densities with smoothness α, in the Wasserstein distance ($\beta = 1$), or $n^{-\frac{\alpha}{2\alpha+d}}$ rate in total variation distance ($\beta = 0$). This result can be interpreted as: for complex nonparametric densities, the curse of dimensionality is unavoidable; the GAN framework provides reasonable solutions mathematically, albeit regularization may improve the rate. However, on the blessing side, for nonparametric densities in a RKHS with nice kernels (say, for densities on the image manifold), GAN may only suffer the curse of dimensionality w.r.t. the intrinsic dimension.

In the parametric case ($\alpha \gg d$, or $\alpha \to \infty$), the optimal rate should be $n^{-\frac{1}{2}}$, the so-called parametric rate. In this section, we show that leaky ReLU neural networks are able to attain similar parametric rates, but with a slight sub-optimal rate, for parametric density class representable by neural network generators.

In this section, we consider when both the generator and discriminator are neural networks. To be specific, let $\mathcal{F} = \{f_\omega(x) : \mathbb{R}^d \to \mathbb{R}\}$ be the discriminator functions realized by a neural network with parameter ω. Let $\mathcal{G} = \{g_\theta(z) : \mathbb{R}^d \to \mathbb{R}^d\}$ be the generator neural network transformation with parameter θ. Consider $Z \sim \pi$ as the random input distribution with density π, and the target density $X \sim \nu$. Denote μ_θ as the density of $g_\theta(Z)$.

Consider the parametrized GAN estimator

$$\widehat{\theta}_{m,n} \in \mathop{\arg\min}_{\theta: g_\theta \in \mathcal{G}} \max_{\omega: f_\omega \in \mathcal{F}} \left\{ \widehat{\mathbb{E}}_m f_\omega(g_\theta(Z)) - \widehat{\mathbb{E}}_n f_\omega(X) \right\}, \tag{3.1}$$

where m and n denote the number of the generator samples and real samples. We will generalize the oracle inequality first (Lemma 2), then show that the oracle approach, when applied to neural networks, sheds light on the choice of generator/discriminator pair as effective regularization in Theorem 3. We will show that with effective regularization, one can obtain parametric rates for learning densities in strong evaluation metrics including total variation, Kullback-Leibler and Wasserstein.

3.1 Generalized Oracle Inequality and Parametric Rate

Lemma 2 (Generalized oracle inequality)**.** *Consider the GAN estimator $\widehat{\theta}_{m,n}$ defined in (3.1). Recall the definition of composition in (1.2), under the condition that $\mathcal{F}$ and $\mathcal{F} \circ \mathcal{G}$ are symmetric, the following oracle inequality holds for any μ_θ with $g_\theta \in \mathcal{G}$,*

$$d_{\mathcal{F}}\left(\mu_{\widehat{\theta}_{m,n}}, \nu\right) \leq d_{\mathcal{F}}(\mu_\theta, \nu) + 2 d_{\mathcal{F}}\left(\widehat{\nu}^n, \nu\right) + d_{\mathcal{F}}\left(\widehat{\mu}_\theta^m, \mu_\theta\right) + d_{\mathcal{F}\circ\mathcal{G}}(\widehat{\pi}^m, \pi).$$

Here for any measure μ, we use $\widehat{\mu}^n$ to denote the empirical measure with n samples.

The above lemma is a generalization of Lemma 1. The main changes are two-folded. First, we show the dependence on the number of generator samples m, besides the number of real samples n. Second, the role and complexity of the generator network is made explicit in the bound. One can view that when $m \to \infty$, the current lemma reduces to Lemma 1. We will show that the oracle inequality helps establish parametric rates for densities representable by neural networks, in the next Theorem 3 and 4 and their corollaries. In addition, it provides theoretical insight on the choice of generator/discriminator pair, as a new form of regularization. Here we emphasize that the evaluation metric d_{TV} is *different* from the GAN discriminator metric $d_{\mathcal{F}}$ in GAN estimator (3.2).

Theorem 3 (GANs: parametric)**.** *Consider the GAN estimator*

$$\widehat{\theta}_{m,n} \in \mathop{\arg\min}_{\theta: g_\theta \in \mathcal{G}} \max_{\omega: f_\omega \in \mathcal{F}, \|f_\omega\|_\infty \leq B} \left\{ \widehat{\mathbb{E}}_m f_\omega(g_\theta(Z)) - \widehat{\mathbb{E}}_n f_\omega(X) \right\}. \tag{3.2}$$

where $B > 0$ is some absolute constant, m and n denote the number of the generator samples and *real samples. Then for total variation distance, and Kullback-Leibler divergence, we have*

$$\begin{aligned}
\mathbb{E}\, d_{TV}^2\left(\nu, \mu_{\widehat{\theta}_{m,n}}\right) &\leq \frac{1}{4}\left[\mathbb{E}\, d_{KL}\left(\nu || \mu_{\widehat{\theta}_{m,n}}\right) + \mathbb{E}\, d_{KL}\left(\mu_{\widehat{\theta}_{m,n}} || \nu\right)\right] \\
&\leq \frac{1}{2}\sup_{\theta}\inf_{\omega}\left\|\log\frac{\nu}{\mu_\theta} - f_\omega\right\|_\infty + \frac{B}{4\sqrt{2}}\inf_{\theta}\left\|\log\frac{\mu_\theta}{\nu}\right\|_\infty^{1/2} \\
&\quad + C\cdot\sqrt{Pdim(\mathcal{F})\left(\frac{\log m}{m}\vee\frac{\log n}{n}\right)}\vee\sqrt{Pdim(\mathcal{F}\circ\mathcal{G})\frac{\log m}{m}},
\end{aligned} \tag{3.3}$$

where $C > 0$ is some universal constant independent of the pseudo-dimension $Pdim(\mathcal{F})$, $Pdim(\mathcal{F}\circ\mathcal{G})$ (see Definition 4) and m, n.

The upper bound in the above theorem consists of two parts: the approximation error $A(\mathcal{F}, \mathcal{G}, \nu)$ and stochastic error $S(\mathcal{F}, \mathcal{G}, n, m)$,

$$\begin{aligned}
A(\mathcal{F}, \mathcal{G}, \nu) &:= \frac{1}{2}\sup_{\theta}\inf_{\omega}\left\|\log\frac{\nu}{\mu_\theta} - f_\omega\right\|_\infty + \frac{B}{4\sqrt{2}}\inf_{\theta}\left\|\log\frac{\mu_\theta}{\nu}\right\|_\infty^{1/2} =: A_1(\mathcal{F}, \mathcal{G}, \nu) + A_2(\mathcal{G}, \nu), \\
S_{n,m}(\mathcal{F}, \mathcal{G}) &:= \sqrt{\mathrm{Pdim}(\mathcal{F})\left(\frac{\log m}{m}\vee\frac{\log n}{n}\right)}\vee\sqrt{\mathrm{Pdim}(\mathcal{F}\circ\mathcal{G})\frac{\log m}{m}}.
\end{aligned} \tag{3.4}$$

As a direct corollary, one can establish similar results for the Wasserstein distance. In the next section, we will elaborate on the interplay among the two approximation error term $A_1(\mathcal{F}, \mathcal{G}, \nu)$, $A_2(\mathcal{G}, \nu)$, and the stochastic error term $S_{n,m}(\mathcal{F}, \mathcal{G})$.

Corollary 2. *Recall the definition in (3.4). Assume $\mathcal{F}$ is with Lipschitz constant $L_\mathcal{F}$ and $\mathcal{G}$ with $L_\mathcal{G}$. Then either (1) $Z \sim N(0, I_d)$, or (2) Z, X lie in $[0,1]^d$, we have*

$$\mathbb{E}\, d_W^2\left(\nu, \mu_{\widehat{\theta}_{m,n}}\right) \leq C_1\cdot A(\mathcal{F}, \mathcal{G}, \nu) + C_2\cdot S_{n,m}(\mathcal{F}, \mathcal{G})$$

where $C_1, C_2 > 0$ are some constants independent of $Pdim(\mathcal{F})$, $Pdim(\mathcal{F}\circ\mathcal{G})$ and m, n.

3.2 Generator/Discriminator Pair Regularization

One fundamental new feature about regularization in GAN is: both the generator and discriminator are choices of "tuning parameters", for users to specify. For a target density of interest, consider the following two thought experiments to explain the intricacies on the interplay between generator/discriminator pair.

1. For a fixed generator class $\mathcal{G}$, as the discriminator class $\mathcal{F}$ becomes more complex, it is easier for discriminator to tell apart good and bad generators (w.r.t. the target distribution) in the TV sense (on the population level). However, the stochastic error becomes worse as one is learning from a large model in GAN. This is also reflected in the upper bound obtained in Theorem 3, along the blue dashed arrow direction in Fig. 1.

2. For a fixed discriminator class $\mathcal{F}$, as the generator $\mathcal{G}$ becomes richer, it is capable of expressing densities that are closer to the target distribution. However, it also adds additional difficulty for two reasons. First, the generator may create densities that are far away from the target in

the TV sense, but not distinguishable by the discriminator (on the population level). Second, the stochastic error becomes worse as one is learning from a larger generator model. This is shown by the red dashed arrow direction in Fig. 1.

In general, tuning/regularization using the generator/discriminator pair is more subtle than the conventional bias-variance (or approximation-stochastic error) trade-off, illustrated by Fig. 1, with $A_1(\mathcal{F}, \mathcal{G}, \nu), A_2(\mathcal{G}, \nu)$ and $S_{n,m}(\mathcal{F}, \mathcal{G})$ defined in (3.4). Here, the tuning parameters lie in a two dimensional domain, rather than a one dimensional index. As $(\mathcal{G}, \mathcal{F})$ both become richer, $A_2(\mathcal{G}, \mu)$ decreases, $S_{n,m}(\mathcal{F}, \mathcal{G})$ increases, but $A_1(\mathcal{F}, \mathcal{G}, \nu)$ may increase, decrease or stay unchanged. On negative side, one can only eliminate some $(\mathcal{G}, \mathcal{F})$ pairs due to notions of dominance on the two dimensional domain. The simple 'U' shape picture for bias-variance trade-off no longer exists. On the positive side, by stepping into the two dimensional tuning domain, there are more choices for tuning pairs that potentially give rise to better rates, which we will showcase in Theorem 4.

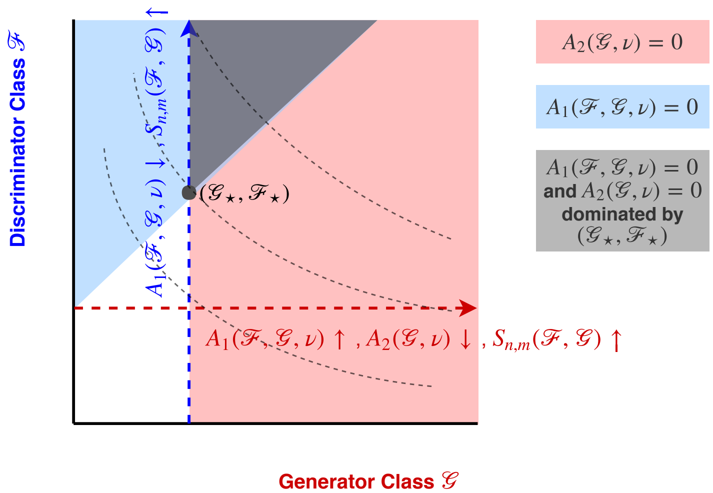

Figure 1: Diagram on how well GAN learns density in TV distance, when tuning with *generator $\mathcal{G}$ and discriminator $\mathcal{F}$ pair*. The diagram is illustrated based on an upper bound on TV distance, $A(\mathcal{F}, \mathcal{G}, \nu) + S_{n,m}(\mathcal{F}, \mathcal{G})$ in Theorem 3. The red shaded region corresponds to $A_2(\mathcal{G}, \nu) = 0$ and the blue shaded region is $A_1(\mathcal{F}, \mathcal{G}, \nu) = 0$. The grey dashed line corresponds to the indifference curve for the statistical error $S_{n,m}(\mathcal{F}, \mathcal{G})$. One can see that the choice $(\mathcal{G}_\star, \mathcal{F}_\star)$ dominates the grey shaded area, and the other choice on the same grey dashed line.

Let us discuss the approximation error term $A(\mathcal{F}, \mathcal{G}, \nu)$ as a step towards understanding the new notion of pair regularization for GAN, through choosing the generator/discriminator pair. The following corollary illustrates that the grey shaded area is dominated by the choice $(\mathcal{G}_\star, \mathcal{F}_\star)$ in Fig. 1.

Corollary 3 (Choice of generator/discriminator)**.** *Consider the target density class* $\log \nu \in \mathcal{D}_R$*, and the generator class* $\log \mu_\theta \in \mathcal{D}_G$*. With the discriminator chosen as*

$$\mathcal{F}_D = \mathcal{D}_R - \mathcal{D}_G := \{\log \nu - \log \mu_\theta \mid \textit{for all } \log \nu \in \mathcal{D}_R, \ \log \mu_\theta \in \mathcal{D}_G\},$$

then

$$A_1(\mathcal{F}, \mathcal{G}, \nu) = 0. \tag{3.5}$$

In addition, if the generator is well-specified in the sense $\mathcal{D}_G = \mathcal{D}_R$, then

$$A_2(\mathcal{G}, \nu) = 0. \tag{3.6}$$

And (3.5) and (3.6) altogether imply $\mathbb{E}\, d_{TV}^2\left(\nu, \mu_{\widehat{\theta}_{m,n}}\right) \precsim S_{n,m}(\mathcal{F}, \mathcal{G})$.

Remark 4 (Pair regularization and diagram)**.** Let us illustrate the above corollary using Fig. 1. Eqn. (3.5) corresponds to the blue shaded region in the diagram, Eqn. (3.6) represents the red shaded region, and the intersection is highlighted by the grey shaded region. At the intersection, the approximation error $A(\mathcal{F}, \mathcal{G}, \nu)$ is zero, so all pairs are dominated by the choice $(\mathcal{G}_\star, \mathcal{F}_\star)$ (as other pairs have a larger variance $S_{n,m}(\mathcal{F}, \mathcal{G})$). We would also like to argue that $(\mathcal{G}_\star, \mathcal{F}_\star)$ is the best solution along the indifference curve for $S_{n,m}(\mathcal{F}, \mathcal{G})$, denoted by the grey dashed line. To see this, moving $(\mathcal{G}_\star, \mathcal{F}_\star)$ towards the northwest direction on the indifference curve away from $(\mathcal{G}_\star, \mathcal{F}_\star)$, $A_1, S_{m,n}$ stay unchanged, but we have $A_2(\mathcal{G}_\star, \nu) \leq A_2(\mathcal{G}', \nu)$; moving $(\mathcal{G}', \mathcal{F}')$ towards the southeast direction, $A_2, S_{m,n}$ stay the same, but $A_1(\mathcal{G}_\star, \mathcal{F}_\star, \nu) \leq A_1(\mathcal{G}', \mathcal{F}', \nu)$. Similarly, one can argue that all pairs above the indifference curve is dominated by $(\mathcal{G}_\star, \mathcal{F}_\star)$.

We acknowledge that the diagram is illustrated using an upper bound on the TV distance, however, qualitatively, similar phenomenon extends to $\mathbb{E}\, d_{TV}^2\left(\nu, \mu_{\widehat{\theta}_{m,n}}\right)$ (see the first paragraph in Section 3.2). We defer the further discussion on the pair regularization versus classic regularization to Section 4.

3.3 Applications to Leaky ReLU Networks

In this section we consider two special cases of leaky ReLU generator and discriminator, to make explicit the rates for estimating parametric densities. We make use of the Theorem 3 and the notion of generator/discriminator pair regularization, to choose good pair $(\mathcal{G}_\star, \mathcal{F}_\star)$ for particular generative/adversarial networks problems to obtain parametric rates.

The *generator* $x = g_\theta(z) : \mathbb{R}^d \to \mathbb{R}^d$ is parametrized by a multi-layer perceptron (MLP):

$$\begin{aligned} h_0 &= z, \\ h_l &= \sigma_a(W_l h_{l-1} + b_l),\ 0 < l < L \\ x &= W_L h_{L-1} + b_L, \end{aligned}$$

where h_l denotes the output of hidden units, and x is the transformed final output of the MLP. Here the activation is leaky ReLU

$$\sigma_a(t) = \max\{t, at\},\ \text{for some fixed } 0 < a \leq 1. \tag{3.7}$$

Denote the parameter space for generator

$$\Theta(d, L) := \{(W_l \in \mathbb{R}^{d\times d}, b_l \in \mathbb{R}^d), 1 \leq l \leq L \mid \text{rank}(W_l) = d, \forall 1 \leq l \leq L\}.$$

One can verify, when the input distribution $Z \sim U([0,1]^d)$ is uniform, the class of densities realizable by $g_\theta(Z)$, for $\theta \in \Theta(d, L)$ has the following form (hierarchy of indicators),

$$\log \mu_\theta(x) = c_1 \sum_{l=1}^{L-1} \sum_{i=1}^{d} \mathbb{1}_{m_{li}(x) \geq 0} + c_0, \tag{3.8}$$

with some proper choice of c_1, c_0. Here $m_{li}(x)$ is the function computed by the i-th hidden unit in l-th layer in a MLP with depth L and d hidden units in each layer, with dual leaky ReLU activation (defined in next paragraph) and weights properly chosen as a function of θ. For details, see (5.6) and (5.8).

The *discriminator* $f_\omega(x) : \mathbb{R}^d \to \mathbb{R}$ is parametrized by a feedforward neural network with activation functions include dual leaky ReLU activation

$$\sigma_a^\star(t) := \min\{t, at\}, \text{ for } a \geq 1, \tag{3.9}$$

and threshold activation $\sigma_\infty^\star(t) := \mathbb{1}_{t \leq 0}$. The structural feature of a feedforward network is that hidden units are grouped in a sequence of L layers (the depth of the network), where a node is in layer $1 \leq l \leq L$, if it has a predecessor in layer $l-1$ and no predecessor in any layer $l' \geq l$. Computation of the final output unit proceeds layer-by-layer: at any layer $l < L$, each hidden unit u receives an input in the form of a linear combination $\widetilde{x}_u' w_u + b_u$, and then outputs $\sigma_a(\widetilde{x}_u' w_u + b_u)$, where the vector $\widetilde{x}_u$ collects the output of all the units with a directed edge into u (i.e., from prior layers).

Theorem 4 (Leaky-ReLU generator and discriminator, uniform as input)**.** *Consider a multi-layer perceptron generator $g_\theta : \mathbb{R}^d \to \mathbb{R}^d$, $\theta \in \Theta(d, L)$ with depth L and width d, using leaky ReLU $\sigma_a(\cdot)$ activation* (3.7)*. Consider the class of realizable densities, i.e., $X \sim \nu$ enjoys the same distribution as $g_{\theta_*}(Z)$ with a $\theta_* \in \Theta(d, L)$ and $Z \sim U([0,1]^d)$. Choose the discriminator $f_\omega : \mathbb{R}^d \to \mathbb{R}$ as a feedforward neural network with architecture shown in Fig. 2 with depth $L+2$, using dual leaky ReLU $\sigma_a^\star()$* (3.9) *and threshold activations.*

Then the GAN estimator $\mu_{\widehat{\theta}_{m,n}}$ defined in (3.2)*, satisfies the following parametric rates for total variation distance,*

$$\mathbb{E}\, d_{TV}^2\left(\nu, \mu_{\widehat{\theta}_{m,n}}\right) \precsim \sqrt{d^2 L^2 \log(dL) \left(\frac{\log m}{m} \vee \frac{\log n}{n}\right)}.$$

Remark 5 (Relations to literature)**.** The above theorem is a direct application of Theorem 3 and Corollary 3. Remark here we use the architecture design as pair regularization. Investigations on the parametric rates for GANs have been considered in Bai, Ma, and Risteski (2018), based on spectral norm-based capacity controls as regularization of networks, i.e. $\forall l \in [L], \|W_l\|_{\text{op}}, \|W_l^{-1}\|_{\text{op}} \leq C$. The approach they are taking is to establish multiplicative equivalence on $d_{\mathcal{F}}(\mu, \nu) \asymp d_W(\mu, \nu)$ for $\mu, \nu \in \mathcal{G}$.

In contrast, we use the oracle inequality approach developed in an early version of the current paper (Liang, 2017). We study through the angle of pseudo-dimensions, without requiring that the spectral radius of each W_l, W_l^{-1} is bounded. This has two advantages. First, the generator class can express a wider range of densities, as we only require that W_l has full rank. Second, we make explicit the dependence of the depth of the neural networks L in the rate. Remark that in our setting, we can allow for *very deep* network with $L \precsim \sqrt{n \wedge m / \log(n \wedge m)}$, with generator width being as small as the dimension d. In addition, we are able to obtain a better polynomial dependence on the dimension d, in the error.

Finally, as a sanity check, we show that GANs can also achieve the correct sample complexity (w.r.t. the dependence on dimension d) in estimating multivariate Gaussian.

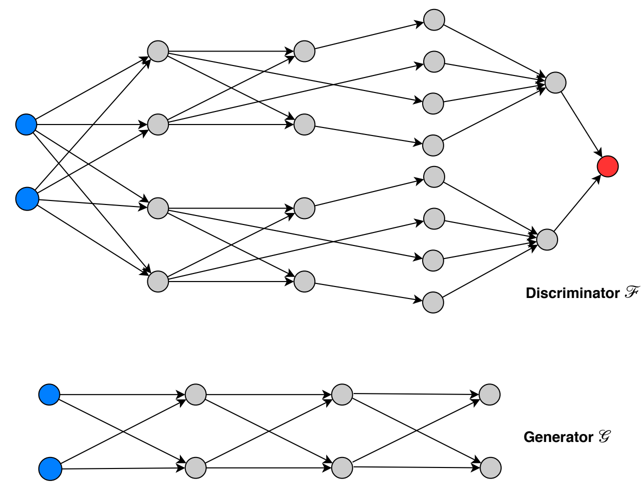

Figure 2: Illustration of discriminator $\mathcal{F}$ (feed-forward network) and generator $\mathcal{G}$ (multi-layer perceptron) in Theorem 4, for $L = 3$.

Corollary 4 (Multivariate Gaussian estimation, isotropic Gaussian as input)**.** *Consider $\nu \sim N(b_*, \Sigma_*)$ to be a multivariate Gaussian in $\mathbb{R}^d$. Consider a linear generator (neural network with no hidden layer) with input distribution $N(0, I_p)$ $(p \geq d)$, and the discriminator to be a one hidden layer neural network with quadratic activation $\sigma(t) = t^2$, the GAN estimator $\mu_{\widehat{\theta}_{m,n}}$ defined in (3.2), satisfies the following rates,*

$$\mathbb{E}\, d_{TV}^2 \left(\nu, \mu_{\widehat{\theta}_{m,n}}\right) \precsim \sqrt{\frac{d^2 \log d}{n} + \frac{(pd + d^2)\log(p+d)}{m}}.$$

4 Discussions

We would like to further discuss on the following question: even overlooking computation, what is the advantage of GANs compared to classic nonparametric density estimation, and the classic parametric models (where one reduces the problem of learning densities to parameter estimation). We would like to use the diagram as in Fig. 1 to point out some conclusions (based on Theorems 1-3) and conjectures.

- Classic parametric models: can be viewed as the left interval (along y-axis) in Fig. 3, where the generator class $\mathcal{G}$ is simple and limited. The discriminator can be viewed as assessing how well we are estimating the finite parameters, which relates to how well we are learning densities in the parametric class. More advanced discriminator won't help. The pair regularization effectively reduces to one dimensional tuning on the discriminator: what is a good loss function on the parameter set.

- Classic nonparametric density estimation: can be viewed as the top interval (along x-axis) in Fig. 3, here by tuning the generator class $\mathcal{G}$ (using sieves, kernels, etc.), one can achieve the optimal rates when the target density lies in a certain nonparametric class. The minimax theory for the adversarial framework (Theorem 1) informs us, when the target is truly nonparametric,

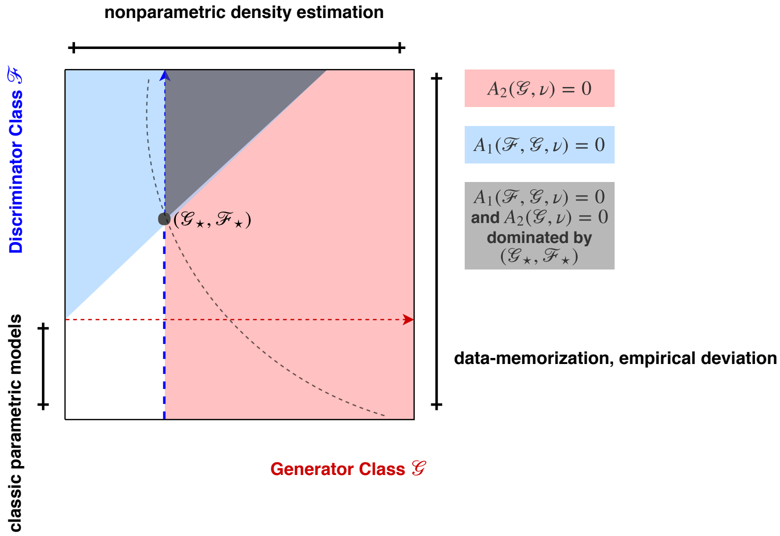

Figure 3: Diagram for generator/discriminator pair regularization.

tuning with the generator class is optimal: there is no theoretical gain in utilizing the generator/discriminator pair to tune.

- Empirical density, or data memorization: can be viewed as the right interval (along y-axis) in Fig. 3. Here the generator class is flexible enough to memorize the training data, and one should try to avoid this by means of regularization (Theorem 2).

- For a certain target density ν (in between parametric and nonparametric for many realistic cases), optimal tuning with the generator/discriminator pair $(\mathcal{G}_\star, \mathcal{F}_\star)$ as illustrated in Fig. 3 could potentially do better than both the parametric and nonparametric approaches. For the parametric approach, the approximation error $A_2(\mathcal{G}, \nu)$ is too large. For the nonparametric approach, as shown by the indifference curve, the stochastic error $S_{n,m}(\mathcal{F}, \mathcal{G})$ dominates. Therefore we *conjecture* that the full *tuning with the generator/discriminator pair* $(\mathcal{G}_\star, \mathcal{F}_\star)$ could potentially explain the empirical success of GANs, on the statistical side: as one has the choice of flexibly tuning the generator/discriminator pair with deep neural networks, in the two dimensional domain, balancing $A_1(\mathcal{F}, \mathcal{G}, \nu), A_2(\mathcal{G}, \nu), S_{n,m}(\mathcal{F}, \mathcal{G})$ simultaneously.

We remark again that tuning with the generator/discriminator pair is more general and flexible than the conventional nonparametric approach where one only tune the "generator" class, and the classic parametric approach where no tuning is needed (the "discriminator" is effectively specified by the parametric model). As illustrated by Fig. 3 in a non-rigorous way, the generator/discriminator pair $(\mathcal{G}_\star, \mathcal{F}_\star)$ dominates the grey shaded area and the one above the indifference curve (noted by the grey dashed line). And because of the flexibility, one could envision cases for ν when the optimal tuning pair dominates the parametric line and nonparametric line.

In this paper, we only consider the statistical problem of how well GANs learn density, assuming the optimization, say (3.2), can be done. Admittedly, computation of GANs is a considerably harder question (Mescheder, Nowozin, and Geiger, 2017; Daskalakis, Ilyas, Syrgkanis, and Zeng, 2017; Liang and Stokes, 2018; Arbel, Sutherland, Bińkowski, and Gretton, 2018; Lucic, Kurach, Michalski, Gelly, and Bousquet, 2017), which we leave as future work.

5 Proof of Main Results

We start with an equivalent definition of Sobolev space for $W^{\alpha,q}(r)$ for $q = 2$ is through the coefficients of the generalized Fourier series, which is also referred to as the Sobolev ellipsoid.

Definition 5 (Sobolev ellipsoid)**.** *Let* $\theta = \{\theta_\xi, \xi = (\xi_1, \dots, \xi_d) \in \mathbb{N}^d\}$ *collects the coefficients of the generalized Fourier series, define*

$$\Theta^\alpha(r) := \left\{ \theta \in \mathbb{N}^d \to \mathbb{R} : \sum_{\xi \in \mathbb{N}^d} (1 + \sum_{i=1}^d \xi_i^2)^\alpha \theta_\xi^2 \le r^2 \right\}.$$

It is clear that $\Theta^\alpha(r)$ (frequency domain) is an equivalent representation of $W^{\alpha,2}(r)$ (spatial domain) in $L^2(\mathbb{N}^d)$ for trigonometric Fourier series. For more details on Sobolev spaces, we refer to Nemirovski (2000); Tsybakov (2009); Nickl and Pötscher (2007).

Proof of Lemma 1. For any $\mu \in \mu_G$, we know that due to the optimality of GAN

$$d_{\mathcal{F}_D}(\mu, \nu_n) - d_{\mathcal{F}_D}(\mu_n, \nu_n) \ge 0.$$

$$\begin{aligned} d_{\mathcal{F}_D}(\mu_n, \nu) &\le d_{\mathcal{F}_D}(\mu_n, \nu_n) + d_{\mathcal{F}_D}(\nu_n, \nu) \\ &\le d_{\mathcal{F}_D}(\mu, \nu_n) + d_{\mathcal{F}_D}(\nu_n, \nu) \\ &\le d_{\mathcal{F}_D}(\mu, \nu) + d_{\mathcal{F}_D}(\nu, \nu_n) + d_{\mathcal{F}_D}(\nu_n, \nu). \end{aligned}$$

Now take $\mu = \arg\min_{\mu \in \mu_G} d_{\mathcal{F}_D}(\mu, \nu)$, and recall that $\mathcal{F}_D$ is symmetric around 0, we have

$$d_{\mathcal{F}_D}(\mu_n, \nu) \le \min_{\mu \in \mu_G} d_{\mathcal{F}_D}(\mu, \nu) + 2 d_{\mathcal{F}_D}(\nu, \nu_n).$$

□

Proof of Lemma 2. We abbreviate $\widehat{\theta}_{m,n}$ as $\widehat{\theta}$ in this proof when there is no confusion. Recall the definition of $d_{\mathcal{F}}\left(\mu_{\widehat{\theta}_{m,n}}, \nu\right)$, we have

$$\begin{aligned} & d_{\mathcal{F}}\left(\mu_{\widehat{\theta}_{m,n}}, \nu\right) = \sup_{f_\omega} \left\{ \mathbb{E}\, f_\omega \circ g_{\widehat{\theta}}(Z) - \mathbb{E}\, f_\omega(X) \right\} \\ & \le \sup_{f_\omega} \left\{ \mathbb{E}\, f_\omega \circ g_{\widehat{\theta}}(Z) - \widehat{\mathbb{E}}_n f_\omega(X) \right\} + \sup_{f_\omega} \left\{ \widehat{\mathbb{E}}_n f_\omega(X) - \mathbb{E}\, f_\omega(X) \right\} \\ & \le \sup_{f_\omega} \left\{ \widehat{\mathbb{E}}_m f_\omega \circ g_{\widehat{\theta}}(Z) - \widehat{\mathbb{E}}_n f_\omega(X) \right\} + \sup_{f_\omega} \left\{ \mathbb{E}\, f_\omega \circ g_{\widehat{\theta}}(Z) - \widehat{\mathbb{E}}_m f_\omega \circ g_{\widehat{\theta}}(Z) \right\} + \sup_{f_\omega} \left\{ \widehat{\mathbb{E}}_n f_\omega(X) - \mathbb{E}\, f_\omega(X) \right\}. \end{aligned}$$

For any θ such that $g_\theta \in \mathcal{G}$, we recall the optimality condition of GAN procedure

$$\sup_{f_\omega} \left\{ \widehat{\mathbb{E}}_m f_\omega \circ g_{\widehat{\theta}_{m,n}}(Z) - \widehat{\mathbb{E}}_n f_\omega(X) \right\} \le \sup_{f_\omega} \left\{ \widehat{\mathbb{E}}_m f_\omega \circ g_\theta(Z) - \widehat{\mathbb{E}}_n f_\omega(X) \right\},$$

then one can proceed with

$$
\begin{aligned}
& d_{\mathcal{F}}\left(\mu_{\widehat{\theta}_{m,n}}, \nu\right) \\
& \leq \sup_{f_\omega} \left\{ \widehat{\mathbb{E}}_m f_\omega \circ g_\theta(Z) - \widehat{\mathbb{E}}_n f_\omega(X) \right\} + \sup_{f_\omega} \left\{ \mathbb{E}\, f_\omega \circ g_{\widehat{\theta}}(Z) - \widehat{\mathbb{E}}_m f_\omega \circ g_{\widehat{\theta}}(Z) \right\} + \sup_{f_\omega} \left\{ \widehat{\mathbb{E}}_n f_\omega(X) - \mathbb{E}\, f_\omega(X) \right\} \\
& \leq \sup_{f_\omega} \left\{ \widehat{\mathbb{E}}_m f_\omega \circ g_\theta(Z) - \mathbb{E}\, f_\omega \circ g_\theta(Z) \right\} + \sup_{f_\omega} \left\{ \mathbb{E}\, f_\omega \circ g_\theta(Z) - \mathbb{E}\, f_\omega(X) \right\} + \sup_{f_\omega} \left\{ \mathbb{E}\, f_\omega(X) - \widehat{\mathbb{E}}_n f_\omega(X) \right\} \\
& \quad + \sup_{f_\omega} \left\{ \mathbb{E}[f_\omega \circ g_{\widehat{\theta}}(Z)] - \widehat{\mathbb{E}}_m [f_\omega \circ g_{\widehat{\theta}}(Z)] \right\} + \sup_{f_\omega} \left\{ \widehat{\mathbb{E}}_n [f_\omega(X) - \mathbb{E}\, f_\omega(X) \right\} \\
& \leq 2 \sup_{f_\omega} \left\{ \widehat{\mathbb{E}}_n f_\omega(X) - \mathbb{E}\, f_\omega(X) \right\} + \sup_{f_\omega} \left\{ \widehat{\mathbb{E}}_m f_\omega \circ g_\theta(Z) - \mathbb{E}\, f_\omega \circ g_\theta(Z) \right\} + \sup_{f_\omega} \left\{ \mathbb{E}\, f_\omega \circ g_{\widehat{\theta}}(Z) - \widehat{\mathbb{E}}_m f_\omega \circ g_{\widehat{\theta}}(Z) \right\} \\
& \quad + \sup_{f_\omega} \left\{ \mathbb{E}\, f_\omega \circ g_\theta(Z) - \mathbb{E}\, f_\omega(X) \right\}
\end{aligned}
$$

where the last step uses the fact that $f_\omega \in \mathcal{F}$ then $-f_\omega \in \mathcal{F}$. As the above holds for any θ such that $g_\theta \in \mathcal{G}$, we know then

$$
\begin{aligned}
d_{\mathcal{F}}\left(\mu_{\widehat{\theta}_{m,n}}, \nu\right) - d_{\mathcal{F}}(\mu_\theta, \nu) & \leq 2 d_{\mathcal{F}}\left(\widehat{\nu}^n, \nu\right) + d_{\mathcal{F}}\left(\widehat{\mu}_\theta^m, \mu_\theta\right) + \sup_{f_\omega} \left\{ \mathbb{E}\, f_\omega \circ g_{\widehat{\theta}}(Z) - \widehat{\mathbb{E}}_m f_\omega \circ g_{\widehat{\theta}}(Z) \right\} \\
& \leq 2 d_{\mathcal{F}}\left(\widehat{\nu}^n, \nu\right) + d_{\mathcal{F}}\left(\widehat{\mu}_\theta^m, \mu_\theta\right) + \sup_{f_\omega, g_\theta} \left\{ \mathbb{E}\, f_\omega \circ g_\theta(Z) - \widehat{\mathbb{E}}_m f_\omega \circ g_\theta(Z) \right\} \\
& \leq 2 d_{\mathcal{F}}\left(\widehat{\nu}^n, \nu\right) + d_{\mathcal{F}}\left(\widehat{\mu}_\theta^m, \mu_\theta\right) + d_{\mathcal{F} \circ \mathcal{G}}(\widehat{\pi}^m, \pi).
\end{aligned}
$$

□

Proof of Theorem 1. In this proof we consider the base measure $\pi(x)$ to be either a uniform measure on $[0,1]^d$ (bounded domain). Recall the density $\nu(x)$, we can represent the function in the Fourier/trigonometric series form

$$
\nu(x) = \sum_{\xi \in \mathbb{N}^d} \theta_\xi(\nu) \psi_\xi(x), \text{ where } \theta(\xi) \in \mathbb{N}^d \text{ denotes the coefficients of } \nu.
$$

with the basis $\psi_\xi(x) = \prod_{i=1}^d \psi_{\xi_i}(x_i)$. We construct the following smoothed/regularized version $\widetilde{\nu}_n$, with a cut-off parameter M to be determined later,

$$
\widetilde{\nu}_n(x) := \sum_{\xi \in \mathbb{N}^d} \widetilde{\theta}_\xi(\nu) \psi_\xi(x),
$$

where based on i.i.d. samples $X^{(1)}, X^{(2)}, \ldots X^{(n)} \sim \nu$

$$
\widetilde{\theta}_\xi(\nu) := \begin{cases} \frac{1}{n} \sum_{j=1}^n \prod_{i=1}^d \psi_{\xi_i}(X_i^{(j)}), & \text{for } \xi \text{ satisfies } \|\xi\|_\infty \leq M \\ 0, & \text{otherwise} \end{cases}.
$$

In other words, $\widetilde{\nu}_n$ filters out all the high frequency (less smooth) components, when the multi-index ξ has largest coordinate larger than M. Expand $f \in \mathcal{F}$ in Fourier series,

$$
f(x) = \sum_{\xi \in \mathbb{N}^d} \theta_\xi(f) \psi_\xi(x).
$$

For any $\nu(x) \in W^\alpha(r)$, we have

$$
\begin{aligned}
\mathbb{E}\, d_{\mathcal{F}}(\nu, \widetilde{\nu}_n) &= \mathbb{E} \sup_{f \in \mathcal{F}} \int f(x) \left(\nu(x) - \widetilde{\nu}_n(x)\right) dx \\
&= \mathbb{E} \sup_{f \in \mathcal{F}} \sum_{\xi \in \mathbb{N}^d} \theta_\xi(f) \left(\widetilde{\theta}_\xi(\nu) - \theta_\xi(\nu)\right) \\
&= \mathbb{E} \sup_{f \in \mathcal{F}} \left\{ \sum_{\xi \in [M]^d} \theta_\xi(f) \left(\widetilde{\theta}_\xi(\nu) - \theta_\xi(\nu)\right) + \sum_{\xi \in \mathbb{N}^d \backslash [M]^d} \theta_\xi(f) \theta_\xi(\nu) \right\} \\
&\leq \mathbb{E} \sup_{f \in \mathcal{F}} \sum_{\xi \in [M]^d} \theta_\xi(f) \left(\widetilde{\theta}_\xi(\nu) - \theta_\xi(\nu)\right) + \mathbb{E} \sup_{f \in \mathcal{F}} \sum_{\xi \in \mathbb{N}^d \backslash [M]^d} \theta_\xi(f) \theta_\xi(\nu).
\end{aligned}
$$

For the truncated first term, we know

$$
\begin{aligned}
&\mathbb{E} \sup_{f \in \mathcal{F}} \sum_{\xi \in [M]^d} \theta_\xi(f) \left(\widetilde{\theta}_\xi(\nu) - \theta_\xi(\nu)\right) \\
&\leq \mathbb{E} \sup_{f \in \mathcal{F}} \left\{ \sum_{\xi \in [M]^d} (1 + \|\xi\|_2^2)^\beta \theta_\xi^2(f) \right\}^{1/2} \left\{ \sum_{\xi \in [M]^d} (1 + \|\xi\|_2^2)^{-\beta} \left(\widetilde{\theta}_\xi(\nu) - \theta_\xi(\nu)\right)^2 \right\}^{1/2} \\
&\leq \mathbb{E} \left\{ \sum_{\xi \in [M]^d} (1 + \|\xi\|_2^2)^{-\beta} \left(\widetilde{\theta}_\xi(\nu) - \theta_\xi(\nu)\right)^2 \right\}^{1/2} \quad \text{as } \sup_{f \in \mathcal{F}} \sum_{\xi \in [M]^d} (1 + \|\xi\|_2^2)^\beta \theta_\xi^2(f) \leq 1 \qquad (5.1) \\
&\leq \left\{ \sum_{\xi \in [M]^d} (1 + \|\xi\|_2^2)^{-\beta}\, \mathbb{E} \left(\widetilde{\theta}_\xi(\nu) - \theta_\xi(\nu)\right)^2 \right\}^{1/2} \quad \text{Jensen's inequality} \qquad (5.2) \\
&\leq \sqrt{C_{d,\beta} \frac{M^{d-2\beta} \vee 1}{n}}
\end{aligned}
$$

where the last line $\mathbb{E} \left(\widetilde{\theta}_\xi(\nu) - \theta_\xi(\nu)\right)^2 \leq \frac{1}{n} \mathbb{E}_{X \sim \nu} \psi_\xi^2(X) \leq 1$ for trigonometric series $\forall \xi$. And $\sum_{\xi \in [M]^d} (1 + \|\xi\|_2^2)^{-\beta} \leq C_{d,\beta} \left(M^{d-2\beta} \vee 1\right)$.

For the second term, the following inequality holds

$$
\begin{aligned}
&\mathbb{E} \sup_{f \in \mathcal{F}} \sum_{\xi \in \mathbb{N}^d \backslash [M]^d} \theta_\xi(f) \theta_\xi(g) \leq \mathbb{E} \sup_{f \in \mathcal{F}} \left\{ \sum_{\xi \in [M]^d} \theta_\xi^2(f) \right\}^{1/2} \cdot \left\{ \sum_{\xi \in [M]^d} \theta_\xi^2(g) \right\}^{1/2} \\
&\leq \sup_{f \in \mathcal{F}} \left\{ (1 + M^2)^{-\beta} \sum_{\xi \in [M]^d} (1 + \|\xi\|_2^2)^\beta \theta_\xi^2(f) \right\}^{1/2} \left\{ (1 + M^2)^{-\alpha} \sum_{\xi \in [M]^d} (1 + \|\xi\|_2^2)^\alpha \theta_\xi^2(g) \right\}^{1/2} \\
&\leq r \sqrt{\frac{1}{M^{2(\alpha+\beta)}}}.
\end{aligned}
$$

Combining two terms, we have for any $\nu(x) \in \mathcal{G}$, with the optimal choice of $M \asymp n^{\frac{1}{2\alpha+d}}$

$$\sup_{\nu \in \mathcal{G}} \mathbb{E}\, d_{\mathcal{F}}(\nu, \widetilde{\nu}_n) \leq \inf_{M \in \mathbb{N}} \left\{ \sqrt{C \frac{M^{d-2\beta} \vee 1}{n}} + r\sqrt{\frac{1}{M^{2(\alpha+\beta)}}} \right\} \tag{5.3}$$
$$\precsim n^{-\frac{\alpha+\beta}{2\alpha+d}} \vee n^{-\frac{1}{2}}.$$

Let us now establish the lower bound. Again we consider only the $[0,1]^d$ as the domain. We need to make sure each hypothesis is indeed a valid density function in $W^{\alpha,\infty}(1)$ in $\mathbb{R}^d$. Choose a kernel function $K(u) = (a_1 \exp(-\frac{1}{1-4u^2}) - a_2) I(|u| < 1/2)$ for some small fixed $a_1, a_2 > 0$ to ensure that $K(x) \in W^{\alpha \vee \beta, \infty}(1)$, the Hölder class, and $\int K(u) du = 0$. Let m be a parameter (depends on the sample size n) to be determined later, and denote $h_m = 1/m$.

Define the hypothesis class to be

$$\Omega_\alpha = \left\{ g_w(x) = 1 + \sum_{\xi \in [m]^d} w_\xi h_m^\alpha \varphi_\xi(x), w \in \{0,1\}^{m^d} \right\},$$

$$\Lambda_\beta = \left\{ f_v(x) = \sum_{\xi \in [m]^d} v_\xi h_m^\beta \varphi_\xi(x), v \in \{-1,1\}^{m^d} \right\},$$

where

$$\varphi_\xi(x) = \prod_{i=1}^d K\left(\frac{x_i - \frac{\xi_i - 1/2}{m}}{h_m} \right), \quad \text{recall } h_m = 1/m.$$

Let us verify (1) $\Omega_\alpha \subset W^{\alpha,\infty}(r)$ for some r, and each element in the set is a valid density; (2) $\Lambda_\beta \subset W^{\beta,\infty}(1)$. For any multi-index γ,

$$\|D^{(\gamma)} g_w\|_\infty \leq \sup_{\xi \in [m]^d} h_m^\alpha \|D^{(\gamma)} \varphi_\xi\|_\infty = h_m^{\alpha-|\gamma|} \|D^{(\gamma)} K(u)\|_\infty \leq h_m^{\alpha-|\gamma|} \leq 1,$$
$$\|D^{(\gamma)} f_v(x)\|_\infty \leq h_m^{\beta-|\gamma|} \leq 1.$$

We also need to bound $\|g_w\|_\infty$, for any w

$$\|g_w\|_\infty \leq 1 + h_m^\alpha \sup_{\xi \in [m]^d} \|\varphi_\xi(x)\|_\infty \leq 1 + h_m^\alpha \leq 1 + 1/100,$$

as long as m is large enough. Therefore we know $\|(g_w(x) - g_0(x))/g_w(x)\|_\infty \leq \frac{1/100}{1-1/100} \leq 1/50$. Last, we can check g_ω is a proper density as

$$\int \varphi_\xi(x) dx = \prod_{i=1}^d \int K\left(\frac{x_i - \frac{\xi_i - 1/2}{m}}{h_m} \right) dx_i = 0.$$

The construction is due to Varshamov-Gilbert construction in conjunction with Fano's Lemma 9. To apply the Fano's Lemma, let's first construct multiple hypothesis that are separated in terms of the loss, then we will show that the hypothesis are close in statistical sense. Let's use the construction credited to Varshamov-Gilbert (Lemma 2.9 in Tsybakov (2009)): we know that there exists a subset $\{w^{(0)}, \ldots, w^{(H)}\} \subset \{0,1\}^h$ such that $w^{(0)} = (0, \ldots, 0)$,

$$\rho(w^{(j)}, w^{(k)}) \geq \frac{h}{8}, \ \forall\, j,k \in [H], \ j \neq k,$$
$$\log H \geq \frac{h}{8} \log 2,$$

where $\rho(w, w')$ denotes the Hamming distance between w and w' on the hypercube. In our case $h = m^d$. For the loss function, any $w, w' \in \{w^0, \ldots, w^H\}$

$$\begin{aligned}
d_{\mathcal{F}}(g_w, g_{w'}) &:= \sup_{f \in W^{\beta,\infty}(1)} \int f(x) g_w(x) dx - \int f(x) g_{w'}(x) dx \\
&\geq \sup_{f \in \Lambda_\beta} \int f(x) \left(g_w(x) - g_{w'}(x)\right) dx \\
&= \sup_{v} h_m^{\alpha+\beta} \sum_{\xi \in [m]^d} v_\xi (w_\xi - w'_\xi) \int \varphi_\xi^2(x) dx \\
&= h_m^{\alpha+\beta+d} \sum_{\xi \in [m]^d} I(w_\xi \neq w'_\xi) \int \prod_{i=1}^d K^2(u_i)\, du \\
&\geq c_{a_1,a_2} h_m^{\alpha+\beta+d} \rho(w, w') \geq c_{a_1,a_2} \frac{m^d}{8} h_m^{\alpha+\beta+d} \asymp h_m^{\alpha+\beta}.
\end{aligned}$$

Now let's show that based n i.i.d. data generated from density $g_w(x)$, it is hard to distinguish the hypothesis. Note that for $|t| < 1/50$, we know that $\log(1+t) \geq t - t^2$. Therefore

$$\begin{aligned}
d_{KL}\left(P_{w^{(j)}}^{\otimes n}, P_{w^{(0)}}^{\otimes n}\right) &= n \cdot d_{KL}\left(P_{w^{(j)}}, P_{w^{(0)}}\right) \\
&= n \int -\log\left(1 + \frac{g_0 - g_{w^{(j)}}}{g_{w^{(j)}}}\right) g_{w^{(j)}} dx \\
&\leq n \int \frac{(g_0 - g_{w^{(j)}})^2}{g_{w^{(j)}}} dx \leq 1.01 n \sum_{\xi \in [m]^d} \int h_m^{2\alpha} \varphi_\xi^2(x) dx \\
&\leq 1.01 n \sum_{\xi \in [m]^d} \int h_m^{2\alpha+d} \prod_{i=1}^d K^2(u_i)\, du \precsim n h_m^{2\alpha+d} m^d.
\end{aligned}$$

Therefore if we choose $m \asymp n^{-\frac{1}{2\alpha+d}}$, we know $\frac{1}{H}\sum_{j=1}^H D_{\mathrm{KL}}(P_{w^{(j)}}^{\otimes n}, P_{w^{(0)}}^{\otimes n}) \leq c \log H = c' m^d$. Using the Fano's Lemma, the lower bound for density estimation is of the order $h_m^{\alpha+\beta} = n^{-\frac{\alpha+\beta}{2\alpha+d}}$, as

$$\begin{aligned}
\inf_{\widetilde{\nu}_n} \sup_{\nu \in W^{\alpha,\infty}(r)} \mathbb{E}\, d_{\mathcal{F}}\left(\widetilde{\nu}_n, \nu\right) &= \inf_{\hat{g}} \sup_{g \in W^{\alpha,\infty}(r)} \mathbb{E} \sup_{f \in W^{\beta,\infty}(1)} \int f(x) \left(\hat{g}(x) - g(x)\right) dx \\
&\geq \inf_{\hat{w}} \sup_{w \in \{w^{(0)},\ldots,w^{(H)}\}} \mathbb{E}\, d_{\mathcal{F}}\left(g_{\hat{w}}, g_w\right) \\
&\geq c h_m^{\alpha+\beta} \cdot \inf_{\hat{w}} \sup_{w \in \{w^{(0)},\ldots,w^{(H)}\}} P_w \left(d_{\mathcal{F}}\left(g_{\hat{w}}, g_w\right) \geq c h_m^{\alpha+\beta}\right) \\
&\geq c h_m^{\alpha+\beta} \frac{\sqrt{H}}{1+\sqrt{H}} \left(1 - 2c' - \sqrt{\frac{2c'}{\log H}}\right) \\
&\geq c n^{-\frac{\alpha+\beta}{2\alpha+d}}.
\end{aligned}$$

The parametric rate lower bound $n^{-1/2}$ can be obtained by reducing to estimation of Gaussian. Proof completed. □

Proof of Corollary 1. The proof logic of this corollary follows similarly as in Theorem 1. Express $f \in \mathcal{F}$ under the eigenfunctions

$$f(x) = \sum_{i \in \mathbb{N}} f_i \psi_i(x), \text{ with } \sum_i t_i f_i^2 \leq 1$$

where $t_i \asymp i^{-\kappa}$ and $f_i = \int f(x)\psi_i(x)\pi(x)dx$ are the coefficients. Consider the target density class

$$\nu(x) = \sum_{i \in \mathbb{N}} \nu_i \psi_i(x), \text{ then}$$

$$\|\mathcal{T}_\pi^{-(\alpha-1)/2} \nu\|_{\mathcal{H}} \leq r \text{ is equivalent to } \sum_i t_i^\alpha \nu_i^2 \leq r^2.$$

Define

$$\widetilde{\nu}_n(x) := \sum_{i \in \mathbb{N}} \widetilde{\nu}_i \psi_i(x),$$

where based on i.i.d. samples $X^{(1)}, X^{(2)}, \ldots X^{(n)} \sim \nu$

$$\widetilde{\theta}_\xi(\nu) := \begin{cases} \frac{1}{n}\sum_{j=1}^n \psi_i(X^{(j)}), & \text{for } i \leq M \\ 0, & \text{otherwise} \end{cases}.$$

Follow the sample logic as in the Proof of Theorem 1, we have for any $\nu(x) \in \mathcal{G}$, with the optimal choice of $M \asymp n^{\frac{1}{\alpha\kappa+1}}$

$$\begin{aligned}
\mathbb{E}\, d_{\mathcal{F}}(\nu, \widetilde{\nu}_n) &\leq \mathbb{E} \sup_{f \in \mathcal{F}} \sum_{i \leq M} f_i \left(\widetilde{\nu}_i - \nu_i\right) + \mathbb{E} \sup_{f \in \mathcal{F}} \sum_{i > M} f_i \nu_i. \\
&\leq \sqrt{\sum_{i \leq M} t_i f_i^2} \sqrt{\sum_{i \leq M} t_i^{-1}\, \mathbb{E}(\widetilde{\nu}_i - \nu_i)^2} + t_M^{\frac{\alpha+1}{2}} \\
&\leq \inf_{M \in \mathbb{N}} \left\{ \sqrt{C \frac{M^{1-\kappa} \vee 1}{n}} + Cr \sqrt{\frac{1}{M^{\kappa(\alpha+1)}}} \right\} \\
&\precsim n^{-\frac{(\alpha+1)\kappa}{2\alpha\kappa+2}} \vee n^{-\frac{1}{2}}.
\end{aligned}$$

□

Proof of Theorem 2. If $\mathcal{F}_D$ consists of L-Lipschitz functions (Wasserstein GAN) on $\mathbb{R}^d$, $d \geq 2$, plug in the ℓ_∞-covering number bound for Lipschitz functions,

$$
\begin{aligned}
&\log \mathcal{N}(\epsilon, \mathcal{F}_D, \|\cdot\|_\infty) \leq C \left(\frac{L}{\epsilon}\right)^d, \\
&\mathbb{E}\, d_{\mathcal{F}_D}(\nu, \widehat{\nu}^n) \leq 2 \inf_{0<\delta<1/2} \left(4\delta + \frac{8\sqrt{2}}{\sqrt{n}} \int_\delta^{1/2} \sqrt{\log \mathcal{N}(\epsilon, \mathcal{F}_D, \|\cdot\|_\infty)} d\epsilon\right) \leq 16 \left(\frac{4\sqrt{2C}}{d-2}\right)^{\frac{2}{d}} L n^{-\frac{1}{d}}, \\
&\mathbb{E}\, d_{\mathcal{F}_D}(\nu, \widehat{\nu}^n) \leq \mathcal{O}\left(\left(\frac{C}{d^2 n}\right)^{-\frac{1}{d}}\right).
\end{aligned}
$$

This matches the best known bound as in Canas and Rosasco (2012) (Section 2.1.1).

Let's consider when $\mathcal{F}_D$ denotes Sobolev space $W^{\beta,2}$ on $\mathbb{R}^d$. Recall the entropy number estimate for $W^{\beta,2}$ (Nickl and Pötscher, 2007), we have

$$
\begin{aligned}
&\log \mathcal{N}(\epsilon, \mathcal{F}_D, \|\cdot\|_\infty) \leq C \left(\frac{1}{\epsilon}\right)^{\frac{d}{\beta} \vee 2}, \\
&\mathbb{E}\, d_{\mathcal{F}_D}(\nu, \widehat{\nu}^n) \leq \mathcal{O}\left(n^{-\frac{\beta}{d}} + \frac{\log n}{\sqrt{n}}\right).
\end{aligned}
$$

Remark in addition that the parametric rate $\frac{1}{\sqrt{n}}$ is inevitable, which can be easily seen from the Sudakov minoration,

$$
\mathbb{E}_\epsilon \sup_{f \in \mathcal{F}_D} \frac{1}{n} \sum_{i=1}^n \epsilon_i f(X_i) \geq \sup_\epsilon \frac{\epsilon}{2} \sqrt{\frac{\log \mathcal{M}(\epsilon, \mathcal{F}_D, \|\cdot\|_n)}{n}} \geq \frac{1}{\sqrt{n}}.
$$

For the smoothed/regularized density, one can apply Lemma 1 and Theorem 1 to obtain the claimed result. □

Proof of Theorem 3. For any distribution $g_{\widehat{\theta}_{m,n}}(Z)$ (we abbreviate $\widehat{\theta}_{m,n}$ as $\widehat{\theta}$ in this proof), by Pinsker's inequality (Lemma 8),

$$
d_{TV}^2 \left(X, g_{\widehat{\theta}}(Z)\right) \leq \frac{1}{2} d_{KL} \left(X || g_\theta(Z)\right)
$$

which implies that for any $X \sim \nu$

$$
\begin{aligned}
4d_{TV}^2\left(X, g_{\widehat{\theta}}(Z)\right) &\leq d_{KL}\left(X || g_{\widehat{\theta}}(Z)\right) + d_{KL}\left(g_{\widehat{\theta}}(Z) || X\right) \\
&= \int \log \frac{\nu(x)}{\mu_{\widehat{\theta}}(x)} \left(\nu(x) - \mu_{\widehat{\theta}}(x)\right) dx \qquad \text{for any } f_\omega \in \mathcal{F} \\
&= \int \left(\log \frac{\nu(x)}{\mu_{\widehat{\theta}}(x)} - f_\omega(x)\right) \left(\nu(x) - \mu_{\widehat{\theta}}(x)\right) dx + \int f_\omega(x) \left(\nu(x) - \mu_{\widehat{\theta}}(x)\right) dx \\
&= \int \left(\log \frac{\nu(x)}{\mu_{\widehat{\theta}}(x)} - f_\omega(x)\right) \left(\nu(x) - \mu_{\widehat{\theta}}(x)\right) dx + d_{\mathcal{F}}\left(\mu_{\widehat{\theta}}, \nu\right) \\
&= \left\| \log \frac{\nu(x)}{\mu_{\widehat{\theta}}(x)} - f_\omega(x) \right\|_\infty \left\| \nu(x) - \mu_{\widehat{\theta}}(x) \right\|_1 + d_{\mathcal{F}}\left(\mu_{\widehat{\theta}}, \nu\right) \\
&\leq 2 \left\| \log \frac{\nu}{\mu_{\widehat{\theta}}} - f_\omega \right\|_\infty + d_{\mathcal{F}}\left(\mu_{\widehat{\theta}}, \nu\right)
\end{aligned}
$$

where the last line is due to the fact that $\mu_{\widehat{\theta}}, \nu(x)$ are both proper densities, so $\left\|\nu(x) - \mu_{\widehat{\theta}}(x)\right\|_1 \leq 2$.

Due to the oracle inequality Lemma 2, one has for any θ

$$
\begin{aligned}
d_{\mathcal{F}}\left(\mu_{\widehat{\theta}}, \nu\right) &\leq d_{\mathcal{F}}(\mu_\theta, \nu) + 2d_{\mathcal{F}}\left(\widehat{\nu}^n, \nu\right) + d_{\mathcal{F}}\left(\widehat{\mu}_\theta^m, \mu_\theta\right) + d_{\mathcal{F}\circ\mathcal{G}}(\widehat{\pi}^m, \pi) \\
&\leq B d_{TV}(\mu, \nu) + 2d_{\mathcal{F}}\left(\widehat{\nu}^n, \nu\right) + d_{\mathcal{F}}\left(\widehat{\mu}_\theta^m, \mu_\theta\right) + d_{\mathcal{F}\circ\mathcal{G}}(\widehat{\pi}^m, \pi) \\
&\leq B\sqrt{\frac{1}{4}\left[d_{KL}\left(\mu_\theta || \nu\right) + d_{KL}\left(\nu || \mu_\theta\right)\right]} \\
&\quad + 2d_{\mathcal{F}}\left(\widehat{\nu}^n, \nu\right) + d_{\mathcal{F}}\left(\widehat{\mu}_\theta^m, \mu_\theta\right) + d_{\mathcal{F}\circ\mathcal{G}}(\widehat{\pi}^m, \pi) \\
&\leq B\sqrt{\frac{1}{2}\left\|\log \frac{\mu_\theta}{\nu}\right\|_\infty} + 2d_{\mathcal{F}}\left(\widehat{\nu}^n, \nu\right) + d_{\mathcal{F}}\left(\widehat{\mu}_\theta^m, \mu_\theta\right) + d_{\mathcal{F}\circ\mathcal{G}}(\widehat{\pi}^m, \pi)
\end{aligned}
$$

where second line uses the fact that for any $f \in \mathcal{F}$, $\|f\|_\infty \leq B$.

Assemble the bounds, we have for any θ, ω

$$
\begin{aligned}
4d_{TV}^2\left(\nu, \mu_{\widehat{\theta}_{m,n}}\right) \leq 2\left\|\log \frac{\nu}{\mu_{\widehat{\theta}}} - f_\omega\right\|_\infty &+ B\sqrt{\frac{1}{2}\left\|\log \frac{\mu_\theta}{\nu}\right\|_\infty} \\
&+ 2d_{\mathcal{F}}\left(\widehat{\nu}^n, \nu\right) + d_{\mathcal{F}}\left(\widehat{\mu}_\theta^m, \mu_\theta\right) + d_{\mathcal{F}\circ\mathcal{G}}(\widehat{\pi}^m, \pi)
\end{aligned}
$$

Therefore by choosing $\theta_\star$ minimizes $\left\|\log \frac{\mu_\theta}{\nu}\right\|_\infty$

$$
\begin{aligned}
\mathbb{E}\, d_{TV}^2\left(\nu, \mu_{\widehat{\theta}_{m,n}}\right) &\leq \frac{1}{2}\,\mathbb{E}\left\{\inf_\omega \left\|\log \frac{\nu}{\mu_{\widehat{\theta}}} - f_\omega\right\|_\infty\right\} + \frac{B}{4\sqrt{2}}\sqrt{\inf_\theta \left\|\log \frac{\mu_\theta}{\nu}\right\|_\infty} \\
&\quad + \mathbb{E}\left\{2d_{\mathcal{F}}\left(\widehat{\nu}^n, \nu\right) + d_{\mathcal{F}}\left(\widehat{\mu}_{\theta_\star}^m, \mu_{\theta_\star}\right) + d_{\mathcal{F}\circ\mathcal{G}}(\widehat{\pi}^m, \pi)\right\} \\
&\leq \frac{1}{2}\sup_\theta \inf_\omega \left\|\log \frac{\nu}{\mu_\theta} - f_\omega\right\|_\infty + \frac{B}{4\sqrt{2}} \inf_\theta \left\|\log \frac{\mu_\theta}{\nu}\right\|_\infty^{1/2} \\
&\quad + \mathbb{E}\left\{2d_{\mathcal{F}}\left(\widehat{\nu}^n, \nu\right) + d_{\mathcal{F}}\left(\widehat{\mu}_{\theta_\star}^m, \mu_{\theta_\star}\right) + d_{\mathcal{F}\circ\mathcal{G}}(\widehat{\pi}^m, \pi)\right\}.
\end{aligned}
$$

Recall the symmetrization in Lemma 3,

$$\begin{aligned}
&\mathbb{E}\left\{2d_{\mathcal{F}}\left(\widehat{\nu}^n,\nu\right)+d_{\mathcal{F}}\left(\widehat{\mu}^m_{\theta_\star},\mu_{\theta_\star}\right)+d_{\mathcal{F}\circ\mathcal{G}}(\widehat{\pi}^m,\pi)\right\}\\
&\leq 4\,\mathbb{E}\,\mathcal{R}_n\left(\mathcal{F}\right)+2\,\mathbb{E}\,\mathcal{R}_m\left(\mathcal{F}\right)+2\,\mathbb{E}\,\mathcal{R}_m\left(\mathcal{F}\circ\mathcal{G}\right)\\
&\leq C\sqrt{\mathrm{Pdim}(\mathcal{F})\left(\frac{\log m}{m}\vee\frac{\log n}{n}\right)}+C\sqrt{\mathrm{Pdim}(\mathcal{F}\circ\mathcal{G})\frac{\log m}{m}},
\end{aligned}$$

where the last step uses the relationship between Rademacher complexity and pseudo-dimension, as in Lemma 6. $\square$

Proof of Corollary 2. Now let's consider Wasserstein distance. Consider in addition the Lipschitz constants of $\mathcal{F}$ to be $L_{\mathcal{F}}$, and $\mathcal{G}$ to be $L_{\mathcal{G}}$, namely

$$\begin{aligned}
|f_\omega(x)-f_\omega(x')|&\leq L_{\mathcal{F}}\|x-x'\|\\
\|g_\theta(z)-g_\theta(z')\|&\leq L_{\mathcal{G}}\|z-z'\|
\end{aligned}$$

Consider first the case when $Z\sim N(0,I_d)$ (unbounded). Then for any $f\in Lip(1)$, we know

$$f(g_\theta(z))\in Lip(L_{\mathcal{G}}). \tag{5.4}$$

In other words, $f\circ g_\theta(Z)$ are $L_{\mathcal{G}}^2$ sub-Gaussian (Lemma 8), therefore

$$d_W^2\left(\nu,\mu_{\widehat{\theta}}\right)\leq 2L_{\mathcal{G}}^2\cdot d_{KL}\left(\nu||\mu_{\widehat{\theta}}\right)$$

and

$$\begin{aligned}
d_{\mathcal{F}}\left(\nu,\mu_\theta\right)&\leq L_{\mathcal{F}}\cdot d_W\left(\nu,\mu_\theta\right)\\
&\leq\sqrt{2}L_{\mathcal{F}}L_{\mathcal{G}}\sqrt{d_{KL}\left(\nu||\mu_\theta\right)}.
\end{aligned}$$

Follow the analysis with as in the TV distance, we have

$$\begin{aligned}
\mathbb{E}\,d_W^2\left(\nu,\mu_{\widehat{\theta}}\right)\leq{}& L_{\mathcal{G}}^2\sup_\theta\inf_\omega\left\|\log\frac{\nu}{\mu_\theta}-f_\omega\right\|_\infty+L_{\mathcal{G}}^3L_{\mathcal{F}}\inf_\theta\left\|\log\frac{\mu_\theta}{\nu}\right\|_\infty^{1/2}\\
&+C\sqrt{\mathrm{Pdim}(\mathcal{F})\left(\frac{\log m}{m}\vee\frac{\log n}{n}\right)}+C\sqrt{\mathrm{Pdim}(\mathcal{F}\circ\mathcal{G})\frac{\log m}{m}}.
\end{aligned}$$

Consider then the case when $z,x\in[0,1]^d$ is bounded, we know

$$\|g_\theta(z)-g_\theta(z')\|\leq L_{\mathcal{G}}\sqrt{d} \tag{5.5}$$

Therefore $\|g_\theta(z)\|\leq M+L_{\mathcal{G}}\sqrt{d}$, and the support of $g_\theta(Z)$ and X lies in $R:=M+(L_{\mathcal{G}}+1)\sqrt{d}$. Hence

$$\mathbb{E}\,d_W^2\left(\nu,\mu_{\widehat{\theta}}\right)\leq R^2\,\mathbb{E}\,d_{TV}^2\left(\nu,\mu_{\widehat{\theta}}\right).$$

The last line is because for any $f(x)$ that has Lipchitz constant 1 with $f(0)=0$ (as $\sup_{f:f\in Lip(1)}\int f(\mu-\nu)dx=\sup_{f-f(0):f\in Lip(1)}\int(f-f(0))(\mu-\nu)dx$, for probability measures μ,ν), it must be true that $f(x)$ is bounded in a bounded domain with radius R.

$\square$

Proof of Theorem 4. Consider the generator network to be realized by a multi-layer perceptron.

$$\begin{aligned} h_1 &= \sigma(W_1 z + b_1) \\ &\dots \\ h_l &= \sigma(W_l h_{l-1} + b_l) \\ &\dots \\ x &= W_L h_{L-1} + b_L \end{aligned}$$

Denote

$$\Theta := \{(W_l \in \mathbb{R}^{d\times d}, b_l \in \mathbb{R}^d), 1 \le l \le L \mid \mathrm{rank}(W_l) = d, \forall 1 \le l \le L\}$$

Consider the density evolution from layer l to layer $l-1$ (change of measure)

$$\begin{aligned} \log \mu_l(h_l) &= \log \mu_{l-1}(h_{l-1}) + \log\det\left(\frac{\partial h_{l-1}}{\partial h_l}\right) \\ &= \log \mu_{l-1}(h_{l-1}) - \log\det W_l - \sum_{i=1}^{d} \log \sigma'(\sigma^{-1}(h_l(i))). \end{aligned}$$

Therefore if one recursively apply the above equality (change of measure) to track the density of X

$$\begin{aligned} \log \mu_\theta(x) &= \log \mu_{L-1}(h_{L-1}) - \log\det W_L, \quad \text{where } h_{L-1} = W_L^{-1}(x - b_L) \\ &= \log \mu_{L-2}(h_{L-2}) - \sum_{j=L-1}^{L} \log\det W_j - \sum_{i=1}^{d} \log \sigma'(\sigma^{-1}(h_{L-1}(i))), \\ \dots &\qquad \text{where } h_{L-2} = W_{L-1}^{-1}(\sigma^{-1}(h_{L-1}) - b_{L-1}) \\ &= \log \mu(z) - \sum_{j=1}^{L} \log\det W_j - \sum_{j=1}^{L-1}\sum_{i=1}^{d} \log \sigma'(\sigma^{-1}(h_j(i))), \quad \text{where } z = W_1^{-1}(\sigma^{-1}(h_1) - b_1) \end{aligned}$$

Now consider $\mu(z) = 1$ to be the uniform measure on $z \in [0,1]^d$. Consider leaky ReLU activation $\sigma(t) = \max(t, at)$ for $0 < t \le 1$, then $\sigma^{-1}(t) = \min(t, t/a)$, and $\log \sigma'(t) = \log(a) \cdot 1_{t \le 0}$.

Let's consider the realizable case when $\log \nu(x) = \log \mu_{\theta_*}(x)$ for some $\theta_* \in \Theta$

Denote $m_l := \sigma^{-1}(h_{L-l})$, for any $1 \le l \le L-1$. Then it follows that

$$m_1 = \sigma^{-1}(W_L^{-1}x - W_L^{-1}b_L) \tag{5.6}$$

$$m_l = \sigma^{-1}(W_{L-l+1}^{-1} m_{l-1} - W_{L-l+1}^{-1} b_{L-l+1}) \tag{5.7}$$

$$\log \mu_\theta(x) = -\sum_{j=1}^{L} \log\det W_j - \sum_{j=1}^{L-1}\sum_{i=1}^{d} \log \sigma'(m_{L-j}(i)) \tag{5.8}$$

$$= -\sum_{j=1}^{L} \log\det W_j - \sum_{j=1}^{L-1}\sum_{i=1}^{d} \log \sigma'(m_j(i)) \tag{5.9}$$

Now consider a discriminator network which follows

$$\begin{aligned}
m_1 &= \sigma^{-1}(V_1 x + c_1) \\
&\cdots \\
m_{L-1} &= \sigma^{-1}(V_{L-1} m_{L-2} + c_{L-1}) \\
h_\omega(x) &= \sum_{j=1}^{L-1} \sum_{i=1}^{d} \log(1/a) 1_{m_j(i) \le 0} + c_L.
\end{aligned}$$

Here the parameter set is,

$$\omega \in \Omega := \{(V_l \in \mathbb{R}^{d\times d}, c_l \in \mathbb{R}^d), c_L \in \mathbb{R}, 1 \le l \le L-1 \mid \mathrm{rank}(V_l) = d, \forall 1 \le l \le L-1\}.$$

Choose the discriminator function $w = (w_1, w_2)$ where $w_1, w_2 \in \Omega$

$$f_\omega(x) = h_{\omega_1}(x) - h_{\omega_2}(x).$$

Then we can verify that Corollary 3 follows. Recall the upper bound in Theorem 4, we can see that for the choice of generator and discriminator

$$\begin{aligned}
\frac{1}{2} \sup_\theta \inf_\omega \left\| \log \frac{\nu}{\mu_\theta} - f_\omega \right\|_\infty &= 0 \\
\frac{B}{4\sqrt{2}} \inf_\theta \left\| \log \frac{\mu_\theta}{\nu} \right\|_\infty^{1/2} &= 0
\end{aligned}$$

as $\log v(x)$ can be realized by $\log \mu_{\theta_*}(x)$, and that for any $\theta \in \Theta$, there exist an ω such that

$$f_\omega(x) = \log \nu(x) - \log \mu_\theta(x).$$

Recall Bartlett, Harvey, Liaw, and Mehrabian (2017)'s result on Vapnik-Chervonenkis dimension of feed-forward neural networks (Lemma 7 with degree at most 1 and number of pieces $p+1=2$), we know for neural networks $\mathcal{F}$ and $\mathcal{F}\circ\mathcal{G}$ respectively

$$\begin{aligned}
\text{for } \mathcal{F}: &\quad \text{weights } W_{\mathcal{F}} \le 2(d^2L + 2dL) + 2, \text{ units } U_{\mathcal{F}} \le 4dL, \text{ depth } L_{\mathcal{F}} \le L+2 \ ; \\
\text{for } \mathcal{F}\circ\mathcal{G}: &\quad \text{weights } W_{\mathcal{F}\circ\mathcal{G}} \le W_{\mathcal{F}} + d^2L, \text{ units } U_{\mathcal{F}\circ\mathcal{G}} \le U_{\mathcal{F}} + dL, \text{ depth } L_{\mathcal{F}\circ\mathcal{G}} \le L_{\mathcal{F}} + L \ .
\end{aligned}$$

Therefore, we have the following upper bound on VC-dimension,

$$\begin{aligned}
\mathrm{VCdim}(\mathcal{F}) &\le C \cdot L_{\mathcal{F}} W_{\mathcal{F}} \log U_{\mathcal{F}} = C d^2 L^2 \log(dL) \\
\mathrm{VCdim}(\mathcal{F}\circ\mathcal{G}) &\le C \cdot L_{\mathcal{F}\circ\mathcal{G}} W_{\mathcal{F}\circ\mathcal{G}} \log U_{\mathcal{F}\circ\mathcal{G}} \le C' d^2 L^2 \log(dL).
\end{aligned}$$

□

Proof of Corollary 4. Suppose $\log \nu(x) = -\frac{1}{2}(x-b_*)'\Sigma_*^{-1}(x-b_*) + \frac{1}{2}\log\det(\Sigma_*^{-1}) - \frac{d}{2}\log(2\pi)$. And the generator class is depth-one NN, with weights $\theta = (W, b)$, $X = WZ + b$, then $\log \mu_\theta(x) = -\frac{1}{2}(x-b)'(WW')^{-1}(x-b) + \frac{1}{2}\log\det((WW')^{-1}) - \frac{d}{2}\log(2\pi)$.

For the discriminator, if one is allowed to use $\sigma(t) = t^2$, then one can have $O(d)$ units in discriminator network with depth 2, so that the two approximation error term is zero (Note one can also realize with ReLU activation in a bounded domain, using saw construction, as in Yarotsky (2017)). By Lemma 7 with degree at most 2, $\mathrm{VCdim}(\mathcal{F}) \precsim d^2 \log d$, $\mathrm{VCdim}(\mathcal{F}\circ\mathcal{G}) \precsim (pd + d^2)\log(p+d)$. Therefore $\mathbb{E}\, d_{TV}^2\left(g_\theta(Z), X\right) \le C\left(\frac{d^2 \log d}{n \wedge m} + \frac{(pd+d^2)\log(p+d)}{m}\right)^{1/2}$. □

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A Supporting Lemmas

Let's define the empirical Rademacher complexity

$$\mathcal{R}_n(\mathcal{F}) := \mathbb{E}_{\epsilon} \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \epsilon_i f(X_i). \tag{A.1}$$

Lemma 3 (Symmetrization and entropy integral)**.** *For $\widehat{\nu}^n(x) = \frac{1}{n}\sum_{i=1}^n \delta_{X_i}(x)$, then*

$$\mathbb{E}\, d_{\mathcal{F}}(\nu, \widehat{\nu}^n) \leq 2\, \mathbb{E}\, \mathcal{R}_n(\mathcal{F}). \tag{A.2}$$

Assuming $\sup_{f\in\mathcal{F}} \|f\|_\infty \leq 1$, one has the standard entropy integral bound,

$$\mathbb{E}\, d_{\mathcal{F}}(\nu, \widehat{\nu}^n) \leq 2\, \mathbb{E} \inf_{0<\delta<1/2} \left(4\delta + \frac{8\sqrt{2}}{\sqrt{n}} \int_{\delta}^{1/2} \sqrt{\log \mathcal{N}(\epsilon, \mathcal{F}, \|\cdot\|_n)} d\epsilon \right),$$

where $\|f\|_n := \sqrt{1/n \sum_{i=1}^n f(X_i)^2}$ is the empirical ℓ_2-metric on data $\{X_i\}_{i=1}^n$. Furthermore, because $\|f\|_n \leq \max_i |f(X_i)|$, and therefore $\mathcal{N}(\epsilon, \mathcal{F}, \|\cdot\|_n) \leq \mathcal{N}(\epsilon, \mathcal{F}|_{X_1 \ldots, X_n}, \infty)$ and so the upper bound in the conclusions also holds with $\mathcal{N}(\epsilon, \mathcal{F}|_{X_1 \ldots, X_n}, \infty)$.

The next two results, Theorems 12.2 and 14.1 in Anthony and Bartlett (2009), show that the metric entropy may be bounded in terms of the pseudo-dimension and that the latter is bounded by the Vapnik-Chervonenkis (VC) dimension.

Lemma 4. *Assume for all $f \in \mathcal{F}$, $\|f\|_\infty \leq M$. Denote the pseudo-dimension of $\mathcal{F}$ as $\mathrm{Pdim}(\mathcal{F})$, then for $n \geq \mathrm{Pdim}(\mathcal{F})$, we have for any ϵ and any $X_1, \ldots, X_n$,*

$$\mathcal{N}(\epsilon, \mathcal{F}|_{X_1,\ldots,X_n}, \infty) \leq \left(\frac{2eM \cdot n}{\epsilon \cdot \mathrm{Pdim}(\mathcal{F})} \right)^{\mathrm{Pdim}(\mathcal{F})}.$$

Lemma 5. *If $\mathcal{F}$ is the class of functions generated by a neural network with a fixed architecture and fixed activation functions, then*

$$\mathrm{Pdim}(\mathcal{F}) \leq \mathrm{VCdim}(\tilde{\mathcal{F}})$$

where $\tilde{\mathcal{F}}$ has only one extra input unit and one extra computation unit compared to $\mathcal{F}$.

Lemma 6 (Rademacher complexity and Pseudo-dimension)**.** *Under the condition $\max_i |f(X_i)| \leq B$, then for any $n \geq \mathrm{Pdim}(\mathcal{F})$,*

$$\mathcal{R}_n(\mathcal{F}) \leq C \cdot B \sqrt{\frac{\mathrm{Pdim}(\mathcal{F}) \log n}{n}}$$

for some universal constant $C > 0$.

Proof. The proof is a direct application of the Dudley entropy integral in Lemma 3 and the covering number bound by pseudo-dimension in Lemma 4. See A.2.2 in Farrell, Liang, and Misra (2018) for details. □

Lemma 7 (Theorem 6 in Bartlett et al. (2017), Vapnik-Chervonenkis dimension)**.** *Consider function class computed by a feed-forward neural network architecture with W parameters and U computation units arranged in L layer. Suppose that all non-output units have piecewise-polynomial activation functions with $p+1$ pieces and degree no more than d, and the output unit has the identity function as its activation function. Then the VC-dimension and pseudo-dimension is upper bounded*

$$\mathrm{VCdim}(\mathcal{F}) \leq C \cdot \Big(LW \log(pU) + L^2 W \log d \Big),$$

with some universal constants $C > 0$. The same result holds for pseudo-dimension $\mathrm{Pdim}(\mathcal{F})$.

Lemma 8 (van Handel (2014), special case of Theorem 4.8 and Example 4.9)**.** *For any two random variables $g_\theta(Z), X \in \mathbb{R}^d$, Pinsker's inequality asserts that*

$$2 d_{TV}^2 \left(g_\theta(Z), X \right) \leq d_{KL} \left(X || g_\theta(Z) \right).$$

Assume in addition that $Z \sim N(0, I_d)$ to be isotropic Gaussian and for all θ, $\|g_\theta(z) - g_\theta(z')\| \leq L\|z - z'\|$ is L-Lipschitz. Then for any X we know

$$d_W^2 \left(g_\theta(Z), X \right) \leq 2L^2 d_{KL} \left(X || g_\theta(Z) \right).$$

Proof. Consider any 1-Lipchitz function $f : \mathbb{R}^d \to \mathbb{R}$, then $f \circ g_\theta$ is L-Lipschitz, which implies $f \circ g_\theta$ is L^2-subGaussian due to Gaussian concentration Theorem 3.25 in van Handel (2014). Therefore we know $f(g_\theta(Z))$ is L^2-subGaussian for any f that is 1-Lipchitz, together with Theorem 4.8 in van Handel (2014), the proof completes. □

Lemma 9 (Theorem 2.5 in Tsybakov (2009))**.** *Assume that $H \geq 2$ and suppose Θ contains $\theta_0, \theta_1, \ldots, \theta_H$ such that:*

1. *$d(\theta_j, \theta_k) \geq 2s > 0$, for all $j, k \in [H]$ and $j \neq k$.*
2. *$\frac{1}{H} \sum_{j=1}^{H} D_{\mathrm{KL}}(P_j, P_0) \leq c \log H$ with $0 < c < 1/8$ and $P_j = P_{\theta_j}$ for $j \in [H]$.*

Then for any estimator $\hat{\theta}$,

$$\sup_{\theta \in \Theta} P_\theta(d(\hat{\theta}, \theta) \geq s) \geq \frac{\sqrt{H}}{1 + \sqrt{H}} \left(1 - 2c - \sqrt{\frac{2c}{\log H}} \right) > 0.$$