On Hospice Operations under Medicare Reimbursement Policies

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Abstract

This paper analyzes the United States Medicare hospice reimbursement policy. The existing policy consists of a daily payment for each patient under care with a global cap of revenues accrued during the Medicare year, which increases for each newly admitted patient. We investigate the hospice’s expected profit and provide reasons for a spate of recent provider bankruptcies related to the reimbursement policy; recommendations to alleviate these problems are given. We also analyze a hospice’s incentives for seeking out new patients, finding several unintended consequences of the Medicare reimbursement policy. Specifically, a hospice may seek short-lived patients (such as cancer sufferers) over longer-living patients particularly towards the end of the Medicare year. Also, the optimal recruitment rates for all patient types will vary during the year. These phenomena are unintended and undesirable but are a direct consequence of the Medicare reimbursement policy. We propose an alternative reimbursement policy which overcomes these shortcomings.
1 Introduction and Problem Motivation

Hospices are health care providers that cater to patients in the end phases of their lives. Patients who are admitted to hospices undertake palliative care in lieu of further curative care. Since 1983 Medicare has reimbursed hospices for the care provided to eligible patients. This reimbursement consists of a daily payment for care but the American federal government’s exposure is limited by an annual *cap* that depends on the number of patients admitted in a year (the full reimbursement policy is described in §2.1). The Medicare hospice benefit is generally regarded as a success since it improves the quality of life while saving Medicare (compared with a patient continuing curative care) an average of $2309 per hospice user (Taylor et al., 2007). However, some recent issues have arisen to question the efficacy of this reimbursement policy. Specifically an increasing number of providers have entered bankruptcy, and the blame for this is attributed to the structure of the reimbursement policy (Sack, 2007). We investigate this claim analytically, study hospice provider incentives under the current policy, and explore alternative Medicare policy possibilities.

There have been numerous instances of hospices ending the Medicare year with negative caps, which need to be repaid to the Federal government (MedPAC, 2006). The consensus appears to be that the primary cause of this is patients living longer than expected. MedPAC (2006) indicates these longer than expected stays are not uniform across the country, although no ready explanation for this is apparent. However, a repayment request from the government can be devastating to a small hospice provider. Hometown Hospice, a small provider in Alabama, has struggled repaying the excess payments, seeking bank credit to do so and dreading a pending bill: “If they hit us with a number in the several hundred-thousand range, I just don’t see how we can survive” says a Hometown owner (Sack, 2007). The total US cap overpayment in 2008 was $208 million with an average of $612,000 amongst the hospices exceeding their caps.

A hospice provider faces uncertainty concerning the patients’ lifespan after admission (known as the Length of Stay, LOS). It is recognized that these uncertain LOSs differ by patient disease (Christakis et al., 1996), although the Medicare reimbursement policy (described in §2.1) is independent of disease-type. In our models we restrict attention to two disease types, although the results are robust to this characterization. Patients are classified as type 1 or type 2, characterized by the former suffering shorter mean LOS than the latter while the hospice incurs a lower marginal cost for the latter. In a broad sense, one could think of type 1 patients as those suffering from cancer and type 2 as non-cancer, although this categorization is far from perfect as there are several
noncancerous diseases with shorter LOS than several cancer diseases (e.g., chronic kidney disease has a mean LOS of 28-30 days, CMMS, 2009). For the remainder of the paper, we refer to two patient types.

We consider elements of the industry and market that may affect hospice profitability, including patient census, patient disease mix, and LOS uncertainty. The detrimental nature of poor mix realization, lack of scale, and uncertainty are well recognized in the operations management literature. For example, Eppen (1979) recognized the value of pooling inventory in the context of warehouses. Such lessons are instructive for analyzing hospice operations and we create a model to do so. In particular, we formulate a model for hospice profit and use it to examine the potential causes for hospices receiving payments exceeding the cap and the reasons behind potential bankruptcies.

However, the payment scheme elapsing over a finite horizon raises further issues beyond the profitability of the provider. The complex accounting involved may be leading to some undesirable traits in the rate of hospice admissions, such as patient recruiting rates which differ across diseases and change during the year. If this is the case, it is likely contrary to equity objectives of the social planner. Indeed, it is helpful to speculate what the “optimization problem” might look like if the United States Government was planning and operating the hospices. The government (i.e., the social planner) desires that any eligible person who wishes to be admitted to a hospice, can be readily admitted, regardless of disease type. However, there is also a prevailing budgetary constraint (which in reality would be the portion of the Medicare budget allocated to the Hospice Benefit) under which they operate. Thus, the social planner’s optimal plan should maximize the common rate of recruiting across disease types by using the entire budget, which will be consumed with recruiting search costs and daily costs of caring for the patients. A simple mathematical program could capture this logic but we do not pursue this approach, instead our focus is on the rational behavior of decentralized hospice operators.

There is indeed evidence to suggest that some untoward recruiting practices occur in the hospice industry, some of which are compelled by the reimbursement policy’s cap. The Medicare Payment Advisory Commission advises the Secretary of Health and Human Services to direct the Office of the Inspector General to investigate the financial relationships between “hospices and long-term care facilities such as nursing facilities and assisted living facilities that may represent a conflict of interest and influence admissions to hospice,” and “the appropriateness of hospice marketing materials and other admissions practices and potential correlations between length of stay and deficiencies in marketing or admissions practices” (MedPAC, 2010). For example, Richard R.
Slager, chairman and chief executive of VistaCare, a large hospice provider in the western US, says his company aims their marketing towards cancer patients: “In communities where we have had cap issues, we have to really look hard for shorter-length-of-stay patients to offset it. It’s a never-ending nightmare” (Sack, 2007). This paper investigates the hospice manager’s optimal recruitment problem and finds that, indeed, the manager has an incentive to purposefully seek cancer and other short-lived patients particularly towards the end of the Medicare year (when the hospice’s cap might be exhausted). We propose some remedies to address these shortcomings.

Since the creation of the Medicare hospice benefit in 1983, there has been much research on many aspects of hospices but, to the best of our knowledge, none have directly addressed the issues focused on here. In fact, much of the literature does not address the reimbursement policy at all. An exception is Fraser (1985) who describes the ethical and policy implications of Medicare’s hospice reimbursement policy but does not identify the issues of provider bankruptcy or non-stationary and disease-dependent recruitment, the foci of our study. GAO (2004) investigates whether modifications to the reimbursement policy are warranted but limit their focus to the comparison of the per diem rates and the costs of care. Huskamp et al. (2001) consider how the Medicare rules affect care. A variety of papers consider factors related to hospice access such as race (Gordon, 1995; Han, Remsburg, and Iwashyna, 2006; Ngo-Metzger et al., 2003), the elderly (Virnig et al., 2004), insurance (McCarthy et al., 2003), complex care (Lorenz et al., 2004), geography (Connor et al., 2007), and particular diseases such as dementia (Schonwetter et al., 2003; Luchins et al., 1997; McCarty and Volicer, 2009; Hanrahan et al., 1999), lung disease (Abrahm and Hansen-Fraschen, 2002), kidney disease (Murray et al., 2006), or cancer (Moinpour and Polissar, 1989; McCarthy et al., 2003). Still others consider the operating costs (More and Kidder, 1985; Huskamp et al., 2008; Killaly et al., 2007; Taylor, 2009) or the economics of end of life (Buntin and Huskamp, 2002). There is consistent evidence in the literature that the hospice benefit reduces Medicare costs (e.g., Campbell et al., 2004) while enhancing end of life care (Pyensen et al., 2004; Taylor et al., 2007). Other literature describes the demographics of hospice beneficiaries (Christakis and Escarce, 1996; Banaszak-Holl and Mor, 1996; Miller and Mor, 2001).

The remainder of the paper is structured as follows. In §2 the Medicare hospice reimbursement policy is described (§2.1), the hospice profitability model is formulated and analyzed (§2.2), the profitability model is studied numerically (§2.3), and remedies to the policy’s shortcomings are considered (§2.4). In §3 we formulate and analyze a dynamic recruitment model and §4 presents a policy remedy to overcome the unintended consequences of the current policy. Concluding remarks
are provided in §5. Proofs are relegated to an appendix throughout.

2 Model of the Current Policy

This section presents an annualized model of the existing policy. In particular, §2.1 precisely describes the Medicare hospice reimbursement policy operating in the United States and gives some data on trends under this policy. In §2.2 we develop an expected profit model for a hospice subject to a patient mix of two diseases exhibiting different stochasticity, which is numerically exercised in §2.3 where the vulnerabilities of the reimbursement policy are exposed. A mechanism to alleviate these vulnerabilities is given in §2.4.

2.1 Medicare’s Existing Reimbursement Policy

This subsection describes the current Medicare reimbursement policy. The Medicare year runs from November 1 through October 31 of the following year. To be admitted to a hospice, a patient needs the signature of two physicians (typically, one will be the patients primary attending physician and the other will be employed by the hospice) that the patient is not expected to live more than six months from admission\(^1\) and that the patient agrees to forgo any curative care and undertake palliative care only.

During the Medicare year, a hospice will receive a payment for a patient under hospice care for a part or entire day. This payment will differ according to whether the patient is receiving routine home care ($142.91 per day), continuous home care ($834.10 per day), in-patient respite care ($147.83 per day), or general inpatient care ($635.74 per day).\(^2\) These payment rates can differ slightly by region in the United States depending on estimates of the cost of operation in these regions but do not depend upon the disease afflicting the patient. In 2002 and 2003, 93% of reimbursed hospice days were paid at the routine home care rate, 4.1% were continuous home care days, 2.7% were inpatient respite care days, and 0.2% were general inpatient care days (MedPAC, 2006). We focus primarily on routine home care in our models, which are the large majority of reimbursed days.

\(^1\)If the patient lives beyond the initial six month period, the patient can be recertified for two sequential 90 day periods, following by an unlimited number of sequential 60 day periods. Until 1990, there was a limit of 210 days over which a hospice could receive payments for a patient.

The second part of the Medicare hospice reimbursement policy is a payment cap, applied to the entire hospice (i.e., the cap is not patient specific),\textsuperscript{3} intended to limit the government’s exposure. At the beginning of the Medicare year, this cap is zero but it increases by $23,014.50\textsuperscript{4} for every newly admitted patient and decreases for every daily payment the hospice receives. Clearly, this cap can fluctuate up (upon the admission of a new patient) and down (upon receiving a daily payment from Medicare). The daily payment rates and cap increase quantities are adjusted from year to year, but unlike the daily rates, the cap does not vary by geography. At the end of the Medicare year, if the hospice’s cap is negative, the provider has received payments greater than the number of patients admitted multiplied by $23,014.50. This excess amount must be repaid to Medicare. If the cap is positive, the hospice did not receive as many payments as they were entitled to, and the cap is reset to zero in the new Medicare year. Anecdotally, we have heard that hospice managers are loath to “leave money on the table” (ending the Medicare year with a positive cap).

There is an increasing trend of hospices receiving payments that exceed the cap and having to repay this excess. For example, MedPAC (2010) reports the percentage of all hospices exceeding the cap rose from 2.6\% in 2002 to 10.4\% in 2007 and points to two explanatory factors. Firstly, that there has been an increase in the proportion of longer stay patients and secondly, that there has been an increase in the length of stay (LOS) of the longer stay patients. When Medicare instituted the hospice benefit, it was envisioned it would be primarily for patients suffering from cancer or similarly short LOS diseases (MedPAC, 2006). In 1998 forty-seven percent of all hospice users had noncancer diagnoses, whereas this has risen to 69 percent in 2008 (MedPAC, 2010). Over the same time period, the number of users with “debility” increased from 8,500 to 107,000, the number with Alzheimer’s diseases or non Alzheimer’s dementia grew from 28,000 to 174,000. Part of this increase is simply an overall increase in the usage of Medicare hospice benefit, where 22.9\% of Medicare decedents used hospice in 2000 increasing to 40.1\% in 2008 (MedPAC, 2010), but these diseases typically have longer LOS than cancer diagnoses.

Medicare reports hospice spending nearly quadrupled from 2000 to 2008 with hospice and home health accounting for 6\% of Medicare expenditures in 2008 ($11.2 billion). Regarding the increasing LOS, MedPAC (2010) reports the median LOS remained steady at 17 days between 2000 and 2008 whereas the 90th percentile grew from 141 days to 235 days. In brief, the short stays kept at a

\textsuperscript{3}Medicare applies a second cap, which is rarely enacted. It limits the proportion of inpatient care days to 20\% of all reimbursed days. This is to encourage hospice care in the patient’s home to be the primary method of delivery. Any days exceeding the 20\% level will be reimbursed at the routine home care rate.

\textsuperscript{4}Again for the 2009 year.
similar LOS but the long stays grew longer. There appear to be no definitive explanations for these extended LOS, although MedPAC is sufficiently concerned that they recommended Congress direct the Secretary of Health and Human Services require a medical review of all stays exceeding 180 days in hospices where such stays make up 40% or more of all cases (MedPAC, 2010). However, there is a suggestion that the LOS of longer living patients is less predictable (MedPAC, 2010; Schonwetter et al., 2003), and a hospice medical director is quoted “Doing this for 40-something years, every time I think somebody is going to die tomorrow, damned if they don’t live for a year and a half,” reflecting the challenge in determining the appropriate time of hospice admission (Sack, 2007).

It is reported that the states with the highest share of hospices exceeding the cap (Mississippi, Alabama, Arizona, and Oklahoma) have a higher LOS for all disease types, compared with other states. MedPAC (2010) reports that hospices in Alabama and Mississippi have a markedly higher LOS than other states, across all diseases. For example, MedPAC (2006) states the mean LOS in South Dakota is 41 days whereas it is 122 days in Mississippi. MedPAC (2010) states the growth in the number of providers ranged from 62 to 160 percent between 2001 and 2008 for the four states with the highest share of hospices reaching the cap in 2007 (Mississippi, Alabama, Arizona, and Oklahoma). This suggests a dramatic increase in the number of providers, outstripping the growth in the number of Medicare decedents (average annual percentage point change of Medicare decedents using hospice from 2000 to 2007 is 2.3%). MedPAC (2010) describes hospices’ access to capital is generally favorable, primarily due to them not being as capital intensive as other providers, because their physical infrastructure is not extensive. Although a direct comparison of these numbers is difficult, it might suggest an increase in the number of smaller hospice providers (MedPAC suggests there is no state-by-state relationship between the supply of hospices and rate of hospice usage). As might be expected, and as will be seen in our models, sufficient patient volume is an important factor in hospice profitability.

It might be constructive to examine some common characteristics of hospices that exceed the cap. MedPAC (2010) reports the following attributes. They tend to: be for-profit, freestanding facilities\(^5\); have a smaller patient census; treat a larger share of patients with Alzheimer’s disease and other neurological conditions; exhibit significantly longer LOS, even when patient mix is taken into account; and have a proportion of patients with stays exceeding 180 days (for particular diseases) substantially higher than those hospices below the cap.

Concerning hospice costs, MedPAC (2010) finds the average provider costs per day can vary

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\(^5\)Freestanding facilities are those not operated by a hospital, home health agency, or skilled nursing facility.
by hospice type, finding that for-profit based hospices are less costly than non-profit hospices; rural hospices are less costly than urban hospices; and, curiously, hospices exceeding the payment cap are less costly than those below the cap. It is also found that the daily costs are higher at admission and discharge than regular care, so providers with longer average LOS have lower daily costs, which may explain why hospices exceeding the payment cap are less costly. Killaly et al. (2007) find the providers’ marginal and average costs are higher for cancer patients than non-cancer patients. MedPAC also examines hospice margins and generally all hospice types are profitable except hospital based providers (which consistently have negative margins from 2001 to 2007, presumably the result of inflated overhead allocation) and providers with the lowest patient volumes. Indeed, hospice margins increase with patient volume in every year under study (MedPAC, 2010). MedPAC (2010) projects the aggregate margin to be 4.6% across all hospices in 2010, whereas it was 5.9% in 2007. Next we consider a model which considers the profitability of hospices given patient arrival rates, cap, LOS, etc.

2.2 The Hospice’s Profit Model

This subsection models a hospice’s expected annual profit function. We first express the general form for profit given the actual Medicare reimbursement policy and then simplify the model of demand to make the model tractable. We also examine a model where the cap is applied per patient rather than across patients and discuss the implications of such a policy change. As mentioned earlier, the focus is on “routine home care” service offered by hospices, which is the reimbursed service for over 90% of reimbursed hospice days.

The year is considered as a whole; specific dynamics caused by year-end effects will be discussed in Section 3. In particular, all patients are assumed to arrive, be cared for, and be reimbursed for in the same fiscal year. While in practice there will be patients that live from one Medicare year to the next, we feel this model captures the key effects regarding hospice profitability at a high level without losing tractability or insights.

Let $N_1$ be the number of type 1 (short LOS) patients admitted in a year and $N_2$ be the number of type 2 patients admitted. Assume that $X_{1i}$ is the remaining life of type 1 patient $i$, and $X_{2j}$ is the remaining life of type 2 patient $j$, $i = 1, \ldots, N_1$ and $j = 1, \ldots, N_2$. Further, assume type 1 and type 2 patients cost $c_1$ and $c_2$ per day to treat respectively, and let $A$ be the fixed cost of operating the hospice. Let $r$ be the daily pre-cap reimbursement rate for patients and $K$ the cap adjustment per patient. Then
Profit Rate / year = $r \left( \sum_{i=1}^{N_1} X_i^1 + \sum_{i=1}^{N_2} X_i^2 \right) \wedge K(N_1 + N_2) - c_1 \sum_{i=1}^{N_1} X_i^1 - c_2 \sum_{i=1}^{N_2} X_i^2 - A$

As discussed above, this ignores beginning and end-of-horizon effects. We will refer to this model as our static model and relaxation of this assumption will occur with the dynamic model of Section 3.

While uncapped revenue might be modeled as a compound Poisson process for example, we instead consider the simpler case where $N_1$ and $N_2$ are deterministic constants: $N_1 = \lambda_1$ and $N_2 = \lambda_2$. Assume $X_i^1 \sim N(m_1, \sigma_1^2)$ and $X_i^2 \sim N(m_2, \sigma_2^2)$ are independent and identically distributed (i.i.d.), and independent of each other. We are not aware of any research on the true distribution of LOS but an assumption of independence appears reasonable on the surface. Then,

uncapped revenues = $r \left( \sum_{i=1}^{\lambda_1} X_i^1 + \sum_{i=1}^{\lambda_2} X_i^2 \right) \sim rN(m, \sigma^2)$,

where $m = (\lambda_1 m_1 + \lambda_2 m_2)$ and $\sigma^2 = (\lambda_1 \sigma_1^2 + \lambda_2 \sigma_2^2)$. Note that, if volume is large, then $rN(m, \sigma^2)$ may form a reasonable approximation for uncapped revenues (by the central limit theorem) even if individually a normal approximation is not a good model for LOS and arrivals are stochastic. Let $\phi(\cdot)$ and $\Phi(\cdot)$ denote the pdf and cdf of the standard normal distribution, respectively.

Whether a hospice provider becomes bankrupt could occur regardless of whether they exceed the cap or not, and we argue is a more pressing concern than exceeding the cap. Consider three hospices. Firstly, consider a hospice which does not exceed the cap, but has marginal costs so high that they cannot make a sufficient net contribution to cover the fixed overhead\(^6\); this hospice will make a loss. Secondly, consider a hospice which has substantial margins but has payments exceeding the cap. With patients who live substantially longer than expected, and thus incur the continuing cost of care but receive payments which will be repaid, this hospice will potentially make a loss, too. MedPAC (2010) suggests that rural providers and providers catering to longer LOS patients tend to have healthier than average margins, two circumstances which apply to Hometown Hospice from the Introduction. A third hospice could be one with a patient population with very short LOS, thus inhibiting the provider from gaining as many revenues as they are entitled to and not enough to cover their overhead. Thus, there are several avenues for a hospice to incur a loss. The following proposition provides an explicit expression for determining whether expected profit is positive.

\(^6\)The Medicare payment is intended to “encompass not only the cost of visits but also other costs a hospice incurs related to on-call services, care planning, drugs, medical equipment, and supplies related to the patient’s terminal condition, patient transportation between hospice care sites, and other less frequently used services” (MedPAC, 2010).
Proposition 1 Under deterministic patient numbers and i.i.d. normal patient lifetimes, the expected annual (static) profit is given by

\[ \pi = \lambda \left\{ K - (K - r\bar{m}) \Phi \left( \frac{K/r - \bar{m}}{\sigma/\sqrt{\lambda}} \right) - \frac{r\bar{\sigma}}{\sqrt{\lambda}} \phi \left( \frac{K/r - \bar{m}}{\sigma/\sqrt{\lambda}} \right) - \bar{c} - A \right\} \]  

(1)

where \( \lambda = \lambda_1 + \lambda_2 \), \( \bar{m} = \frac{\lambda_1}{\lambda} m_1 + \frac{\lambda_2}{\lambda} m_2 \), \( \bar{\sigma} = \sqrt{\frac{\lambda_1}{\lambda} \sigma_1^2 + \frac{\lambda_2}{\lambda} \sigma_2^2} \) and \( \bar{c} = \frac{\lambda_1}{\lambda} (c_1 m_1) + \frac{\lambda_2}{\lambda} (c_2 m_2) \). Static profit \( \pi \) is concave increasing in \( K \), concave in \( r \), decreasing in \( \sigma_1 \) and \( \sigma_2 \), and linearly decreasing in \( c_1 \), \( c_2 \), and \( A \). Further, the per patient contribution margin

\[ K - (K - r\bar{m}) \Phi \left( \frac{K/r - \bar{m}}{\sigma/\sqrt{\lambda}} \right) - \frac{r\bar{\sigma}}{\sqrt{\lambda}} \phi \left( \frac{K/r - \bar{m}}{\sigma/\sqrt{\lambda}} \right) - \bar{c} \]  

(2)

is increasing in \( \lambda \), as is total profit \( \pi \) whenever (2) is positive. Finally, if \( \sigma_1 = \sigma_2 = 0 \) then

\[ \pi = (\lambda_1 + \lambda_2) [(r\bar{m}) \wedge K - \bar{c}] - A, \]

which is the limit of (2) as \( \bar{\sigma} \to 0 \).

Some comments are in order on these results. Firstly, high costs or low revenues obviously affect profits adversely. Secondly, so long as the per patient margin is positive then large volumes are better for the provider. Without variability \( (r\bar{m}) \wedge K > \bar{c} \) is necessary for per patient margins to be positive, but with variability this is not sufficient, a large enough volume of patients is also required (due to risk pooling effects). Finally, the effects of patient mix and LOS are also important in determining profitability and will be studied numerically in §2.3, where it will be seen that profit is not monotone in either. In fact, correct patient mix is critical to profitability and this will be studied under the assumption that mix is exogenous in the next subsection and as an endogenous decision variable in §3. We now study the effects of the cap.

Corollary 1 If \( K = rm \), then

\[ \pi = \lambda(K - r\bar{\sigma}\phi(0)/\sqrt{\lambda} - \bar{c}) - A. \]

Further, as \( K \to \infty \), \( \pi \to \lambda(r\bar{m} - \bar{c}) - A. \)

Corollary 1 show that if the cap is set equal to expected per patient revenue, then the effects of pooling can be seen yet more clearly. However, when the cap is large, variability in patient LOS becomes irrelevant. It is only in the presence of a restrictive cap that high LOS variability can negatively impact a provider’s average profit.

One may ask the question, why set the cap across patients, why not provide a per patient cap? Indeed, this was partially in effect prior to 1990 when there was a 210 day limit on the number of reimbursement days per patient. The following proposition examines the provider’s profit in the presence of a per patient cap.
Proposition 2. Under deterministic patient numbers and iid patient lifetimes, if the cap is per
patient, then the expected annual (static) profit is given by

\[ \lambda_1 \left\{ K - (K - rm_1)\Phi \left( \frac{K/r - m_1}{\sigma_1} \right) - r\sigma_1\phi \left( \frac{K/r - m_1}{\sigma_1} \right) \right\} + \]

\[ \lambda_2 \left\{ K - (K - rm_2)\Phi \left( \frac{K/r - m_2}{\sigma_2} \right) - r\sigma_2\phi \left( \frac{K/r - m_2}{\sigma_2} \right) \right\} - \lambda \bar{c} - A. \]

Even though Section 3 shows that the annualized cap provides incentives for undesirable behavior,
this model highlights the need for a pooled cap. In particular, notice how the expected per patient
margin (contained within the braces in the expression above) compares with the margin in the
pooled cap model in equation (2). If the cap was simply per patient then the presence of variability
in LOS could be extremely detrimental to the hospice because this variability is no longer moderated
by \( \sqrt{\lambda} \) in the denominator. This is yet another example of the beneficial effects of pooling.

2.3 Numerical Exercises

This section numerically exercises the stochastic profit model from the previous subsection to illus-
trate some behavior of the reimbursement scheme. We first consider two specific hospice examples
and then examine more general comparative statics of hospice profitability.

We start with a specific hospice that exceeded the cap. Consider Hometown Hospice in Wilcox
County, Alabama (Sack, 2007), founded in 2003. It is a small, for-profit provider serving the
rural poor, mostly in their homes. After its first two years of operation, Hometown was required
to repay $900,000 which represented 27% of revenues. Of Hometown’s 56 patients on October
31, 2007, seventeen had been under care for at least six months but two for more than 500 days
(Sack, 2007). Accurate estimation of a prospective admittee’s LOS is frequently difficult and such
estimation appears to be even more challenging for noncancer diagnoses (MedPAC, 2010). For
example, Fox et al. (1999) describe how some seriously ill hospitalized patients (advanced chronic
obstructive pulmonary disease, congestive heart failure, or end-stage liver disease) are denied the
hospice benefit since the “clinical prediction criteria are not effective in identifying a population
with a survival prognosis of 6 months or less.” Grady (2010) describes the reluctance of doctors,
patients, and the patients’ families to discontinue curative treatment, which further contributes to
LOS estimation issues.

Note that there is a distinction between a provider receiving payments exceeding the cap and
a hospice that goes bankrupt. The former is simply a comparison of the payments received from
Medicare and the prevailing cap reflecting the number of patients admitted during the year. The
latter will account for the daily costs and the fixed overhead. In the absence of variability, whether the hospice exceeds the cap is a relatively straightforward calculation of simply summing the expected payments for each patient-day \( r(\lambda_1 m_1 + \lambda_2 m_2) \) compared with the cap associated with admissions \( (K(\lambda_1 + \lambda_2)) \). Thus, whether a provider exceeds the cap is clearly driven by the weighted LOS of all admitted patients. In the case of Hometown Hospice with a greater than average proportion of higher LOS patients and with those patients exhibiting longer LOS, it is clear they are in jeopardy of exceeding the cap, which indeed has occurred in reality.

Now consider the following numerical examples. Recall that \( K \) and \( r \) are parameters of the reimbursement policy whereas the other model parameters are specific to the hospice. The hospice specific parameters are drawn from Sack (2007) and MedPAC (2010), as much as possible. For example, taking the reported 58 patients as a steady state census at Hometown Hospice (Sack, 2007) and initially estimating their average LOS as 200 days, we used Little’s Law to estimate that the total patient arrivals \( \lambda_1 + \lambda_2 = 58/200 = 0.29 \) patients per day. The article also implies many of the patients under care at Hometown would be type 2 patients which we assumed to be 90% of all patients. We have taken the daily costs of caring, \( c_1 \) and \( c_2 \), as the daily marginal costs of care for cancer and non-cancer patients, respectively, from Killaly et al. (2007). The values of \( K \) and \( r \) are obtained for the 2007 fiscal year where \( r = \$130.79 \) is the 2007 per diem rate for routine home care.

**Example 1.** Consider a rural hospice in Alabama, inspired by Sack (2007). It is a provider characterized by a small patient census, a large proportion of patients with noncancerous diseases, and with long mean LOS. Let \( K = \$21,410 \), \( r = \$130.79 \) per day, \( m_1 = 100 \) days, \( m_2 = 300 \) days, \( \sigma_1 = 50 \) days, \( \sigma_2 = 100 \) days, \( c_1 = \$60 \) per day, \( c_2 = \$45 \) per day, and \( A = \$600,000 \). The patient arrival rates have means of \( \lambda_1 + \lambda_2 = 0.29 \) per day and 10 percent of all patients are type 1 patients. The expected annual profit for this hospice is \(-\$66,731\) based on expected annual (capped) revenues of \$1,882,856 and expected annual (uncapped) revenues of \$3,876,354.

The problem this hospice faces is a combination of a lack of scale, a propensity of noncancerous patients, and a high mean LOS for those patients. These factors in combination result in the hospice losing \$66,731, after paying the fixed overhead of \$600,000. The effect of the cap limiting their revenues is acutely shown by noticing that their revenues were curtailed by almost \$2 million. This hospice has insufficient volume to reach positive profit. Figure 1(a) shows the expected profit function for Example 1, truncated by the zero profit plane. Clearly, the expected profit increases...
Figure 1: The expected profit curves for Examples 1 and 2 as a function of $\theta = \lambda_1/(\lambda_1 + \lambda_2)$ and $\lambda = \lambda_1 + \lambda_2$. The flat region reflects zero (or less) profit.

with $\lambda_1 + \lambda_2$ (as shown in Proposition 1) but is unimodal in mix. A population that provides short LOSs results in low revenues collected, while with a population with long LOSs there is a high probability of exceeding the cap; both effects are detrimental to hospice profitability. Thus, either too high or too low a proportion of type 1 (short LOS) patients may be problematic.

**Example 2.** Consider a hospice in South Dakota, again inspired by Sack (2007). It is a hospice with a high proportion of short-living patients. Also, the patients generally, regardless of disease, have low mean LOS. Let $K = $21,410, $r = $130.79 per day, $m_1 = 40$ days, $m_2 = 60$ days, $\sigma_1 = 20$ days, $\sigma_2 = 80$ days, $c_1 = $60 per day, $c_2 = $45 per day, and $A = $600,000. The patient arrival rates have means of $\lambda_1 + \lambda_2 = 0.5$ per day and 90 percent of all patients are type 1 patients. The expected annual profit for this hospice is -$42,025 based on expected annual (capped) revenues of $1,001,450 and expected annual (uncapped) revenues of $1,002,505.

The hospice in Example 2 suffers from insufficient scale and the problem that their patients do not live long enough for the hospice to accrue sufficient revenue to cover the fixed overhead of $600,000. Figure 1(b) shows the expected profit function for Example 1, truncated by the zero profit plane. Although this hospice has a greater scale than that of Example 1, the higher proportion of short living patients does not allow them to gain sufficient revenues to reach profitability. So, this hospice also suffers from an inappropriate patient mix. Notice that the cap barely has an effect upon this hospice, reducing their expected revenues by only $1055 (variability in LOS is the reason the cap is occasionally in effect). However, their patients do not live long enough for them
to accrue all the revenues implied by the $K = 21,410$ cap adjustment. Indeed, the hospice has $0.5 \times 365 = 182.5$ patients and should be entitled to $182.5 \times 21,410 = 3.9$ million in revenues under the cap but instead receives a little over $1$ million.

Finally, consider the effect of LOS on hospice profitability. Figure 2 illustrates the effect of mean and standard deviation of the type 2 patients’ LOS in Example 1. As the example has a positive margin, the expected profit is decreasing in the standard deviation (the flat plane represents zero profit). The expected profit is not monotone in mean LOS, however. When the mean LOS is low, the hospice cannot gain sufficient revenues to pay the overhead; when the mean LOS is high, the hospice will exhaust the cap (thus, truncating revenues) while continuing to incur the daily costs of care. These reasons are identical to those behind non-monotonicity in mix as explained earlier.

### 2.4 Remedies

The hospices in the two examples from §2.3 suffer from different problems. The hospice in Example 1 had poor volumes and patients with excessive LOSs resulting in their revenues being limited by the cap. In contrast, the hospice in Example 2 experienced healthier volumes but the mix of patients resulted in comparatively short LOSs, limiting the scope for gaining sufficient revenues to cover the fixed costs.

One remedy we propose is for appropriate hospices to merge. There are several potential benefits from this. Firstly, the merged hospice will have larger volumes and expected profits are increasing in scale. Secondly, if the constituent hospices are chosen well, the resulting patient
mix in the merged provider could result in a more robust operating mix. The beauty of this proposal is that the overwhelming majority of patients receive routine home care and thus there are few physical facilities that need to be accounted for. Thus, the merged hospice can be a virtual organization possibly spanning disparate geographies. Specifically, MedPAC (2010) highlights the issues of hospices in Alabama and Mississippi exceeding the cap, due primarily to the exceedingly long LOSs, and Sack (2007) highlights the dramatically shorter LOSs in South Dakota. This leads to our third example, where the means and standard deviations of the patient LOSs and arrival rates for each disease type result are drawn from the merged hospices from Examples 1 and 2, appropriately weighted. Specifically, 

\[ m_1 = \frac{100 \times 0.029}{0.029 + 0.45} + \frac{40 \times 0.45}{0.029 + 0.45} = 43.63, \]

\[ m_2 = \frac{300 \times 0.261}{0.261 + 0.05} + \frac{60 \times 0.05}{0.261 + 0.05} = 261.41, \] and so on.

**Example 3.** Consider a merged hospice consisting of operations in Alabama and South Dakota. The patient characteristics in each location follow those of Examples 1 and 2. However, in combination, the characteristics of the patient census is a blend of these extremities. Let \( K = \$21,410, \)

\( r = \$130.79 \) per day, \( m_1 = 43.63 \) days, \( m_2 = 261.41 \) days, \( \sigma_1 = 22.95 \) days, \( \sigma_2 = 97.06 \) days, \( c_1 = \$60 \) per day, \( c_2 = \$45 \) per day, and \( A = \$1,200,000. \) The patient arrival rates have means of \( \lambda_1 + \lambda_2 = 0.79 \) per day and 61 percent of all patients are cancer patients. The expected annual profit for this hospice is \( \$1,329,810 \) based on expected annual (capped) revenues of \( \$4,313,140 \) and expected annual (uncapped) revenues of \( \$4,848,580. \)

Figure 3 shows the expected profit function for Example 3, truncated by the zero profit plane.
This is an example where the merged hospice is making a profit whereas the individual hospices were losing money (for different reasons). Notice the assumed fixed overhead in this example is simply the sum of the individual overheads from Examples 1 and 2, whereas in reality the merged hospice fixed overhead would be less than $1,200,000. The government is in the prime position to act as a “match-maker” to encourage such mergers and to remove any regulatory hurdles to inhibit the mergers. Of course, any mergers would be subject to the usual antitrust scrutiny but this would be balanced by the public good of viable hospices providing services to vulnerable communities. Of course, it is simple to see that the revenues of the merged hospice in Example 3 ($4,313,140) exceed the sum of the individual revenues of the hospices in Examples 1 and 2 ($1,882,856+$1,001,450=$2,884,306), suggesting the government will be paying an extra $4,955.21 per patient. These extra payments easily dwarf the per patient savings of $2309 (Taylor et al., 2007) the Medicare hospice benefit yields (compared with regular curative Medicare treatment). However, under the auspices of such mergers, Medicare could reduce the per diem without jeopardizing the profitability of the hospices, and thus make such a recommendation revenue neutral. For instance, in the current example if the routine home care per diem was reduced to $98.54, the merged hospice would make expected profit of $566,914, and the extra per patient reimbursements would be $2309, thus making the hospice benefit not more or less costly than the curative Medicare benefits while still enabling the merged hospices to reach profitability.

There are three aspects which contribute to improved profitability through merging hospices. The first is scale. When the hospices have a positive contribution margin, their profitability increases with greater volumes; the volume of the merged hospice has the sum of the individual hospices’ volumes, so this increases profitability. Secondly, the merged hospice enjoys the well-known benefits of pooling (see, e.g., Eppen, 1979) where the relative variability is reduced and Proposition 1 shows the hospice profit decreases in variability. The third benefit is that patient mix of the merged hospice is a blend (based on relative volumes) of the patient mixes of the individual hospices.

When carefully chosen, such mergers could potentially alleviate unfortunate systemic patient mix issues. For example, the characteristics of Example 2 suggests the hospice would prefer no cancer patients but this is unlikely to be the case in their natural arrivals, so a deliberate search for another hospice with a complementary patient mix could enhance their overall profitability. Further, we feel that correcting unprofitable patient mix through hospice mergers is a more socially

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7Medicare has sometimes reduced the hospice benefit per diem for some types of care.
equitable solution than allowing hospices to adjust their mix by actively searching for patients of the “right” type; under this latter solution all patient types may no longer receive equal hospice access. Indeed, a reliance upon the natural arrivals of certified terminally ill patients at the merged hospice would appear to have fewer ethical ramifications than a solution that may at the extreme try to induce non-terminally ill patients to join the hospice. In the following section we show that hospices indeed have an incentive to actively search for the right patient type, that there is anecdotal evidence that this occurs in practice, and that the optimal search strategies will change over the course of a year, further exacerbating equal access concerns.

3 The Hospice Manager’s Problem

This section presents the patient recruitment challenge a hospice manager faces in order to preserve profitability. We formulate and analyze a dynamic model where there are regular arrivals of hospice patients of two types during the Medicare year; in addition to these regular arrivals (at differing rates) of patients, hospices are allowed to seek, or “recruit,” additional admissions of these patient types during the year. These recruiting rates of additional patients (of each type) are the decision variables of this dynamic model.

There is significant evidence in the literature that hospices do indeed search for, or recruit, patients. For example, as reported in Sack (2007), Ms. Youngblood, the Hometown Hospice nurse, said that the nursing home and senior center nurses joke about her “marketing” forays: “They’ll say, ‘Here comes Nurse Kevorkian. She has no shame.’ And I’ll say, ‘Look, I have to have a paycheck, too.’ ” Further, there is evidence that hospices do care about the type of patients admitted, sometime targeting their materials towards cancer patients (Sack, 2007). As mentioned earlier, MedPAC (2010) is sufficiently concerned about the financial relationships between hospices and long-term care facilities that they recommend further investigation.

Nursing home operators control which hospices offer services to their patients and investigators have found instances in which hospices and nursing homes have struck (potentially illegal) agreements in which the hospice pays extra money to the nursing home in exchange for exclusive access to its patients (Frantz, 1998). Frantz (1998) describes a concurrent expansion of fraud in the Medicare hospice benefit with legitimate expansion of the program, highlighting the case of Joseph Kirschenbaum, an Illinois-based hospice operator federally indicted for fraud and other offences, who paid nursing homes $10 for each new hospice patient and $89 per month to doctors who would
certify patients as terminally ill without examining them or reviewing their medical records. Our model assumes that hospices do have the ability to recruit beyond their natural arrival rates but that such recruiting has a convex increasing cost, because diseconomies of scale appear natural in a limited and competitive market.

### 3.1 A Fluid model

The model advanced in this section treats arrivals of patients to each class as fluid arriving at the system at a constant rate. In particular, class $i$ customers arrive at rate $\lambda_i$ ($i = 1, 2$) per unit of time. The average lifespan of a class $i$ patient is $m_i$ and total lifespan has density $f_i(\cdot)$ and hazard rate $h_i(\cdot)$. However, the lifespan is not random and it simply decays at a rate consistent with these distributions (see the proof of Proposition 3 for the precise definition of lifespan). Consider a class $i$ patient arriving to the system at time $t \in [0, T]$, and let $r_i(t)$ and $c_i(t)$ denote the potential revenues to be collected from Medicare and the cost of caring for that patient over the fiscal year $[t, T]$, respectively. Similarly, let $v_i(t)$ denote the (terminal) value attributed to a class $i$ patient arriving at time $t$ at the end of the planning horizon. As before, let $r$ and $c_i$ denote the daily revenue and cost of caring for a class $i$ patient ($i = 1, 2$), respectively. Similarly, let $v_i$ denote the terminal value of a class $i$ patient ($i = 1, 2$). Then the following proposition characterizes $r_i(t), c_i(t),$ and $v_i(t)$.

**Proposition 3** For $t \in [0, T]$ and $i = 1, 2$,

$$r_i(t) = r \int_0^\infty x \land (T-t) f_i(x) dx,$$

$$c_i(t) = c_i \int_0^\infty x \land (T-t) f_i(x) dx,$$

and

$$v_i(t) = v_i \int_{T-t}^{\infty} f_i(x) dx.$$

For the case of exponential LOS, $f_i(x) = e^{-x/m_i}/m_i$ and

$$r_i(t) = rm_i[1 - e^{-(T-t)/m_i}],$$

$$c_i(t) = cm_i[1 - e^{-(T-t)/m_i}],$$

and

$$v_i(t) = ve^{-(T-t)/m_i}.$$

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8 Kirschenbaum received over $11.5 million over three years of operation but was accused of defrauding Medicare and others of $28 million. He was sentenced to three years in prison, paid a $7500 fine, paid the Illinois Department of Revenue $569,000, and settled civil lawsuits for $22 million (Mowatt, 1999).
The hospice manager is required to admit all arriving patients but also faces a decision as to whether or not to actively recruit more patients. Let $\alpha_i(t)$ denote the rate at which the hospice manager recruits class $i$ patients at time $t$. There is a convex increasing cost $s_i(\alpha)$ associated with recruiting (or searching for) class $i$ patients at rate $\alpha$. For concreteness, assume $s_i(\alpha) = \frac{1}{2}\eta_i\alpha^2$ for $i = 1, 2$ and $\alpha \geq 0$, where $\eta_1, \eta_2 > 0$ are given parameters.

Suppose that there are $n_i$ class $i$ patients in the hospice at the beginning of the fiscal year. Then, given the hospice manager’s recruiting policy $\alpha_1(\cdot), \alpha_2(\cdot)$, the cumulative (potential) revenues up to time $t$, denoted by $R(t)$, is given by

$$R(t) = \sum_{i=1}^{2} \int_0^t r_i(x)\left[\lambda_i + \alpha_i(x)\right]dx + \sum_{i=1}^{2} r_i(0)n_i.$$  \hfill (3)

Similarly, the cumulative care cost incurred by the hospice manager up to time $t$, is denoted by

$$C(t) = \sum_{i=1}^{2} \int_0^t c_i(x)\left[\lambda_i + \alpha_i(x)\right]dx + \sum_{i=1}^{2} c_i(0)n_i,$$ \hfill (4)

and the cumulative terminal value associated with patients arriving (and recruited) up to time $t$ is given by

$$V(t) = \sum_{i=1}^{2} \int_0^t v_i(x)\left[\lambda_i + \alpha_i(x)\right]dx.$$ \hfill (5)

The cumulative recruiting costs $S(t)$ up to time $t$ is given by

$$S(t) = \sum_{i=1}^{2} \int_0^t s_i(\alpha_i(x))dx.$$ \hfill (6)

As mentioned earlier, a crucial feature of the Medicare reimbursement policy is that the hospice’s revenue is constrained by a cap, which increases with the number of patients admitted during the fiscal year. To be specific, the cap at time $t$ is given by

$$K(t) = K \sum_{i=1}^{2} \int_0^t (\lambda_i + \alpha_i(x))dx.$$ \hfill (7)

Therefore, the realized revenue at the end of the fiscal year is given by $\min\{K(T), R(T)\}$, and the hospice manager’s problem (P) can be written as follows: Choose search rates $\alpha_1(\cdot), \alpha_2(\cdot)$ dynamically so as to

$$\text{maximize } \min(K(T), R(T)) + V(T) - C(T) - S(T)$$

subject to $\alpha_i(t) \geq 0$ for all $i, t$. \hfill (P)

To facilitate our analysis, define for a given $q$, $0 \leq q \leq 1$,

$$\delta_i^q(t) = (1-q)K + qr_i(t) - c_i(t) + v_i(t) \text{ for } t \in [0, T] \text{ and } i = 1, 2.$$ \hfill (8)
It is straightforward to show that $\delta^q_i(\cdot)$ is strictly increasing on $[0, T]$ if

$$-qr + c_i + v_i h_i(T - t)$$

is positive for all $0 \leq t \leq T$ (and decreasing if it is negative). Call this first case INC($q$) and the second DEC($q$). For either INC($q$) or DEC($q$) the inverse $(\delta^q_i)^{-1}(\cdot)$ is well defined and we restrict attention to these cases. For exponential lifespans the hazard rate is constant so all values of $q$ will always fall in either INC($q$) or DEC($q$) and no restriction is necessary. Using that inverse, define the trigger time functions $(\bar{t}_i(q), \tilde{t}_i(q))$ for class $i$ ($i = 1, 2$) as follows:

$$t_i(q) = \begin{cases} 
(\delta^q_i)^{-1}(0) & \text{if INC($q$), } \delta^q_i(0) < 0, \delta^q_i(T) > 0 \\
0 & \text{if } \delta^q_i(0) \geq 0 \\
T & \text{otherwise.}
\end{cases}$$  \hfill (9)

$$\bar{t}_i(q) = \begin{cases} 
(\delta^q_i)^{-1}(0) & \text{if DEC($q$), } \delta^q_i(0) > 0, \delta^q_i(T) < 0 \\
T & \text{otherwise.}
\end{cases}$$  \hfill (10)

These trigger functions will determine when (if at all) the hospice begins recruiting and when they end recruiting for a given patient type.

Although the hospice manager’s problem (P) is an optimal control problem (and hence, infinite dimensional), its dual is much simpler. Indeed, in Appendix B, we show that the dual formulation can be reduced to a one-dimensional convex optimization problem, enabling an explicit solution to both the dual and the hospice manager’s original problem. The proof of this relies on the duality theory for optimal control problems developed by Rockafellar (1970), which is also introduced in Appendix B. For $q \in [0, 1]$, define

$$F(q) = -K(\lambda_1 + \lambda_2)T + r_1(0)n_1 + r_2(0)n_2 + \sum_{i=1}^{2} \lambda_i \int_0^T r_i(s)ds + \sum_{i=1}^{2} \int_{\bar{t}_i(q)}^{\tilde{t}_i(q)} \frac{r_i(t) - K}{\eta_i} (K + (r_i(t) - K)q - c_i(t) + v_i(t))dt.$$  \hfill (11)

In Appendix B, $F(\cdot)$ will be shown to be the derivative of the dual objective function. The following proposition shows that the inverse $F^{-1}$ of $F$ is well defined.

**Proposition 4** $F$ is continuously differentiable and strictly increasing on $q$ such that $t_i(q) < T$.

We are now ready to state our main result.
**Theorem 1** If $F(0) > 0$ then let $q^* = 0$, if $F(1) < 0$ then let $q^* = 1$, otherwise let $q^* = F^{-1}(0)$.

The hospice manager’s optimal recruiting rates for $i = 1, 2$ and $t \in [0, T]$ are given by

$$
\alpha_i^*(t) = \begin{cases} 
K(1-q^*)+q^*r_i(t)-c_i(t)+v_i(t) & \text{if } \bar{t}_i(q^*) \leq t \leq t_i(q^*), \\
0 & \text{otherwise.}
\end{cases}
$$

(12)

These rates are increasing in $t$ under case $INC(q^*)$ and decreasing under $DEC(q^*)$.

We have thus explicitly characterized the optimal recruiting rates for a given hospice. For either type of patient, the hospice may choose to never actively recruit those patients, relying entirely on their natural arrival rates, they may recruit patients throughout the year, only at the beginning of the year, or they may only recruit towards the end of the Medicare year.

Figure 4 illustrates specific optimal $\alpha_i(t)$ functions, using the parameters of Examples 1, 2, and 3, setting $\eta_1 = 30000$ and $\eta_2 = 30000$, and using an exponential distribution for the patient lifespans (given the dramatic decline of the exponential distribution, we adjust $K$ to $13,750$ ensuring Example 1 has a binding cap; an alternative would be to increase the patient lifespans). The recruiting costs are set so that recruiting rates were of a similar magnitude to the natural arrival rates.

In Example 1 we see the hospice starts the year recruiting patients of both types, which is unsurprising as each is profitable (the daily margin of patient type 1 is $130.79-60 = 70.79$ and of patient type 2 is $130.79-45 = 85.79$). However, while we observe the hospice will recruit type 1 (e.g., cancer) patients throughout the year, they will halt the recruitment of type 2 (e.g., dementia) patients part-way during the year (around day 250). Moreover, the type 1 patients are recruited at a higher rate than type 2 patients. This behavior is consistent with the intuition that type 1 (short LOS) patients can be used to increase the cap, particularly towards the end of the Medicare year. This is verified by examining Figure 5. In Figure 5(a) the difference between the cap and the revenues for the natural arrivals (i.e., absent any recruiting) is displayed, demonstrating a shortfall of $221,768$ at time $T$. Once the hospice’s recruiting is also included, the gap between the cap and the revenues, displayed in Figure 5(b), will be closed at time $T$. Observe that Example 1 has longer LOS (weighted across the two patient types) than Example 2, resulting in the cap binding in Example 1 and $\alpha_1(t)$ is consistently positive (and greater than $\alpha_2(t)$) during the Medicare year to achieve a perfect balance between the cap and revenues. Given this is a cap constrained example, we use terminal values of $v_i = -c_i m_i$ for $i = 1, 2$, which reflects the lifetime costs for any patients living at the end of the year.
(a) Example 1.

(b) Example 2.

(c) Example 3.

Figure 4: The optimal recruiting rates for Examples 1, 2, and 3 ($\eta_1 = \eta_2 = 30000$).
Example 2 is one where the cap is never in any danger of binding, but it is the short LOS of their patients which limits the accumulation of revenues. Without any recruiting, the revenues will be $884,858 and the cap will be $2.5 million. But with recruiting, the revenues are $1,456,200 and the cap is more than $3.8 million. Given there are no cap issues, we apply terminal values of $v_i = (r - c_i)m_i$ for $i = 1, 2$ which reflects the lifetime margins for those patients alive at the end of the year (these are applied for Example 3 also). Further, in both Examples 2 and 3 the hospice will actively recruit the more profitable type 2 patients at a higher rate than type 1 patients (margins given above). The convex increasing recruiting cost limits this recruiting, however. Example 3 sees a blend of Examples 1 and 2; without any recruiting, the hospice would receive $2.66 million in revenues compared with their cap of $3.96 million. However, with recruiting, the hospice is able to increase their revenues ($7.15 million) and use a greater portion of the corresponding cap ($8.23 million).

Anecdotal evidence suggests that a hospice manager’s focus at the end of the year is to “manage the cap” rather than worry about the patients who will live into the new Medicare year. Moreover, there is the suggestion that the cap issues are largely ignored during the year and the managers will “scramble” to address the cap problems through cancer patient recruiting only towards the end of the Medicare year (this would be reflected as an increasing $\alpha_1(\cdot)$ towards the end of the year). However, our model is deterministic which endows the hospice manager with perfect foresight, purging the managers’ myopia which could otherwise result in type 1 recruitment at the end of the year; our model distributes this type 1 recruiting throughout the year, which is encouraged by the convex increasing recruiting cost. If less negative (or even zero) terminal values are assigned to patients in Example 1 then indeed we can find situations where $\alpha_1(\cdot)$ increases over the horizon while $\alpha_2(\cdot)$ decreases.

Our fluid model does not include any model of randomness but in such a case one would expect recruiting rates to be highly state dependent (and likely intractable for exact analysis). However, we speculate that if LOS uncertainty were included, it would amplify the effects of the cap due to extreme random shocks, resulting in more type 1 patients being recruited, especially towards the end of the horizon, and perhaps the type 2 recruiting stopping earlier than indicated in Figure 4.

We believe that nonstationary recruiting rates are undesirable from a policy perspective because they do not align with the equal access objective of a social planner. The immediate implication of nonstationary recruiting is that a prospective patient’s desirability to a hospice depends on the calendar. A cancer patient would not be as highly sought in December than they might be in
October (the Medicare year runs from November 1 to October 31). The government’s motivation in establishing the Medicare hospice benefit was to facilitate comfortable palliative care for patients during their end-of-life by third party providers. The nonstationary active recruiting which can arise as the optimal policy and is observed in practice, is clearly an unintended consequence of the government’s Medicare reimbursement policy. The following section presents a new policy for Medicare to consider, which alleviates this unintended behavior.

4 The Legacy Policy and its Analysis

This section introduces and analyzes a policy to overcome the nonstationary behavior observed under the current Medicare reimbursement policy seen in Section 3. We label this policy the legacy policy as it explicitly addresses the possibility of beneficiaries living into the next Medicare year. The implementation of this policy requires the hospice to segregate the tracking of the patients admitted in each Medicare year until they expire, but is otherwise no more burdensome in terms of administration than the current Medicare policy. Indeed, our goal is a policy that maintains the same fundamental framework of the Medicare policy but does not have an inherent incentive for non-stationary recruiting. That is, we assume that Medicare wishes to maintain a cap (to limit its exposure, especially as the declaration of a patient being terminally ill can be difficult), to keep the cap pooled (to mitigate risk to the hospice), and to keep payment rates constant across disease types and time frame (for ease of implementation).

The policy consists of allowing the hospice to continue receiving revenues for all the patients living at the end of the year until any remaining cap is exhausted or all these patients expire, whichever occurs first. If the cap is exhausted before these patients expire, then the hospice...
receives no further revenues for these patients but continues to incur the costs of caring for them.\textsuperscript{9}

In practice, this policy is likely most simply implemented by allowing exactly one extra legacy year and assuming that the probability of having patients live into a third Medicare year with the cap still not exhausted is negligible. Notice that this policy would mean that a start-up hospice would have no cap adjustment to pay-back until the end of its second year of operation.

Under the modified policy, because the accounting is over the entire patient life rather than just over the Medicare year, \( r_i(s) = r m_i \) and \( c_i(s) = c_i m_i \) for all \( i, s \). Similarly, \( n_i = 0, v_i = 0, \) and \( v_i(s) = 0 \). Then the fluid model of the hospice manager’s problem can be adapted to choosing recruiting rates \( \dot{z}(\cdot) \) in order to

\[
\max \min \left\{ KT \left[ (\lambda_1 + \lambda_2 + z_1(T) + z_2(T)) \right], r \left[ (\lambda_1 m_1 + \lambda_2 m_2) T + z_1(T) m_1 + z_2(T) m_2 \right] \right. \\
- \sum_{i=1}^{2} c_i m_i \left[ \lambda_i T + z_i(T) \right] dt - \sum_{i=1}^{2} \int_{0}^{T} s_i(\dot{z}_i(t)) dt \right. \\
\left. - \sum_{i=1}^{2} c_i m_i (\lambda_i + \alpha_i) - T \sum_{i=1}^{2} s_i(\alpha_i) \right\} \tag{13}
\]

subject to

\[
z_i(t) = z_i(0) + \int_{0}^{t} \dot{z}_i(s) ds \text{ with } z_i(0) = 0 \text{ for all } i, t, \tag{14}
\]

\[
\dot{z}_i(s) \geq 0. \tag{15}
\]

The following result significantly simplifies the above control problem.

**Proposition 5** Any optimal solution of the formulation (13)-(15) is a stationary recruiting policy, i.e., \( \alpha_i(t) = \dot{z}_i(t) = \bar{\alpha}_i \) for all \( i, t \) and some \( \bar{\alpha}_i \geq 0 \).

Therefore, the hospice’s manager’s problem can be stated without loss of generality as follows: Choose the stationary recruiting rates \( \alpha_1, \alpha_2 \) so as to

\[
\max \left\{ \min \left[ KT \sum_{i=1}^{2} (\lambda_i + \alpha_i), r T \sum_{i=1}^{2} (\lambda_i + \alpha_i) m_i \right] - T \sum_{i=1}^{2} c_i m_i (\lambda_i + \alpha_i) - T \sum_{i=1}^{2} s_i(\alpha_i) \right\} \]

subject to \( \alpha_i \geq 0, i = 1, 2 \).

Finally, observe that without loss of generality we can set \( T = 1 \) so that the problem becomes

\[
\max \left\{ \min \left[ K \sum_{i=1}^{2} (\lambda_i + \alpha_i), r \sum_{i=1}^{2} (\lambda_i + \alpha_i) m_i \right] - \sum_{i=1}^{2} c_i m_i (\lambda_i + \alpha_i) - \sum_{i=1}^{2} s_i(\alpha_i) \right\} \tag{16}
\]

\textsuperscript{9}It is important that Medicare prohibit the hospice from live discharging any of these remnant patients, as the provider would have every incentive to jettison them once the cap is exhausted. Incentives for live discharges are also an issue under the current Medicare policy and this is described further in the Conclusions section.
subject to $\alpha_i \geq 0, i = 1, 2$.

We are now ready to state our main result for the section.

**Proposition 6** Let $\alpha^*_i$ ($i = 1, 2$) be the optimal solution to (16).

1. If $\sum_{i=1}^2 (rm_i - K) \left( \frac{(r-c_i)m_i}{\eta_i} + \lambda_i \right) \leq 0$, then
   \[
   \alpha^*_i = \frac{m_i}{\eta_i} (r - c_i) > 0, i = 1, 2.
   \]

2. If $0 < \sum_{i=1}^2 (rm_i - K) \left( \frac{(r-c_i)m_i}{\eta_i} + \lambda_i \right) < \sum_{i=1}^2 \frac{1}{\eta_i} (K - rm_i)^2$, then
   \[
   \alpha^*_i = \frac{1}{\eta_i} \left[ \gamma^*_1 (K - c_i m_i) + (1 - \gamma^*_1) m_i (r - c_i) \right], i = 1, 2
   \]
   where
   \[
   \gamma^*_1 = \frac{(rm_1 - K) \left( \frac{(r-c_1)m_1}{\eta_1} + \lambda_1 \right) + (rm_2 - K) \left( \frac{(r-c_2)m_2}{\eta_2} + \lambda_2 \right)}{\frac{1}{\eta_1} (K - rm_1)^2 + \frac{1}{\eta_2} (K - rm_2)^2}.
   \]

3. If $\sum_{i=1}^2 (rm_i - K) \left( \frac{(r-c_i)m_i}{\eta_i} + \lambda_i \right) \geq \sum_{i=1}^2 \frac{1}{\eta_i} (K - rm_i)^2$, then
   \[
   \alpha^*_i = \frac{1}{\eta_i} (K - c_i m_i), i = 1, 2.
   \]

Depending on the parameters, Proposition 6 has three cases for the solutions of the optimal recruiting rate under the legacy policy.

- In case 1, the reimbursements determine the revenue rate (i.e., the cap does not bind).
- In case 2, the cap equals the reimbursement rate (perfectly balanced).
- In case 3, the cap binds in the optimal solution (i.e., the cap determines the revenue rate).

In case 1 of Proposition 6 the potential revenues of the hospice will not use the available cap and the recommended recruiting rate will then simply solve the resulting first order condition of (16). This results in rates which simply balance out the contribution over the remaining life of the patient against the cost of recruiting them. These rates are increasing in $r$ and $m_i$, and decreasing in $\eta_i$ and $c_i$. In case 3 the cap is binding and the resulting optimal recruiting rate simply takes the adjustable cap rate as the total revenue, subtracts the total costs of caring, and balances this quantity against the cost of recruiting these patients. Thus, the recruiting rates are increasing in $K$ but decreasing in $\eta_i$, $m_i$, and $c_i$. In case 2 the recruiting rates of cases 1 and 3 are blended in such a way to create a perfectly balanced situation where the weighted total revenues equals the available cap.
An observation is that although Proposition 6 states that the legacy policy will deliver a stationary policy, it does not imply the recruiting rates will be identical across diseases or even equal in proportion to volumes. Indeed, Proposition 6 confirms the recruiting rates will differ, according to the underlying characteristics and economics for each disease. So, while the legacy policy addresses the problem of nonstationarity of the existing reimbursement policy, it may not affect the issue of the relative profitability across diseases (accounting for the cost of care, the cost of recruiting, and the lifespans of patients). This is consistent with the existing policy and could only be addressed by Medicare instigating a disease-specific reimbursement policy. Indeed, Killaly et al. (2007) recommends revisiting Medicare’s disease-invariant per diem reimbursement, but we believe a disease-dependent payment rate would need to be very carefully chosen; otherwise the system will be prone to gaming such as hospices classifying admissions based on the higher margin disease in the case of co-morbid patients. Thus, more careful consideration is merited.

Table 1 presents the legacy policy’s optimal recruiting rates for Examples 1, 2, and 3 for various values of $\eta_1$ and $\eta_2$. There are several observations. Firstly, for the scenarios where there are

<table>
<thead>
<tr>
<th>$\eta_1, \eta_2$</th>
<th>Example</th>
<th>$\alpha^*_1$</th>
<th>$\alpha^*_2$</th>
<th>Proposition 6</th>
<th>$\frac{1}{T} \int_0^T \alpha_1(t)dt$</th>
<th>$\frac{1}{T} \int_0^T \alpha_2(t)dt$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000,1000</td>
<td>1</td>
<td>7.75</td>
<td>0.25</td>
<td>case 3</td>
<td>6.137</td>
<td>1.332</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2.83</td>
<td>5.15</td>
<td>case 1</td>
<td>2.832</td>
<td>5.147</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>10.36</td>
<td>3.96</td>
<td>case 2</td>
<td>4.034</td>
<td>22.21</td>
</tr>
<tr>
<td>10000,10000</td>
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<td>0.7750</td>
<td>0.0250</td>
<td>case 3</td>
<td>0.6427</td>
<td>0.1137</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.2832</td>
<td>0.5147</td>
<td>case 1</td>
<td>0.2832</td>
<td>0.5147</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1.072</td>
<td>0.3034</td>
<td>case 2</td>
<td>0.3089</td>
<td>2.2426</td>
</tr>
<tr>
<td>30000,30000</td>
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<td>0.2583</td>
<td>0.0083</td>
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<td>0.2357</td>
<td>0.0235</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.0944</td>
<td>0.1716</td>
<td>case 1</td>
<td>0.0944</td>
<td>0.1716</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.3711</td>
<td>0.0662</td>
<td>case 2</td>
<td>0.1030</td>
<td>0.7475</td>
</tr>
<tr>
<td>1000,10000</td>
<td>1</td>
<td>7.75</td>
<td>0.0250</td>
<td>case 1</td>
<td>6.137</td>
<td>0.1332</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2.832</td>
<td>0.5147</td>
<td>case 1</td>
<td>2.832</td>
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<tr>
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<td>0.2500</td>
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<td>1.137</td>
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<td>5.147</td>
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<td>5.147</td>
</tr>
<tr>
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<td>1.113</td>
<td>1.987</td>
<td>case 3</td>
<td>0.3089</td>
<td>22.43</td>
</tr>
</tbody>
</table>

Table 1: The legacy policy optimal stationary recruiting rates.
equal costs of recruiting type 1 and type 2 patients, Example 1 falls within Proposition 6’s case 3, Example 2 falls within Proposition 6’s case 1, and Example 3 falls within Proposition 6’s case 2, consistently. This is natural since Example 1 deals with the hospice with lengthy LOS and the cap is exhausted, Example 2 deals with a hospice which struggles gaining sufficient revenues before their patients die and leave some of the cap unused, and Example 3 consists of merging the hospices of Examples 1 and 2. Example 3 falls into a range where the cap would be exhausted if using case 1 recruiting rates but the recruiting rates here would exactly balance the potential revenues and the admissible cap.

As might be expected, as the costs of recruiting increase the legacy policy recruiting rates decrease in a proportional manner. Also, when $\eta_1 = \eta_2$, $\alpha_1^* > \alpha_2^*$ for Example 1 (case 3) and Example 3 and $\alpha_1^* < \alpha_2^*$ for Example 2. The reason for this is that type 1 patients have lower LOS costs than type 2 patients ($c_1 m_1 = 6000 < c_2 m_2 = 13500$) and so the policy in case 3 will favor spending more of the cap to that lower cost patient type since the revenue is effectively taken out of the equation because all the cap will be used. However, for Examples 2 and 3 not all of the cap will necessarily used so the per diem rate plays a role. In Example 2 (case 1) because the cap will not be reached, the lifetime contribution ($m_1(r - c_1)$) will determine the recruiting rate, which means the type 2 (the more profitable type) will be more heavily recruited. In Example 3, as already suggested, the recruiting rates will be adjusted in order to precisely exhaust the cap, which results in the type 1 recruiting rates being somewhat higher than those for type 2 patients. For our numerical case, in Example 3 (which falls in Proposition 6’s case 2) as $K$ reduces, $\alpha_2^*$ decreases and $\alpha_1^*$ is non-monotone (although it seems to decrease over much of the range). This suggests that there may be less emphasis on type 2 patients as the cap limit looms as a threat, although it is clear there is much adjustment between the rates in order to achieve the perfect balance implied by case 2.

For comparison, we also include the time-average recruiting rates from the existing reimbursement policy (the final two columns of Table 1). In the cases where the recruiting costs are symmetrical, for Example 1 the legacy policy results in higher overall (summed) recruiting rates than the time average total recruiting from the fluid model. For Example 2, the legacy policy and the time average recruiting rates are identical; this is due to the fluid model’s terminal values being set at the lifetime profit, which coincide with Proposition 6’s case 1 recruiting levels. In Examples 1 and 3 we observe that the legacy policy recommends a higher recruiting level than the fluid model’s time average for type 1 patients while this is reversed for type 2 patients.
While more complicated policy structures could be designed, there is benefit derived from the relative simplicity of the existing and legacy policies from an implementation perspective. Even for these simple policies, an opportunity has grown for Regional Home Health & Hospice Intermediary (RHHI) firms to act as Medicare billing agencies on behalf of the hospices. If the policies became more complicated, then the likelihood is that such intermediaries will accrue even more channel profits. A benefit of the legacy policy is that it strongly resembles the existing policy and is unlikely to involve much more difficulty in implementation (the difference is that the hospice will need to segregate any revenues received on behalf of those beneficiaries living into the following year from those newly admitted in that following year).

5 Concluding Remarks and Discussion

We have examined Medicare’s hospice reimbursement policy, both from profitability and patient recruitment perspectives. From the profitability perspective, we are able to discern the aspects of the policy or market conditions which lead to potential losses. Specifically, we find that if a provider has a lack of scale and/or an imbalance in the mix of patients, they run a risk of not receiving sufficient revenues to reach profitability. This could be because the patient census lived too long and the cap limited the revenues gained from Medicare while the provider continued to incur the costs of caring for the patients, or possibly because the patient census lived too short for the provider to gain sufficient revenues to cover the fixed overhead of operation. We suggest a potential remedy is for the government to encourage the merging of appropriate providers.

Merging providers (or encouraging acquisition) from disparate localities can avoid the geographic element which contributes to an imbalanced patient mix, resulting in a patient mixture and scale that increases the expected revenue while consolidating the fixed overheads. By providing a more attractive patient mix for the hospice, it may also alleviate some of the concerns of differing patient recruiting rates by time of year and disease type, which while not explicitly addressed in our initial static model do become apparent in our dynamic model. This recommendation is viable because hospices frequently will be providing care in the homes of the patients rather than in their own physical facilities and thus a merged organization could operate over disparate geographies.

Another aspect of the current Medicare hospice reimbursement policy we investigated was the manager’s optimal recruitment policy. Using a dynamic fluid model, we demonstrate that the manager has an incentive to recruit patients of differing diseases at rates which differ and which
change during the Medicare year. For example, the manager might seek to recruit type 1 (short LOS) patients at a rate which increases towards the end of the year while the recruitment rate for type 2 (long LOS) patients might decrease. The basic reason for this is that type 1 patients can increase the cap by the same amount as type 2 patients but are expected to live for shorter durations, which is an important consideration when patients living into the next Medicare year are taken into consideration. This is clearly an unintended and negative consequence of the current policy and there is strong anecdotal evidence to suggest that such behavior indeed occurs in practice.

We design and analyze an alternative policy, labeled the *legacy policy*, which allows the hospice to continue receiving revenues for these remnant patients providing any positive remnant cap exists at the end of the year, until the last patient expires or the remnant cap is exhausted, whichever occurs first. Importantly, the remnant cap and the remnant patients are tracked separately from both the new cap launched at the start of the new year and any newly admitted patients. We show this alternative policy restores stationarity into the manager’s problem, which is compatible with an objective of equal access to hospice care. An attractive attribute of the alternative policy is that it closely resembles the existing policy so that its implementation is not expected to be disruptive to Medicare or hospice providers.

Even with the legacy policy in place however, the optimal recruitment rates for different disease types will differ (albeit in a stationary fashion). There are a number of possible remedies for this issue. In particular, Medicare could reimburse at different rates for different diseases, they could adjust the cap increment for different disease types, or they could move to a fixed plus variable reimbursement for differing patients. All of these suggestions raise significant new issues, such as the classifying of patients with co-morbidities into a single class, the incentives for patient churn inherent in a fixed payment system, and the difficulty involved in calibrating payment rates to actual costs. Our focus has been on highlighting and alleviating the calendar-based recruitment incentives inherent in the current policy. We leave a broader policy study of all incentives under the current scheme as the subject of future research.

Although, we believe our fluid model does capture the foundational properties (and shortfalls) of the Medicare hospice reimbursement policy, we recognize it has some limitations. Firstly, the fluid model only considers two patient classes. This greatly facilitates the analysis, particularly in consideration of the dual transformation. However, each of these patient classes encapsulates the primary tradeoff between patient afflictions: a longer length of stay versus a higher daily cost of care. We do not believe any further major insights would be gleaned if the analysis were extended...
to additional patient classes. The second limitation is that the fluid model is entirely deterministic, in patient arrivals and in patient lifespans. We do allow the lifespan of a patient to follow general distributions but these are deterministic distributions which the hospice manager could perfectly anticipate. If stochastic patient lifespans were permitted, we believe the issue of the cap binding is likely to arise more frequently, akin to a production capacity constraint in a stochastic inventory system, resulting in even higher rates of recruiting cancer patients and perhaps later in the year. The third related limitation relates to the calibration of the model, specifically the modifications to the parameters necessary for the numerical study. In particular, when using the exponential distribution, the patient lifespan distribution plunges rapidly into a long tail. In order to capture the example where the cap is binding, either the patient lifespans \((m_1, m_2)\) must be dramatically increased or the cap adjustment \((K)\) decreased. We chose the latter. While this does not take away from the analytical results, other more empirically derived distributions could fit the problem parameters better. The final issue is the treatment of the terminal values in the numerical examples. For Example 1 in Section 3 we were aware this was a situation where the cap binds and thus any patients living at the end of the period would be incurring costs for the hospice but not generating revenues so a terminal value of \(v_i = -c_i m_i\) was applied. In all likelihood, this value was probably too severe (partly demonstrated by the decreasing nature of the curves in Figure 4(a); a more appropriate terminal value might be between \(-c_i m_i\) and 0. Similar reasoning applies to Examples 2 and 3, but our observations are not contingent upon specific values.

Another disturbing trend in the hospice industry is the practice of “live discharges.” A live discharge is a living patient who is released from a hospice. Given the unpredictable trajectory of terminal diseases, some patients’ diseases simply do not follow expectations, a patient may recover, or a patient may simply elect to resume conventional medical treatment, which of course results in their leaving the hospice. However, a hospice caring for patients who have exhausted their contribution to the cap may feel the pressure to discharge such patients, however unethical (and potentially illegal) such a practice may be. Some hospice providers state that while they will not “live discharge” such unprofitable patients, they “sometimes delayed admission for those patients with illnesses that might result in longer stays” (Sack, 2007). Taylor et al. (2008) suggests 15.5% of hospice users were discharged alive from years 1993 to 2000, although the costs per day survived are similar for all patients (live discharges or not). MedPAC (2010) finds live discharges are far more prevalent amongst providers exceeding their caps (46% of all discharges) than those not exceeding their caps (16%) in 2007, and consequently recommended the Office of the Inspector General
investigate the “appropriateness of enrollment practices among hospices with unusual utilization patterns.” Carlson et al. (2009) find that live discharges are more prevalent for smaller hospices and that long-stay patients may be more susceptible to this practice, two factors which our findings suggest can lead to diminished profitability, although newer hospices were also commonly found to be live-dischargers, perhaps implying some inexperience in judging LOS may play a factor. There would thus appear to be a practice amongst hospices to jettison unprofitable patients, although we do not incorporate this custom in our models and leave such investigation as the subject of future research.

To sum up, under any government policy there will always be unintended consequences; this paper sheds light on some of these under the Medicare hospice reimbursement program. In particular, we studied both the efficacy of the program with respect to hospice profitability and the hospice providers’ incentives for patient recruitment under the program. The primary remedies we suggest, namely the merging of appropriate providers and the new legacy policy we propose, seek to mitigate the consequences of undesirable patient mix at a hospice. With respect to implementing the legacy policy, Congress may wish to run a pilot program with the new policy or they could use historical data (if available) to estimate its financial impact. Finally, we believe that by removing end-of-year recruitment pressures, the legacy policy could also improve job satisfaction for hospice managers, perhaps leading to greater hospice access in the long run.
A Proofs

Proof of Proposition 1. Expected patient revenue equals $E[R \land \kappa]$, where $R \sim rN(m, \sigma^2)$ and $\kappa = (\lambda_1 + \lambda_2)K$. Then,

$$E[R \land \kappa] = r\sigma E\left[\frac{R - rm}{r\sigma} \land \frac{\kappa - rm}{r\sigma}\right] + rm = r\sigma E[Z \land z] + rm$$

where $Z$ is the standard normal r.v. and $z = (\kappa - rm)/(r\sigma)$. Now,

$$E[Z \land z] = \int_{-\infty}^{z} x\phi(x)dx + z[1 - \Phi(z)]$$

$$= -\phi(z) + z[1 - \Phi(z)]$$

where the first equality is by definition and the second equality follows directly from the properties of the normal distribution (e.g., Zipkin, 2000, p.459). Then

$$E[R \land \kappa] = r\sigma[-\phi(z) + z(1 - \Phi(z))] + rm$$

Substituting $\bar{m} = m/\lambda$, $\bar{\sigma}^2 = \sigma^2/\lambda$, and $\kappa = \lambda K$ and rearranging terms yields total expected revenue as

$$E[R \land \kappa] = \lambda \left[K - (K - \bar{m})\Phi\left(\frac{K/r - \bar{m}}{\bar{\sigma}/\sqrt{\lambda}}\right) - \frac{r\bar{\sigma}}{\sqrt{\lambda}}\phi\left(\frac{K/r - \bar{m}}{\bar{\sigma}/\sqrt{\lambda}}\right)\right].$$

The expected cost is $c_1m_1\lambda_1 + c_2m_2\lambda_2 + A = c\lambda + A$ and the result follows.

For the comparative static results,

$$\frac{\partial \pi}{\partial K} = -\frac{1}{2}\lambda \left(\frac{2}{\sqrt{\pi}} \int_{0}^{\bar{z}} e^{-t^2} dt - 1\right)$$

where $\bar{z} = \frac{\lambda(K - \bar{rm})}{\sqrt{2r\bar{\sigma}}}$. Since $\frac{2}{\sqrt{\pi}} \int_{0}^{\bar{z}} e^{-t^2} dt$ is the Gaussian error function, bounded above by 1, $\frac{\partial \pi}{\partial K} > 0$. Moreover, some additional algebra shows $\frac{\partial^2 \pi}{\partial K^2} < 0$. So, $\pi$ is concave increasing in $K$. Further,

$$\frac{\partial^2 \pi}{\partial r^2} = -\frac{e^{\frac{\lambda(K-r\lambda_0)^2}{2r^2\sigma^2}} K^2 \lambda^{3/2}}{\sqrt{2\pi}r^3\sigma} < 0, \quad \frac{\partial \pi}{\partial \sigma_1} = -\frac{\lambda_1 e^{\frac{\lambda(K-r\lambda_0)^2}{2r^2\sigma^2}} r\sigma_1}{\sqrt{2\pi}\lambda\sigma} < 0, \quad \frac{\partial \pi}{\partial \sigma_2} = -\frac{\lambda_2 e^{\frac{\lambda(K-r\lambda_0)^2}{2r^2\sigma^2}} r\sigma_2}{\sqrt{2\pi}\lambda\sigma} < 0,$$

$$\frac{\partial \pi}{\partial c_1} = -\lambda_1 m_1 < 0, \quad \frac{\partial \pi}{\partial c_2} = -\lambda_2 m_2 < 0, \quad \frac{\partial \pi}{\partial A} = -1 < 0.$$
which is clearly non-negative. Also,  

\[ \frac{\partial \pi}{\partial \lambda} = PPCM + \lambda \left( \frac{\partial PPCM}{\partial \lambda} \right) + PPCM + \frac{r\bar{\sigma}}{2\sqrt{2\pi}\lambda} e^{-\frac{(K-rm)^2}{2\sigma^2}}, \]

which is non-negative when the PPCM is positive. Finally, if \( \sigma_1 = \sigma_2 = 0 \), then the uncapped revenue random variable reduces to \( rm \), and 

\[ \pi = (rm) \land (K(\lambda_1 + \lambda_2)) - c_1\lambda_1m_1 - c_2\lambda_2m_2 - A \]

\[ = (\lambda_1 + \lambda_2)[(r\bar{m}) \land K - \bar{c}] - A. \]

This is clearly the limit of (1) as \( \bar{\sigma} \rightarrow 0 \). 

**Proof of Proposition 2.** Expected revenues for each type 1 patient equal \( E[R_1 \land \kappa_1] \), where \( R_1 \sim rN(m_1, \sigma_1^2) \) and \( \kappa_1 = \lambda_1K \). Then,

\[ E[R_1 \land \kappa_1] = r\sigma_1 E \left[ \frac{R_1 - rm_1}{r\sigma_1} \land \frac{\kappa_1 - rm_1}{r\sigma_1} \right] + rm_1 = r\sigma_1 E[Z \land z] + rm_1 \]

where \( Z \) is the standard normal r.v. and \( z = (\kappa_1 - rm_1)/(r\sigma_1) \). Now,

\[ E[Z \land z] = \int_{-\infty}^{z} x\phi(x)dx + z[1 - \Phi(z)] \]

\[ = -\phi(z) + z[1 - \Phi(z)] \]

where the first equality is by definition and the second equality follows directly from the properties of the normal distribution (e.g., Zipkin, 2000, p.459). Then

\[ E[R_1 \land \kappa_1] = r\sigma_1[-\phi(z) + z(1 - \Phi(z))] + rm_1 \]

\[ = r\sigma_1 \left[ -\phi \left( \frac{\kappa_1 - rm_1}{r\sigma_1} \right) + \frac{\kappa_1 - rm_1}{r\sigma_1} \left( 1 - \Phi \left( \frac{\kappa_1 - rm_1}{r\sigma_1} \right) \right) \right] + rm_1 \]

Substituting \( \kappa_1 = \lambda_1K \) and rearranging terms yields

\[ E[R_1 \land \kappa_1] = K - (K - rm_1)\Phi \left( \frac{K/r - m_1}{\sigma_1} \right) - r\sigma_1\phi \left( \frac{K/r - m_1}{\sigma_1} \right). \]

A similar result holds for the per patient revenue for type 2 patients. Combining this with the per patient costs and fixed costs the result follows. 

**Proof of Proposition 3.** Fix \( t \) and \( i \), and consider one unit of class \( i \) fluid arriving at time \( t \), the fraction remaining in the system at time \( s \geq t \) equals \( \Pr(X_i > s - t) \), where \( X_i \) is a generic random variable with density \( f_i(\cdot) \). Note that lifetimes are not random in this model so \( \Pr(X_i > s - t) \) is
the exact fraction that remains, not the expected fraction. Then the terminal value \( v_i(t) \) associated with one unit of class \( i \) fluid arriving at time \( t \) is given by

\[
v_i(t) = v_i \Pr(X_i > T - t).
\]

The cumulative (potential) revenue \( r_i(t) \) over \([0, T]\) from one unit of class \( i \) fluid arriving at time \( t \) is given by

\[
r_i(t) = r_i \int_t^T \Pr(X_i > s - t) ds = r_i \int_0^{T-t} \Pr(X_i > x) dx = r_i \int_0^{\infty} (x \wedge (T - t)) f_i(x) dx,
\]

where the final equality is achieved with an interchange of integrals. The derivation of \( c_i(\cdot) \) follows similarly. The results for exponential follow immediately by substituting the exponential density for \( f_i(\cdot) \).

Proof of Proposition 4. Note that \( t_i(\cdot) \) and \( \tilde{t}_i(\cdot) \) may not be differentiable at all points. However, their right and left derivatives exist. Therefore, a viable proof strategy to show differentiability is to establish that the right and left derivatives of \( F \) are equal. In the interest of brevity, we will proceed as if \( t_i(\cdot) \) and \( \tilde{t}_i(\cdot) \) are differentiable, but it will be clear in calculating \( F' \) that whether we use the left derivatives of \( t_i(\cdot) \) and \( \tilde{t}_i(\cdot) \) or the right derivatives makes no difference in calculating \( F' \) because the terms involving \( t_i' \) and \( \tilde{t}_i' \) will vanish.

To be more specific, note by Leibnitz’s differentiation rule that

\[
F'(q) = \sum_{i=1}^{2} \left( \int_{\underline{t}_i(q)}^{t_i(q)} \frac{(r_i(t) - K)^2}{\eta_i} dt + \frac{r_i(\tilde{t}_i(q)) - K}{\eta_i} \delta_i^q(\tilde{t}_i(q)) \tilde{t}_i'(q) - \frac{r_i(t_i(q)) - K}{\eta_i} \delta_i^q(t_i(q)) t_i'(q) \right),
\]

where the last two terms are zero regardless of whether one uses the left or the right derivative of \( t_i(\cdot) \) and \( \tilde{t}_i(\cdot) \). This follows because for \( i = 1, 2 \), \( t_i(\cdot) \) and \( \tilde{t}_i(\cdot) \) are constant (with zero derivative) except when they equal \((\delta_i^q)^{-1}(0)\). Then, because \( \underline{t}_i(\cdot) \) and \( \tilde{t}_i(\cdot) \) are continuous for \( i = 1, 2 \), \( F \) is continuously differentiable with

\[
F'(q) = \sum_{i=1}^{2} \left( \int_{\underline{t}_i(q)}^{t_i(q)} \frac{(r_i(t) - K)^2}{\eta_i} dt \right).
\]

This is strictly positive whenever the integral is non-empty, which holds when \( t_i(q) < T \).

Proof of Theorem 1. Note that the hospice manager’s problem (P) is equivalent to the optimal control problem (44)-(48) presented in Appendix B, where \( \alpha_i(t) = \dot{z}_i(t) \) for all \( i, t \). Similarly,
the dual optimal control problem (60)-(64) is equivalent to the dual problem (D), introduced in
Appendix B, with \( \dot{p}(t) = 0 \) and \( p(t) = (K(1 - q), q) \) for all \( t \). Rockafellar (1970) provides a duality
relationship between (44)-(48) and (60)-(64), whereby the two formulations have the same optimal
objective. Moreover, by Theorem 5 of Rockafellar (1970), the optimal solutions to (44)-(48) and
(60)-(64), must satisfy

\[
(\dot{p}(t), p(t)) \in \partial L(t, (z(t), \zeta(t)), (\dot{z}(t), \dot{\zeta}(t)))
\]  

(18)

for \( t \in [0, T] \). Also note by Proposition 8.12 of Rockafellar and Wets (1997) that for any proper
convex function \( f \), its subgradient set \( \partial f(\bar{x}) \) at \( \bar{x} \) is given by

\[
\partial f(\bar{x}) = \{ u : f(x) \geq f(\bar{x}) + \langle u, x - \bar{x} \rangle \text{ for all } x \}.
\]

Namely, for \( v \in \partial f(\bar{x}) \), we must have that \( f(x) \geq f(\bar{x}) + \langle v, x - \bar{x} \rangle \) for all \( x \). Rearranging the
terms gives

\[
\langle v, \bar{x} \rangle - f(\bar{x}) \geq \langle v, x \rangle - f(x) \text{ for all } x,
\]

which holds with equality for \( x = \bar{x} \). Therefore,

\[
\langle v, \bar{x} \rangle - f(\bar{x}) = \sup_{x} \{ \langle v, x \rangle - f(x) \} = f^*(v).
\]  

(19)

Hence, we conclude that \( v \in \partial f(\bar{x}) \) if and only if \( \bar{x} \) is an element of the set \( \arg \max_{x} \{ \langle v, x \rangle - f(x) \} \)
in defining \( f^*(v) \), c.f., (19). Using this observation, (18) holds if and only if, for \( t \in [0, T], \)

\[
(z(t), \zeta(t), \dot{z}(t), \dot{\zeta}(t)) \in \arg \max_{x,y} \{ (\dot{p}(t), p(t)) \cdot (x, y) - L(t, x, y) \}.
\]  

(20)

By Proposition 7, (20) is equivalent to having

\[
\dot{z}_i(t) = \left[ \frac{p^z_i + r_i(t)p^\zeta_i + v_i(t) - c_i(t)}{\eta_i} \right] \quad \text{for } i = 1, 2,
\]

(21)

\[
\dot{\zeta}_i(t) = r_i(t)\dot{z}_i(t), \quad \text{for } i = 1, 2.
\]

Also, note by equivalence of the formulations (60)-(64) and (D) that

\[
p^z = K(1 - q) \quad \text{and} \quad p^\zeta = q.
\]

Similarly, by the equivalence of (44)-(48) and (P) we have that

\[
\alpha(t) = \dot{z}(t) \text{ for } t \in [0, T].
\]
Therefore, (21) gives the hospice manager’s optimal recruiting rate as

$$\alpha_i(t) = \frac{[K(1-q) + r_i(t)q + v_i(t) - c_i(t)]^+}{\eta_i}, i = 1, 2.$$  

(22)

Then using the definitions of $\ell_i(\cdot)$ and $\bar{t}_i(\cdot)$, c.f. Equations (9) and (10), we conclude that

$$\alpha^*_i(t) = \begin{cases} 
\frac{K(1-q^*)+q^*r_i(t)-c_i(t)+v_i(t)}{\eta_i} & \text{if } t \leq \bar{t}_i(q^*), \\
0 & \text{otherwise}
\end{cases}$$

where $q^*$ is the optimal solution to the dual problem as characterized in Proposition 8.

The monotonicity results follow directly from (22) and the reasoning below equation (8). □

**Proof of Proposition 5.** Let $\alpha(\cdot)$ denote a feasible nonstationary policy. Then let

$$\bar{\alpha}_i = \frac{1}{T} \int_0^T \alpha_i(s)ds,$$

and observe that for $i = 1, 2$,

$$\int_0^T \frac{1}{T} s_i(\alpha_i(t))dt > s_i \left( \int_0^T \frac{1}{T} \alpha_i(t)dt \right) = s_i(\bar{\alpha}_i)$$

(23)

by Jensen’s inequality and strict convexity of $s_i(\cdot)$. Multiplying both sides of (23) by $T$ gives

$$\int_0^T s_i(\alpha_i(t))dt > Ts_i(\bar{\alpha}_i)$$

so that the recruiting costs are strictly larger for the nonstationary policy $\alpha(\cdot)$ than its stationary counterpart $\bar{\alpha}$ while all other costs and the revenues are the same for the two policies. Thus, switching over to the stationary policy $\bar{\alpha}$ strictly improves the hospice’s profit. □

**Proof of Proposition 6.** We can make this optimization convex and put in the form of Boyd and Vandenberghe (2004) as follows:

$$\min -\xi + \sum_{i=1}^{2} c_i m_i (\lambda_i + \alpha_i) + \sum_{i=1}^{2} s_i (\alpha_i)$$

subject to $\xi - K \sum_{i=1}^{2} (\lambda_i + \alpha_i) \leq 0,$

$$\xi - r \sum_{i=1}^{2} (\lambda_i + \alpha_i) m_i \leq 0,$$

$$-\alpha_i \leq 0, i = 1, 2.$$
Writing the KKT conditions (see, e.g., p. 243 of Boyd and Vandenberghe, 2004) for this optimization, let \( \gamma_i \) be the Lagrange multiplier associated with the \( i \)th constraint, \( i = 1, \ldots, 4 \), then setting the gradients equal to zero implies

\[
\begin{align*}
-1 + \gamma_1 + \gamma_2 &= 0, \\
c_1m_1 + \eta_1\alpha_1 - K\gamma_1 - rm_1\gamma_2 - \gamma_3 &= 0, \\
c_2m_2 + \eta_2\alpha_2 - K\gamma_1 - rm_2\gamma_2 - \gamma_4 &= 0.
\end{align*}
\]

That is,

\[
\begin{align*}
\alpha_1 &= \frac{1}{\eta_1} \left( K\gamma_1 + rm_1\gamma_2 + \gamma_3 - c_1m_1 \right) \quad (24) \\
\alpha_2 &= \frac{1}{\eta_2} \left( K\gamma_1 + rm_2\gamma_2 + \gamma_4 - c_2m_2 \right) \quad (25) \\
\gamma_1 + \gamma_2 &= 1 \quad (26)
\end{align*}
\]

We also note that the coextremality conditions give the following:

\[
\begin{align*}
\xi - K(\alpha_1 + \alpha_2) - K(\lambda_1 + \lambda_2) &\leq 0, \quad (27) \\
\xi - r(\alpha_1m_1 + \alpha_2m_2) - r(\lambda_1m_1 + \lambda_2m_2) &\leq 0, \quad (28) \\
\alpha_1 &\geq 0, \quad (29) \\
\alpha_2 &\geq 0, \quad (30) \\
\gamma_1(\xi - K(\alpha_1 + \alpha_2) - K(\lambda_1 + \lambda_2)) &= 0, \quad (31) \\
\gamma_2(\xi - r(\alpha_1m_1 + \alpha_2m_2) - r(\lambda_1m_1 + \lambda_2m_2)) &= 0, \quad (32) \\
\alpha_1\gamma_3 &= 0, \quad (33) \\
\alpha_2\gamma_4 &= 0, \quad (34) \\
\gamma_1, \gamma_2, \gamma_3, \gamma_4 &\geq 0. \quad (35)
\end{align*}
\]

Then equations (24)-(35) will pin down the optimal solution.

It is easy to see that at least one of the equations (27) and (28) will bind. Therefore, we have three main cases to consider:

- Case 1: Constraint (27) does not bind.
- Case 2: Both (27)-(28) bind.
- Case 3: Constraint (28) does not bind.
We also need to consider the following four subcases (in each case):

- (a) \( \alpha_1 > 0, \alpha_2 > 0; \)
- (b) \( \alpha_1 > 0, \alpha_2 = 0; \)
- (c) \( \alpha_1 = 0, \alpha_2 > 0; \)
- (d) \( \alpha_1 = 0, \alpha_2 = 0. \)

In total we have 12 cases to analyze. To illustrate the approach consider Case 1(a).

Case 1(a): Equation (27) does not bind, \( \alpha_1 > 0, \alpha_2 > 0; \) Since \( \alpha_1 > 0, \alpha_2 > 0, \) equations (33) and (34) give \( \gamma_3 = \gamma_4 = 0. \) Moreover, since equation (27) does not bind, equation (31) gives \( \gamma_1 = 0. \) Then by equation (26), we have that \( \gamma_2 = 1. \) That is,

\[
(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (0, 1, 0, 0).
\]

Then it follows from (24)-(25) that

\[
\alpha_1^* = \frac{1}{\eta_1} (r - c_1)m_1 > 0,
\]

\[
\alpha_2^* = \frac{1}{\eta_2} (r - c_2)m_2 > 0.
\]

Since (27) does not bind and \( \gamma_2 = 1, \) (28) must bind. That is,

\[
\xi^* = r (\alpha_1m_1 + \alpha_2m_2) + r (\lambda_1m_1 + \lambda_2m_2)
= r \left( \frac{1}{\eta_1} (r - c_1)m_1^2 + \frac{1}{\eta_2} (r - c_2)m_2^2 \right) + r (\lambda_1m_1 + \lambda_2m_2).
\]

To be consistent, we must have (27) not bind. That is,

\[
\xi^* < K(\alpha_1 + \alpha_2) + K(\lambda_1 + \lambda_2)
\]

or, we must have

\[
r \left( \frac{1}{\eta_1} (r - c_1)m_1^2 + \frac{1}{\eta_2} (r - c_2)m_2^2 \right) + r (\lambda_1m_1 + \lambda_2m_2) < K \left( \frac{1}{\eta_1} (r - c_1)m_1 + \frac{1}{\eta_2} (r - c_2)m_2 \right) + K(\lambda_1 + \lambda_2)
\]

which is the condition in the proposition statement.

Case 3(a): Equation (28) does not bind, \( \alpha_1 > 0, \alpha_2 > 0. \) Thus, \( \gamma_2 = 0 \) by (32), then \( \gamma_1 = 1 \) by (26). Further, \( \alpha_1 > 0 \) implies \( \gamma_3 = 0 \) by (33) and \( \alpha_2 > 0 \) implies \( \gamma_4 = 0 \) by (34). Thus,
\((\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (1, 0, 0, 0)\). Then by (24)-(25) we have
\[
\begin{align*}
\alpha_1^* &= \frac{1}{\eta_1} (K - c_1m_1) > 0, \\
\alpha_2^* &= \frac{1}{\eta_2} (K - c_2m_2) > 0.
\end{align*}
\]
Because (28) does not bind, (27) must bind. That is,
\[
\xi^* = K \left( \frac{1}{\eta_1} (K - c_1m_1) + \frac{1}{\eta_2} (K - c_2m_2) \right) + K(\lambda_1 + \lambda_2).
\]
To be consistent, (28) must not bind. That is,
\[
\xi^* < r(\alpha_1m_1 + \alpha_2m_2) + r(\lambda_1m_1 + \lambda_2m_2)
\]
resulting in the condition,
\[
K \left( \frac{1}{\eta_1} (K - c_1m_1) + \frac{1}{\eta_2} (K - c_2m_2) \right) + K(\lambda_1 + \lambda_2) < r \left( \frac{m_1}{\eta_1} (K - c_1m_1) + \frac{m_2}{\eta_2} (K - c_2m_2) \right) + r(\lambda_1m_1 + \lambda_2m_2)
\]
which is seen in the proposition statement.

Case 2(a): Equations (27)-(28) both bind, \(\alpha_1 > 0, \alpha_2 > 0\). First, \(\alpha_1 > 0\) implies \(\gamma_3 = 0\) by (33) and \(\alpha_2 > 0\) implies \(\gamma_4 = 0\) by (34). Thus from (24)-(25),
\[
\begin{align*}
\alpha_1 &= \frac{1}{\eta_1} (K\gamma_1 + rm_1\gamma_2 - c_1m_1) = \frac{1}{\eta_1} (K\gamma_1 + rm_1(1 - \gamma_1) - c_1m_1) \quad (40) \\
\alpha_2 &= \frac{1}{\eta_2} (K\gamma_1 + rm_2\gamma_2 - c_2m_2) = \frac{1}{\eta_2} (K\gamma_1 + rm_2(1 - \gamma_1) - c_2m_2). \quad (41)
\end{align*}
\]
Because (27) and (28) bind,
\[
K(\alpha_1 + \alpha_2) + K(\lambda_1 + \lambda_2) = r(\alpha_1m_1 + \alpha_2m_2) + r(\lambda_1m_1 + \lambda_2m_2).
\]
Substituting (40) and (41) and solving for \(\gamma_1\), results in
\[
\gamma_1^* = \frac{(rm_1 - K)(r-c_1)m_1 + \lambda_1) + (rm_2 - K)(r-c_2)m_2 + \lambda_2)}{\frac{1}{\eta_1}(K - rm_1)^2 + \frac{1}{\eta_2}(K - rm_2)^2}
\]
and
\[
\begin{align*}
\alpha_1^* &= \frac{1}{\eta_1} (K\gamma_1^* + rm_1(1 - \gamma_1^*) - c_1m_1) \\
\alpha_2^* &= \frac{1}{\eta_2} (K\gamma_1^* + rm_2(1 - \gamma_1^*) - c_2m_2),
\end{align*}
\]
which is the result in the proposition statement. Cases 1(b), 1(c), 1(d), 2(b), 2(c), 2(d), 3(b), 3(c), and 3(d) all have internal contradictions and thus are invalid. For example, consider Case 1(b).

Case 1(b): Equation (27) does not bind, \(\alpha_1 > 0, \alpha_2 = 0\). If (27) does not bind, then \(\gamma_1 = 0\) by (31)), and therefore \(\gamma_2 = 1\) (by (26)). Since \(\alpha_1 > 0, \gamma_3 = 0\) by (33). By (25), \(\alpha_2 = \frac{1}{\eta_2} (rm_2 + \gamma_4 - c_2m_2) = 0\) which implies \(\gamma_4 = m_2(c_2 - r) < 0\), which is a contradiction with (35).
B Duality Analysis

In this appendix, we derive the dual formulation and some auxiliary results. To facilitate the statement of the dual formulation, let

\[ \beta_1 = K(\lambda_1 + \lambda_2)T \]  \hspace{1cm} (42)
\[ \beta_2 = \sum_{i=1}^{2} \int_0^T \lambda_i r_i(s) ds + r_1(0)n_1 + r_2(0)n_2 \]  \hspace{1cm} (43)

It will be shown below that the dual formulation (D) of (P) can be stated as follows: Choose \( q \) so as to

\[ \min \beta_1 (1 - q) + \beta_2 q + \sum_{i=1}^{2} \int_{\xi_i(q)}^{\eta_i(q)} \frac{1}{2\pi} (K + (r_i(t) - K)q + v_i(t) - c_i(t))^2 dt \]

as to

\[ \text{subject to } 0 \leq q \leq 1, i = 1, 2. \]  \hspace{1cm} (D)

This appendix proves this statement and (in Proposition 8) provides the optimal solution to this dual formulation.

For \( i = 1, 2 \) and \( t \in [0, T] \), let

\[ z_i(t) = \int_0^t \alpha_i(s) ds \]

(in particular \( \dot{z}_i(s) = \alpha_i(s) \)). That is, \( z_i(t) \) is the cumulative number of recruited patients of type \( i \) until time \( t \) and \( \dot{z}_i(t) \) is the rate of recruiting of patients of type \( i \) at time \( t \). Observe that the hospice manager’s problem (P) can be written as follows: Choose \( \dot{z}(-) \) and \( \dot{\zeta}(-) \) so as to

\[ \min \{- \min(\beta_1 + Kz_1(T) + Kz_2(T), \beta_2 + \zeta_1(T) + \zeta_2(T)) \} + \int_0^T \sum_{i=1}^{2} [(c_i(t) - v_i(t))\dot{z}_i(t) + s_i(\dot{z}_i(t))] dt \]  \hspace{1cm} (44)

subject to

\[ z_i(t) = z_i(0) + \int_0^t \dot{z}_i(s) ds, z_i(0) = 0, \]  \hspace{1cm} (45)
\[ \zeta_i(t) = \zeta_i(0) + \int_0^t \dot{\zeta}_i(s) ds, \zeta_i(0) = 0, \]  \hspace{1cm} (46)
\[ \dot{z}_i(s) \geq 0, \]  \hspace{1cm} (47)
\[ \dot{\zeta}_i(s) = r_i(s)\dot{z}_i(s). \]  \hspace{1cm} (48)

The cumulative revenue accrued until time \( t \) from patients of type \( i \) is \( \zeta_i(t) \) and \( \dot{\zeta}_i(t) \) is the revenue rate at time \( t \) from patients of type \( i \). To put this in the framework of Rockafellar (1970), define

\[ L(t, x, y) = \sum_{i=1}^{2} [(c_i(t) - v_i(t))y^*_i + s_i(y^*_i) + \chi_{\{\dot{y}_i^* = r_i(t)y_i^*\}}, \]  \hspace{1cm} (49)
\[ l(x, y) = \chi_{\{x = 0, x < 0\}} - \min \{\beta_1 + Ky_1^* + Ky_2^*, \beta_2 + y_1^* + y_2^*\} \]  \hspace{1cm} (50)
for \( x_i = (x_i^x, x_i^y) \in \mathbb{R}^2, x = (x_1, x_2) \in \mathbb{R}^2, y_i = (y_i^x, y_i^y) \in \mathbb{R}^2, y = (y_1, y_2) \in \mathbb{R}^4 \) and \( t \in [0, T] \), where \( \chi_F(\cdot) \) is an “indicator” function taking values zero or infinity. Namely,

\[
\chi_F(a) = \begin{cases} 
\infty & \text{if } a \notin F, \\
0 & \text{otherwise.}
\end{cases}
\]

Notice \( L(t, x, y) \) is independent of \( x \) because the profit function is independent of the cumulative number of recruited patients and cumulative revenue within the year; the cumulative number of patients and cumulative revenue is relevant only at time \( T \) which are represented by \( y \) in \( l(x, y) \). Then the hospice manager’s problem can be written as follows: Choose the functions \( \dot{z}(\cdot), \zeta(\cdot) \) so as to

\[
\min_t l((z(0), \zeta(0)), (z(T), \zeta(T))) + \int_0^T L(t, (z(t), \zeta(t)), (\dot{z}(t), \dot{\zeta}(t))) dt. \tag{51}
\]

Following Rockafellar (1970) to derive the dual problem \(^{10} \)

\[
m(d(0), d(T)) = l^*(d(0), -d(T)), \tag{52}
\]

\[
M(t, p, s) = L^*(t, s, p), \tag{53}
\]

where \( l^* \) and \( L^* \) are convex conjugates of \( l \) and \( L \), respectively, \( m \) is the boundary function dual to \( l \), \( M \) is the Lagrangian function dual to \( L \), and \( d(t) = (d^x(t), d^y(t)) \in \mathbb{R}^4 \). The dual problem can then be stated as follows: Choose \( p(\cdot) \) and \( \dot{p}(\cdot) \) so as to

\[
\min_t m(p(0), p(T)) + \int_0^T M(t, p(t), \dot{p}(t)) dt. \tag{54}
\]

The next step is to characterize \( m, M \) which is done in the next proposition.

**Proposition 7** We have that

\[
m(d^x(0), d^y(0), d^x(T), d^y(T)) = \beta_1 d^x_1(T) / K + \beta_2 d^y_2(T) + \chi_{\{d^x_1(T) = d^y_2(T), j = z, \zeta\}} + \chi_{\{d^x_1(T) + K d^y_2(T) = K\}} + \chi_{\{d^x_j(T) \geq 0, i = 1, 2, j = z, \zeta\}}, \tag{55}
\]

\[
M(t, p, s) = \sum_{i=1}^2 \frac{1}{2\eta_i} ([p^x_i + r_i(t)p^y_i + v_i(t) - c_i(t)]^2 + \chi_{\{s = 0\}}). \tag{56}
\]

Moreover,

\[
\arg \max_{x, y} \{(s, p) \cdot (x, y) - L(t, x, y)\} = \left\{ (x, y) : y_i^x = \left[ \frac{p^x_i + r_i(t)p^y_i + v_i(t) - c_i(t)}{\eta_i} \right]^+ \text{ and } y_i^y = r_i(t) y_i^x \right\}. \tag{57}
\]

\(^{10}\)Here we are following Rockafellar’s (1970) notation as closely as possible to facilitate the use of his results. For example, the swapping of the order of arguments in \( M \) and \( L^* \) in (53) matches with his equation (5.5) on p. 190.
Moreover, its objective is given by 
\[ m(d^2(0), d^0(T), d^z(T), d^\zeta(T)) = l^*(d^2(0), d^0(T), -d^2(T), -d^z(T)) \]

\[ = \sup_c \{ c_1(T)d_1^z(T) + c_2(T)d_2^z(T) + c_1^\zeta(T)d_1^\zeta(T) + c_2^\zeta(T)d_2^\zeta(T) - \chi_{\{c=0\}} \}
+ \min \{ \beta_1 + Kc_1(T) + Kc_2(T), \beta_2 + c_1^\zeta(T) + c_2^\zeta(T) \} \}

\[ = \sup_{c(T)} \{ -c_1(T)d_1^z(T) - c_2(T)d_2^z(T) - c_1^\zeta(T)d_1^\zeta(T) - c_2^\zeta(T)d_2^\zeta(T) \}
+ \min \{ \beta_1 + Kc_1(T) + Kc_2(T), \beta_2 + c_1^\zeta(T) + c_2^\zeta(T) \} \}

The optimization problem is equivalent to the following linear program:
\[
\max_{c(T)} \xi - c_1^z(T)d_1^z(T) - c_2^z(T)d_2^z(T) - c_1^\zeta(T)d_1^\zeta(T) - c_2^\zeta(T)d_2^\zeta(T)
\]
subject to
\[
\xi \leq \beta_1 + Kc_1(T) + Kc_2(T),
\]
\[
\xi \leq \beta_2 + c_1^\zeta(T) + c_2^\zeta(T).
\]

The dual linear program is given by
\[
\min \beta_1 y_1 + \beta_2 y_2
\]
subject to
\[
\begin{bmatrix}
-K & 0 \\
-K & 0 \\
0 & -1 \\
0 & -1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix}
= \begin{bmatrix}
-d_1^z(T) \\
-d_2^z(T) \\
-d_1^\zeta(T) \\
-d_2^\zeta(T) \\
1
\end{bmatrix},
\]
\[ y \geq 0, \]
whose constraints are equivalent to the following:
\[
d_1^z(T) = d_2^z(T) \text{ for } j = z, \zeta,
\]
\[
d_1^z(T)/K + d_1^\zeta(T) = 1,
\]
\[
d_1^i(T) \geq 0 \text{ for } i = 1, 2, j = z, \zeta.
\]
Moreover, its objective is given by \( \beta_1 d_1^z(T)/K + \beta_2 d_1^\zeta(T). \) Therefore,
\[
m(d^2(0), d^0(0), d^z(T), d^\zeta(T)) = \beta_1 d_1^z(T)/K + \beta_2 d_1^\zeta(T) + \chi_{\{d_1^z(T)=d_2^z(T), j=z,\zeta\}} + \chi_{\{d_1^z(T)+Kd_1^\zeta(T)=K\}}
+ \chi_{\{d_1^i(T)\geq0, i=1,2, j=z,\zeta\}}.
\]
Similarly,

\[ M(t, p, s) = L^*(t, s, p), \]

where

\[
L^*(t, s, p) = \sup_{x,y} \{ (s,p) \cdot (x,y) - L(t, x, y) \} \\
= \sup_{x, s, y, p} \left\{ x_1^2 + x_2^2 + y_1^2 + y_2^2 + y_1^2 p_1 + y_2^2 p_2 \right\} \\
- \sum_{i=1}^{2} \left[ (c_i(t) - v_i(t))y_i^2 + \frac{1}{2} \eta_i(y_i^2)^2 \right] : y_i^2 \geq 0, y_i^* = r_i(t)y_i^2.
\]

It is clear that we must have \( s \equiv 0 \) (and that \( x^\sharp, x^\zeta \) can take any value). Then

\[
L^*(t, s, p) = \chi_{\{s=0\}} + \sup_{y^\sharp} \left\{ y_1^2 p_1 + y_2^2 p_2 + r_1(t)y_1^2 p_1 + r_2(t)y_2^2 p_2 \right\} \\
- \sum_{i=1}^{2} \left[ (c_i(t) - v_i(t))y_i^2 + \frac{1}{2} \eta_i(y_i^2)^2 \right] : y_i^2 \geq 0 \right\},
\]

where we substituted \( y_i^* = r_i(t)y_i^2 \). Notice that the optimization problem on the right-hand side decomposes so that

\[
L^*(t, s, p) = \sum_{i=1}^{2} \sup_{y_i^\zeta} \left\{ y_i^\zeta (p_i^\zeta + r_i(t)p_i^\zeta - c_i(t) + v_i(t)) - \frac{1}{2} \eta_i(y_i^\zeta)^2 : y_i^\zeta \geq 0 \right\} + \chi_{\{s=0\}}. \tag{58}
\]

Then the first order conditions give

\[
y_i^\zeta = \left[ \frac{p_i^\zeta + r_i(t)p_i^\zeta + v_i(t) - c_i(t)}{\eta_i} \right]^+ . \tag{59}
\]

Substituting this into (58) and using the definition \( M(t, p, s) = L^*(t, s, p) \) gives

\[
M(t, p, s) = \sum_{i=1}^{2} \frac{1}{2\eta_i} (|p_i^\zeta + r_i(t)p_i^\zeta + v_i(t) - c_i(t)|^2)^2 + \chi_{\{s=0\}}.
\]

Moreover, (57) follows from (59), and the fact that \( y_i^\zeta = r_i(t)y_i^2 \) and that \( (x^\sharp, z^\zeta) \) can take any value because \( s \equiv 0 \).

Then combining Proposition 7 with (54) gives the dual formulation: Choose \( p(\cdot), \dot{p}(\cdot) \) so as to

\[
\min \beta_1 p_1(T)/K + \beta_2 p_1(T) + \sum_{i=1}^{2} \int_{0}^{T} \frac{1}{2\eta_i} (|p_i^\zeta(t) + r_i(t)p_i^\zeta(t) + v_i(t) - c_i(t)|^2)^2 dt \tag{60}
\]

subject to

\[
p_j(T) = p_j(T) \text{ for } j = z, \zeta, \tag{61}
\]
\[
p_1(T) + Kp_1(T) = K, \tag{62}
\]
\[
p_i(T) \geq 0 \text{ for } i = 1, 2, j = z, \zeta, \tag{63}
\]
\[
\dot{p}_i(t) = 0 \text{ for } i = 1, 2. \tag{64}
\]

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Note that \( p(t) = p(T) \) for all \( t \in [0, T] \) because \( \dot{p}(t) = 0 \). Then letting \( q = p_1^r(T) \), using the constraint that \( p_1^r(T) + Kp_1^r(T) = K \), and using the definition of \( \delta_q^r(\cdot) \), c.f. Equations (8)-(10), we arrive at the dual formulation (D) introduced earlier.

\[
\min \beta_1 (1 - q) + \beta_2 q + \sum_{i=1}^{2} \int_{L_i(q)}^{\tilde{t}_i(q)} \frac{1}{2\eta_i} (\delta_q^r(t))^2 dt
\]

subject to

\[ q \geq 0. \]

**Proposition 8** The optimal solution \( q^* \) of the dual formulation is given as follows:

\[
q^* = \begin{cases} 
0 & \text{if } F(0) \geq 0, \\
F^{-1}(0) & \text{if } F(0) < 0 < F(1), \\
1 & \text{if } F(1) \leq 0, 
\end{cases} \tag{65}
\]

**Proof.** Define

\[
G(q) = \beta_1 (1 - q) + \beta_2 q + \sum_{i=1}^{2} \int_{L_i(q)}^{\tilde{t}_i(q)} \frac{1}{2\eta_i} (\delta_q^r(t))^2 dt
\]

then the dual formulation can be written as:

\[
\min G(q) \text{ subject to } 0 \leq q \leq 1. \tag{67}
\]

It is straightforward to show that for \( q \in [0, 1] \),

\[
G'(q) = -\beta_1 + \beta_2 + \sum_{i=1}^{2} \int_{L_i(q)}^{\tilde{t}_i(q)} \frac{r_i(t) - K}{\eta_i} \delta_q^r(t) dt,
\]

which makes it clear that \( F \equiv G' \). There are two cases to consider. First, suppose that \( \tilde{t}_i(q) < T \) for \( 0 \leq q \leq 1 \), then \( F' > 0 \), c.f. Proposition 4, \( G \) is strictly convex and the first order conditions are sufficient to pin down the unique optimal solution. Moreover, because \( F \) is strictly increasing, it follows that if \( F(0) \geq 0 \), then \( q^* = 0 \); and if \( F(1) \leq 0 \), then \( q^* = 1 \). Otherwise, we have an interior solution characterized by \( F(q) = 0 \), which yields \( q^* = F^{-1}(0) \).

If \( \tilde{t}_i(q) = T \) for some \( \tilde{q} \) we must consider two subcases. First suppose \( INC(\tilde{q}) \), then we have that \( \delta_q^r(T) \leq 0 \) (because \( \tilde{t}_i(\tilde{q}) = T \)). Further, for any \( q < \tilde{q} \) we have \( INC(q), \delta_q^r(T) < 0 \), and \( G(q) = \beta_1 (1 - q) + \beta_2 q \). Further, if \( F(0) \geq 0 \) then \( G(q) \) is minimized (over the range \([0, \tilde{q}]\)) at \( q^* = 0 \). If \( F(0) < 0 \) then \( G(q) \) is minimized (over the range \([0, \tilde{q}]\)) at \( \tilde{q} \) so it suffices to restrict attention to \( q \geq \tilde{q} \) and the above convexity argument can be repeated by restricting attention to
the region where $t_i(q) < T$. Now suppose $DEC(\tilde{q})$, then $t_i(\tilde{q}) = T$ implies that $\delta_\tilde{q}_i(0) < 0$. Further, for any $q > \tilde{q}$ we have $DEC(q)$, $\delta_q^i(0) < 0$, and $G(q) = \beta_1(1 - q) + \beta_2q$. In this case, if $F(1) \leq 0$ then $G(q)$ is minimized (over the range $[\tilde{q}, T]$) at $q^* = 1$. If $F(1) > 0$ then $G(q)$ is minimized (over the range $[\tilde{q}, T]$) at $\tilde{q}$ so if suffices to restrict attention to $q \leq \tilde{q}$ and the above convexity argument can again be repeated.

References


