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Seemingly Unrelated Regressions

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The Seemingly Unrelated Regressions (SUR) model explains the variation of not just one dependent variable, as in the univariate multiple regression model, but the variation of a set of m dependent variables, e.g. the monthly consumption expenditures of m consumers or the annual voting behavior of m voters, in terms of the variation of general and specific input or independent variables and error terms specific to each individual, problems that are frequently encountered in many sciences. Indeed, Geweke (2003, p. 162) has written, "The seemingly unrelated regressions (SUR) model developed in Zellner (1962) is perhaps the most widely used econometric model after linear regressions. The reason is that it provides a simple and useful representation of systems of demand equations that arise in neoclassical static theories of producer and consumer behavior."

It is the case that a SUR model is a collection of two or more regression relations that can be analyzed with data on the dependent and independent variables. For many years, the individual regression relations were fitted one by one, usually using least squares techniques and justified by an appeal to single equation estimation optimality properties, e.g. the least squares estimators are best linear unbiased estimators according to the well known Gauss Markov theorem and maximum likelihood estimators when single equation normal likelihood functions are employed. What was overlooked in this pre-1962 literature is the fact that when the error terms in the different regression equations are correlated the regression equations are related, not unrelated and that the sample information in other regressions can be employed to improve the precision of estimation of parameters in any given regression equation under a wide range of conditions. That is, new, operational SUR best linear unbiased estimators for the parameters of a set of say m regression equations were put forward that uniformly dominate the single equation least squares estimators under a broad range of conditions. It was shown that these SUR or generalized least squares (GLS) estimators are best linear unbiased and maximum likelihood and Bayesian estimators under frequently encountered conditions. And in addition, by joint analysis of the set of regression equations rather than equation by equation analysis, more precise estimates and predictions are obtained that lead to better solutions to many applied problems, e.g. portfolio formation procedures in work by Quintana, et al (2003) in which dynamic regression equations with time varying parameters and various input variables were employed to explain the variation of monthly stock prices. By taking account of the fact that the regression equations were related and not unrelated, SUR estimation, prediction and portfolio formation procedures were utilized to yield improved analyses of the variation of stock prices and to form optimal portfolios with very good rates of return. For textbook and other analyses of the SUR model and applications of it, see. e.g., Geweke (2003), Greene (2003), Judge et al. (1985), Meng and Rubin (1996), Percy (1992, 1996), Rossi, Allenby and McCulloch (2005), Quintana, Putnam and Wilford (1998), Srivastava and Giles (1987), Theil (1971), Zellner (1962,1963), and Zellner and Huang (1962). Also, in Zellner and Theil (1962) similar techniques were applied to simultaneous equations models to yield a new joint estimator, the three stage least squares estimator that dominates single equation estimators by taking account of the correlation of error terms in equations of the system by use of joint estimation of coefficients in equations of structural models.

The simplest version of a linear, constant parameter SUR system is one that contains $m \geq 2$ linear regression equations, $y_i = X_i \beta_i + u_i$, i = 1,2,…, m, where y_i is an nx1 vector of observations on the i'th dependent variable, X_i is an nxk_i matrix with full column rank of observations on the k_i independent variables in the $i'th$ regression equation, β_i is a $k_i x1$ vector of regression parameters and u_i is an nx1 vector of zero mean error terms. The usual method of estimating the regression coefficients was to estimate the equations individually by least squares to obtain, $\hat{\beta}_i = (X_i' X_i)^{-1} X_i' y_i$, i=1,2,…,m. However, in Zellner (1962), it was shown that when the error terms are correlated across the equations, the equations are related and joint estimation, rather than equation-by-equation estimation, leads to more precise estimates of the regression coefficients and predictions of future values of the dependent variables. Indeed, as explained in the articles and texts cited below, these joint SUR estimators are generalized best linear unbiased estimators and with a normality assumption for the error terms, maximum likelihood and "diffuse prior" Bayesian estimators. Further they reduce to single equation least squares estimators when error terms in the different equations are mutually uncorrelated, i.e., the equations are unrelated. In addition, use of SUR techniques leads to improved tests of hypotheses regarding regression coefficients' and other parameters' values. Similarly, taking account of the error terms' correlations across equations leads to better predictions of future values of the dependent variables; see, e.g., Zellner and Israilevich (2005) for use of SUR techniques in forecasting U.S. economic sectors' output growth rates and aggregate output growth rates. And in Chib and Greenberg (1995), Geweke (2003), and Rossi et al (2005), modern Bayesian methods, are described that yield optimal finite sample estimation, testing and prediction techniques for many variants of the SUR model, e.g. SUR models with time varying parameters, auto correlated error terms, etc. Similarly, when the dependent variables are discrete random variables as in multivariate logit or probit models with correlated error terms, the SUR joint estimation, testing and prediction techniques have been found to be useful; see, e.g. Lee et al. (1977).

References

Chib, Sid and Greenberg, Edward (1995), "Hierarchical Analysis of SUR Models with Extensions to Correlated Serial Errors and Time-Varying Parameter Models," J. of Econometrics 68, 339-360.

Geweke, John (2003), Contemporary Bayesian Econometrics and Statistics, Hoboken, NJ: Wiley.

Greene, William H. (2003), Econometric Analysis, 5th ed., Upper Saddle, NJ: Prentice-Hall.

Judge, George G., Griffiths, William E., Hill, Robin C., Lutkepohl, Helmut and Lee, T.C., (1985), The Theory and Practice of Econometrics, New York: Wiley.

Lee, T.C., Judge, George G. and Zellner, Arnold (1977), Estimating the Parameters of the Markov Probability Model from Aggregate Time Series Data, Amsterdam: North-Holland, 2nd rev. ed.

Meng, Xiao-Li and Rubin, Donald B. (1996), "Efficient Methods for Estimation and Testing with Seemingly Unrelated Regressions in the Presence of Latent Variables and Missing Observations," pp. 215-227 in Berry, Donald A., Chaloner, Katheryn M. and Geweke, John, eds. (1996), Bayesian Analysis in Statistics and Econometrics: Essays in Honor of Arnold Zellner, New York: Wiley.

Percy, David F. (1996), "Zellner's Influence on Multivariate Linear Models," pp. 203-213, in Berry, Donald A., Chaloner, Kathryn, M. and Geweke, John, eds., Bayesian Analysis in Statistics and Econometrics: Essays in Honor of Arnold Zellner, New York: Wiley.

Quintana, Jose M., Putnam, Bluford H., Wilford, David S. (1996), "Mutual and Pension Funds Management: Beating the Markets Using a Global Bayesian Investment Strategy," published in 1996 joint Sect. On Bayesian Statistical Science, Amer. Stat. Assoc. (www.amstat.org), and Int. Soc. for Bayesian Analysis (www.bayesian.org) Proceedings Volume and presented at ISBA Meeting in Istanbul, Turkey, 1995.

Rossi, Peter E., Allenby, Greg M., and McCulloch, Robert (2005), Bayesian Statistics and Marketing, Hoboken, NJ: Wiley

Srivastava, V.K. and Giles, David E. A. (1987), Seemingly Unrelated Regression Equations Models, New York: Dekker.

Theil, Henri (1971), Principles of Econometrics, New York: Wiley

Zellner, Arnold (1962), "An Efficient Method of Estimating Seemingly Unrelated Regressions and Tests for Aggregation Bias," J. Am. Stat. Assoc., 57, 348-368.

Zellner, Arnold (1963), "Estimators for Seemingly Unrelated Regressions: Some Exact Finite Sample Results," J. Am. Stat. Assoc., 58, 977-992; corrigendum, (1972), 67, 255.

Zellner, Arnold and Huang, David S. (1962), "Further Properties of Efficient Estimators for Seemingly Unrelated Regression Equations," International Economic Review, 3, 300-313.

Zellner, Arnold and Theil, Henri (1962), “Three-Stage Least Squares: Simultaneous Estimation of Simultaneous Equations,” Econometrica, 54-78.

Zellner, Arnold and Israilevich, Guillermo (2005), “The Marshallian Macroeconomic Model: A Progress Report,” Macroeconomic Dynamics, 9, 220-243 and International Journal of Forecasting 21 (2005), 627-645.