Dynamic Power Control in a Wireless Static Channel Subject to a Quality-of-Service Constraint

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A controller dynamically chooses a state-dependent transmission rate on a static, point-to-point wireless link by varying transmission power over time. The transmitter is modeled as a finite-buffer Markovian queue with adjustable service rates. That is, data packets arrive to the system according to a Poisson process, and packet size is exponentially distributed. The controller chooses a transmission rate from a fixed set \( A \) of available values, depending on the backlog in the system. The objective is to minimize long-run average energy consumption subject to a quality-of-service constraint, which is expressed as an upper bound on the packet drop rate. An explicit formula is developed for the optimal transmission rate as a function of the packet queue length.

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1. Introduction and Summary

This paper is concerned with dynamic power control in wireless communication. Energy is often a constraining factor in a wireless network. It is observed in Uysal-Biyikoglu et al. (2002) that the energy required to transmit a packet over a wireless link can be significantly reduced by lowering transmission power, which in turn results in lower transmission rates, and hence degradation in quality of service. We formulate the dynamic power control problem in the context of a static, point-to-point wireless link, which is modeled as a single-server, finite-buffer Markovian queue (see Figure 1). The objective is to minimize average energy consumption per time unit (i.e., average power) over an infinite planning horizon, subject to a quality-of-service constraint.

Data packets arrive at the system according to a Poisson process with rate \( \lambda = 1 \). An arriving packet is admitted to the system if the buffer is not full; otherwise, it is dropped. Packets have exponentially distributed size. That is, transmission of a packet at a specified “nominal” transmission rate is exponentially distributed. Throughout this paper, adapting terminology that is standard in queueing theory, the term “jobs” will be used to mean “packets,” and “serving” a job will refer to transmitting a packet. As usual, the term “queue length” will be used to mean the number of jobs in the system, including the job being served if there is one.

The queue length evolves as a birth-and-death process with constant arrival rate \( \lambda = 1 \) (unless the buffer is full) and with state-dependent service rates \( \mu_n \) that can be chosen from the set \( A = [0, \bar{\mu}] \) of available service rates. Also given is a function \( c(\cdot) \) on \( A \), where \( c(x) \) is a cost rate associated with service rate \( x \). In the context of the power control problem, readers should think of \( \mu_n \in A \) as the transmission rate to be used when there are \( n \) packets in the system, and \( c(\mu_n) \) as the power needed to induce transmission rate \( \mu_n \). It is assumed that \( c(\cdot) \) is increasing, strictly convex, and continuously differentiable on \( A \) with \( c(0) = 0 \). (The last of these assumptions is just a matter of convenient normalization.) It is not necessary for our purposes to assume a specific functional form for \( c(\cdot) \). However, Uysal-Biyikoglu et al. (2002) uses information theory to derive the following functional form.

\[
c(x) = e^{\alpha x} - 1, \quad x \in A,
\]

where \( \alpha \) is a positive constant. Of course, this particular cost function satisfies our assumptions on the cost of control \( c(\cdot) \).

For us a policy is a vector \( \mu = (\mu_1, \ldots, \mu_N) \) with all components belonging to the set \( A \), where \( N \) is the buffer size. One interprets \( \mu_n \) as the service rate to be used when the queue length is \( n \). The problem is to choose a policy \( \mu \) that minimizes average cost per time unit—i.e., average power—over an infinite planning horizon, subject to a quality-of-service constraint. In particular, we impose a lower bound on the long-run average throughput rate, which equivalently corresponds to imposing an upper bound on long-run average packet drop rate, or an upper bound on buffer overflow probability. In the language of dynamic programming, we are restricting attention to stationary Markov policies. To be more specific, we are considering a constrained...
Markov decision process with continuous time parameter, finite state space, compact action space, time-invariant data, and a long-run average-cost criterion.

Altman (1999) develops a general approach to solving constrained Markov decision problems in a discrete-time framework. Relaxing the constraints that appear in the original model formulation, he derives an equivalent “Lagrangian” problem that is itself equivalent to an (infinite) linear program. In principle, the solution to this (infinite) linear program provides a solution to the Lagrangian problem, which in turn provides a solution to the constrained Markov decision problem. A minor modification of that general method could be applied in our setting, yielding an equivalent infinite linear program similar to what one sees in Altman (1999, pp. 178–179), and presumably that approach would eventually lead to the same solution developed in this paper. Sennott (2001) develops an alternative general theory of constrained Markov decision processes, although the models there are rather special. Her general approach, like Altman’s, is potentially relevant to the problem studied here.

Rather than applying existing theory described immediately above, our analysis proceeds from first principles, which ultimately takes less space, especially when one considers the need for minor technical changes in the general theory. The arguments used here rely heavily on the special character of the problem being considered, and no claim is made of a contribution to general methodology; the contribution of this paper lies in its formulation and solution of the power control problem.

However, we do use the general theory to motivate the Lagrangian problem studied in §4, where congestion concerns are expressed through a cost component rather than a constraint. To be more specific, we replace the constraint on packet drop rate with a penalty cost for dropping packets in formulating the Lagrangian problem. By solving the Lagrangian problem parametrically for each value of the penalty rate, we develop an “explicit” solution for the power control problem, imposing no assumptions beyond the ones already set forth. To repeat, our mathematical treatment is self-contained, making no use of general theory except to motivate the solution strategy. The optimal policy ultimately obtained for the power control problem is monotone, meaning that the optimal service rate increases as a function of queue length, which is consistent with known results that will be reviewed shortly.

The rest of this paper is organized as follows. Section 2 reviews the relevant literature. The precise statement of the mathematical problem to be solved in this paper is given in §3. In §4, motivated by the general theory of constrained Markov decision processes, we introduce the Lagrangian problem, and solve it explicitly from first principles. The main result of the paper, Theorem 2, is presented in §5. Finally, a numerical example illustrating the solution method is presented in §6.

2. Background

Power control problems have been studied extensively in the wireless communication literature, cf. Zander (1993), Biglieri et al. (1998), and Uysal-Biyikoglu (2003). Research in this area can be divided into three major groups, each focusing on a different aspect of the problem. Motivated primarily by spread-spectrum cellular networks, the first group focuses on a setting where many users coexist in a medium. Existence of multiple users results in channel interference, and the goal is to alleviate channel interference by power control. A generic objective is to maximize the number of users that can be accommodated in a cell at a certain signal quality or information rate, cf. Gilhausen et al. (1991), Fleming et al. (1994), Zander (1993), and Bambos et al. (2000). Papers in the second group consider a single transmitter-receiver pair, so there is no channel interference. However, the status of the channel is varying over time, which affects the quality of transmission. Hereafter, such a channel will be referred to as a time-varying channel. A typical objective in the second group is to maximize long-run average achievable throughput rate in a time-varying channel; cf. Biglieri et al. (1998), Goldsmith (1994), and Goldsmith and Varaiya (1997). Finally, the last group, which is the most relevant to the problem studied in this paper, explores the trade-off between energy and delay; cf. Uysal-Biyikoglu et al. (2002a, b), Berry (2000), and Berry and Gallagher (2002).

Energy is often a constraining factor in a wireless network. It is observed in Uysal-Biyikoglu et al. (2002b) that the energy required to transmit a packet over a wireless link can be significantly reduced by lowering transmission power, which in turn results in longer transmission times. Because information is often time critical, it is natural to consider a problem formulation where the goal is to minimize energy subject to delay constraints. Indeed, Uysal-Biyikoglu et al. (2002b) studies the problem of scheduling packet transmissions to minimize energy consumption in the presence of constraints on packet delays. The authors consider a static channel (that is, the channel status is
not varying over time) and develop optimal schedules for
deterministic input streams, which they refer to as optimal
offline schedules. They also propose online schedules for
the actual system based on the insight gained from these
off-line schedules. Uysal-Biyikoglu et al. (2002a) extends
the formulation to time-varying channels, and advocates
scheduling policies that use information on both the queue
state and the channel state. As in the static channel case, the
authors first derive optimal offline schedules, and then pro-
pose heuristic online schedules based on the insight gained
from those.

To the best of our knowledge, the first study that explores
adaptive control policies using information on both the
queue state and the channel state is the Ph.D. thesis of
Berry (2000). He uses a discrete-time Markov chain model
to study a dynamic power control problem in the context
of a time-varying wireless channel. Actually, Berry (2000)
studies two different formulations, both of which have the
objective of minimizing the long-run average energy con-
sumption, that is, average power. In one formulation, he
imposes a constraint on expected packet delays. In his sec-
ond formulation, an upper bound on the buffer overflow
probability is imposed. Using dynamic programming, Berry
develops structural results regarding the optimal control
policy, such as the monotonicity of the control in the num-
ber packets in the system. The problem studied in this paper
is fundamentally simpler than Berry’s because we restrict
attention to static channels. Consequently, we are able to
characterize the optimal control policy explicitly.

One can also view the problem studied in this paper as
generic operations research problem of dynamic service
rate control in a Markovian queue. Many papers have been
written on different versions of the service rate control
problem over the last 30 years; cf. Crabill (1972, 1974),
Weber and Stidham (1987), Stidham and Weber (1989), and
George and Harrison (2001), and see Stidham (1988, 2002)
for a survey. Also, readers interested in the historical
development of the subject of controlling queueing systems
via dynamic programming methods are referred to the bib-
lography of a recent book by Sennott (1999). Like the
authors named in the preceeding sentences, we consider
the characterization of optimal policies in the context of
a semi-Markov decision process (SMDP) (cf. Bertsekas
1995), which is obtained when we allow the decision maker
to choose a new service rate whenever a new job arrives
or a service is completed, but not at any other time. The
monotonicity of the optimal policy we derive is consistent
with earlier results. Although formulations considered
in earlier work are somewhat related to the power control
problem that we study, our formulation is fundamentally
different: In the operations research literature, congestion
concerns are modeled typically by means of holding costs,
and one tries to optimally balance the trade-off between
holding costs and the cost of control. In contrast, we for-
mulate congestion concerns through a constraint on the
long-run average packet drop rate, which is more natural
in the context of power control in wireless communication;
cf. Berry (2000). Finally, two important antecedents of this
paper are George and Harrison (2001) and Ata et al. (2002).
Although the formulations in those papers are quite differ-
ent, some aspects of their analysis carry over directly to
our setting, as the readers will see in §4.

3. Problem Formulation

This section provides a precise statement of the mathe-
atical problem to be solved. The following derivations rely
on readers’ familiarity with the theory of birth-and-death
processes; cf. Karlin and Taylor (1997). By convention,
it is assumed that the service rate is zero whenever the
system is empty. Therefore, for us a policy is a vector \( \mu = \left( \mu_1, \ldots, \mu_N \right) \) with all components belonging to the set \( A = [0, \bar{\mu}] \). It is intuitively clear that any work-nonconserving
policy would not be optimal, as the workload needs to be
drained with minimal energy. Thus, work-nonconserving
policies would be dominated by work-conserving poli-
cies. Therefore, we require for any policy that \( \mu_n > 0 \),
\( n = 1, \ldots, N \), which will also simplify the exposition.
Given any such policy \( \mu \), it is straightforward to derive
the steady-state distribution \( \pi(\mu) = (\pi_0(\mu), \ldots, \pi_N(\mu)) \)
of the queue-length process evolving under policy \( \mu \). In
particular, \( \pi(\mu) \) satisfies the following balance equations
(recall that \( \lambda = 1 \) by convention):

\[
\pi_n(\mu) = \mu_{n+1} \pi_{n+1}(\mu) \quad \text{for } n = 0, 1, \ldots, N - 1,
\]

and the usual condition

\[
\sum_{n=0}^{N} \pi_n(\mu) = 1.
\]

By using (1) and (2), it is straightforward to find \( \pi(\mu) \)
explicitly. In particular,

\[
\pi_n(\mu) = \left[ \prod_{i=n+1}^{N} \mu_i \right] \pi_N(\mu), \quad n = 0, 1, \ldots, N - 1,
\]

where

\[
\pi_N(\mu) = \left[ 1 + \mu_N + \mu_N \mu_{N-1} + \cdots + \mu_N + \cdots + \mu_1 \right]^{-1}.
\]

As mentioned earlier, for a policy to be admissible we also require that the associated long-run average packet
drop rate be less than or equal to a certain upper bound. To
be more specific, we impose an upper bound \( \beta \) with \( 0 < \beta < 1 \) on the long-run average fraction of packets dropped,
that is, buffer overflow probability. That is, we require

\[
\pi_N(\mu) \leq \beta
\]

for any admissible policy \( \mu \). Therefore, the class of admis-
sible policies is given by the set \( \mathcal{A} \), where

\[
\mathcal{A} = \{ \mu \in \mathbb{R}^n : \mu_n \in A \setminus \{0\} \text{ for } n = 1, \ldots, N, \pi_N(\mu) \leq \beta \}.
\]
Assuming that $\mathcal{A}$ is not empty, the long-run average cost rate associated with an admissible policy $\mu$ is given by

$$z_\mu = \sum_{n=1}^{N} \pi_n(\mu)c(\mu_n).$$

Next, define

$$z^* = \inf\{z_\mu: \mu \in \mathcal{A}\}$$

with the understanding that the infimum is $+\infty$ if $\mathcal{A}$ is empty. An admissible policy $\mu$ is said to be optimal if $z_\mu = z^*$.

Because our dynamic control problem involves a constraint on the long-run average packet drop rate, the standard methods of dynamic programming do not apply immediately. Therefore, motivated by the general theory of constrained Markov decision processes (cf. Altman 1999), we first study a closely related problem (the Lagrangian problem) in §4, which can be solved explicitly by standard dynamic programming techniques. Next, by using that solution and exploiting a “duality” relationship, an explicit solution is advanced for the problem described in this section.

4. A Closely Related Problem and Its Solution

In this section, motivated by the general theory of constrained Markov decision processes (cf. Altman 1999), we consider a problem formulation with rejection penalties (the so-called Lagrangian problem), where congestion concerns are expressed through a cost component rather than a constraint. It turns out that the solution to this problem formulation can be used to solve the power control problem.

In §§4.1 through 4.3, the Lagrangian problem and its solution are presented. In particular, in §4.1 the Lagrangian problem is stated precisely, and the associated optimality equation is introduced in §4.2. Subsection 4.3 presents the solution to the optimality equation along with several preliminary results that are used in proving Theorem 1 as well as in proving the results in §5.

4.1. The Lagrangian Problem

In the model formulation that we consider in this section, congestion concerns are expressed through a cost component in the objective rather than a constraint. That is, new arrivals are rejected when the buffer is full, and congestion costs come in the form of rejection penalties (see Figure 2).

![Figure 2. A system model for the formulation with rejection penalties.](image)

In particular, there is a fixed cost $p > 0$ per rejected job. To be specific, it suffices to focus on the case $p > c'(0)$, because we restrict attention to work-conserving policies. Moreover, we do not impose any constraints on the packet drop rate. Our assumptions regarding the set of available service rates $A$, and the cost of control $c(\cdot)$ are the same as in §1; and the problem is to choose the service rate dynamically as a function of the current queue length so as to minimize long-run average cost incurred per time unit.

In this problem formulation, an admissible policy is given by a vector $\mu = (\mu_1, \ldots, \mu_N)$ with all components belonging to the set $A$. We also require $\mu_n > 0$ for $n = 1, \ldots, N$ as in §3. Therefore, the class of admissible policies $\mathcal{A}$ is given by

$$\mathcal{A} = \{\mu \in \mathbb{R}^N: \mu_n \in A \setminus \{0\} \text{ for } n = 1, \ldots, N\}.$$  

Clearly, the steady-state distribution of the queue-length process under an admissible policy $\mu$ is given by (3) and (4), and the long-run average cost $\gamma_\mu$ associated with $\mu$ is (recall that $\lambda = 1$ by convention)

$$\gamma_\mu = \sum_{n=1}^{N} \pi_n(\mu)c(\mu_n) + \pi_N(\mu)p.$$  

We also define

$$\gamma^* = \inf\{\gamma_\mu: \mu \in \mathcal{A}\},$$

and an admissible policy $\mu$ is said to be optimal if $\gamma_\mu = \gamma^*$.

4.2. Optimality Equation

The standard way of solving the problem introduced in the previous subsection is to consider the associated optimality equation, or Bellman equation, which provides a means for characterizing an optimal policy analytically. Specializing the general form of the optimality equation for a semi-Markov decision process with average cost criterion and using the uniformization technique (cf. Bertsekas 1995), one arrives at the following set of equations (recall that $\lambda = 1$ by convention):

$$v_1 = v_0 + \gamma,$$

$$v_n = \min_{x \in A} \left\{ \frac{c(x) + x}{\mu} - \frac{\gamma}{\mu} + \frac{1}{1 + \mu} v_{n+1} + \frac{x}{1 + \mu} v_{n-1} \right\}, \quad n = 1, \ldots, N - 1,$$

$$v_N = \min_{x \in A} \left\{ \frac{c(x) + p}{\mu} - \frac{\gamma}{\mu} + \frac{x}{\mu} v_{N-1} + \frac{\mu - x}{\mu} v_N \right\}.$$
Here one interprets $\gamma$ as a guess at the minimum average cost, which was denoted $\gamma^*$ in §4.1. The vector of unknowns $(v_0, v_1, \ldots, v_N)$ is often called a relative cost function in average-cost dynamic programming. One can interpret $v_n$ as the minimum expected cost incurred until the next entry into an arbitrary reference state $m \geq 0$, starting in state $n$, under a certain revised cost structure; see Bertsekas (1995) for further discussion.

It is easy to see that Equations (10)–(12) determine the relative costs only up to an additive constant even if $\gamma$ is treated as a known constant. Therefore, as in George and Harrison (2001), it is natural to define relative cost differences
\[
y_n = v_n - v_{n+1} \quad \text{for } n = 1, \ldots, N,
\]
and then one can re-express (11) and (12) as follows:
\[
y_{n+1} = \max_{x \in A} \{xy_n - c(x)\} + \gamma, \quad n = 1, \ldots, N - 1,
\]
\[
p = \max_{x \in A} \{xy_N - c(x)\} + \gamma.
\]

It should be emphasized that the derivation sketched above serves only as motivation in our treatment; the only property of this optimality equation that we require (Proposition 1) will be proved from first principles.

As in George and Harrison (2001), to further reduce our optimality equation it is natural to define the function
\[
\phi(y) = \sup_{x \in A} \{yx - c(x)\}, \quad y \geq 0.
\]

Then, (10), (14), and (15) can be equivalently represented as follows:
\[
y_1 = \gamma, 
\]
\[
y_{n+1} = y + \phi(y_n), \quad n = 1, \ldots, N - 1,
\]
\[
p = y + \phi(y_N).
\]

Given the assumptions on the cost of control $c(\cdot)$ and the set of available service rates $A$ that were set forth in §1, it is straightforward to prove the following: First, the supremum in (16) is finite for all $y \geq 0$, and second, there exists a unique maximizer $\psi(y)$ for each $y \geq 0$. Several other important properties of $\phi$ and $\psi$ are proved in George and Harrison (2001) and Ata et al. (2002), and some of those properties will be compiled in the beginning of the next subsection.

We now provide a “verification lemma,” which allows one to rigorously prove the optimality of a policy derived from a solution of (17)–(19).

**Proposition 1.** Let $\gamma$ and $(y_1, \ldots, y_N)$ be a solution of the optimality equation (17)–(19). Provided $y_n \geq 0$ for $n = 1, \ldots, N$, and the candidate policy $\mu^*$ defined by $\mu^*_n = \psi(y_n)$ is admissible—i.e., $\mu^*_n > 0$ for $n = 1, \ldots, N$—it is also optimal. Moreover, $\gamma_\mu = \gamma = \gamma^*$.

**Proof.** Let $\mu = (\mu_1, \ldots, \mu_N)$ be an arbitrary admissible policy. By definition of admissibility, $\mu_n > 0$ for $n = 1, \ldots, N$, and the steady-state distribution $\pi(\mu)$ associated with $\mu$ satisfies the balance equations (1). Also, by definition of $\phi(\cdot)$, one writes
\[
\mu_n y_n - c(\mu_n) \leq \phi(y_n) \quad \text{for } n = 1, \ldots, N - 1.
\]

Then, by (18) it follows that
\[
\mu_n y_n - c(\mu_n) \leq y_{n+1} - \gamma \quad \text{for } n = 1, \ldots, N - 1.
\]

Multiplying both sides of this by $\pi_n$ and summing over $n = 1, \ldots, N - 1$, and using the balance equations (1), give the following:
\[
\gamma \sum_{n=1}^{N-1} \pi_n(\mu) \leq \sum_{n=1}^{N-1} \pi_n(\mu) c(\mu_n) \\
+ \pi_N(\mu) y_N - \pi_1(\mu) y_1.
\]

On the other hand, (19) and the definition of $\phi(\cdot)$ give
\[
\mu_n y_n - c(\mu_N) \leq \phi(y_N) = p - \gamma.
\]

Equivalently, one writes
\[
\gamma \leq p + c(\mu_N) - \mu_N y_N.
\]

Multiplying both sides of (21) by $\pi_n(\mu)$, and combining that with (20) and the balance equations (1), give
\[
\gamma \sum_{n=0}^{N} \pi_n(\mu) \leq \sum_{n=1}^{N} \pi_n(\mu) c(\mu_n) + \pi_N(\mu) p.
\]

Therefore,
\[
\gamma_\mu \equiv \sum_{n=1}^{N} \pi_n(\mu) c(\mu_n) + \pi_N(\mu) p \geq \gamma.
\]

Moreover, because the candidate policy $\mu^*$ is admissible by assumption, one can repeat the steps above for the candidate policy, in which case all inequalities in the preceding derivation are replaced with equalities. Therefore, it follows that
\[
\gamma_\mu^* \equiv \sum_{n=1}^{N} \pi_n(\mu^*) c(\mu^*_n) + \pi_N(\mu^*) p = \gamma = \gamma^*. \quad \square
\]

Our next task is to solve the optimality equation, which is undertaken in §4.3, where we emphasize the parametric dependence of our solution on the penalty rate $p$ to facilitate future analysis.

### 4.3. Solving the Optimality Equation

In this subsection, we solve the optimality equation explicitly. To facilitate our analysis, we first present several
important properties of the functions $\phi$ and $\psi$, which are straightforward to derive. Indeed, most of these properties are standard as $\phi$ is the convex conjugate of $c(\cdot)$ (or equivalently, the Legendre transformation of $c(\cdot)$); cf. pages 104 and 256 of Rockafellar (1970). Recall that

$$
\psi(y) = \arg \max_{x \in A} \{yx - c(x)\}, \quad y \geq 0. \tag{22}
$$

Because the cost of control $c(\cdot)$ is assumed to be continuously differentiable, increasing, and strictly convex, its derivative $c'(\cdot)$ is also continuous and strictly increasing on $A$. Therefore, its inverse $(c')^{-1}(\cdot)$ is well defined, continuous, and increasing. Combining these, it is straightforward to arrive at the following representation of the function $\psi$:

$$
\psi(y) = \begin{cases} 
0 & \text{if } 0 \leq y \leq c'(0), \\
(c')^{-1}(y) & \text{if } c'(0) < y < c'(\bar{\mu}), \\
\bar{\mu} & \text{if } y \geq c'(\bar{\mu}). 
\end{cases} \tag{23}
$$

Figure 3 portrays an illustrative function $\psi(\cdot)$.

It has been established (indeed under weaker assumptions) in George and Harrison (2001) that

$$
\phi(y) = \int_0^y \psi(u) \, du, \quad y \geq 0. \tag{24}
$$

The following proposition is an immediate consequence of (23) and (24). Figure 4 portrays an illustrative function $\phi$.

**Proposition 2.** $\phi(\cdot)$ is continuous, nondecreasing, and convex on $[0, \infty)$ with $\phi(0) = 0$ and $\lim_{y \to \infty} \phi(y) = \infty$. In particular, $\phi \equiv 0$ on $[0, c'(0)]$; it is strictly increasing on $(c'(0), \infty)$; and it is affine on $(c'(\bar{\mu}), \infty)$ with slope $\bar{\mu}$.

Having compiled the important properties of the functions $\phi$ and $\psi$, we now solve the optimality equation. For each $\gamma > c'(0)$, define $y_n(\gamma)$ for $n = 1, \ldots, N$ inductively as follows:

$$
y_1(\gamma) \equiv \gamma \quad \text{and} \quad y_{n+1}(\gamma) \equiv \gamma + \phi(y_n(\gamma))
$$

for $n = 1, \ldots, N - 1$. \tag{25}

We also define

$$
f(\gamma) \equiv \gamma + \phi(y_N(\gamma)), \quad \gamma > c'(0). \tag{26}
$$

The following observation lies at the heart of our solution approach. For each $\gamma > c'(0)$, the vector $(y_1(\gamma), \ldots, y_N(\gamma))$ and $\gamma$ jointly solve the optimality equation (17)–(19) associated with the penalty rate $f(\gamma)$. Propositions 4, 5, and 6 below are devoted to showing that for each $p > c'(0)$ one can choose $\bar{\gamma} > c'(0)$ uniquely, such that $f(\bar{\gamma}) = p$, and therefore, the vector $(y_1(\bar{\gamma}), \ldots, y_N(\bar{\gamma}))$ and $\bar{\gamma}$ solve the optimality equation for the penalty rate $p$. (In particular, $\bar{\gamma}$ is equal to $\gamma(p)$; cf. (28).) Then, Theorem 1 constructs a policy, whose admissibility and monotonicity are established by Proposition 3, and proves that this policy is optimal. We now make this observation precise by the following propositions, whose proofs are straightforward, and hence, are omitted. First, the following proposition is immediate from the definition (25) and Proposition 2.

**Proposition 3.** For each $\gamma > c'(0)$, one has the following:

(i) $0 \leq c'(0) < y_1(\gamma) < y_2(\gamma) < \cdots < y_N(\gamma)$;

(ii) $0 < \psi(y_1(\gamma)) < \psi(y_2(\gamma)) < \cdots < \psi(y_N(\gamma)) < \bar{\mu}$.

Using definition (25) and Proposition 2, it is also straightforward to prove the following proposition by induction.

**Proposition 4.** For each $n = 1, \ldots, N$, the function $y_n(\cdot)$ is continuous, strictly increasing, and convex on $(c'(0), \infty)$ with

$$
\lim_{\gamma \downarrow c'(0)} y_n(\gamma) = c'(0) \quad \text{and} \quad \lim_{\gamma \to \infty} y_n(\gamma) = \infty. \tag{27}
$$

Along the lines of Proposition 4, it is easy to see that $f(\cdot)$ is continuous, strictly increasing, and convex with $\lim_{\gamma \downarrow c'(0)} f(\gamma) = c'(0)$ and $\lim_{\gamma \to \infty} f(\gamma) = \infty$. Therefore, $f^{-1}$, the inverse of $f$, is well defined, continuous, and strictly increasing on $(c'(0), \infty)$. Consequently, for each $p > c'(0)$, we define

$$
\gamma(p) \equiv f^{-1}(p). \tag{28}
$$

The following proposition is then immediate. An illustrative function $\gamma(\cdot)$ is portrayed in Figure 5.
5. Solution to the Power Control Problem

This section develops an explicit solution to the power control problem described in §3. First, letting $\beta(p)$ denote the long-run average packet drop rate under the optimal policy $\mu^*$, which obviously is equal to $\pi_\gamma(\mu^*(p))$, we analyze its parametric dependence on the penalty rate $p$. It is immediate from (3) that

$$\beta(p) = \left[1 + \mu_n^*(p) + \mu_1^*(p)\mu_n^*(p)\right]^{-1}$$

for $p > c'(0)$, (30)

where $\mu_n^*(p) = \psi(y_n(\gamma(p)))$ for $n = 1, \ldots, N$. Defining

$$\tilde{\beta} = \frac{1 - \tilde{\mu}}{1 - \tilde{\mu}^{N+1}}$$

and $\tilde{p} = f(c'(\tilde{\mu}))$,

one has the following proposition, which is the key to constructing an optimal policy for the power control problem.

**Proposition 7.** $\beta(\cdot)$ is continuous on $(c'(0), \infty)$; it equals $\tilde{\beta}$ over $[\tilde{p}, \infty)$, and it is strictly decreasing over $(c'(0), \tilde{p})$ with $\lim_{p \to c'(0)} \beta(p) = 1$.

**Proof.** First, we prove that $\beta(\cdot)$ is continuous on $(c'(0), \infty)$. Because $\mu_n^*(p) = \psi(y_n(\gamma(p)))$, $\beta(\cdot)$ is continuous on $[0, \infty)$, and $y_n(\gamma(p))$ is continuous on $(c'(0), \infty)$ with image $(c'(0), \infty)$, it follows that $\mu_n^*(\cdot)$ is continuous on $(c'(0), \infty)$ for $n = 1, \ldots, N$. It is then immediate from (30) that $\beta(\cdot)$ is continuous on $(c'(0), \infty)$.

Second, we prove that $\beta(p) = \tilde{\beta}$ for $p \geq \tilde{p}$. Fixing $p \geq \tilde{p}$, we observe that $\gamma(p) \geq y(\tilde{\beta}) = c'(0)$ by Proposition 5, and hence, $y_n(\gamma(p)) \geq y_n(\gamma(\tilde{\beta}))$ for $n = 1, \ldots, N$ by Proposition 4. Therefore, it follows by monotonicity of $\psi(\cdot)$ that

$$\mu_n^*(p) = \psi(y_n(\gamma(p))) \geq \psi(y_n(\gamma(\tilde{\beta}))) = \mu_n^*(\tilde{\beta})$$

for $n = 1, \ldots, N$.

Also, by (ii) of Proposition 3 we have that $0 < \mu_1^*(\tilde{\beta}) \leq \mu_2^*(\tilde{\beta}) \leq \cdots \leq \mu_n^*(\tilde{\beta}) \leq \tilde{\mu}$. In particular, by (25) it follows that

$$\mu_1^*(\tilde{\beta}) = \psi(y_1(\gamma(\tilde{\beta}))) = \psi(y(\tilde{\beta})) = \psi(f^{-1}(\tilde{p}))$$

$$= \psi(f^{-1}(f(c'(\tilde{\mu})))) = \psi(c'(\tilde{\mu})) = \tilde{\mu}.$$

Therefore, we conclude that $\mu_n^*(\tilde{\beta}) = \tilde{\mu}$ for $n = 1, \ldots, N$, which also implies that $\mu_n^*(p) = \tilde{\mu}$ for $n = 1, \ldots, N$. Then, it is straightforward to conclude by (30) that $\beta(p) = \tilde{\beta}$ for $p \geq \tilde{p}$.

Next, we prove that $\beta(\cdot)$ is strictly decreasing over $(c'(0), \tilde{p})$. Because $\psi(\cdot)$, $y_n(\cdot)$ are nondecreasing functions and $\mu_n(p) = \psi(y_n(p))$, it is clear that $\mu_n(\cdot)$ is nondecreasing over $(c'(0), \tilde{p})$ for $n = 1, \ldots, N$. Clearly, this implies that $\beta(\cdot)$ is nonincreasing over $(c'(0), \tilde{p})$. Moreover, because $\mu_1(\cdot) = \psi(y_1(\gamma(\tilde{p}))) = \psi(y(\tilde{p}))$, and $\gamma(\cdot)$ is strictly increasing over $(c'(0), \infty)$, and $\psi(\cdot)$ is strictly increasing over $(0, c'(\tilde{\mu}))$, it is straightforward to conclude...
that $\mu_1(\cdot)$ is strictly increasing over $(c'(0), \bar{p})$. This in turn proves that $\beta(\cdot)$ is strictly decreasing over $(c'(0), \bar{p})$. Finally, as $p < c'(0)$, we have that $p(\gamma) \downarrow c'(0)$, and hence, $\gamma_n(\gamma(p)) \downarrow c'(0)$ and $\psi(\gamma_n(\gamma(p))) \downarrow 0$ for $n = 1, \ldots, N$. It is then immediate from (30) that $\beta(p) \uparrow 1$ as $p \downarrow c'(0)$. $\square$

The following corollary is immediate from Proposition 7. Figure 6 displays an illustrative $\beta(\cdot)$-function.$^2$

**Corollary 2.** If $\beta \in (\bar{\beta}, 1)$, then there exists a unique $p^* \in (c'(0), \bar{p})$ such that $\beta(p^*) = \beta$.

The analysis of the power control problem divides into various cases, one of which is interesting mathematically. To be specific, the following two conclusions are more or less obvious. First, if $\beta < \bar{\beta}$, then there exists no admissible policy; second, if $\beta = \bar{\beta}$, then the policy $\mu$ with $\mu_n = \bar{\mu}$ for $n = 1, \ldots, N$ is optimal. Also, the case $\beta = 1$ is not considered simply because it leads to a degenerate problem. Therefore, we focus attention on the case $\beta \in (\bar{\beta}, 1)$. Then, the following theorem characterizes an optimal policy.

**Theorem 2.** Suppose that $\beta \in (\bar{\beta}, 1)$. Then, there exists a unique $p^* > c'(0)$ such that $\beta(p^*) = \beta$, and the policy $\mu^*(p^*) = \psi(\gamma_n(\gamma(p^*)))$ for $n = 1, \ldots, N$ is optimal for the power control problem. Moreover, $z_{\mu^*}(p^*) = z^* = \gamma^* - p^*\beta$.

**Proof.** The existence and uniqueness of $p^*$ such that $p^* > c'(0)$ and $\beta(p^*) = \beta$ follows from Corollary 2. Also, because $p^* > c'(0)$, it follows by Proposition 3 that $\mu^*_n(p^*) > 0$ for $n = 1, \ldots, N$. Therefore, the policy $\mu^*(p^*)$ is admissible. Let $\mu$ be any other admissible policy. Then, by (i) of Corollary 1 it follows that

$$\sum_{n=1}^{N} \pi_n(\mu) c(\mu_n) \geq \gamma(p^*) - p^* \pi_N(p^*).$$

Clearly, $\pi_N(\mu) \leq \beta$ because $\mu$ is admissible. Then, it follows trivially that

$$\sum_{n=1}^{N} \pi_n(\mu) c(\mu_n) \geq \gamma(p^*) - p^* \beta.$$

On the other hand, one also has by (ii) of Corollary 1 that

$$\sum_{n=1}^{N} \pi_n(\mu^*(p^*)) c(\mu^*_n(p^*)) = \gamma(p^*) - p^* \beta.$$

Therefore, the policy $\mu^*(p^*)$ is optimal. Also, it follows by (5), (6), and Theorem 1 that $z_{\mu^*}(p^*) = z^* = \gamma^* - p^*\beta$. $\square$

### 6. A Numerical Example

In this section, we present a numerical example to illustrate the solution method. Taking millisecond as the unit of measurement, we let $\lambda = 1$ packet per msec, and $\bar{\mu} = 10$ packets per msec so that $A = [0, 10]$. (If one takes average packet length to be 10 Kb as in Uysal-Biyikoglu et al. 2002b, then $\lambda = 10$ Mbs.) For simplicity, we let $\alpha = 1$. Therefore,

$$c(x) = e^x - 1, \quad x \in [0, 10].$$

In particular, $c'(0) = 1$ and $c'(\bar{\mu}) = e^{10}$. Then, it is straightforward to derive

$$\phi(y) = \begin{cases} 0 & \text{if } 0 \leq y \leq 1, \\ y \ln(y) - y + 1 & \text{if } 1 < y < e^{10}, \\ 10y - e^{10} + 1 & \text{if } y \geq e^{10}, \end{cases}$$

and

$$\psi(y) = \begin{cases} 0 & \text{if } 0 \leq y \leq 1, \\ \ln(y) & \text{if } 1 < y < e^{10}, \\ 10 & \text{if } y \geq e^{10}. \end{cases}$$

Finally, we take the buffer size $N = 1,000$, and the upper bound on average packet drop rate $\beta = 10^{-3}$.

Given this data, the goal is to construct the function $\beta(\cdot)$ and choose $p^*$ as in Corollary 2, that is, $\beta(p^*) = 10^{-3}$. Then, given this penalty rate $p^*$, the policy in Theorem 2 is optimal. Of course, finding that policy requires determining the optimal average cost $\gamma^*$ of the Lagrangian problem associated with the penalty rate $p^*$. However, the following observation will enable us to determine $\gamma^*$ and $p^*$ simultaneously, simplifying the solution method.

As observed in §4.3, for each $\gamma > c'(0)$ the vector $(\gamma_1(\gamma), \ldots, \gamma_N(\gamma))$ and $\gamma$ jointly solve the optimality equation (17)–(19) associated with the penalty rate $f(\gamma)$. Given the properties of $f(\cdot)$ and Proposition 7, it is straightforward to conclude the following. First, the composite function $\beta \circ f$ is continuous on $(c'(0), \infty)$. It equals $\beta$ over $[c'(\bar{\mu}), \infty)$, and it is strictly decreasing over $(c'(0), c'(\bar{\mu}))$ with $\lim_{x \to c'(0)} \beta(f(x)) = 1$. In particular, there exists a unique $\gamma^* > c'(0)$ such that $\beta(f(\gamma^*)) = \beta$. Letting $p^* = f(\gamma^*)$, it is clear that $\beta(p^*) = \beta$. Moreover, by construction $(\gamma_1(\gamma^*), \ldots, \gamma_N(\gamma^*))$ and $\gamma^*$ solve the Lagrangian problem associated with the penalty rate $p^*$. In particular, $\gamma^*$ is the optimal average cost of the Lagrangian problem associated with the penalty rate $p^*$, that is, $\gamma^* = \gamma^*$. Therefore, $p^*$ is the optimal rejection penalty (that is, $\beta(p^*) = \beta$), the policy $\mu^*(p^*)$ with $\mu^*_n(p^*) = \psi(\gamma_n(\gamma^*))$ for $n = 1, \ldots, N$ is optimal, and the optimal average cost is $z^* = \gamma^* - p^*\beta$. 

![Figure 6. An illustrative $\beta$ function.](image-url)
To find the optimal policy for the numerical example, one needs to construct the function $\beta \circ f$ on $(c'(0), \infty)$, which can be done as follows. For each $\gamma > c'(0)$, interpreting $f(\gamma)$ as a rejection penalty, the associated optimal service rates (in the Lagrangian problem) are as follows:

$$\mu^*_n(f(\gamma)) = \psi(y_n(\gamma)), \quad n = 1, \ldots, N,$$

where $f(\gamma)$ is given by (25) and (26). Then, one writes by (30) that

$$\beta(f(\gamma)) = 1 \left[ 1 + \mu^*_N(f(\gamma)) + \mu^*_N(f(\gamma))\mu^*_{N-1}(f(\gamma)) + \cdots + \mu^*_N(f(\gamma)), \ldots, \mu^*_1(f(\gamma)) \right]. \quad (33)$$

The next task is to solve the equation $\beta(f(\gamma)) = 10^{-3}$. This is done by a one-dimensional search over values of $\gamma$ (which is trivial because the function $\beta \circ f$ is monotone). By an initial test run, the relevant range for the search is found to be $[1.718, 1.719]$. Then, by choosing a much finer grid $\gamma^*$, $p^*$, and $z^*$ are found to have the following values:

$$\gamma^* = 1.71830841,$$
$$p^* = 2.72776505,$$
$$z^* = 1.71541698,$$

with $\beta(p^*) = 0.00100352$. Figure 7 displays the optimal packet drop rate $\beta(\cdot)$ as a function of the penalty rate $p$. (The vertical axis is plotted in base 10 log scale.) Of course, Figure 7 is produced by plotting pairs of $(f(\gamma), \beta(f(\gamma)))$ as $\gamma$ varies over the relevant range $[1.718, 1.719]$. Finally, Figure 8 displays the optimal service rate as a function of number of packets in the system. Several points are worth mentioning regarding the solution to the numerical example. Clearly, the optimal service rates are monotone in the number of packets in the system. Moreover, the service rates take values between $[0.5413, 1.0034]$. An interesting observation is that the optimal service rates are in the range $[0.9045, 1.0034]$ when the number of packets is greater than 15; and the optimal service rates are in the range $[0.9900, 1.0034]$ when the number of packets in the buffer is larger than 179. In particular, the last observation seems consistent with the assertion made in Ata (2003) that “small” deviations from the nominal service rate 1 is sufficient to achieve “good” performance, which supports the validity of the heavy traffic approximations in this context.

**Endnotes**

1. An important special case is when the function $c(\cdot)$ is linear, which arises when the signal-to-noise ratio on the channel is low. This case is not addressed here. However, it is plausible that one can adopt the framework of George and Harrison (2001) to address that case.

2. Clearly, $\hat{\beta}$ tends to zero as $\bar{\mu}$ tends to $\infty$. Therefore, it seems plausible that the analysis can be extended to the case where $A$ is unbounded (under additional growth assumptions on $c(\cdot)$ along the lines of George and Harrison 2001).

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**References**


