Analysis on the forward market equilibrium model

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Abstract

We establish the existence results for the Allaz–Vila [B. Allaz, J.-L. Vila, Cournot competition, forward markets and efficiency, J. Econ. Theory 59 (1993) 1–16] forward market equilibrium model when the M producers have different linear cost functions. We also consider an example with three asymmetric producers. The computational results supplement the conclusion in that the forward trading would increase market efficiency.

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1. Introduction

Allaz and Vila [1] presented a forward market model with identical Cournot duopolists. They showed that even with certainty and perfect foresight, forward trading can improve market efficiency. Each of the producers will sell forward so as to make them worse off and make consumers better off than would be the case if the forward market did not exist. This phenomenon is similar to that of the prisoners’ dilemma.

In the forward market model mentioned above, the inverse demand function is affine and the producers have the same linear cost function. Hence, one can solve for a Nash equilibrium of the forward market in closed form; see [1]. However, it is not clear that a Nash equilibrium would exist if the producers had nonidentical cost functions. Indeed, one can construct a simple example of Cournot duopolists with nonidentical linear cost functions for which the Allaz–Vila approach is not valid. If fact, the two-period forward market model belongs to a new class of mathematical programs called Equilibrium Problems with Equilibrium Constraints (EPECs), where each player solves a nonconvex mathematical program with equilibrium constraints (MPEC) [2], and a Nash equilibrium for an EPEC may not exist because of the nonconvexity in each player’s problem. Pang and Fukushima [4] give a simple numerical example of such a case.

We observe that the mathematical structure of the two-period forward market model is similar to that of the multiple leader Stackelberg model analyzed by
The remainder of this paper is organized as follows. In the next section, we give a general formulation for the two-period forward market model with producers. In Section 3, we reformulate the forward market equilibrium model by introducing new variables on spot market sales. Assuming that the inverse demand function is affine and allowing the producers to have nonidentical linear cost functions, we establish the existence of a forward market equilibrium. In Section 4, we use the sequential nonlinear complementarity (SNCP) algorithm proposed in [8] to compute a forward market Nash equilibrium for a three-producer example.

2. The two-period forward market model

We use the following notation throughout this paper:

\[M\] number of producers
\[f_i\] producer \(i\)'s forward sales in the first period
\[Q_f\] the total forward sales in the first period
\[x_i\] the production of producer \(i\) in the second period
\[s_i\] producer \(i\)'s spot sales in the second period
\[Q_s\] the total spot sales in the second period
\[c_i(\cdot)\] the cost function of producer \(i\)
\[u_i(\cdot)\] the payoff function of producer \(i\) from the spot market in the second period
\[\pi_i(\cdot)\] the overall profit function of producer \(i\)
\[p_f(\cdot)\] the forward price (or inverse demand function) in the first period
\[p(\cdot)\] the spot price (or inverse demand function) in the second period

2.1. The production game

Given the producers’ forward position vector \(f = (f_1, \ldots, f_M)\), the producers are playing a Cournot game in production quantities in the second period. For each \(i = 1, \ldots, M\), producer \(i\), assuming the production quantities \(x_{-i} = (x_j)_{j=1, j \neq i}^M\) of the other producers are fixed at \(\bar{x}_{-i}\), chooses the nonnegative production quantity \(x_i\) to maximize the payoff function in the second period:

\[
[u_i(f_1, \ldots, f_M)](x_i, \bar{x}_{-i}) = p\left(x_i + \sum_{j=1, j \neq i}^M \bar{x}_j\right)(x_i - f_i) - c_i(x_i). \tag{1}
\]

Indeed, if the producer \(i\) has already sold \(f_i\) in the forward market, it can only sell quantity \((x_i - f_i)\) in the spot market. The vector of production quantities \(x^* = (x_1^*, \ldots, x_M^*)\) is said to be a Nash–Cournot equilibrium for the production game, if for each \(i = 1, \ldots, M\), \(x_i^*\) solves

\[
\text{maximize}_{x_i \geq f_i} \left\{ p\left(x_i + \sum_{j=1, j \neq i}^M x_j^*\right)(x_i - f_i) - c_i(x_i) \right\}. \tag{2}
\]

Accordingly, we use \(x(f) = (x_1(f), \ldots, x_M(f))\) to denote a Nash–Cournot production equilibrium \(x^*\) corresponding to the forward position vector \(f\).

2.2. Forward market equilibrium

In the first period, the producers are playing a Cournot game in forward quantities. Assuming the forward position of other producers are fixed, producer \(i\) chooses his forward position \(f_i(\geq 0)\) to maximize the overall profit function:

\[
\pi_i(f_i, \bar{f}_{-i}) = p_f\left(f_i + \sum_{j=1, j \neq i}^M \bar{f}_j\right)(f_i) + [u_i(f_i, \bar{f}_{-i})](x(f_i, \bar{f}_{-i}))
\]

\[
= p\left(\sum_{j=1}^M x_j (f_i, \bar{f}_{-i})\right)(x_i (f_i, \bar{f}_{-i})) - c_i (x_i (f_i, \bar{f}_{-i})), \tag{3}
\]

since \(p_f(f_i + \sum_{j=1, j \neq i}^M \bar{f}_j) = p(\sum_{j=1}^M x_j (f_i, \bar{f}_{-i}))\) under perfect foresight.
A vector \( f^* = (f^*_1, \ldots, f^*_M) \) is said to be a forward market equilibrium if for \( i = 1, \ldots, M \), \( f^*_i \) solves

\[
\begin{aligned}
\text{maximize} & \quad \left\{ p \left( \sum_{j=1}^M x_j(f_i, f^*_j) \right) (x_i(f_i, f^*_j)) \\
& \quad - c_i(x_i(f_i, f^*_j)) \right\}.
\end{aligned}
\]

(4)

Moreover, \( x(f^*) = (x_j(f^*_j, f^*_{-j}), j = 1, \ldots, M) \) is a Nash–Cournot equilibrium for the production game corresponding to the forward equilibrium \( f^* \), as defined in (2).

3. Existence of a forward market equilibrium

In [1], Allaz and Vila showed that one can solve for the forward market Nash equilibrium in closed form when demand and cost functions are affine and the producers have the same cost function, i.e., \( c_i(x_i) = c x_i \), for \( i = 1, \ldots, M \). In particular, in the case of Cournot duopolists, and the demand function \( p(q) = a - q \), with \( 0 < c < a \), the unique forward market equilibrium outcome is

\[
\begin{aligned}
x_1 = x_2 = \frac{2(a - c)}{5}; & \quad f_1 = f_2 = \frac{a - c}{5}; \\
p = c + \frac{a - c}{5}.
\end{aligned}
\]

The purpose of this paper is to establish an existence theorem for the forward market equilibrium when the \( M \) producers have nonidentical linear cost functions. Since the producers would only produce what they can sell, the production quantity equals the sum of forward sales and spot market sales. We then introduce new variable \( s_i \) for producer \( i \)'s spot market sales and replace \( x_i \) by \( f_i + s_i \) in the model of the production game to obtain the following equivalent formulation:

\[
\begin{aligned}
[u_i(f_1, \ldots, f_M)](s_i, \bar{s}_{-i}) & \equiv p \left( s_i + \sum_{j=1, j \neq i}^M \bar{s}_j + \sum_{j=1}^M f_j \right) (s_i) \\
& \quad - c_i(s_i + f_i).
\end{aligned}
\]

(5)

With the new variables \( s_i \), we can define the spot market equilibrium. In particular, given a vector of forward positions \( f \in \mathbb{R}^M \), a vector of spot market sales \( s^* \in \mathbb{R}^M \) is said to be a spot market equilibrium if for each \( i = 1, \ldots, M \), \( s^*_i \) solves

\[
\begin{aligned}
\text{maximize} & \quad \left\{ p \left( s_i + \sum_{j=1, j \neq i}^M s^*_j + \sum_{j=1}^M f_j \right) (s_i) \\
& \quad - c_i(s_i + f_i) \right\}.
\end{aligned}
\]

(6)

Similarly, we use \( s(f) = (s_1(f), \ldots, s_M(f)) \) to denote the spot market equilibrium given the forward position \( f \).

Lemma 1. Given the forward sales \( f \), if a vector \( s(f) \) is a spot market equilibrium, then \( x(f) = s(f) + f \) is a Nash equilibrium for the production game and vice versa.

Proof. This is clear. \( \square \)

Following Lemma 1, an equivalent formulation for the forward market equilibrium (4) is as follows. A vector \( f^* = (f^*_1, \ldots, f^*_M) \) is said to be a forward market equilibrium if for \( i = 1, \ldots, M \), \( f^*_i \) solves

\[
\begin{aligned}
\text{maximize} & \quad p \left( f_i + \sum_{j=1, j \neq i}^M f^*_j + \sum_{j=1}^M s_j(f_i, f^*_i) \right) \\
& \quad \times (f_i + s_i(f_i, f^*_i)) - c_i(f_i + s_i(f_i, f^*_i)).
\end{aligned}
\]

(7)

We observe that the spot and forward market equilibrium models ((6) and (7)) are similar to the multiple-leader Stackelberg model analyzed by Sherali [6]. The intuition is that in the forward market, every producer is a leader, while in the spot market, every producer becomes a follower, implementing his best response, given every producer's forward position in the first period. The two differences between the forward-spot market model and the multiple-leader Stackelberg model are:

(i) the cost function of a producer in the forward-spot market model is a function of both forward and spot sales;
(ii) each producer's revenue in the first period includes the spot market sales \( s_i(f_i, f^*_i) \) from the second period.
It turns out that these two differences are problematic. For (i), in contrast to the case in [6], the spot market sales \( s(f) \) cannot be simplified to a function of total forward sales \( Q_f = \sum_{j=1}^{M} f_j \) only; this is due to the \( f_j \) term in the cost function \( c_i(\cdot) \). Hence, the aggregate spot market reaction curve \( Q_s(f) = \sum_{i=1}^{M} s_i(f) \) is a function of the forward sales \( f \) and, in general, cannot be reduced to a function of the total forward market sales, \( Q_f \), a variable in \( R^1 \). The aggregate spot market sales, \( Q_s \), being a function of total forward sales only, is crucial in the analysis. Below, we show that the aggregate spot market sales \( Q_s \) is indeed a function of the aggregate forward market sales \( Q_f \) when the inverse demand function is affine and producers’ cost functions are linear.

**Assumption 2.** We assume that the inverse demand function is \( p(z) := a - bz \) for \( z \geq 0 \) with \( a, b > 0 \), and for each \( i = 1, \ldots, M \), producer \( i \)'s cost function is \( c_i(z) := c_i z \) with \( c_i > 0 \).

**Remark.** Since the producers are maximizing their profits, and \( p(z) < 0 \) for all \( z > a/b \), Assumption 2 also implies that no producer will produce more than \( a/b \) units.

**Proposition 3.** Let Assumption 2 hold. Given producers’ forward sales \( f \), the spot market sales \( s_i(f) \), for \( i = 1, \ldots, M \) and the aggregate spot market sales \( Q_s(f) \) can be simplified to a function of \( Q_f \), and denoted as \( s_i(Q_f) \) and \( Q_s(Q_f) \), respectively.

**Proof.** Substitute \( p(z) = a - bz \), \( c_i(z) = c_i z \), and \( Q_f = \sum_{j=1}^{M} f_j \) into (6); then producer \( i \)'s profit maximization problem in the spot market equilibrium model becomes

\[
\begin{aligned}
\text{maximize}_{s_i \geq 0} & \quad -c_i f_i - \left[ b \left( \sum_{j=1, j \neq i}^{M} s_j + Q_f \right) \right] \\
& + c_i - a \left( s_i - bs_i^2 \right).
\end{aligned}
\]

Since the objective function in (8) is strictly concave, the spot equilibrium \( s^* \) exists and is unique. Furthermore, it must satisfy the (necessary and sufficient)

KKT conditions:

\[
0 \leq s_i \perp b \left( \sum_{j=1}^{M} s_j + Q_f \right) + c_i - a + bs_i \geq 0,
\]

\[
i = 1, \ldots, M.
\]

Since \( f_i \) does not appear explicitly in (9) and \( s_i^* \) is unique, we can denote \( s_i^* \) as \( s_i(Q_f) \) for each \( i = 1, \ldots, M \). Consequently, the aggregate spot market sales \( Q_s = \sum_{i=1}^{M} s_i(Q_f) \) is a function of the total forward sales \( Q_f \) and is denoted as \( Q_s(Q_f) \).

Define \( I(Q_f) := \{ i \mid s_i(Q_f) > 0 \} \) and let \( |I(Q_f)| \) be the cardinality of index set \( I(Q_f) \). Let \( s_i^+(Q_f) \) and \( Q_i^+(Q_f) \) denote the right-hand derivatives of \( s_i(Q_f) \) and \( Q_i(Q_f) \), respectively. The following theorem states the properties of \( s_i(Q_f) \) and \( Q_s(Q_f) \) corresponding to the forward sales \( Q_f \).

**Theorem 4.** Suppose Assumption 2 holds and let \( Q_f \) be the aggregate forward sales. For each \( i = 1, \ldots, M \), the spot market sales \( s_i(Q_f) \) and the aggregate spot market sales \( Q_s(Q_f) \) satisfy the following properties for each \( Q_f > 0 \):

(i) for \( i = 1, \ldots, M \), each \( s_i(Q_f) \) is a continuous, nonnegative, decreasing, piecewise linear convex function in \( \{ Q_f \geq 0 : s_i(Q_f) > 0 \} \) with

\[
s_i^+(Q_f) = \begin{cases} 
-1/(|I(Q_f)|+1) & \text{if } i \in I(Q_f), \\
0 & \text{otherwise};
\end{cases}
\]

(ii) the aggregate spot sales \( Q_s(Q_f) \) is a continuous, nonnegative, decreasing, piecewise linear convex function in \( Q_f \) for \( \{ Q_f \geq 0 : Q_s(Q_f) > 0 \} \) with

\[
s_s^+(Q_f) = \begin{cases} 
-1/(|I(Q_f)|+1) & \text{if } Q_s(Q_f) > 0, \\
0 & \text{if } Q_s(Q_f) = 0;
\end{cases}
\]

(iii) the function \( T(Q_f) := Q_f + Q_s(Q_f) \) is an increasing, piecewise linear convex function in \( Q_f \geq 0 \).

**Proof.** Notice that with the inverse demand function \( p(\cdot) \) and cost function \( c_i(\cdot) \) stated in Assumption 2, the objective function in (8) for the spot market equilibrium model is identical to the oligopolistic market
equilibrium model studied in [3], except for an extra constant term \( c_i f_i \) in (8). However, adding a constant to the objective function will not change the optimal solution. Indeed, we solve the same KKT system (9) for the equilibrium solutions of these two models.

For (i) and (ii), Sherali et al. [7] proved that for \( i = 1, \ldots, M, s_i(Q_f) \) and \( Q_s(Q_f) \) are continuous and nonnegative. For a fixed \( Q_f \), if \( \mathcal{I}(Q_f) \neq \emptyset \), then \( Q_s(Q_f) > 0 \) and from the KKT conditions (9), we have

\[
b(Q_s(Q_f) + Q_f) + c_i - a + b s_i(Q_f) = 0, \quad \forall i \in \mathcal{I}(Q_f).
\]

Summing over \( i \in \mathcal{I}(Q_f) \), we obtain

\[
b |\mathcal{I}(Q_f)| (Q_s(Q_f) + Q_f) + \sum_{i \in \mathcal{I}(Q_f)} c_i
\]

\[
- |\mathcal{I}(Q_f)| a + b Q_s(Q_f) = 0,
\]

which gives

\[
Q_s(Q_f) = \frac{|\mathcal{I}(Q_f)| a - b |\mathcal{I}(Q_f)| Q_f - \sum_{i \in \mathcal{I}(Q_f)} c_i}{b |\mathcal{I}(Q_f)| + 1}, \tag{12}
\]

and

\[
s_i(Q_f) = \frac{a - b Q_f + \sum_{j \in \mathcal{I}(Q_f)} c_j}{b |\mathcal{I}(Q_f)| + 1} - \frac{c_i}{b}, \quad \forall i \in \mathcal{I}(Q_f). \tag{13}
\]

Taking the right-hand derivatives of \( Q_s(Q_f) \) and \( s_i(Q_f) \) with respect to \( Q_f \), we obtain

\[
Q_s^+(Q_f) = \frac{-|\mathcal{I}(Q_f)|}{|\mathcal{I}(Q_f)| + 1} < 0 \quad \text{if} \ Q_s(Q_f) > 0, \tag{14}
\]

and

\[
s_i^+(Q_f) = -\frac{1}{|\mathcal{I}(Q_f)| + 1} < 0 \quad \text{if} \ i \in \mathcal{I}(Q_f). \tag{15}
\]

Now, suppose that \( s_i(Q_f) = 0 \) and \( s_i^+(Q_f) > 0 \) for some \( Q_f \geq 0 \). Then there must exist a point \( \tilde{Q}_f \) near \( Q_f \) such that \( s_i(\tilde{Q}_f) > 0 \) and \( s_i^+(\tilde{Q}_f) > 0 \); this leads to a contradiction. Hence, if \( s_i(Q_f) = 0 \), then \( s_i^+(Q_f) = 0 \), and \( s_i(Q_f) = 0 \) for all \( Q_f \geq Q_f \). In other words, if it is not profitable for the producer \( i \) to sell in the spot market for a given aggregate forward market sales \( Q_f \), then he will not be active in the spot market for any aggregate forward sales \( \tilde{Q}_f \) greater than \( Q_f \).

The same implication also holds for the aggregate spot market sales \( Q_s(Q_f) \), i.e., if \( Q_s(Q_f) = 0 \), then \( Q_s^+(Q_f) = 0 \) and \( Q_s(Q_f) = 0 \) for all \( Q_f \geq Q_f \).

Observe that \( Q_f^+ > Q_f^2 \) implies \( \mathcal{I}(Q_f^2) \supseteq \mathcal{I}(Q_f^+) \). Furthermore, for \( i = 1, \ldots, M \), we have

\[
s_i^+(Q_f^2) > s_i^+(Q_f^+) \quad \text{if} \quad \mathcal{I}(Q_f^2) \supsetneq \mathcal{I}(Q_f^+), \]

\[
s_i^+(Q_f^2) = s_i^+(Q_f^+) \quad \text{if} \quad \mathcal{I}(Q_f^2) = \mathcal{I}(Q_f^+).
\]

This proves that for each \( i = 1, \ldots, M, s_i(Q_f) \) is nonincreasing, piecewise linear and concave in \( Q_f \geq 0 \). Similarly, we can establish that \( Q_s^+(Q_f) = 0 \) if \( Q_s(Q_f) = 0 \) and \( Q_s(Q_f) \) is nonincreasing, piecewise linear and convex in \( Q_f \geq 0 \). This completes the proof for (ii).

For (iii), since \( T^+(Q_f) = 1 + Q_s^+(Q_f) = 1/|\mathcal{I}(Q_f)| + 1 > 0 \), the function \( T(Q_f) \) is increasing in \( Q_f \) for \( Q_f \geq 0 \). \( \square \)

**Lemma 5.** Let Assumption 2 hold. For all \( i = 1, \ldots, M \) and any fixed vector \( \bar{f}_{-i} = (\bar{f}_j)_{j \neq i} \), the objective function in (7) is strictly concave in \( f_i \). Furthermore, there exists a unique optimal solution \( f_i^*(\bar{f}_{-i}) \) for the problem (7), with \( 0 \leq f_i^*(\bar{f}_{-i}) \leq a/b \).

**Proof.** The proof closely follows the analysis of Lemma 1 in [6]. To ease the notation, we define \( \bar{y}_i = \sum_j (\bar{f}_{-i})_j \) and

\[
g_i(f_i, \bar{y}_i) = (a - f_i - \bar{y}_i - Q_s(f_i + \bar{y}_i)) \times \left( f_i + s_i(f_i + \bar{y}_i) \right) - (c_i f_i + c_i s_i(f_i + \bar{y}_i)). \tag{16}
\]

Then for a fixed vector \( \bar{y}_i \), producer \( i \)'s profit maximization problem (7) can be written as

\[
\text{maximize } \{g_i(f_i, \bar{y}_i)\}, \quad f_i \geq 0 \tag{17}
\]

Let \( g_i^+(f_i, \bar{y}_i) \) denote the right-hand derivative of \( g_i(f_i, \bar{y}_i) \) with respect to \( f_i \). To show that \( g_i(f_i, \bar{y}_i) \) is strictly concave in \( f_i \), it suffices to show that for any \( f_i \geq 0 \), there exists \( \delta > 0 \) such that \( g_i^+(f_i + \delta, \bar{y}_i) < g_i^+(f_i, \bar{y}_i) \) for all \( 0 \leq \delta < \delta \).

Without loss of generality, we assume \( \mathcal{I}(f_i + \bar{y}_i) = \{i : s_i(f_i + \bar{y}_i) > 0\} \neq \emptyset \) at a given point \( (f_i, \bar{y}_i) \); the assertion of the lemma on strict concavity of \( g_i(f_i, \bar{y}_i) \)
Theorem 6. If Assumption 2 holds, then there exists a forward market equilibrium.

Proof. For a given vector \( f = (f_1, \ldots, f_M) \), we define a point-to-point map
\[
 F(f) = (f_1^*(f_1), \ldots, f_M^*(f_M)).
\]

In light of Lemma 5, \( f_i^*(f_{-i}) \) is the unique optimal solution of (7) for the corresponding vector \( f_{-i} \) for all \( i = 1, \ldots, M \). If we can show that the map \( F \) is continuous, then by Brouwer’s fixed point theorem, it will have a fixed point on the compact convex set
\[
 C = \{ f : 0 \leq f_i \leq a/b, \forall i = 1, \ldots, M \}.
\]
Furthermore, this fixed point is a forward market equilibrium.

To show \( F \) is continuous, it suffices to show that \( f_i^*(f_{-i}) \) is continuous for \( i = 1, \ldots, M \). Consider a sequence \( (f_i^k) \) converging to \( f_i \), and let \( f_i^k \) denote \( f_i^*(f_{-i}^k) \) of (7) for the corresponding fixed vector \( f_{-i}^k \).

Choose any \( \hat{f}_i \geq 0 \). Since \( f_i^k \) is the optimal solution of (17), for a fixed vector \( f_{-i}^k \), we have
\[
 g_i \left( f_i^k, \sum_{j} (f_{-i}^k)_j \right) \geq g_i \left( \hat{f}_i, \sum_{j} (f_{-i}^k)_j \right), \quad \forall k,
\]
where \( g_i(\cdot, \cdot) \) is defined in (16). As \( k \to \infty \), by the continuity of \( g_i(\cdot, \cdot) \), we obtain
\[
 g_i \left( \hat{f}_i, \sum_{j} (f_{-i}^k)_j \right) \geq g_i \left( \hat{f}_i, \sum_{j} f_{-i}^k_j \right).
\]

Since \( \hat{f}_i \) is chosen arbitrarily, the above inequality holds for any \( \hat{f}_i \geq 0 \). This implies that \( \hat{f}_i \) is an optimal solution of (7) for the corresponding \( f_{-i} \). It follows that \( \hat{f}_i = f_i^*(f_{-i}) \), since from Lemma 5, the optimal solution is unique. This completes the proof. \( \square \)

4. An EPEC approach for computing a forward market equilibrium

The computation of a forward market equilibrium involves solving a family of concave maximization problems, as defined in (7). However, the objective functions are nonsmooth in these problems because \( s_i(Q_f) \) is piecewise linear concave in \( Q_f \) on \( (Q_f : s_i(Q_f) > 0) \); see Theorem 4. One might encounter difficulties in solving these nonsmooth problems with nonlinear programming solvers. An alternative to avoid the nonsmooth objective functions is to formulate the forward market equilibrium problem as...
an equilibrium problem with equilibrium constraints (EPEC). Since we do not know the explicit representation of $s_i$ as a function of $Q_f$, we return to the original formulation of the forward market equilibrium model (4) and embed the necessary and sufficient KKT

\[-a + b e^T (f + s) + b f_i + c_i - \lambda f, i - b e^T \lambda c, i + b s^T \lambda x, i = 0,\]

\[b f_i e - 2b(e_i o s - \lambda f, i - b \lambda c, i + b [s \circ \lambda c, i]) + (e e^T + I)[s \circ \lambda c, i] = 0,\]

\[0 \leq c - (a - b(e e^T + e^T s))e + b s \perp \lambda c, i \geq 0,\]

\[0 \leq -s \circ [c - (a - b(e e^T + e^T s))e + b s], \quad \lambda x, i \geq 0.\]

(23)

conditions for spot market equilibrium $s^*$ as a set of constraints in producer $i$'s profit maximization problem in the forward market:

 maximize $(\theta_i - c_i)(f_i + s_i)$

subject to $\theta_i = a - b \left( f_i + e^T s + \sum_{j \neq i} \bar{f}_j \right)$,

$0 \leq s \perp c - \theta_i e + b s \geq 0,$

$f_i \geq 0,$

(20)

where $c = (c_1, \ldots, c_M)$, $f = (f_1, \ldots, f_M)$, and $e$ is a vector of all ones of the proper dimension.

Observe that producer $i$'s profit maximization problem (20) is an MPEC because it includes complementarity constraints

$0 \leq s \perp c - \theta_i e + b s \geq 0.$

(21)

Furthermore, each MPEC is parameterized by other producers' forward sales $\bar{f}_i$. Hereafter, we denote producer $i$'s maximization problem by MPEC($\bar{f}_i$).

The problem of finding a forward market equilibrium solution is formulated as an EPEC:

Find $(f^*, s^*, \theta^*)$ such that for all $i = 1, \ldots, M$,

$(f^*_i, s^*_i, \theta^*_i)$ solves MPEC($f^*_{-i}$) (20).

(22)

The following theorem expresses the strong stationarity conditions [8] for a solution to the forward market equilibrium model (22) in which the $i$th MPEC takes the form of (20). For the definition of MPEC-LICQ, see [5].

Theorem 7. Suppose $(f^*, s^*, \theta^*)$ is a solution for the forward market equilibrium model (22). If for each $i = 1, \ldots, M$, the MPEC-LICQ holds at a feasible point $(f^*_i, s^*_i)$ for the $i$th MPEC (20), then $(f^*, s^*)$ is an EPEC strongly stationary point. In particular, there exist vectors $\lambda^* = (\lambda^*_i, \ldots, \lambda^*_M)$ with $\lambda^*_i = (\lambda f, i^*, \lambda c, i^*, \lambda x, i^*)$ such that $(f^*, s^*, \lambda^*)$ solves the system

Conversely, if $(f^*, x^*, \lambda^*)$ is a solution to system (23), then $(f^*, s^*)$ is an EPEC B(ouligand)-stationary point of the forward market equilibrium model (22).

Proof. See Theorem 4.2 in [8]. \qed

In what follows, we apply the SNCP algorithm proposed in [8] to solve the EPEC formulation for the forward market equilibrium model (22) for each of the four scenarios. The SNCP algorithm finds a solution of the system (23), if one exists.

4.1. An example with three producers

Consider the case with three producers in the market. The cost function of producer $i$ is

$c_i(z) = c_i z \quad \text{with} \quad (c_1, c_2, c_3) = (2, 3, 4).$

The inverse demand function is $p(z) = 10 - z$.

The following four scenarios:

(1) no producer contracts forward sales;

(2) only one producer is allowed to contract forward sales;

(3) only two producers are allowed to contract forward sales;

(4) all three producers can participate in the forward market.

The computational results for each scenario are summarized in Table 2. The notation used in that table is explained in Table 1.

We summarize some observations based on the computational results for the four scenarios. First, we
Table 1
Notation used for computational results

<table>
<thead>
<tr>
<th>Cases</th>
<th>$f = (f_1, f_2, f_3)$</th>
<th>$x = (x_1, x_2, x_3)$</th>
<th>$\pi = (\pi_1, \pi_2, \pi_3)$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>List of producers allowed to sell in the forward market</td>
<td>The vector of producers’ forward sales</td>
<td>The vector of producers’ production quantities</td>
<td>The vector of producers’ overall profits</td>
<td>The spot (and forward) price</td>
</tr>
</tbody>
</table>

Table 2
Computational results on four scenarios

<table>
<thead>
<tr>
<th>Cases</th>
<th>$f$</th>
<th>$x$</th>
<th>$\pi$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No forward sales</td>
<td>(0, 0, 0)</td>
<td>(2.75, 1.75, 0.75)</td>
<td>(7.563, 3.063, 0.563)</td>
<td>4.75</td>
</tr>
<tr>
<td>Only producer 1</td>
<td>(3, 0, 0)</td>
<td>(5, 1, 0)</td>
<td>(10, 1, 0)</td>
<td>4</td>
</tr>
<tr>
<td>Only producer 2</td>
<td>(0, 3, 0)</td>
<td>(2, 4, 0)</td>
<td>(4, 4, 0)</td>
<td>4</td>
</tr>
<tr>
<td>Only producer 3</td>
<td>(0, 0, 1)</td>
<td>(2.5, 1.5, 1.5)</td>
<td>(6.25, 2.25, 0.75)</td>
<td>4.5</td>
</tr>
<tr>
<td>Only producer 1 &amp; 2</td>
<td>(2, 1, 0)</td>
<td>(4, 2, 0)</td>
<td>(8, 2, 0)</td>
<td>4</td>
</tr>
<tr>
<td>Only producer 1 &amp; 3</td>
<td>(2.333, 0, 0.333)</td>
<td>(4.444, 1.111, 0.333)</td>
<td>(9.383, 1.235, 0.037)</td>
<td>4.111</td>
</tr>
<tr>
<td>Only producer 2 &amp; 3</td>
<td>(0, 2.25, 0.25)</td>
<td>(2.125, 3.375, 0.375)</td>
<td>(4.516, 3.797, 0.047)</td>
<td>4.125</td>
</tr>
<tr>
<td>All producers</td>
<td>(2, 1, 0)</td>
<td>(4, 2, 0)</td>
<td>(8, 2, 0)</td>
<td>4</td>
</tr>
</tbody>
</table>

notice that allowing more producers to participate in the forward market will not necessarily decrease the market clearing price. For example, the price is 4 when only producer 2 is allowed to produce in the forward market and it increases to 4.125 when producer 3 joins producer 2 to produce in the forward market. In contrast to the outcome of the Cournot game, each producer can increase production by being the only player in the forward market.

If the producers have the chance to sell forward, they can do so profitably. For example, if only producer 1 is allowed to sell forward, he can increase his profit from 7.563 (the profit in the Cournot game) to 10; if only producer 2 is allowed to contract forward sales, producer 1 can increase his profit from 4 to 8 if he starts selling in the forward market, and similarly for producer 3. This is similar to the conclusion on the emergence of a forward market in [1] for the case of identical producers. However, if all producers participate in the forward market, producer 1 is better off and producers 2 and 3 are worse off than would be the case if the forward market did not exist; this phenomenon is in contrast to that of the prisoners’ dilemma observed for the case of identical producers in [1].

Finally, the results suggest that the market is the most efficient (in terms of the clearing price) when producers 1 and 2 both participate in the forward market, in which case, producer 3 will not produce in the forward market even if he is allowed to do so. This supplements the conclusion in [1] that the forward trading would increase market efficiency.

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