TECHNICAL APPENDIX FOR GENERALIZED LEAST SQUARES INFERENCE IN PANEL AND MULTILEVEL MODELS WITH SERIAL CORRELATION AND FIXED EFFECTS

CHRISTIAN B. HANSEN

Let

$$C_{st} = x_{st}'\beta_1 + z_{st}'\beta_2^s + v_{st}, \tag{1}$$

or, in vector notation, $C_s = X_s\beta_1 + Z_s\beta_2^s + V_s$, where $C_s = [C_{s1}, \dots, C_{sT}]'$ is $T \times 1$, $X_s = [x_{s1}, \dots, x_{sT}]'$ is $T \times k_1$, $Z_s = [z_{s1}, \dots, z_{sT}]'$ is $T \times k_2$, and $V_s = [v_{s1}, \dots, v_{sT}]'$ is $T \times 1$. Also, let x_{sth} be the h^{th} element of x_{st} so that $x_{st}' = [x_{st1}, \dots, x_{stk_1}]$, and define z_{sth} similarly. Define $\ddot{v}_{st} = v_{st} - z_{st}'(Z_s'Z_s)^{-1}Z_s'V_s$, $\ddot{x}_{st}' = x_{st}' - z_{st}'(Z_s'Z_s)^{-1}Z_s'X_s$, $\ddot{V}_s = [\ddot{v}_{s1}, \dots, \ddot{v}_{sT}]'$, and $\ddot{X}_s = [\ddot{x}_{s1}, \dots, \ddot{x}_{sT}]'$. Let v_{st}^- be a $p \times 1$ vector with $v_{st}^- = [v_{s(t-p)}, \dots, v_{s(t-1)}]'$, and define $\ddot{v}_{st}^-$ similarly.

Assumption 1 ($S \to \infty$, T fixed)**.** Suppose the data are generated by model (1) and

- N1. $v_{st} = v_{st}^{-\prime}\alpha + \eta_{st}$, where η_{st} is strictly stationary in t for each s, $\mathrm{E}[\eta_{st}^2] = \sigma_\eta^2$, $\mathrm{E}[\eta_{st}\eta_{s\tau}] = 0$ for $t \neq \tau$, and the roots of $1 - \alpha_1 z - \alpha_2 z^2 - \dots - \alpha_p z^p = 0$ have modulus greater than 1.
- N2. $\{X_s, V_s, \eta_s\}$ are iid across s. $\{Z_s\}$ are nonstochastic and identical across s.
- N3. (i) $\mathrm{Rank}(\sum_{t=1}^T \mathrm{E}[\ddot{x}_{st}\ddot{x}_{st}']) = \mathrm{Rank}(\mathrm{E}[\ddot{X}_s'\ddot{X}_s]) = k_1$. (ii) $\mathrm{Rank}(Z_s'Z_s) = k_2 \ \forall\ s$.
- N4. $\mathrm{E}[V_s|X_s] = 0$, $\mathrm{E}[V_sV_s'|X_s] = \Gamma(\alpha)$.
- N5. $\mathrm{E}[\eta_{st}^4] = \mu_4 < \infty$ and $\mathrm{E}[x_{sth}^4] \leq \Delta < \infty\ \forall\ s,\ t,\ h$.

Proposition 1. *If Assumption 1 is satisfied,* $\widehat{\alpha} \xrightarrow{p} \alpha_T(\alpha)$, *where*

$$\alpha_T(\alpha) = \mathrm{E}[\sum_{t=p+1}^{T} \ddot{v}_{st}^-\ddot{v}_{st}^{-\prime}]^{-1}\mathrm{E}[\sum_{t=p+1}^{T} \ddot{v}_{st}^-\ddot{v}_{st}] = (\Gamma_p(\alpha) + \frac{1}{T-p}\Delta_\Gamma(\alpha))^{-1}(A(\alpha) + \frac{1}{T-p}\Delta_A(\alpha)).$$

Proposition 2. *Suppose* $\alpha_T(\alpha)$ *is continuously differentiable in* α *and that* $H = D\alpha_T(\alpha)$ *is invertible for all* α *such that N1 is satisfied, where* $D\alpha_T(\alpha)$ *is the derivative matrix of* $\alpha_T(\alpha)$ *in* α. *Define* $\widehat{\alpha}^{(\infty)} = \alpha_T^{-1}(\widehat{\alpha})$. *Then, if Assumption 1 is satisfied,* $\widehat{\alpha}^{(\infty)} - \alpha \xrightarrow{p} 0$ *and* $\sqrt{S}(\widehat{\alpha}^{(\infty)} - \alpha) \xrightarrow{d} \frac{1}{T-p}H^{-1}(\Gamma_p(\alpha) + \frac{1}{T-p}\Delta_\Gamma(\alpha))^{-1}N(0, \Xi)$, *where*

$$\Xi = \mathrm{E}[\sum_{t_1=p+1}^{T}\sum_{t_2=p+1}^{T} \ddot{v}_{st_1}^-\ddot{\mu}_{st_1}\ddot{\mu}_{st_2}\ddot{v}_{st_2}^{-\prime}]$$

and $\ddot{\mu}_{st} = \ddot{v}_{st} - \ddot{v}_{st}^{-\prime}\alpha_T(\alpha)$.

Assumption 2 ($S,\ T \to \infty$)**.** Suppose the data are generated by model (1) and

- NT1. $v_{st} = v_{st}^{-\prime}\alpha + \eta_{st}$, where η_{st} is strictly stationary in t for each s, $\mathrm{E}[\eta_{st}^2] = \sigma_\eta^2$, $\mathrm{E}[\eta_{st}\eta_{s\tau}] = 0$ for $t \neq \tau$, and the roots of $1 - \alpha_1 z - \alpha_2 z^2 - \ldots - \alpha_p z^p = 0$ have modulus greater than 1.
- NT2. $\{X_s, V_s, \eta_s\}$ are iid across s. $\{Z_s\}$ are nonstochastic and identical across s.
- NT3. (i) $[X, Z]$, where $X = [X_1', \ldots, X_S']'$ and $Z = \mathrm{diag}(Z_1, ..., Z_S)$ has full rank. (ii) $Z_s'Z_s$ is uniformly positive definite with minimum eigenvalue $\lambda_s \geq \lambda > 0$ for all s.
- NT4. $\mathrm{E}[V_s|X_s] = 0$, $\mathrm{E}[V_sV_s'|X_s] = \Gamma(\alpha)$.
- NT5. $\{X_{st}, V_{st}, \eta_{st}\}$ is α-mixing of size $\frac{-3r}{r-4}$, $r > 4$, and $z_{ith}^2 \leq \Delta < \infty$, $\mathrm{E}|x_{ith}^2|^{r+\delta} \leq \Delta < \infty$, and $\mathrm{E}|\eta_{it}^2|^{r+\delta} \leq \Delta < \infty$ for some $\delta > 0$ and all i, t, h.

Proposition 3. *If Assumption 2 is satisfied,* $\sqrt{ST}(\widehat{\alpha} - \alpha) \xrightarrow{d} N(\rho B(\alpha), \Gamma_p^{-1}\Xi\Gamma_p^{-1})$ *if* $\frac{S}{T} \to \rho \geq 0$*, where* $\Xi = \lim_{T\to\infty} \frac{1}{T-p}\sum_{t_1=p+1}^{T}\sum_{t_2=p+1}^{T} \mathrm{E}[v_{st_1}^{-}\eta_{st_1}\eta_{st_2}v_{st_2}^{-\prime}]$*. In addition, if* η_{st} *are independent for all* s *and* t*,* $\Xi = \sigma_\eta^2\Gamma_p$*.*

Assumption 3. For $B(\alpha, T)$ defined in Lemma C.5,

- NT6. $B(\alpha, T)$ is continuously differentiable in α with bounded derivative uniformly in T for all α satisfying the stationarity condition given in NT1.

Proposition 4. *If Assumption 2 is satisfied,* $\sqrt{ST}(\widehat{\alpha}^{(1)} - \alpha) \xrightarrow{d} N(0, \Gamma_p^{-1}\Xi\Gamma_p^{-1})$ *for* Ξ *defined in Proposition 3 if* $\frac{S}{T} \to \rho \geq 0$*. If Assumption 3 is also satisfied, then* $\sqrt{ST}(\widehat{\alpha}^{(1)} - \alpha) \xrightarrow{d} N(0, \Gamma_p^{-1}\Xi\Gamma_p^{-1})$ *if* $\frac{S}{T^3} \to 0$*.*

Proposition 5. *If Assumptions 2 and 3 are satisfied and* $H = D\alpha_T(\alpha)$ *is invertible for all* α *satisfying the stationarity condition given in NT1 uniformly in* T*, where* $D\alpha_T(\alpha)$ *is the derivative matrix of* $\alpha_T(\alpha)$ *in* α*,* $\sqrt{ST}(\widehat{\alpha}^{(\infty)} - \alpha) \xrightarrow{d} N(0, \Gamma_p^{-1}\Xi\Gamma_p^{-1})$ *for* Ξ *defined in Proposition 3.*

Assumption 4 (Higher-Order Asymptotics)**.** Suppose the data are generated by model (1) and

- HO1. $v_{st} = v_{st}^{-\prime}\alpha + \eta_{st}$, where η_{st} are iid $\mathrm{N}(0, \sigma_\eta^2)$ random variables, and $\mathrm{E}[V_sV_s'] = \Gamma(\alpha)$ is a $T \times T$ positive definite matrix with minimum eigenvalue bounded away from 0 and maximum eigenvalue bounded away from infinity uniformly in T.
- HO2. (i) $\{X_s, Z_s\}$ are nonstochastic with $\{Z_s\}$ identical across s. (ii) $[X, Z]$, where $X = [X_1', \ldots, X_S']'$ and $Z = \mathrm{diag}(Z_1, ..., Z_S)$ has full rank, and the minimum eigenvalue of $X'X - X'Z(Z'Z)^{-1}Z'X$ is bounded away from zero. (iii) $|x_{sth}| \leq \Delta$ and $|z_{sth}| \leq \Delta$.

HO3. Let $M_z(\widetilde{\alpha}) = (I \otimes \Gamma(\widetilde{\alpha})^{-1}) - (I \otimes \Gamma(\widetilde{\alpha})^{-1})Z(Z'(I \otimes \Gamma(\widetilde{\alpha})^{-1})Z)^{-1}Z'(I \otimes \Gamma(\widetilde{\alpha})^{-1})$, $M = I - X(X'M_z(\alpha)X)^{-1}X'M_z(\alpha)$, $A(\widetilde{\alpha}) = \frac{1}{ST}X'M_z(\widetilde{\alpha})X$, and $\Psi(\widetilde{\alpha}) = \frac{1}{\sqrt{ST}}X'M_z(\widetilde{\alpha})MV$. For $i = 1, ..., p$, $j = 1, ..., p$, and $k = 1, ..., p$, (i) each matrix in $A(\alpha)$, $A_i(\alpha) = \frac{\partial A}{\partial \widetilde{\alpha}_i}|_\alpha$, and $A_{ij} = \frac{\partial^2 A}{\partial \widetilde{\alpha}_i \partial \widetilde{\alpha}_j}|_\alpha$ approaches a limit as $S,\ T \to \infty$, and $\lim A(\alpha)$ is nonsingular, (ii) the covariance matrices for the vectors in $\Psi_i(\alpha) = \frac{\partial \Psi}{\partial \widetilde{\alpha}_i}|_\alpha$, and $\Psi_{ij}(\alpha) = \frac{\partial^2 \Psi}{\partial \widetilde{\alpha}_i \partial \widetilde{\alpha}_j}|_\alpha$ approach limits as $S,\ T \to \infty$, and (iii) all the matrices in $A(\widetilde{\alpha})$, $A_i(\widetilde{\alpha})$, $A_{ij}(\widetilde{\alpha})$, $A_{ijk}(\widetilde{\alpha}) = \frac{\partial^3 A}{\partial \widetilde{\alpha}_i \partial \widetilde{\alpha}_j \partial \widetilde{\alpha}_k}|_{\widetilde{\alpha}}$, $\Psi(\widetilde{\alpha})$, $\Psi_i(\widetilde{\alpha})$, $\Psi_{ij}(\widetilde{\alpha})$, and $\Psi_{ijk}(\widetilde{\alpha}) = \frac{\partial^3 \Psi}{\partial \widetilde{\alpha}_i \partial \widetilde{\alpha}_j \partial \widetilde{\alpha}_k}|_{\widetilde{\alpha}}$ are bounded in probability uniformly for $\widetilde{\alpha} \in \mathcal{A}$ where $\mathcal{A}$ is a neighborhood of α.

HO4. $B(\alpha, T)$ is three times continuously differentiable in α with the first three derivatives bounded uniformly in T for all α satisfying the stationarity condition given in NT1.

Proposition 6. *Let $\widehat{\alpha}$ be the least squares estimator of α, $\widehat{\alpha}^{(\infty)}$ be the iteratively bias-corrected estimator, and $\widehat{\beta}(\widehat{\alpha})$ and $\widehat{\beta}(\widehat{\alpha}^{(\infty)})$ be the corresponding FGLS estimators. If Assumption 4 is satisfied and $\frac{S}{T} \to \rho$ as $\{S,\ T\} \to \infty$ jointly, the higher-order bias and variance of $\widehat{\beta}(\widehat{\alpha})$ and $\widehat{\beta}(\widehat{\alpha}^{(\infty)})$ are*

$$Bias(\widehat{\beta}(\widehat{\alpha})) = Bias(\widehat{\beta}(\widehat{\alpha}^{(\infty)})) = 0 \tag{2}$$

$$\begin{aligned} Var(\sqrt{ST}(\widehat{\beta}(\widehat{\alpha}) - \beta)) = {} & A^{-1} + \Upsilon/ST \\ & + \frac{1}{ST}\sum_{i=1}^{p}\sum_{j=1}^{p}\zeta_{ij}\frac{S}{T}B(\alpha, T)_i B(\alpha, T)_j + O(1/ST^2) \end{aligned} \tag{3}$$

$$Var(\sqrt{ST}(\widehat{\beta}(\widehat{\alpha}^{(\infty)}) - \beta)) = A^{-1} + \Upsilon/ST + O(1/ST^2). \tag{4}$$

for $\zeta_{ij} = \mathrm{E}[\Psi_i \Psi_j']$ and $\Upsilon = \sum_{i=1}^{p}\sum_{j=1}^{p}\zeta_{ij}(\frac{ST}{S(T-p)}\Gamma_p^{-1})_{[i,j]}$ where $(\frac{ST}{S(T-p)}\Gamma_p^{-1})_{[i,j]}$ is the $[i, j]$ element of matrix $\frac{ST}{S(T-p)}\Gamma_p^{-1}$. Also,

$$\begin{aligned} Var(\sqrt{ST}(\widehat{\beta}(\widehat{\alpha}) - \beta)) - & Var(\sqrt{ST}(\widehat{\beta}(\widehat{\alpha}^{(\infty)}) - \beta)) \\ & = \frac{1}{ST}\sum_{i=1}^{p}\sum_{j=1}^{p}\zeta_{ij}\frac{S}{T}B(\alpha, T)_i B(\alpha, T)_j \geq 0. \end{aligned}$$

Appendix A. Notation

The following notation will be used throughout the appendix.

Suppose the panel has a cross-sectional dimension $s = 1, \ldots, S$ and a time-series dimension $t = 1, \ldots, T$ with $T > 2p$ where p is the order of the autoregressive process. Let

$$C_{st} = x_{st}'\beta_1 + z_{st}'\beta_2^s + v_{st}, \tag{5}$$

or, in vector notation, $C_s = X_s\beta_1 + Z_s\beta_2^s + V_s$, where $C_s = [C_{s1}, \ldots, C_{sT}]'$ is $T\times 1$, $X_s = [x_{s1}, \ldots, x_{sT}]'$ is $T \times k_1$, $Z_s = [z_{s1}, \ldots, z_{sT}]'$ is $T \times k_2$, and $V_s = [v_{s1}, \ldots, v_{sT}]'$ is $T \times 1$. Also, let x_{sth} be the h^{th} element of x_{st} so that $x'_{st} = [x_{st1}, \ldots, x_{stk_1}]$, and define z_{sth} similarly.

Let v_{st}^{-} be a $p \times 1$ vector with $v_{st}^{-} = [v_{s(t-p)}, \ldots, v_{s(t-1)}]'$.

Define $\ddot{v}_{st} = v_{st} - z'_{st}(Z'_sZ_s)^{-1}Z'_sV_s$, $\ddot{x}'_{st} = x'_{st} - z'_{st}(Z'_sZ_s)^{-1}Z'_sX_s$, $\ddot{V}_s = [\ddot{v}_{s1}, \ldots, \ddot{v}_{sT}]'$, and $\ddot{X}_s = [\ddot{x}_{s1}, \ldots, \ddot{x}_{sT}]'$.

Throughout, let $\|A\| = [\text{trace}(A'A)]^{1/2}$ be the Euclidean norm of a matrix A.

Appendix B. Proof of Proposition 1 and Proposition 2

Proposition 1 is verified by combining Lemmas B.3 and B.4 below, and Proposition 2 follows from Proposition 1 and Lemma B.6.

All results presented below are for asymptotics where $S \to \infty$ with T fixed.

Proof of Proposition 1. Immediate from Lemma B.3 and Lemma B.4. ■

Proof of Proposition 2. That $\alpha_T(\alpha)$ is continuously differentiable in α and that $H = D\alpha_T(\alpha)$ is invertible for all α such that N1 is satisfied imply that $\alpha_T(\alpha)$ is invertible for all α such that N1 is satisfied by the Inverse Function Theorem. (See, e.g. Fitzpatrick (1996) Theorem 16.9.) $\widehat{\alpha}^{(\infty)} - \alpha \xrightarrow{p} 0$ then follows immediately from the definition of $\widehat{\alpha}^{(\infty)}$ and Proposition 1.

To verify the asymptotic normality, expand $\widehat{\alpha}^{(\infty)}$ about $\widehat{\alpha} = \alpha_T(\alpha)$. This gives

$$\widehat{\alpha}^{(\infty)} = \alpha_T^{-1}(\alpha_T(\alpha)) + H^{-1}|_{\widetilde{\alpha_T(\alpha)}}(\widehat{\alpha} - \alpha_T(\alpha)),$$

where $\widetilde{\alpha_T(\alpha)}$ is an intermediate value between $\widehat{\alpha}$ and $\alpha_T(\alpha)$. The conclusion then follows from continuity of H, Proposition 1, and Lemma B.6. ■

B.1. **Lemmas.**

Lemma B.1. *Let $\widehat{\beta}_1$ be the ordinary least squares estimate of β_1. Then if the conditions of Assumption 1 are satisfied, $\widehat{\beta}_1 - \beta_1 \xrightarrow{p} 0$ and $\sqrt{S}(\widehat{\beta}_1 - \beta_1) \xrightarrow{d} N(0, M^{-1}\Omega M^{-1})$, where $M = \mathrm{E}[\ddot{X}'_s\ddot{X}_s]$ and $\Omega = \mathrm{E}[\ddot{X}'_s\Gamma_(\alpha)\ddot{X}_s]$.*

Proof. Using $\|AB\| \leq \|A\|\|B\|$ and $I - Z_s(Z'_sZ_s)^{-1}Z_s$ positive semi-definite, $\mathrm{E}\|\ddot{X}'_s\ddot{X}_s\| \leq \mathrm{E}\|\ddot{X}_s\|^2 = \mathrm{E}[\text{trace}(X'_sX_s - X'_sZ_s(Z'_sZ_s)^{-1}Z'_sX_s)] \leq \text{trace}(\mathrm{E}X'_sX_s) = \text{trace}(\sum_{t=1}^{T}\mathrm{E}[x_{st}x'_{st}]) < \infty$ by N5. Also, $\|\ddot{X}'_s\ddot{V}_s\| \leq (\mathrm{E}\|\ddot{X}'_s\|^2\mathrm{E}\|\ddot{V}_s\|^2)^{1/2} < \infty$ from the Cauchy-Schwarz inequality, N5, and the same argument as above. The Khinchin LLN then yields $\frac{1}{S}\sum_{s=1}^{S}\ddot{X}'_s\ddot{X}_s \xrightarrow{p} M$ and $\frac{1}{S}\sum_{s=1}^{S}\ddot{X}'_s\ddot{V}_s \xrightarrow{p} 0$, from which $\widehat{\beta}_1 - \beta_1 \xrightarrow{p} 0$ follows.

To show asymptotic normality of $\widehat{\beta}_1$ first note that $\ddot{X}_s'\ddot{V}_s$ is iid and has mean zero by N2 and N4. Also, $\mathrm{E}\|\ddot{X}_s'\ddot{V}_s\ddot{V}_s'\ddot{X}_s\| \leq (2\mathrm{E}\|X_s\|^4\mathrm{E}\|V_s\|^4)^{1/2} < \infty$ by N5, $\|AB\| \leq \|A\|\|B\|$, the Cauchy-Schwarz inequality, and $\mathrm{E}\|\ddot{X}_s\|^4 = \mathrm{E}[(\mathrm{trace}(X_s'X_s))^2 - 2\mathrm{trace}(X_s'X_s)\mathrm{trace}(X_s'Z_s(Z_s'Z_s)^{-1}Z_s'X_s) + (\mathrm{trace}(X_s'Z_s(Z_s'Z_s)^{-1}Z_s'X_s))^2] \leq \mathrm{E}[2(\mathrm{trace}(X_s'X_s))^2] = 2\mathrm{E}\|X_s\|^4$, where the inequality follows from $X_s'X_s$, $X_s'Z_s(Z_s'Z_s)^{-1}Z_s'X_s$, and $I - Z_s(Z_s'Z_s)^{-1}Z_s$ positive semi-definite. It then follows from the Lindeberg-Levy CLT that $\frac{1}{\sqrt{S}}\sum_{s=1}^{S}\ddot{X}_s'\ddot{V}_s \xrightarrow{d} N(0,\Omega)$ since $\mathrm{E}[\ddot{X}_s'\ddot{V}_s\ddot{V}_s'\ddot{X}_s] = \mathrm{E}[\ddot{X}_s'V_sV_s'\ddot{X}_s] = \mathrm{E}[\ddot{X}_s'\Gamma(\alpha)\ddot{X}_s]$, from which $\sqrt{S}(\widehat{\beta}_1 - \beta_1) \xrightarrow{d} N(0, M^{-1}\Omega M^{-1})$ is obtained. ■

Lemma B.2. *Define $\widetilde{v}_{st}$ to be the residual from least squares regression of (1); i.e. $\widetilde{v}_{st} = C_{st} - x_{st}'\widehat{\beta}_1 - z_{st}'\widehat{\beta}_2^s = v_{st} - x_{st}'(\widehat{\beta}_1 - \beta_1) - z_{st}'(\widehat{\beta}_2^s - \beta_2) = \ddot{v}_{st} - \ddot{x}_{st}'(\widehat{\beta}_1 - \beta_1)$, where $\widehat{\beta}_1$ and $\widehat{\beta}_2^s$ are least squares estimates of β_1 and β_2^s. Let $\widetilde{v}_{st}^-$ be a $p \times 1$ vector with $\widetilde{v}_{st}^- = [\widetilde{v}_{s(t-p)}, \ldots, \widetilde{v}_{s(t-1)}]'$, and let $\ddot{v}_{st}^-$ be a $p \times 1$ vector with $\ddot{v}_{st}^- = [\ddot{v}_{s(t-p)}, \ldots, \ddot{v}_{s(t-1)}]'$. Under the conditions of Assumption 1, $\frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\widetilde{v}_{st}^-\widetilde{v}_{st}^{-\prime} = \frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^-\ddot{v}_{st}^{-\prime} + o_p(S^{-1/2})$, and $\frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\widetilde{v}_{st}^-\widetilde{v}_{st} = \frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^-\ddot{v}_{st} + o_p(S^{-1/2})$.*

Proof.

$$\frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\widetilde{v}_{st}^-\widetilde{v}_{st} = \frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}(\ddot{v}_{st}^-\ddot{v}_{st} - \ddot{x}_{st}^-(\widehat{\beta}_1 - \beta_1)\ddot{v}_{st} - \ddot{v}_{st}^-\ddot{x}_{st}'(\widehat{\beta}_1 - \beta_1) + \ddot{x}_{st}^-(\widehat{\beta}_1 - \beta_1)(\widehat{\beta}_1 - \beta_1)'\ddot{x}_{st}),$$

where $\ddot{x}_{st}^-$ is a $p \times k_1$ matrix with $\ddot{x}_{st}^- = [\ddot{x}_{s(t-p)}, \ldots, \ddot{x}_{s(t-1)}]'$. Note

$$\mathrm{vec}(\frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{x}_{st}^-(\widehat{\beta}_1 - \beta_1)\ddot{v}_{st}) = \{\frac{1}{S}\sum_{s=1}^{S}[\sum_{t=p+1}^{T}(\ddot{v}_{st} \otimes \ddot{x}_{st}^-)]\}(\widehat{\beta}_1 - \beta_1) = o_p(1)O_p(S^{-1/2})$$

by Lemma B.1, Assumption 1, and the Khinchin LLN. Similarly,

$$\mathrm{vec}(\frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^-\ddot{x}_{st}'(\widehat{\beta}_1 - \beta_1)) = o_p(1)O_p(S^{-1/2}),$$

and

$$\mathrm{vec}(\frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{x}_{st}^-(\widehat{\beta}_1 - \beta_1)(\widehat{\beta}_1 - \beta_1)'\ddot{x}_{st}) = O_p(1)O_p(S^{-1/2})O_p(S^{-1/2}).$$

It then follows that $\frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\widetilde{v}_{st}^-\widetilde{v}_{st} = \frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^-\ddot{v}_{st} + o_p(S^{-1/2})$ and, by a similar argument, $\frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\widetilde{v}_{st}^-\widetilde{v}_{st}^{-\prime} = \frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^-\ddot{v}_{st}^{-\prime} + o_p(S^{-1/2})$. ■

Lemma B.3. *Under the conditions of Assumption 1, $\frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^-\ddot{v}_{st}^{-\prime} \xrightarrow{p} \mathrm{E}[\sum_{t=p+1}^{T}\ddot{v}_{st}^-\ddot{v}_{st}^{-\prime}] = (T-p)(\Gamma_p(\alpha) + \frac{1}{T-p}\Delta_\Gamma(\alpha))$, and $\frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^-\ddot{v}_{st} \xrightarrow{p} \mathrm{E}[\sum_{t=p+1}^{T}\ddot{v}_{st}^-\ddot{v}_{st}] = (T-p)(A(\alpha) + \frac{1}{T-p}\Delta_A(\alpha))$, where $\Gamma_p(\alpha) + \frac{1}{T-p}\Delta_\Gamma(\alpha)$ is a $p \times p$ matrix with i,j element*

$$\begin{aligned}
\mathrm{E}\left[\frac{1}{T-p}\sum_{t=p+1}^{T}\ddot{v}_{st}^-\ddot{v}_{st}^{-\prime}\right]_{[i,j]} &= \gamma_{|i-j|}(\alpha) - \frac{1}{T-p}\mathrm{trace}\left(Z_s'\Gamma_{-i}(\alpha)Z_{s,-j}(Z_s'Z_s)^{-1}\right) \\
&- \frac{1}{T-p}\mathrm{trace}\left(Z_s'\Gamma_{-j}(\alpha)Z_{s,-i}(Z_s'Z_s)^{-1}\right) \qquad (6) \\
&+ \frac{1}{T-p}\mathrm{trace}\left(Z_s'\Gamma(\alpha)Z_s(Z_s'Z_s)^{-1}Z_{s,-i}'Z_{s,-j}(Z_s'Z_s)^{-1}\right),
\end{aligned}$$

$A(\alpha) + \frac{1}{T-p}\Delta_A(\alpha)$ is a $p \times 1$ vector with i^{th} element

$$\begin{aligned}
\mathrm{E}\left[\frac{1}{T-p}\sum_{t=p+1}^{T} \ddot{v}_{st}^{-}\ddot{v}_{st}\right]_{[i,1]} &= \gamma_i(\alpha) - \frac{1}{T-p}\mathrm{trace}\left(Z_s'\Gamma_{-i}(\alpha)Z_{s,-0}(Z_s'Z_s)^{-1}\right) \\
&- \frac{1}{T-p}\mathrm{trace}\left(Z_s'\Gamma_{-0}(\alpha)Z_{s,-i}(Z_s'Z_s)^{-1}\right) \\
&+ \frac{1}{T-p}\mathrm{trace}\left(Z_s'\Gamma(\alpha)Z_s(Z_s'Z_s)^{-1}Z_{s,-i}'Z_{s,-0}(Z_s'Z_s)^{-1}\right),
\end{aligned} \tag{7}$$

$\Gamma(\alpha) = \mathrm{E}[V_sV_s']$, $\Gamma_{-k}(\alpha) = \mathrm{E}[V_sV_{s,-k}']$, $\gamma_i(\alpha) = \mathrm{E}[v_{st}v_{s(t-i)}]$, $V_{s,-k} = (v_{s(p+1-k)}, v_{s(p+2-k)}, \ldots, v_{s(T-k)})'$ and $Z_{s,-k}$ is defined similarly.

Proof. The proof is given for $\frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}^{-\prime} \xrightarrow{p} \mathrm{E}[\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}^{-\prime}] = (T-p)(\Gamma_p(\alpha) + \frac{1}{T-p}\Delta_\Gamma(\alpha))$; $\frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st} \xrightarrow{p} \mathrm{E}[\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}] = (T-p)(A(\alpha) + \frac{1}{T-p}\Delta_A(\alpha))$ follows by a similar argument.

$\frac{1}{S}\sum_{s=1}^{S}\frac{1}{T-p}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}^{-\prime}$ is a $p \times p$ matrix with $[i,j]$ element

$$\begin{aligned}
\left[\frac{1}{S}\sum_{s=1}^{S}\frac{1}{T-p}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}^{-\prime}\right]_{[i,j]} &= \frac{1}{S}\sum_{s=1}^{S}\frac{1}{T-p}\sum_{t=p+1}^{T} v_{s(t-i)}v_{s(t-j)} \\
&- \frac{1}{S}\sum_{s=1}^{S}\frac{1}{T-p}\sum_{t=p+1}^{T} v_{s(t-i)}z_{s(t-j)}'(Z_s'Z_s)^{-1}Z_s'V_s \\
&- \frac{1}{S}\sum_{s=1}^{S}\frac{1}{T-p}\sum_{t=p+1}^{T} v_{s(t-j)}z_{s(t-i)}'(Z_s'Z_s)^{-1}Z_s'V_s \\
&+ \frac{1}{S}\sum_{s=1}^{S}\frac{1}{T-p}\sum_{t=p+1}^{T} z_{s(t-i)}'(Z_s'Z_s)^{-1}Z_s'V_sV_s'Z_s(Z_s'Z_s)^{-1}z_{s(t-j)} \\
&= \frac{1}{S}\sum_{s=1}^{S}\frac{1}{T-p}\sum_{t=p+1}^{T} v_{s(t-i)}v_{s(t-j)} \\
&- \frac{1}{S}\sum_{s=1}^{S}\frac{1}{T-p}V_{s,-i}'Z_{s,-j}(Z_s'Z_s)^{-1}Z_s'V_s \\
&- \frac{1}{S}\sum_{s=1}^{S}\frac{1}{T-p}V_{s,-j}'Z_{s,-i}(Z_s'Z_s)^{-1}Z_s'V_s \\
&+ \frac{1}{S}\sum_{s=1}^{S}\frac{1}{T-p}Z_{s,-i}(Z_s'Z_s)^{-1}Z_s'V_sV_s'Z_s'(Z_s'Z_s)^{-1}Z_{s,-j}' \\
&\xrightarrow{p} \mathrm{E}[\frac{1}{T-p}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}^{-\prime}]_{[i,j]}
\end{aligned} \tag{8}$$

by the Khinchin LLN and repeated application of the triangle and Cauchy-Schwarz inequalities since

(i) $[\frac{1}{T-p}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}^{-\prime}]_{[i,j]}$ is iid by N2,

(ii) $\mathrm{E}|v_{s(t-i)}v_{s(t-j)}| < \infty$ by N1,

(iii) $\mathrm{E}|v_{s(t-i)}z'_{s(t-j)}(Z'_sZ_s)^{-1}Z'_sV_s| \leq \left(\mathrm{E}|z'_{s(t-j)}(Z'_sZ_s)^{-1}Z'_sV_s|^2\mathrm{E}|v_{s(t-i)}|^2\right)^{1/2}$,

(iv) $\mathrm{E}|z'_{s(t-i)}(Z'_sZ_s)^{-1}Z'_sV_sV'_sZ_s(Z'_sZ_s)^{-1}z_{s(t-j)}|$
$\leq \left(\mathrm{E}|z'_{s(t-i)}(Z'_sZ_s)^{-1}Z'_sV_s|^2\mathrm{E}|z'_{s(t-j)}(Z'_sZ_s)^{-1}Z'_sV_s|^2\right)^{1/2}$, and

(v) by N1 and N5,

$$\begin{aligned}
\mathrm{E}|z'_{s(t-j)}(Z'_sZ_s)^{-1}Z'_sV_s|^2 &= \mathrm{E}|z'_{s(t-j)}(Z'_sZ_s)^{-1}(Z'_sZ_s)(Z'_sZ_s)^{-1}Z'_sV_s|^2 \\
&\leq \|z'_{s(t-j)}(Z'_sZ_s)^{-1}Z'_s\|^2\mathrm{E}\|Z_s(Z'_sZ_s)^{-1}Z'_sV_s\|^2 \\
&= \mathrm{trace}(z'_{s(t-j)}(Z'_sZ_s)^{-1}z_{s(t-j)}) \times \\
&\qquad \mathrm{E}\left[\mathrm{trace}(V'_sZ_s(Z'_sZ_s)^{-1}Z'_sV_s)\right] \\
&\leq k_2\mathrm{E}[\mathrm{trace}(V'_sV_s)] = k_2\mathrm{E}[\sum_{t=p+1}^{T}v_{st}^2] < \infty,
\end{aligned}$$

from which it follows that $\mathrm{E}|\left[\frac{1}{T-p}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}^{-\prime}\right]_{[i,j]}| < \infty$. ■

Lemma B.4. *Let $\widehat{\alpha} = (\frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\widetilde{v}_{st}^{-}\widetilde{v}_{st}^{-\prime})^{-1}(\frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\widetilde{v}_{st}^{-}\widetilde{v}_{st})$ be the least squares estimate of α using the least squares residuals, $\widetilde{v}_{st}$ from estimating β_1. If Assumption 1 is satisfied,*

$$\widehat{\alpha} = (\frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}^{-\prime})^{-1}(\frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}) + o_p(S^{-1/2}).$$

Proof. By Lemma B.2,

$$\widehat{\alpha} = (\frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}^{-\prime} + M_1)^{-1}(\frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st} + M_2),$$

where $M_1 = o_p(S^{-1/2})$ and $M_2 = o_p(S^{-1/2})$. After some algebra, it then follows that

$$\begin{aligned}
\widehat{\alpha} &= (\frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}^{-\prime})^{-1}(\frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}) \\
&+ (\frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}^{-\prime} + M_1)^{-1}M_2 \\
&- (\frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}^{-\prime} + M_1)^{-1}M_1(\frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}^{-\prime})^{-1}(\frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}) \\
&= (\frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}^{-\prime})^{-1}(\frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}) + O_p(1)o_p(S^{-1/2}) + O_p(1)o_p(S^{-1/2})O_p(1)O_p(1)
\end{aligned}$$

where the last equality is by Lemma B.3, which yields the conclusion. ■

Lemma B.5. *Define* $\ddot{\mu}_{st} = \ddot{v}_{st} - \ddot{v}_{st}^{-\prime}\alpha_T(\alpha)$. *If Assumption 1 is satisfied,* $\frac{1}{\sqrt{S}}\sum_{s=1}^{S}(\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{\mu}_{st}) \xrightarrow{d} N(0, \Xi)$, *where* $\Xi = \mathrm{E}[\sum_{t_1=p+1}^{T}\sum_{t_2=p+1}^{T}\ddot{v}_{st_1}^{-}\ddot{\mu}_{st_1}\ddot{\mu}_{st_2}\ddot{v}_{st_2}^{-\prime}]$.

Proof. $\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{\mu}_{st}$ are iid by N2. Also,

$$\begin{aligned}
\mathrm{E}[\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{\mu}_{st}] &= \mathrm{E}[\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}(\ddot{v}_{st} - \ddot{v}_{st}^{-\prime}\alpha_T(\alpha))] \\
&= \mathrm{E}[\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}] - \mathrm{E}\{\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}^{-\prime}(\mathrm{E}[\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}^{-\prime}])^{-1}\mathrm{E}[\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}]\} \\
&= 0,
\end{aligned}$$

where the second equality comes from $\alpha_T(\alpha) = \mathrm{E}[\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}^{-\prime}]^{-1}\mathrm{E}[\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}]$ from Proposition 1. Also

$$\begin{aligned}
&\mathrm{E}[\sum_{t_1=p+1}^{T}\sum_{t_2=p+1}^{T}\ddot{v}_{s(t_1-i)}\ddot{\mu}_{st_1}\ddot{\mu}_{st_2}\ddot{v}_{s(t_2-j)}] \\
&\quad = \mathrm{E}[\sum_{t_1=p+1}^{T}\sum_{t_2=p+1}^{T}\ddot{v}_{s(t_1-i)}\ddot{v}_{st_1}\ddot{v}_{st_2}\ddot{v}_{s(t_2-j)}] - \mathrm{E}[\sum_{k=1}^{p}\sum_{t_1=p+1}^{T}\sum_{t_2=p+1}^{T}\ddot{v}_{s(t_1-i)}\ddot{v}_{s(t_1-k)}\ddot{v}_{st_2}\ddot{v}_{s(t_2-j)}\alpha_T(\alpha)_k] \\
&\qquad - \mathrm{E}[\sum_{k=1}^{p}\sum_{t_1=p+1}^{T}\sum_{t_2=p+1}^{T}\ddot{v}_{s(t_1-i)}\ddot{v}_{st_1}\ddot{v}_{s(t_2-k)}\ddot{v}_{s(t_2-j)}\alpha_T(\alpha)_k] \\
&\qquad + \mathrm{E}[\sum_{k_1=1}^{p}\sum_{k_2=1}^{p}\sum_{t_1=p+1}^{T}\sum_{t_2=p+1}^{T}\ddot{v}_{s(t_1-i)}\ddot{v}_{s(t_1-k_1)}\ddot{v}_{s(t_2-k_2)}\ddot{v}_{s(t_2-j)}\alpha_T(\alpha)_{k_1}\alpha_T(\alpha)_{k_2}] \\
&\qquad < \infty
\end{aligned}$$

since $\mathrm{E}|\ddot{v}_{st_1}\ddot{v}_{st_2}\ddot{v}_{st_3}\ddot{v}_{st_4}| \leq (\mathrm{E}|\ddot{v}_{st_1}|^4\mathrm{E}|\ddot{v}_{st_2}|^4\mathrm{E}|\ddot{v}_{st_3}|^4\mathrm{E}|\ddot{v}_{st_4}|^4)^{1/4} < \infty$ by repeated application of the Cauchy-Schwarz inequality and $\mathrm{E}\|\ddot{v}_{st}\|^4 = \mathrm{E}\|e_t'(V_s - Z_s(Z_s'Z_s)^{-1}Z_s'V_s)\|^4 \leq \mathrm{E}(\|e_t\|^4\|V_s\|^4\|I_T - Z_s(Z_s'Z_s)^{-1}Z_s'\|^4) = (T-k_2)^2\mathrm{E}\|V_s\|^4 < \infty$ by N5 and T fixed, where e_t is the t^{th} unit vector. Then the conclusion follows by the Lindeberg-Levy CLT since

$$\mathrm{E}[\sum_{t_1=p+1}^{T}\sum_{t_2=p+1}^{T}\ddot{v}_{st_1}^{-}\ddot{\mu}_{st_1}\ddot{\mu}_{st_2}\ddot{v}_{st_2}^{-\prime}]_{[i,j]} = \mathrm{E}[\sum_{t_1=p+1}^{T}\sum_{t_2=p+1}^{T}\ddot{v}_{s(t_1-i)}\ddot{\mu}_{st_1}\ddot{\mu}_{st_2}\ddot{v}_{s(t_2-j)}] < \infty \ \forall\ i, j.$$

■

Lemma B.6. *Suppose Assumption 1 holds, then* $\sqrt{S}(\widehat{\alpha} - \alpha_T(\alpha)) \xrightarrow{d} \frac{1}{T-p}(\Gamma_p(\alpha) + \frac{1}{T-p}\Delta_\Gamma(\alpha))^{-1}N(0, \Xi)$.

Proof. For $\ddot{\mu}_{st}$ defined in Lemma B.5, $\widehat{\alpha} - \alpha_T(\alpha) = (\frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}^{-\prime})^{-1}(\frac{1}{S}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{\mu}_{st}) + o_p(S^{-1/2})$ by Lemma B.4. The conclusion is then immediate from Lemmas B.3 and B.5. ■

Appendix C. Proof of Propositions 3, 4, and 5

The proofs of Propositions 3, 4, and 5 are collected below. All results presented below are for asymptotics where $S,\ T \to \infty$.

Proof of Proposition 3. $\sqrt{ST}(\widehat{\alpha}-\alpha) = \sqrt{ST}(\widehat{\alpha}-\alpha_T(\alpha)+\alpha_T(\alpha)-\alpha) = \sqrt{ST}(\widehat{\alpha}-\alpha_T(\alpha)) + \sqrt{\frac{S}{T}}B(\alpha,T)$ by Lemma C.5, from which the conclusion follows by Lemmas C.5 and C.8. ■

Proof of Proposition 4.

$$\begin{aligned}\sqrt{ST}(\widehat{\alpha}^{(1)}-\alpha) &= \sqrt{ST}[\widehat{\alpha}-(-\widehat{\alpha}+\alpha_T(\widehat{\alpha}))-\alpha_T(\alpha)+\alpha_T(\alpha)-\alpha]\\ &= \sqrt{ST}(\widehat{\alpha}-\alpha_T(\alpha))+\sqrt{\frac{S}{T}}(B(\alpha,T)-B(\widehat{\alpha},T)).\end{aligned}$$

The first conclusion then follows from Lemmas C.5, C.8, C.9, and the Continuous Mapping Theorem, and the second conclusion follows from Lemmas C.5, C.8, C.9, and a Taylor expansion of $B(\widehat{\alpha},T)$ about $\widehat{\alpha}=\alpha$. ■

Proof of Proposition 5. Recall $\widehat{\alpha}^{(\infty)} = \alpha_T^{-1}(\widehat{\alpha}) = \alpha_T^{-1}(\alpha_T(\alpha)) + H^{-1}|_{\widetilde{\alpha_T(\alpha)}}(\widehat{\alpha}-\alpha_T(\alpha))$. From Lemma C.5, $\alpha_T(\alpha) = \alpha + \frac{1}{T-p}B(\alpha,T)$ which implies $H^{-1}\to I$ as $T\to\infty$ by NT6. The conclusion then follows from Lemma C.8. ■

C.1. Lemmas.

Lemma C.1. *Let $\widehat{\beta}_1$ be the ordinary least squares estimate of β_1. Then if the conditions of Assumption 2 are satisfied, $\widehat{\beta}_1-\beta_1 \xrightarrow{p} 0$ and $\sqrt{ST}(\widehat{\beta}_1-\beta_1)\xrightarrow{d} N(0,M^{-1}\Omega M^{-1})$, where*

$$\begin{aligned}M &= M_{XX}-M_{XZ}M_{ZZ}^{-1}M_{XZ}',\\ \Omega &= \lim_{T\to\infty}\mathrm{E}[\frac{1}{T}X_s'\Gamma(\alpha)X_s]-M_{XZ}M_{ZZ}^{-1}(\lim_{T\to\infty}\mathrm{E}[\frac{1}{T}Z_s'\Gamma(\alpha)X_s])\\ &\quad -(\lim_{T\to\infty}\mathrm{E}[\frac{1}{T}X_s'\Gamma(\alpha)Z_s])M_{ZZ}^{-1}M_{XZ}'+M_{XZ}M_{ZZ}^{-1}(\lim_{T\to\infty}[\frac{1}{T}Z_s'\Gamma(\alpha)Z_s])M_{ZZ}^{-1}M_{XZ}',\end{aligned}$$

with $M_{XX}=\lim_{T\to\infty}\mathrm{E}[\frac{1}{T}X_s'X_s]$, $M_{XZ}=\lim_{T\to\infty}\mathrm{E}[\frac{1}{T}X_s'Z_s]$, and $M_{ZZ}=\lim_{T\to\infty}[\frac{1}{T}Z_s'Z_s]$.

Proof. Let $Q_s = I_T - Z_s(Z_s'Z_s)^{-1}Z_s'$. Then

$$\mathrm{E}\|\frac{1}{T}X_s'Q_sV_s\|^{1+\delta} \le \mathrm{E}(\|\frac{1}{\sqrt{T}}Q_sX_s\|^{1+\delta}\|\frac{1}{\sqrt{T}}Q_sV_s\|^{1+\delta}) \le (\mathrm{E}\|\frac{1}{\sqrt{T}}X_s\|^{2+2\delta}\mathrm{E}\|\frac{1}{\sqrt{T}}V_s\|^{2+2\delta})^{1/2} \le k_1^{\frac{1+\delta}{2}}\Delta,$$

where the first inequality is from $\|AB\| \leq \|A\|\|B\|$, the second from the Cauchy-Schwarz inequality and $\|Q_s A\| \leq \|A\|$, and the third from

$$\begin{aligned}
\mathrm{E}\|\frac{1}{\sqrt{T}}X_s\|^{2+2\delta} &= \mathrm{E}[\frac{1}{T}\mathrm{trace}(X_s'X_s)]^{1+\delta} \\
&= \frac{1}{T^{1+\delta}}\mathrm{E}[\sum_{t=1}^{T}\sum_{h=1}^{k_1} x_{ith}^2]^{1+\delta} \\
&\leq \frac{1}{T^{1+\delta}}[\sum_{t=1}^{T}\sum_{h=1}^{k_1}(\mathrm{E}|x_{ith}^2|^{1+\delta})^{\frac{1}{1+\delta}}]^{1+\delta} \\
&\leq \frac{1}{T^{1+\delta}}(Tk_1\Delta^{\frac{1}{1+\delta}})^{1+\delta} = k_1^{1+\delta}\Delta,
\end{aligned}$$

where the first inequality follows from Minkowski's inequality and the second from NT5, and $\mathrm{E}\|\frac{1}{\sqrt{T}}V_s\|^{2+2\delta} \leq \Delta$ by a similar argument. It also follows that $\mathrm{E}\|\frac{1}{T}X_s'Q_sX_s\|^{1+\delta} \leq k_1^{1+\delta}\Delta$ by the same reasoning. So $\mathrm{E}\|\frac{1}{T}X_s'Q_sX_s\|$ and $\mathrm{E}\|\frac{1}{T}X_s'Q_sV_s\|$ are uniformly integrable in T. (See, e.g. Billingsley (1995).) Also, under NT3-NT5,

$$\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{T}x_{st}x_{st}' - \frac{1}{T}\sum_{t=1}^{T}\mathrm{E}[x_{st}x_{st}'] \xrightarrow{p} 0, \quad \frac{1}{T}\sum_{t=1}^{T}x_{st}z_{st}' - \frac{1}{T}\sum_{t=1}^{T}\mathrm{E}[x_{st}z_{st}'] \xrightarrow{p} 0 \\
&\frac{1}{T}\sum_{t=1}^{T}z_{st}z_{st}' \to \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}z_{st}z_{st}', \quad \frac{1}{T}\sum_{t=1}^{T}x_{st}v_{st}' \xrightarrow{p} 0, \quad \text{and} \quad \frac{1}{T}\sum_{t=1}^{T}z_{st}v_{st}' \xrightarrow{p} 0
\end{aligned}$$

by a LLN, e.g. White (2001) Corollary 3.48. It then follows from Phillips and Moon (1999) Corollary 1 that $\frac{1}{S}\sum_{s=1}^{S}\frac{1}{T}X_s'Q_sX_s \xrightarrow{p} M$ and $\frac{1}{S}\sum_{s=1}^{S}\frac{1}{T}X_s'Q_sV_s \xrightarrow{p} 0$. Hence, $\widehat{\beta}_1 - \beta_1 \xrightarrow{p} 0$.

Let K denote a generic constant. To verify asymptotic normality, note that

$$\begin{aligned}
\mathrm{E}\|\frac{1}{\sqrt{T}}X_s'V_s\|^{2+\frac{\delta}{2}} &\leq \frac{C}{T^{\frac{2+\frac{\delta}{2}}{2}}}\max\{\sum_{t=1}^{T}(\mathrm{E}|x_{st}v_{st}|^{2+\frac{\delta}{2}+\epsilon})^{\frac{2+\frac{\delta}{2}}{2+\frac{\delta}{2}+\epsilon}}, [\sum_{t=1}^{T}(\mathrm{E}|x_{st}v_{st}|^{2+\epsilon})^{\frac{2}{2+\epsilon}}]^{\frac{2+\frac{\delta}{2}}{2}}\} \\
&\leq CK < \infty \qquad (9)
\end{aligned}$$

by NT4-NT5 and Doukhan (1994) Theorem 2. Also,

$$\begin{aligned}
\mathrm{E}\|(\frac{1}{T}X_s'Z_s)(\frac{1}{T}Z_s'Z_s)^{-1}(\frac{1}{\sqrt{T}}Z_s'V_s)\|^{2+\frac{\delta}{2}} &\leq (k_2M)^{2+\frac{\delta}{2}}(\mathrm{E}\|(\frac{1}{T}X_s'Z_s)\|^{4+\delta}\mathrm{E}\|(\frac{1}{\sqrt{T}}Z_s'V_s)\|^{4+\delta})^{1/2} \\
&\leq CK < \infty, \qquad (10)
\end{aligned}$$

where the second inequality follows from $\|AB\| \leq \|A\|\|B\|$, the Cauchy-Schwarz inequality, and NT3, and the third inequality is from NT4-NT5 and Doukhan (1994) Theorem 2. (9) and (10) imply $\mathrm{E}\|\frac{1}{\sqrt{T}}X_s'Q_sV_s\|^{2+\frac{\delta}{2}} < \infty$ since

$$\mathrm{E}\|\frac{1}{\sqrt{T}}X_s'Q_sV_s\|^{2+\frac{\delta}{2}} \leq [(\mathrm{E}\|\frac{1}{\sqrt{T}}X_s'V_s\|^{2+\frac{\delta}{2}})^{\frac{1}{2+\frac{\delta}{2}}} + (\mathrm{E}\|\frac{1}{\sqrt{T}}X_s'Z_s(Z_s'Z_s)^{-1}Z_s'V_s\|^{2+\frac{\delta}{2}})^{\frac{1}{2+\frac{\delta}{2}}}]^{2+\frac{\delta}{2}}$$

by Minkowski's inequality which yields $\mathrm{E}\|\frac{1}{\sqrt{T}}X_s'Q_sV_s\|^2$ uniformly integrable in T. (See, e.g. Billingsley (1995).) $\frac{1}{\sqrt{T}}X_s'Q_sV_s \xrightarrow{d} N(0,\Omega)$ from White (2001) Theorem 5.20, from which it follows that $\Omega_T = \frac{1}{T}\mathrm{E}[X_s'Q_s\Gamma(\alpha)Q_sX_s] \to \Omega$. (E.g. Billingsley (1995) Corollary 25.12.) Then Phillips and

Moon (1999) Theorem 3 gives $\frac{1}{\sqrt{ST}}\sum_{s=1}^{S}\sum_{t=1}^{T} X_s' Q_s V_s \xrightarrow{d} N(0,\Omega)$ which implies $\sqrt{ST}(\widehat{\beta}_1 - \beta_1) \xrightarrow{d} N(0, M^{-1}\Omega M^{-1})$. ■

Lemma C.2. *Define* $\widetilde{v}_{st}$ *to be the residual from least squares regression of (1); i.e.* $\widetilde{v}_{st} = C_{st} - x_{st}'\widehat{\beta}_1 - z_{st}'\widehat{\beta}_2^s = v_{st} - x_{st}'(\widehat{\beta}_1 - \beta_1) - z_{st}'(\widehat{\beta}_2^s - \beta_2) = \ddot{v}_{st} - \ddot{x}_{st}'(\widehat{\beta}_1 - \beta_1)$, *where* $\widehat{\beta}_1$ *and* $\widehat{\beta}_2^s$ *are least squares estimates of* β_1 *and* β_2^s. *Let* $\widetilde{v}_{st}^-$ *be a* $p \times 1$ *vector with* $\widetilde{v}_{st}^- = [\widetilde{v}_{s(t-p)}, \ldots, \widetilde{v}_{s(t-1)}]'$, *and let* $\ddot{v}_{st}^-$ *be a* $p \times 1$ *vector with* $\ddot{v}_{st}^- = [\ddot{v}_{s(t-p)}, \ldots, \ddot{v}_{s(t-1)}]'$. *Under the conditions of Assumption 2,*

$$\frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T} \widetilde{v}_{st}^-\widetilde{v}_{st}^{-\prime} = \frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T} \ddot{v}_{st}^-\ddot{v}_{st}^{-\prime} + o_p((ST)^{-1/2}),$$

and

$$\frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T} \widetilde{v}_{st}^-\widetilde{v}_{st} = \frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T} \ddot{v}_{st}^-\ddot{v}_{st} + o_p((ST)^{-1/2}).$$

Proof. Using calculations similar to those found in the proof of Lemma C.1, it can be shown that $\frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T} \ddot{v}_{st}\ddot{x}_{s(t-j)h} = o_p(1)$ and $\frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T} \ddot{x}_{st}\ddot{x}_{s(t-j)h} = O_p(1)$ for $j = -p, \ldots, p$ and $h = 1, \ldots, k_1$. The conclusion then follows by Lemma C.1 and the same arguments used in proving Lemma B.2. ■

Lemma C.3. *Under the conditions of Assumption 2,*

(i) $\frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T} \ddot{v}_{st}^-\ddot{v}_{st}^{-\prime} \xrightarrow{p} \Gamma_p(\alpha)$, *and* $\frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T} \ddot{v}_{st}^-\ddot{v}_{st} \xrightarrow{p} A(\alpha)$.

(ii) $\frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T} \ddot{v}_{st}^-\ddot{v}_{st}^{-\prime} = \frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T} v_{st}^-v_{st}^{-\prime} + O_p(\frac{1}{T})$, *and* $\frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T} \ddot{v}_{st}^-\ddot{v}_{st} = \frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T} v_{st}^-v_{st} + O_p(\frac{1}{T})$.

Proof. The proof is given for $\frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T} \ddot{v}_{st}^-\ddot{v}_{st}^{-\prime} \xrightarrow{p} \Gamma_p(\alpha)$; $\frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T} \ddot{v}_{st}^-\ddot{v}_{st} \xrightarrow{p} A(\alpha)$ follows by a similar argument.

$\frac{1}{T-p}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}^{-\prime}$ is a $p\times p$ matrix with i,j element

$$\begin{aligned}
\left[\frac{1}{T-p}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}^{-\prime}\right]_{[i,j]} &= \frac{1}{T-p}\sum_{t=p+1}^{T} v_{s(t-i)}v_{s(t-j)} \\
&- \frac{1}{T-p}\sum_{t=p+1}^{T} v_{s(t-i)}z'_{s(t-j)}(Z'_sZ_s)^{-1}Z'_sV_s \\
&- \frac{1}{T-p}\sum_{t=p+1}^{T} v_{s(t-j)}z'_{s(t-i)}(Z'_sZ_s)^{-1}Z'_sV_s \\
&+ \frac{1}{T-p}\sum_{t=p+1}^{T} z'_{s(t-i)}(Z'_sZ_s)^{-1}Z'_sV_sV'_sZ_s(Z'_sZ_s)^{-1}z_{s(t-j)} \\
&= \frac{1}{T-p}\sum_{t=p+1}^{T} v_{s(t-i)}v_{s(t-j)} \\
&- \frac{1}{T-p}V'_{s,-i}Z_{s,-j}(Z'_sZ_s)^{-1}Z'_sV_s \\
&- \frac{1}{T-p}V'_{s,-j}Z_{s,-i}(Z'_sZ_s)^{-1}Z'_sV_s \qquad (11) \\
&+ \frac{1}{T-p}Z_{s,-i}(Z'_sZ_s)^{-1}Z'_sV_sV'_sZ'_s(Z'_sZ_s)^{-1}Z'_{s,-j},
\end{aligned}$$

where $V_{s,-k}=(v_{s(p+1-k)},v_{s(p+2-k)},\ldots,v_{s(T-k)})'$ and $Z_{s,-k}$ is defined similarly.

Consider

$$\begin{aligned}
&\mathrm{E}\|V'_sZ_s(Z'_sZ_s)^{-1}Z'_{s,-i}Z_{s,-j}(Z'_sZ_s)^{-1}Z'_sV_s\|^{1+\frac{\delta}{2}} \\
&\quad= \frac{T-p}{T}\mathrm{E}\|(\frac{1}{\sqrt{T}}V'_sZ_s)(\frac{1}{T}Z'_sZ_s)^{-1}(\frac{1}{T-p}Z'_{s,-i}Z_{s,-j})(\frac{1}{T}Z'_sZ_s)^{-1}(\frac{1}{\sqrt{T}}Z'_sV_s)\|^{1+\frac{\delta}{2}}. \qquad (12)
\end{aligned}$$

Note that

$$\|\frac{1}{T-p}Z'_{s,-i}Z_{s,-j}\| \;\leq\; k_2\Delta<\infty \qquad (13)$$

by NT5. Also,

$$\begin{aligned}
\mathrm{E}\|(\frac{1}{\sqrt{T}}Z'_sV_s)\|^{2+\delta} &= \frac{1}{T^{\frac{2+\delta}{2}}}\mathrm{E}\|(\sum_{t=1}^{T}z_{st}v_{st})\|^{2+\delta} \\
&\leq \frac{1}{T^{\frac{2+\delta}{2}}}C\max\{\sum_{t=1}^{T}(\mathrm{E}|z_{st}v_{st}|^{2+\delta+\epsilon})^{\frac{2+\delta}{2+\delta+\epsilon}},[\sum_{t=1}^{T}(\mathrm{E}|z_{st}v_{st}|^{2+\epsilon})^{\frac{2}{2+\epsilon}}]^{\frac{2+\delta}{2}}\} \\
&\leq K<\infty \qquad (14)
\end{aligned}$$

by Doukhan (1994) Theorem 2 and NT5. It then follows from the Holder's inequality, the Cauchy-Schwarz inequality, $\|AB\|\leq\|A\|\|B\|$, (13), (14), and NT3-NT5 that (12) is bounded away from ∞, which gives $\|V'_sZ_s(Z'_sZ_s)^{-1}Z'_{s,-i}Z_{s,-j}(Z'_sZ_s)^{-1}Z'_sV_s\|$ uniformly integrable in T. Similarly, it can be shown that $\|V'_{s,-i}Z_{s,-j}(Z'_sZ_s)^{-1}Z'_sV_s\|$ and $\|\frac{1}{T-p}\sum_{t=p+1}^{T} v_{s(t-i)}v_{s(t-j)}\|$ are uniformly integrable in T.

Also, as $T \to \infty$,

1. $\frac{1}{T-p}\sum_{t=p+1}^{T} v_{s(t-i)}v_{s(t-j)} \xrightarrow{p} \gamma_{|i-j|}(\alpha)$,
2. $V'_{s,-i}Z_{s,-j}(Z'_sZ_s)^{-1}Z'_sV_s \xrightarrow{d} \Psi'_{ij}M_{ZZ}^{-1}\Psi$,
3. $V'_sZ_s(Z'_sZ_s)^{-1}Z'_{s,-i}Z_{s,-j}(Z'_sZ_s)^{-1}Z'_sV_s \xrightarrow{d} \Psi'M_{ZZ}^{-1}M_{ij}M_{ZZ}^{-1}\Psi$,

where $M_{ZZ} = \lim_{T\to\infty}\frac{1}{T}[Z'_sZ_s]$, $M_{ij} = \lim_{T\to\infty}\frac{1}{T}[Z'_{s,-i}Z_{s,-j}]$, $\Psi \sim N(0, \lim_{T\to\infty}\frac{1}{T}[Z'_s\Gamma(\alpha)Z_s]$, $\Psi_{ij} \sim N(0, \lim_{T\to\infty}\frac{1}{T-p}[Z'_{s,-j}\Gamma(\alpha)Z_{s,-j})$, $\Gamma_{-k} = \mathrm{E}[v_sv'_{s,-k}]$, and $\mathrm{E}[\Psi\Psi'_{ij}] = \lim_{T\to\infty}\frac{1}{T}[Z'_s\Gamma_{-i}(\alpha)Z_{s,-j}]$. (See White (2001) Corollary 3.48 and Theorem 5.20.) Combining 1., 2., and 3. and (11) gives

$$\frac{1}{T-p}\sum_{t=p+1}^{T}\ddot{v}_{s(t-i)}\ddot{v}_{s(t-j)} \xrightarrow{p} \gamma_{|i-j|}(\alpha).$$

It then follows from Phillips and Moon (1999) Corollary 1 that

$$\frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{s(t-i)}\ddot{v}_{s(t-j)} \xrightarrow{p} \gamma_{|i-j|}(\alpha),$$

which yields the first conclusion. Also, using Phillips and Moon (1999) Corollary 1 and considering only 2., 3. and (11) yields the second conclusion. ■

Lemma C.4. *Let* $\widehat{\alpha} = (\frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\widetilde{v}_{st}^{-}\widetilde{v}_{st}^{-\prime})^{-1}(\frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\widetilde{v}_{st}^{-}\widetilde{v}_{st})$ *be the least squares estimate of* α *using the least squares residuals,* $\widetilde{v}_{st}$ *from estimating* β_1*. If Assumption 2 is satisfied,*

$$\begin{aligned}\widehat{\alpha} &= (\frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}^{-\prime})^{-1}(\frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}) + o_p((ST)^{-1/2}) \\ &= (\frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T}v_{st}^{-}v_{st}^{-\prime})^{-1}(\frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T}v_{st}^{-}v_{st}) + O_p(T^{-1}) + o_p((ST)^{-1/2}).\end{aligned}$$

Proof. Follows immediately from Lemmas C.2 and C.3 using the same argument as in Lemma B.4. ■

Lemma C.5. *For* $\alpha_T(\alpha)$ *defined in Lemma 1,* $\alpha_T(\alpha) - \alpha = \frac{1}{T-p}B(\alpha, T)$*, where* $B(\alpha, T) \to B(\alpha)$ *as* $T \to \infty$*, and* $\alpha_T(\widehat{\alpha}) - \widehat{\alpha} = \frac{1}{T-p}B(\widehat{\alpha}, T)$ *if the conditions of Assumption 2 are met.*

Proof. Recall $\alpha_T(\alpha) = (\Gamma_p(\alpha) + \frac{1}{T-p}\Delta_\Gamma(\alpha))^{-1}(A(\alpha) + \frac{1}{T-p}\Delta_A(\alpha))$ and that $\alpha = \Gamma_p(\alpha)^{-1}A(\alpha)$. Then $\alpha_T(\alpha) - \alpha = \frac{1}{T-p}(\Gamma_p(\alpha) + \frac{1}{T-p}\Delta_\Gamma(\alpha))^{-1}(\Delta_A(\alpha) + \Delta_\Gamma(\alpha)\alpha) = \frac{1}{T-p}B(\alpha, T)$. That $B(\alpha, T) \to B(\alpha)$ as $T \to \infty$ follows from the proof of Lemma C.3. $\alpha_T(\widehat{\alpha}) - \widehat{\alpha} = \frac{1}{T-p}B(\widehat{\alpha}, T)$ follows in the same fashion. ■

Lemma C.6. *Define* $\ddot{\mu}_{st} = \ddot{v}_{st} - \ddot{v}_{st}^{-'}\alpha_T(\alpha)$. *If Assumption 2 is satisfied,*

$$\frac{1}{\sqrt{S(T-p)}}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{\mu}_{st} = \frac{1}{\sqrt{S(T-p)}}\sum_{s=1}^{S}\sum_{t=p+1}^{T}v_{st}^{-}\eta_{st} + o_p(1).$$

Proof. $\frac{1}{\sqrt{S(T-p)}}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{\mu}_{st} = \frac{1}{\sqrt{S(T-p)}}\sum_{s=1}^{S}\sum_{t=p+1}^{T}(\ddot{v}_{st}^{-}\ddot{\mu}_{st} - \mathrm{E}[\ddot{v}_{st}^{-}\ddot{\mu}_{st}])$ since

$$\begin{aligned}
\mathrm{E}[\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{\mu}_{st}] &= \mathrm{E}[\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}(\ddot{v}_{st} - \ddot{v}_{st}^{-'}\alpha_T(\alpha))] \\
&= \mathrm{E}[\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}] - \mathrm{E}\{\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}^{-'}(\mathrm{E}[\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}^{-'}])^{-1}\mathrm{E}[\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}]\} \\
&= 0,
\end{aligned}$$

where the second equality comes from $\alpha_T(\alpha) = \mathrm{E}[\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}^{-'}]^{-1}\mathrm{E}[\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}]$ from Propostion 1. Also note that $\ddot{\mu}_{st} = \eta_{st} - (v_{st} - \ddot{v}_{st}) - v_{st}^{-'}(\alpha_T(\alpha) - \alpha) - (\ddot{v}_{st}^{-} - v_{st}^{-})'\alpha_T(\alpha)$ and $v_{st}^{-} = v_{st}^{-} - Z_{st}^{-}(Z_s'Z_s)^{-1}Z_s'V_s$ where $Z_{st}^{-} = [z_{s(t-1)}, \ldots, z_{s(t-p)}]'$. So,

$$\begin{aligned}
\ddot{v}_{st}^{-}\ddot{\mu}_{st} - \mathrm{E}[\ddot{v}_{st}^{-}\ddot{\mu}_{st}] &= (v_{st}^{-} - Z_{st}^{-}(Z_s'Z_s)^{-1}Z_s'V_s) \times \\
&\quad (\eta_{st} - z_{st}'(Z_s'Z_s)^{-1}Z_s'V_s - v_{st}^{-'}(\alpha_T(\alpha) - \alpha) + V_s'Z_s(Z_s'Z_s)^{-1}Z_{st}^{-'}\alpha_T(\alpha)) \\
&\quad -\mathrm{E}[(v_{st}^{-} - Z_{st}^{-}(Z_s'Z_s)^{-1}Z_s'V_s) \times \\
&\quad (\eta_{st} - z_{st}'(Z_s'Z_s)^{-1}Z_s'V_s - v_{st}^{-'}(\alpha_T(\alpha) - \alpha) + V_s'Z_s(Z_s'Z_s)^{-1}Z_{st}^{-'}\alpha_T(\alpha))] \\
&= v_{st}^{-}\eta_{st} - (v_{st}^{-}z_{st}'(Z_s'Z_s)^{-1}Z_s'V_s - \mathrm{E}[v_{st}^{-}z_{st}'(Z_s'Z_s)^{-1}Z_s'V_s]) \\
&\quad -(v_{st}^{-}v_{st}^{-'} - \mathrm{E}[v_{st}^{-}v_{st}^{-'}])(\alpha_T(\alpha) - \alpha) \\
&\quad +(v_{st}^{-}V_s'Z_s(Z_s'Z_s)^{-1}Z_{st}^{-'} - \mathrm{E}[v_{st}^{-}V_s'Z_s(Z_s'Z_s)^{-1}Z_{st}^{-'}])\alpha_T(\alpha) \\
&\quad -(Z_{st}^{-}(Z_s'Z_s)^{-1}Z_s'V_s\eta_{st} - \mathrm{E}[(Z_{st}^{-}(Z_s'Z_s)^{-1}Z_s'V_s\eta_{st}]) \\
&\quad +(Z_{st}^{-}(Z_s'Z_s)^{-1}Z_s'V_sz_{st}'(Z_s'Z_s)^{-1}Z_s'V_s \\
&\qquad -\mathrm{E}[Z_{st}^{-}(Z_s'Z_s)^{-1}Z_s'V_sz_{st}'(Z_s'Z_s)^{-1}Z_s'V_s]) \\
&\quad +(Z_{st}^{-}(Z_s'Z_s)^{-1}Z_s'V_sv_{st}^{-'} - \mathrm{E}[Z_{st}^{-}(Z_s'Z_s)^{-1}Z_s'V_sv_{st}^{-'}])(\alpha_T(\alpha) - \alpha) \\
&\quad -(Z_{st}^{-}(Z_s'Z_s)^{-1}Z_s'V_sV_s'Z_s(Z_s'Z_s)^{-1}Z_{st}^{-'} \\
&\qquad -\mathrm{E}[Z_{st}^{-}(Z_s'Z_s)^{-1}Z_s'V_sV_s'Z_s(Z_s'Z_s)^{-1}Z_{st}^{-'}])\alpha_T(\alpha)).
\end{aligned} \tag{15}$$

Now consider

$$\begin{aligned}
Y_{S,T} &= \frac{1}{\sqrt{S(T-p)}}\sum_{s=1}^{S}\sum_{t=p+1}^{T}(Z_{st}^{-}(Z_s'Z_s)^{-1}Z_s'V_sV_s'Z_s(Z_s'Z_s)^{-1}Z_{st}^{-'} \\
&\quad -\mathrm{E}[Z_{st}^{-}(Z_s'Z_s)^{-1}Z_s'V_sV_s'Z_s(Z_s'Z_s)^{-1}Z_{st}^{-'}]).
\end{aligned}$$

$Y_{S,T}$ is a $p \times p$ matrix with i,j element

$$
\begin{aligned}
[Y_{S,T}]_{[i,j]} &= \frac{1}{\sqrt{S(T-p)}} \sum_{s=1}^{S} \sum_{t=p+1}^{T} (z'_{s(t-i)}(Z'_s Z_s)^{-1} Z'_s V_s V'_s Z_s (Z'_s Z_s)^{-1} z_{s(t-j)} \\
&\quad - \mathrm{E}[z_{s(t-i)}(Z'_s Z_s)^{-1} Z'_s V_s V'_s Z_s (Z'_s Z_s)^{-1} z_{s(t-j)}]) \\
&= \frac{1}{\sqrt{S(T-p)}} \sum_{s=1}^{S} (V'_s Z_s (Z'_s Z_s)^{-1} Z'_{s,-i} Z_{s,-j} (Z'_s Z_s)^{-1} Z'_s V_s \\
&\quad - \mathrm{E}[V'_s Z_s (Z'_s Z_s)^{-1} Z'_{s,-i} Z_{s,-j} (Z'_s Z_s)^{-1} Z'_s V_s]),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{E}\|[Y_{S,T}]_{[i,j]}\|^2 &= \frac{1}{T-p} \mathrm{E}\|V'_s Z_s (Z'_s Z_s)^{-1} Z'_{s,-i} Z_{s,-j} (Z'_s Z_s)^{-1} Z'_s V_s\|^2 \\
&\quad - \frac{1}{T-p} \|\mathrm{E}[V'_s Z_s (Z'_s Z_s)^{-1} Z'_{s,-i} Z_{s,-j} (Z'_s Z_s)^{-1} Z'_s V_s]\|^2 \to 0,
\end{aligned}
$$

where the equality is from independence assumption NT2, since $\mathrm{E}\|A\|^2 \geq \|E[A]\|^2$ and

1. $\mathrm{E}\|V'_s Z_s (Z'_s Z_s)^{-1} Z'_{s,-i} Z_{s,-j} (Z'_s Z_s)^{-1} Z'_s V_s\|^2$
   $\leq \frac{T-p}{T} (\mathrm{E}\|\frac{1}{\sqrt{T}} V'_s Z_s\|^4 \|(\frac{1}{T} Z'_s Z_s)^{-1} (\frac{1}{T-p} Z'_{s,-i} Z_{s,-j}) (\frac{1}{T} Z'_s Z_s)^{-1}\|^4 \mathrm{E}\|\frac{1}{\sqrt{T}} Z'_s V_s\|^4)^{1/2}$
   by the Cauchy-Schwarz inequality.
2. $\|(\frac{1}{T} Z'_s Z_s)^{-1} (\frac{1}{T-p} Z'_{s,-i} Z_{s,-j}) (\frac{1}{T} Z'_s Z_s)^{-1}\|^2 \leq (k_2 M)^2 (\|\frac{1}{T-p} Z'_{s,-i} Z_{s,-j}\|^2) \leq (k_2 M)^2 k_2 \Delta$ by NT3 and NT5.
3. $\mathrm{E}\|\frac{1}{\sqrt{T}} Z'_s V_s\|^4 \leq T^{-2} C \max\{\sum_{t=1}^{T} (\mathrm{E}|z_{st} v_{st}|^{4+\epsilon})^{\frac{4}{4+\epsilon}}, [\sum_{t=1}^{T} (\mathrm{E}|z_{st} v_{st}|^{2+\epsilon})^{\frac{2}{2+\epsilon}}]^2\} \leq K$ by Doukhan (1994) Theorem 2 and NT5.

Hence, $[Y_{S,T}]_{[i,j]} \xrightarrow{p} 0$ by Chebychev's inequality, and it follows that $Y_{S,T} = o_p(1)$. That all other terms, except $\frac{1}{\sqrt{S(T-p)}} \sum_{s=1}^{S} \sum_{t=p+1}^{T} (v^-_{st} v^{-'}_{st} - \mathrm{E}[v^-_{st} v^{-'}_{st}])(\alpha_T(\alpha) - \alpha)$, are $o_p(1)$ follows similarly.

To show $\frac{1}{\sqrt{S(T-p)}} \sum_{s=1}^{S} \sum_{t=p+1}^{T} (v^-_{st} v^{-'}_{st} - \mathrm{E}[v^-_{st} v^{-'}_{st}])(\alpha_T(\alpha) - \alpha) = o_p(1)$, note that $(\alpha_T(\alpha) - \alpha) = O(\frac{1}{T})$ from Lemma C.5. Also, $\mathrm{vec}\{\frac{1}{\sqrt{T-p}} \sum_{t=p+1}^{T} (v^-_{st} v^{-'}_{st} - \mathrm{E}[v^-_{st} v^{-'}_{st}])\} \xrightarrow{d} \Psi \sim N(0, \Omega)$ where

$$
\Omega = \lim_{T \to \infty} \frac{1}{T-p} \sum_{t_1=p+1}^{T} \sum_{t_2=p+1}^{T} [\mathrm{E}(v^-_{st_1} v^{-'}_{st_2} \otimes v^-_{st_1} v^{-'}_{st_2}) - \mathrm{E}(v^-_{st_1} \otimes v^-_{st_1}) \mathrm{E}(v^{-'}_{st_2} \otimes v^{-'}_{st_2})]
$$

as $T \to \infty$ follows from a CLT (e.g. White (2001) Theorem 5.20) and the Cramer-Wold device, and $\|\frac{1}{\sqrt{T-p}} \sum_{t=p+1}^{T} \mathrm{vec}(v^-_{st} v^{-'}_{st} - \mathrm{E}[v^-_{st} v^{-'}_{st}])\|^2 \xrightarrow{d} \|\Psi\|^2$ by the Continuous Mapping Theorem. In addition,

$$\mathrm{E}\|\frac{1}{\sqrt{S(T-p)}}\sum_{s=1}^{S}\sum_{t=p+1}^{T}(v_{st}^{-}v_{st}^{-'}-\mathrm{E}[v_{st}^{-}v_{st}^{-'}])\|^2$$

$$=\mathrm{trace}\{\mathrm{E}[\frac{1}{T-p}\sum_{t_1=p+1}^{T}\sum_{t_2=p+1}^{T}((v_{st_1}^{-}\otimes v_{st_1}^{-})-\mathrm{E}(v_{st_1}^{-}\otimes v_{st_1}^{-}))((v_{st_2}^{-}\otimes v_{st_2}^{-})-\mathrm{E}(v_{st_2}^{-}\otimes v_{st_2}^{-}))']\}$$

$$=\frac{1}{T-p}\sum_{t_1=p+1}^{T}\sum_{t_2=p+1}^{T}[\mathrm{E}(v_{st_1}^{-}v_{st_2}^{-'}\otimes v_{st_1}^{-}v_{st_2}^{-'})-\mathrm{E}(v_{st_1}^{-}\otimes v_{st_1}^{-})\mathrm{E}(v_{st_2}^{-'}\otimes v_{st_2}^{-'})]\rightarrow \mathrm{trace}(\Omega)=\mathrm{E}\|\Psi\|^2$$

as $T\rightarrow\infty$. It follows that $\|\frac{1}{\sqrt{S(T-p)}}\sum_{s=1}^{S}\sum_{t=p+1}^{T}(v_{st}^{-}v_{st}^{-'}-\mathrm{E}[v_{st}^{-}v_{st}^{-'}])\|^2$ is uniformly integrable in T. (For example, Billingsley (1995) 16.14.) Then, using Phillips and Moon (1999) Theorem 3,

$$\frac{1}{\sqrt{S(T-p)}}\sum_{s=1}^{S}\sum_{t=p+1}^{T}(v_{st}^{-}v_{st}^{-'}-\mathrm{E}[v_{st}^{-}v_{st}^{-'}])\xrightarrow{d}N(0,\Omega)\Rightarrow\frac{1}{\sqrt{S(T-p)}}\sum_{s=1}^{S}\sum_{t=p+1}^{T}(v_{st}^{-}v_{st}^{-'}-\mathrm{E}[v_{st}^{-}v_{st}^{-'}])=O_p(1),$$

and the conclusion follows immediately. ■

Lemma C.7. *If Assumption 2 is satisfied,* $\frac{1}{\sqrt{S(T-p)}}\sum_{s=1}^{S}\sum_{t=p+1}^{T}v_{st}^{-}\eta_{st}\xrightarrow{d}N(0,\Xi)$, *where*

$$\Xi=\lim_{T\rightarrow\infty}\frac{1}{T-p}\sum_{t_1=p+1}^{T}\sum_{t_2=p+1}^{T}\mathrm{E}[v_{st_1}^{-}\eta_{st_1}\eta_{st_2}v_{st_2}^{-'}].$$

In addition, if η_{st} *are independent for all* s *and* t, $\Xi=\sigma_\eta^2\Gamma$.

Proof. As $T\rightarrow\infty$, $\frac{1}{\sqrt{T-p}}\sum_{t=p+1}^{T}v_{st}^{-}\eta_{st}\xrightarrow{d}\Psi\sim N(0,\Xi)$ by a CLT (e.g. White (2001) Theorem 5.20) and the Cramer-Wold device, and $\|\frac{1}{\sqrt{T-p}}\sum_{t=p+1}^{T}v_{st}^{-}\eta_{st}\|^2\xrightarrow{d}\|\Psi\|^2$ by the Continuous Mapping Theorem. Also, $\mathrm{E}\|\frac{1}{\sqrt{T-p}}\sum_{t=p+1}^{T}v_{st}^{-}\eta_{st}\|^2=\mathrm{trace}(\frac{1}{T-p}\sum_{t_1=p+1}^{T}\sum_{t_2=p+1}^{T}\mathrm{E}v_{st_1}^{-}\eta_{st_1}\eta_{st_2}v_{st_2}^{-'})\rightarrow$ $\mathrm{trace}(\Xi)=\mathrm{E}\|\Psi\|^2$, which implies that $\mathrm{E}\|\frac{1}{\sqrt{T-p}}\sum_{t=p+1}^{T}v_{st}^{-}\eta_{st}\|^2$ is uniformly integrable in T. (For example, Billingsley (1995) 16.14.) Then, using Phillips and Moon (1999) Theorem 3,

$$\frac{1}{\sqrt{S(T-p)}}\sum_{s=1}^{S}\sum_{t=p+1}^{T}v_{st}^{-}\eta_{st}\xrightarrow{d}N(0,\Xi).$$

Also, if the η_{st} are independent for all s and t, $\mathrm{E}v_{st_1}^{-}\eta_{st_1}\eta_{st_2}v_{st_2}^{-'}=0\;\forall\;t_1\neq t_2$, and $\mathrm{E}v_{st}^{-}\eta_{st}^2v_{st}^{-'}=\sigma_\eta^2\Gamma$. ■

Lemma C.8. $\sqrt{ST}(\widehat{\alpha}-\alpha_T(\alpha))\xrightarrow{d}N(0,\Gamma^{-1}\Xi\Gamma^{-1})$ *for* Ξ *defined in Lemma C.7 if Assumption 2 is satisfied.*

Proof. From Lemma C.4, $\widehat{\alpha}-\alpha_T(\alpha)=(\frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}^{-'})^{-1}(\frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st})-$ $\alpha_T(\alpha)+o_p((ST)^{-1/2})=(\frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}^{-'})^{-1}(\frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{\mu}_{st})+o_p((ST)^{-1/2})$. The conclusion then follows immediately from Lemmas C.3, C.6, and C.7. ■

Lemma C.9. *Under Assumption 2,* $\widehat{\alpha}-\alpha=\max\{O_p(T^{-1}),O_p((ST)^{-1/2})\}$.

Proof. Lemma C.4 gives

$$\begin{aligned}\widehat{\alpha} &= (\frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T} v_{st}^{-}v_{st}^{-'})^{-1}(\frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T} v_{st}^{-}v_{st}) + O_p(T^{-1}) + o_p((ST)^{-1/2}) \\ &= (\frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T} v_{st}^{-}v_{st}^{-'})^{-1}(\frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T} v_{st}^{-}\eta_{st}) + \alpha + O_p(T^{-1}) + o_p((ST)^{-1/2}).\end{aligned}$$

$\frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T} v_{st}^{-}\eta_{st} = O_p((ST)^{-1/2})$ from Lemma C.8, and $\frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T} v_{st}^{-}v_{st}^{-'} = O_p(1)$ follows immediately from a LLN, e.g. White (2001) Corollary 3.48. ■

Appendix D. Proof of Proposition 6

The proof of Proposition 6 is quite similar to the proof provided in Rothenberg (1984) and is sketched below. Throughout, let

$$M_z(\widetilde{\alpha}) = (I \otimes \Gamma(\widetilde{\alpha})^{-1}) - (I \otimes \Gamma(\widetilde{\alpha})^{-1})Z(Z'(I \otimes \Gamma(\widetilde{\alpha})^{-1})Z)^{-1}Z'(I \otimes \Gamma(\widetilde{\alpha})^{-1}).$$

Denote the GLS estimator of β_1 as

$$\widehat{\beta}_1(\alpha) = (X'M_z(\alpha)X)^{-1}X'M_z(\alpha)C$$

and the FGLS estimator of β_1 corresponding to $\widetilde{\alpha}$ as

$$\widehat{\beta}_1(\widetilde{\alpha}) = (X'M_z(\widetilde{\alpha})X)^{-1}X'M_z(\widetilde{\alpha})C.$$

Also, define

$$A(\widetilde{\alpha}) = \frac{1}{ST}X'M_z(\widetilde{\alpha})X,$$

and

$$\Psi(\widetilde{\alpha}) = \frac{1}{\sqrt{ST}}X'M_z(\widetilde{\alpha})MV$$

for $M = I - X(X'M_z(\alpha)X)^{-1}X'M_z(\alpha)$. Then for $i = 1, ..., p$, $j = 1, ..., p$, and $k = 1, ..., p$, let

$$A_i = \frac{\partial A}{\partial \widetilde{\alpha}_i}|_\alpha, \qquad A_{ij} = \frac{\partial^2 A}{\partial \widetilde{\alpha}_i \partial \widetilde{\alpha}_j}|_\alpha, \qquad A_{ijk} = \frac{\partial^3 A}{\partial \widetilde{\alpha}_i \partial \widetilde{\alpha}_j \partial \widetilde{\alpha}_k}|_\alpha,$$

$$\Psi_i = \frac{\partial \Psi}{\partial \widetilde{\alpha}_i}|_\alpha, \qquad \Psi_{ij} = \frac{\partial^2 \Psi}{\partial \widetilde{\alpha}_i \partial \widetilde{\alpha}_j}|_\alpha, \qquad \Psi_{ijk} = \frac{\partial^3 \Psi}{\partial \widetilde{\alpha}_i \partial \widetilde{\alpha}_j \partial \widetilde{\alpha}_k}|_\alpha.$$

Step 1. As in Rothenberg (1984), note that the residual vector

$$\widetilde{V} = C - [X, Z]([X, Z]'[X, Z])^{-1}[X, Z]'C = V - [X, Z]([X, Z]'[X, Z])^{-1}[X, Z]'V$$

does not depend on β_1 and that $\widehat{\alpha}$ and $\widehat{\alpha}^{(\infty)}$ are even functions of $\widetilde{V}$ and hence V. For a given α, $\widehat{\beta}_1(\alpha)$ is a complete sufficient statistic for β_1 since V is normally distributed. Also, by Basu's Theorem (See, for example, Lehmann (1983) p. 46), any statistic whose distribution does not depend on β_1 must be distributed independently of $\widehat{\beta}_1(\alpha)$, so $\widehat{\beta}_1(\alpha)$ is independent of $\widehat{\alpha}$, $\widehat{\alpha}^{(\infty)}$, $\widehat{\beta}_1(\widehat{\alpha}) - \widehat{\beta}_1(\alpha)$, and $\widehat{\beta}_1(\widehat{\alpha}^{(\infty)}) - \widehat{\beta}_1(\alpha)$. It then follows, for $\widetilde{\alpha}$ equal to either $\widehat{\alpha}$ or $\widehat{\alpha}^{(\infty)}$, that

$$\mathrm{E}[(\widehat{\beta}_1(\widetilde{\alpha}) - \beta_1)(\widehat{\beta}_1(\widetilde{\alpha}) - \beta_1)'] = \mathrm{E}[(\widehat{\beta}_1(\alpha) - \beta_1)(\widehat{\beta}_1(\alpha) - \beta_1)'] + \mathrm{E}[(\widehat{\beta}_1(\widetilde{\alpha}) - \widehat{\beta}_1(\alpha))(\widehat{\beta}_1(\widetilde{\alpha}) - \widehat{\beta}_1(\alpha))'].$$

$\widehat{\beta}_1(\alpha) - \beta_1$ is exactly normal with variance $\frac{1}{ST}A^{-1}$, so the higher-order variance of $\widehat{\beta}_1(\widetilde{\alpha})$ is $\frac{1}{ST}A^{-1}$ plus the higher-order variance of $\widehat{\beta}_1(\widetilde{\alpha}) - \widehat{\beta}_1(\alpha)$. In addition, the bias of $\widehat{\beta}_1(\widetilde{\alpha})$ is

$$\mathrm{E}[\widehat{\beta}_1(\widetilde{\alpha}) - \beta_1] = \mathrm{E}[\widehat{\beta}_1(\alpha) - \beta_1] + \mathrm{E}[\widehat{\beta}_1(\widetilde{\alpha}) - \widehat{\beta}_1(\alpha)] = 0 + \mathrm{E}[\widehat{\beta}_1(\widetilde{\alpha}) - \widehat{\beta}_1(\alpha)],$$

so the higher-order bias is also determined by the higher-order bias of $\widehat{\beta}_1(\widetilde{\alpha}) - \widehat{\beta}_1(\alpha)$.

Step 2. Expansion of $\widehat{\beta}_1(\widetilde{\alpha}) - \widehat{\beta}_1(\alpha)$. For $\widetilde{\alpha}$ equal to either $\widehat{\alpha}$ or $\widehat{\alpha}^{(\infty)}$,

$$\widehat{\beta}_1(\widetilde{\alpha}) - \widehat{\beta}_1(\alpha) = A(\widetilde{\alpha})^{-1}\Psi(\widetilde{\alpha}). \tag{16}$$

Let $d_i = \sqrt{ST}(\widetilde{\alpha} - \alpha)_i$ be the i^{th} element of the vector $\sqrt{ST}(\widetilde{\alpha} - \alpha)$. Then expanding (16) about $\widetilde{\alpha} = \alpha$ and noting $\Psi(\alpha) = 0$ yields

$$\begin{aligned}\sqrt{ST}(\widehat{\beta}_1(\widetilde{\alpha}) - \widehat{\beta}_1(\alpha)) = &\sum_{i=1}^{p} A(\alpha)^{-1}\Psi_i(\alpha)d_i/\sqrt{ST} \\ &+ \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}(A(\alpha)^{-1}\Psi_{ij}(\alpha)d_i d_j - A(\alpha)^{-1}A_i(\alpha)A(\alpha)^{-1}\Psi_j(\alpha)d_i d_j \\ &- A(\alpha)^{-1}A_j(\alpha)A(\alpha)^{-1}\Psi_i(\alpha)d_i d_j)/ST + R(\bar{\alpha})/(ST)^{3/2},\end{aligned} \tag{17}$$

where $\bar{\alpha}$ is an intermediate value between α and $\widetilde{\alpha}$ and

$$\begin{aligned}R(\bar{\alpha}) =& \frac{1}{6}\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}[-6A(\bar{\alpha})^{-1}A_i(\bar{\alpha})A(\bar{\alpha})^{-1}A_j(\bar{\alpha})A(\bar{\alpha})^{-1}A_k(\bar{\alpha})A(\bar{\alpha})^{-1}\Psi(\bar{\alpha}) \\ &+ 3A(\bar{\alpha})^{-1}A_{ij}(\bar{\alpha})A(\bar{\alpha})^{-1}A_k(\bar{\alpha})A(\bar{\alpha})^{-1}\Psi(\bar{\alpha}) + 3A(\bar{\alpha})^{-1}A_i(\bar{\alpha})A(\bar{\alpha})^{-1}A_{jk}(\bar{\alpha})A(\bar{\alpha})^{-1}\Psi(\bar{\alpha}) \\ &+ 6A(\bar{\alpha})^{-1}A_i(\bar{\alpha})A(\bar{\alpha})^{-1}A_j(\bar{\alpha})A(\bar{\alpha})^{-1}\Psi_k(\bar{\alpha}) - A(\bar{\alpha})^{-1}A_{ijk}(\bar{\alpha})A(\bar{\alpha})^{-1}\Psi(\bar{\alpha}) \\ &- 3A(\bar{\alpha})^{-1}A_{ij}(\bar{\alpha})A(\bar{\alpha})^{-1}\Psi_k(\bar{\alpha}) - 3A(\bar{\alpha})^{-1}A_i(\bar{\alpha})A(\bar{\alpha})^{-1}\Psi_{jk}(\bar{\alpha}) \\ &+ A(\bar{\alpha})^{-1}\Psi_{ijk}(\bar{\alpha})]d_i d_j d_k.\end{aligned} \tag{18}$$

Step 3. Expansion of $\widehat{\alpha} - \alpha_T(\alpha)$. Recall that the least squares estimator of α using the residuals from the estimation of equation (1), $\widetilde{v}_{st}$, is

$$\widehat{\alpha} = \left(\frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\widetilde{v}_{st}^{-}\widetilde{v}_{st}^{-\prime}\right)^{-1}\left(\frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\widetilde{v}_{st}^{-}\widetilde{v}_{st}\right) \equiv \widehat{G}_1^{-1}\widehat{G}_2$$

where $\widetilde{v}_{st}^{-\prime} = (\widetilde{v}_{s(t-p)}, ..., \widetilde{v}_{s(t-1)})$ and $\widetilde{v}_{st} = \ddot{v}_{st} - \ddot{x}_{st}'(\widehat{\beta}_1 - \beta_1)$ for $\ddot{v}_{st}$ and $\ddot{x}_{st}$ as defined in (**??**) and (**??**) in the text and $\widehat{\beta}_1$ the least squares estimate of β_1. Then

$$\begin{aligned}\widehat{G}_1 =& \frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}^{-\prime} \\ &- \frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T}(\ddot{v}_{st}^{-}(\widehat{\beta}_1 - \beta_1)'\ddot{x}_{st}^{-\prime}\ddot{x}_{st}^{-}(\widehat{\beta}_1 - \beta_1)\ddot{v}_{st}^{-\prime} + \ddot{x}_{st}^{-}(\widehat{\beta}_1 - \beta_1)(\widehat{\beta}_1 - \beta_1)'\ddot{x}_{st}^{-\prime}) \\ =& \widehat{D}_1 + \widehat{M}_1\end{aligned}$$

where $\widehat{D}_1 = \frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st}^{-\prime}$, and

$$\begin{aligned}\widehat{G}_2 &= \frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{v}_{st} \\ &\quad - \frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T}(\ddot{v}_{st}^{-}(\widehat{\beta}_1-\beta_1)'\ddot{x}_{st}\ddot{x}_{st}^{-}(\widehat{\beta}_1-\beta_1)\ddot{v}_{st} + \ddot{x}_{st}^{-}(\widehat{\beta}_1-\beta_1)(\widehat{\beta}_1-\beta_1)'\ddot{x}_{st}) \\ &= \widehat{D}_2 + \widehat{D}_1\alpha_T(\alpha) + \widehat{M}_2\end{aligned}$$

where $\widehat{D}_2 = \frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{\mu}_{st}$ and $\ddot{\mu}_{st} = \ddot{v}_{st} - \ddot{v}_{st}^{-\prime}\alpha_T(\alpha)$. It then follows that

$$\begin{aligned}\widehat{\alpha} - \alpha_T(\alpha) &= \widehat{G}_1^{-1}\widehat{G}_2 - \alpha_T(\alpha) \\ &= (\widehat{D}_1 + \widehat{M}_1)^{-1}(\widehat{D}_2 + \widehat{M}_2 - \widehat{M}_1\alpha_T(\alpha)).\end{aligned}$$

It is shown below that $\widehat{M}_1$ and $\widehat{M}_2$ are $O_p(1/ST)$, and it then follows from Lemmas C.3-C.9 that $\widehat{\alpha}$ and $\widehat{\alpha}^{(\infty)}$ are consistent and have the asymptotic distributions described in Proposition 3.

A Taylor series expansion then yields

$$\begin{aligned}\sqrt{ST}(\widehat{\alpha} - \alpha_T(\alpha)) &= \Gamma_p^{-1}\sqrt{ST}\widehat{D}_2 \qquad (19)\\ &\quad + \Gamma_p^{-1}(\sqrt{ST}\widehat{M}_2 - \sqrt{ST}\widehat{M}_1\alpha_T(\alpha)) - \Gamma_p^{-1}(\widehat{D}_1 - \Gamma_p)\Gamma_p^{-1}\sqrt{ST}\widehat{D}_2 \qquad (20)\\ &\quad - \Gamma_p^{-1}\widehat{M}_1\Gamma_p^{-1}\sqrt{ST}\widehat{D}_2 - \Gamma_p^{-1}(\widehat{D}_1 - \Gamma_p)\Gamma_p^{-1}(\sqrt{ST}\widehat{M}_2 - \sqrt{ST}\widehat{M}_1\alpha_T(\alpha)) \\ &\quad + \frac{1}{2}\Gamma_p^{-1}(\widehat{D}_1 - \Gamma_p)\Gamma_p^{-1}(\widehat{D}_1 - \Gamma_p)\Gamma_p^{-1}\sqrt{ST}\widehat{D}_2 \qquad (21)\\ &\quad - \Gamma_p^{-1}\widehat{M}_1\Gamma_p^{-1}(\sqrt{ST}\widehat{M}_2 - \sqrt{ST}\widehat{M}_1\alpha_T(\alpha)) \\ &\quad + \frac{1}{2}\Gamma_p^{-1}\widehat{M}_1\Gamma_p^{-1}(\widehat{D}_1 - \Gamma_p)\Gamma_p^{-1}\sqrt{ST}\widehat{D}_2 \\ &\quad + \frac{1}{2}\Gamma_p^{-1}(\widehat{D}_1 - \Gamma_p)\Gamma_p^{-1}\widehat{M}_1\Gamma_p^{-1}\sqrt{ST}\widehat{D}_2 \\ &\quad + \frac{1}{2}\Gamma_p^{-1}\widehat{M}_1\Gamma_p^{-1}\widehat{M}_1\Gamma_p^{-1}\sqrt{ST}\widehat{D}_2 \\ &\quad - \frac{1}{6}\widetilde{\Gamma}_p^{-1}(\widehat{D}_1 - \Gamma_p + \widehat{M}_1)\widetilde{\Gamma}_p^{-1}(\widehat{D}_1 - \Gamma_p + \widehat{M}_1)\widetilde{\Gamma}_p^{-1}(\widehat{D}_1 - \Gamma_p + \widehat{M}_1)\widetilde{\Gamma}_p^{-1}\times \\ &\quad (\sqrt{ST}\widehat{D}_2 + \sqrt{ST}\widehat{M}_2 - \sqrt{ST}\widehat{M}_1\alpha_T(\alpha)) \qquad (22)\\ &\equiv \psi_1 + \psi_2/\sqrt{ST} + \psi_3/ST + R/(ST)^{3/2}, \qquad (23)\end{aligned}$$

where $\widetilde{\Gamma}_p$ is a matrix of intermediate values that satisfy $\|\widetilde{\Gamma}_p - \Gamma_p\| \leq \|\widehat{D}_1 - \Gamma_p + \widehat{M}_1\|$.

It is shown in Lemmas C.3 and C.6 that $\widehat{D}_1 - \Gamma_p + \widehat{M}_1 \xrightarrow{p} 0$ and that $\widehat{D}_1 - \Gamma_p = O_p(1/\sqrt{ST})$, and it follows from Lemmas C.6 and C.7 that $\widehat{D}_2 = O_p(1/\sqrt{ST})$.

To show $\widehat{M}_2 = O_p(1/ST)$, let $M_Z = I - Z(Z'Z)^{-1}Z'$ and define M_{Z_s} similarly. Recall that

$$\widehat{M}_2 = -\frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{V}'\ddot{X}(\ddot{X}'\ddot{X})^{-1}\ddot{x}_{st} - \frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{x}_{st}^{-\prime}(\ddot{X}'\ddot{X})^{-1}\ddot{X}'\ddot{V}\ddot{v}_{st}$$
$$+ \frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{x}_{st}^{-\prime}(\ddot{X}'\ddot{X})^{-1}\ddot{X}'\ddot{V}\ddot{V}'\ddot{X}(\ddot{X}'\ddot{X})^{-1}\ddot{x}_{st}. \tag{24}$$

Then

$$\begin{aligned}
\mathrm{E}\|\ddot{V}'\ddot{X}(\ddot{X}'\ddot{X})^{-1}\|^2 &\le (\mathrm{E}\|\frac{1}{ST}V'M_Z X\|^2)\|\frac{1}{ST}(X'M_Z X)^{-1}\|^2 \\
&\le \left(\frac{k_1^{1/2}}{\lambda_{\min}(\frac{1}{ST}X'M_Z X)}\right)^2 \frac{1}{(ST)^2}\sum_{s=1}^{S}\mathrm{E}\|V_s'M_{Z_s}X_s\| \\
&= \left(\frac{k_1^{1/2}}{\lambda_{\min}(\frac{1}{ST}X'M_Z X)}\right)^2 \frac{1}{(ST)^2}\sum_{s=1}^{S}\mathrm{trace}(X_s'M_{Z_s}\Omega M_{Z_s}X_s) \\
&\le \left(\frac{k_1^{1/2}}{\lambda_{\min}(\frac{1}{ST}X'M_Z X)}\right)^2 \frac{\lambda_{\max}(\Omega)}{(ST)^2}\sum_{s=1}^{S}\mathrm{trace}(\Delta^2 i_{k_1}i_T' i_T i_{k_1}') \\
&= \left(\frac{k_1^{1/2}}{\lambda_{\min}(\frac{1}{ST}X'M_Z X)}\right)^2 \lambda_{\max}(\Omega)k_1\Delta^2/ST,
\end{aligned}$$

where the first inequality follows from the definition of $\|A\|$ and the second inequality follows from boundedness of x_{sth} and $c'Ac \le \lambda_{\max}(A)c'c$. Also, defining e_t to be a $T \times 1$ unit vector with t^{th} element equal to 1 and all other elements equal to 0 and $e_t^- = [e_{t-1}, ..., e_{t-p}]'$,

$$\begin{aligned}
\mathrm{E}\|\frac{1}{S(T-p)}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{x}_{st}'\otimes\ddot{v}_{st}^{-}\|^2 &= \frac{1}{(S(T-p))^2}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\sum_{\tau=p+1}^{T}\mathrm{trace}(\mathrm{E}(\ddot{x}_{st}'\ddot{x}_{s\tau}\otimes\ddot{v}_{st}^{-}\ddot{v}_{s\tau}^{-\prime})) \\
&\le \frac{k_1 M^2}{(S(T-p))^2}\sum_{s=1}^{S}\mathrm{trace}((\sum_{t=p+1}^{T}e_t^{-})M_{Z_s}\Omega M_{Z_s}(\sum_{t=p+1}^{T}e_t^{-\prime})) \\
&\le \frac{k_1 M^2\lambda_{\max}(\Omega)}{(S(T-p))^2}\sum_{s=1}^{S}\mathrm{trace}((\sum_{t=p+1}^{T}e_t^{-})(\sum_{t=p+1}^{T}e_t^{-\prime})) \\
&\le \frac{k_1 p M^2\lambda_{\max}(\Omega)}{S(T-p)},
\end{aligned}$$

where the first inequality results from boundedness of x_{sth}, the second from $c'Ac \le \lambda_{\max}(A)c'c$, and the third from the definition of e_t^-. It then follows that

$$\begin{aligned}
\frac{1}{ST}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{v}_{st}^{-}\ddot{V}'\ddot{X}(\ddot{X}'\ddot{X})^{-1}\ddot{x}_{st} &= \left(\frac{1}{ST}\sum_{s=1}^{S}\sum_{t=p+1}^{T}\ddot{x}_{st}'\otimes\ddot{v}_{st}^{-}\right)\mathrm{vec}(V'\ddot{X}(\ddot{X}'\ddot{X})^{-1}) \\
&= O_p(1/\sqrt{ST})O_p(1/\sqrt{ST}) = O_p(1/ST).
\end{aligned}$$

Using similar arguments, it can be shown that the second and third terms of expression (24) are $O_p(1/ST)$. It then follows that $\widehat{M}_2 = O_p(1/ST)$. That $\widehat{M}_1 = O_p(1/ST)$ follows similarly.

Finally, making use of $\widehat{M_1} = O_p(1/ST)$, $\widehat{M_2} = O_p(1/ST)$, $\widehat{D}_1 - \Gamma_p = O_p(1/\sqrt{ST})$, and $\widehat{D}_2 = O_p(1/\sqrt{ST})$ yields that ψ_1, ψ_2, ψ_3, and R are all $O_p(1)$.

Step 4. Expansion of $\widehat{\alpha} - \alpha$ and $\widehat{\alpha}^{(\infty)} - \alpha$.

i. $\sqrt{ST}(\widehat{\alpha}-\alpha) = \sqrt{ST}(\widehat{\alpha}-\alpha_T(\alpha)) + \sqrt{ST}(\alpha_T(\alpha)-\alpha) = \sqrt{ST}(\widehat{\alpha}-\alpha_T(\alpha)) + \sqrt{\frac{S}{T}}B(\alpha,T)$ by Lemma C.5. It then follows that

$$\sqrt{ST}(\widehat{\alpha}-\alpha) = \psi_1 + \sqrt{\frac{S}{T}}B(\alpha,T) + \psi_2/\sqrt{ST} + \psi_3/ST + R/(ST)^{3/2} \tag{25}$$

for ψ_1, ψ_2, ψ_3, and R defined above.

ii. Recall

$$\begin{aligned}
\widehat{\alpha}^{(\infty)} &= \alpha_T^{-1}(\widehat{\alpha}) \\
&= \alpha_T^{-1}(\alpha_T(\alpha)) + H^{-1}|_{\alpha_T(\alpha)}(\widehat{\alpha}-\alpha_T(\alpha)) \\
&\quad + \sum_{i=1}^{p} \frac{\partial H^{-1}}{\partial \alpha_i}|_{\alpha_T(\alpha)}(\widehat{\alpha}-\alpha_T(\alpha))(\widehat{\alpha}-\alpha_T(\alpha))_i \\
&\quad + \sum_{i=1}^{p}\sum_{j=1}^{p} \frac{\partial^2 H^{-1}}{\partial \alpha_i \partial \alpha_j}|_{\widetilde{\alpha_T(\alpha)}}(\widehat{\alpha}-\alpha_T(\alpha))(\widehat{\alpha}-\alpha_T(\alpha))_i(\widehat{\alpha}-\alpha_T(\alpha))_j
\end{aligned} \tag{26}$$

where $\|\widetilde{\alpha_T(\alpha))} - \alpha_T(\alpha)\| \leq \|\widehat{\alpha}-\alpha_T(\alpha)\|$. From Lemma C.5, $\alpha_T(\alpha) = \alpha + \frac{1}{T-p}B(\alpha,T)$, so $H = I + \frac{1}{T-p}\frac{\partial B(\alpha,T)}{\partial \alpha}$, $\frac{\partial H}{\partial \alpha_i} = \frac{1}{T-p}\frac{\partial^2 B(\alpha,T)}{\partial\alpha\partial\alpha_i} = O(1/T)$ by Assumption HO5, and $\frac{\partial^2 H}{\partial\alpha_i\partial\alpha_j} = \frac{1}{T-p}\frac{\partial^3 B(\alpha,T)}{\partial\alpha\partial\alpha_i\partial\alpha_j} = O(1/T)$ by Assumption HO5. Thus, $\frac{\partial H^{-1}}{\partial\alpha_i}|_{\alpha_T(\alpha)} = -H^{-1}\frac{\partial H}{\partial\alpha_i}H^{-1}|_{\alpha_T(\alpha)} = O(1/T)$ and $\frac{\partial^2 H^{-1}}{\partial\alpha_i\partial\alpha_j}|_{\widetilde{\alpha_T(\alpha)}} = H^{-1}\frac{\partial H}{\partial\alpha_i}H^{-1}\frac{\partial H}{\partial\alpha_j}H^{-1} + H^{-1}\frac{\partial H}{\partial\alpha_j}H^{-1}\frac{\partial H}{\partial\alpha_i}H^{-1} - H^{-1}\frac{\partial^2 H}{\partial\alpha_i\partial\alpha_j}H^{-1}|_{\widetilde{\alpha_T(\alpha)}} = O(1/T)$ by Assumption HO5. Also, we have

$$\begin{aligned}
H^{-1} &= I - \frac{1}{T-p}\partial B(\alpha,T) + \frac{1}{(T-p)^2}\partial B(\alpha,T)\partial B(\alpha,T) \\
&\quad - \widetilde{H}^{-1}\frac{1}{T-p}\partial B(\alpha,T)\widetilde{H}^{-1}\frac{1}{T-p}\partial B(\alpha,T)\widetilde{H}^{-1}\frac{1}{T-p}\partial B(\alpha,T)\widetilde{H}^{-1} \\
&\equiv I + \frac{1}{\sqrt{ST}}H_1 + \frac{1}{ST}H_2 + \frac{1}{(ST)^{3/2}}R_H
\end{aligned}$$

where H_1, H_2, and H_3 are $O(1)$ by Assumption HO5 and $\frac{S}{T} \to \rho$. Finally plugging the expansion for H^{-1}, the expressions for the derivatives of H and H^{-1}, and (23) into (26) and collecting terms of the same orders yields

$$\begin{aligned}
\sqrt{ST}(\widehat{\alpha}^{(\infty)}-\alpha) &= \psi_1 + \psi_2/\sqrt{ST} + H_1|_{\alpha_T(\alpha)}\psi_1/\sqrt{ST} \\
&\quad + \psi_3/ST + H_1|_{\alpha_T(\alpha)}\psi_2/ST + H_2|_{\alpha_T(\alpha)}\psi_1/ST \\
&\quad - \sum_{i=1}^{p} H^{-1}\sqrt{ST}\frac{\partial^2 B(\alpha,T)}{\partial\alpha\partial\alpha_i}H^{-1}|_{\alpha_T(\alpha)}\psi_1(\psi_1)_i/ST + R^{(\infty)}/(ST)^{3/2} \\
&= \psi_1 + \psi_2^{(\infty)}/\sqrt{ST} + \psi_3^{(\infty)}/ST + R^{(\infty)}/(ST)^{3/2}.
\end{aligned} \tag{27}$$

Step 5. Higher Order Bias and Variance. Recall

$$\widehat{\beta}_1(\widetilde{\alpha}) - \beta_1 = \widehat{\beta}_1(\widetilde{\alpha}) - \widehat{\beta}_1(\alpha) + \widehat{\beta}_1(\alpha) - \beta_1. \tag{28}$$

(i) Plugging (17) in for $\widehat{\beta}_1(\widetilde{\alpha}) - \widehat{\beta}_1(\alpha)$ and (25) in for the d_i in (28) yields

$$\begin{aligned}\sqrt{ST}(\widehat{\beta}_1(\widehat{\alpha}) - \beta_1) = \sqrt{ST}(\widehat{\beta}_1(\alpha) - \beta_1) &+ \sum_{i=1}^{p} A^{-1}\Psi_i(\psi_1 + \sqrt{\frac{S}{T}}B(\alpha,T))_i/\sqrt{ST} \\ &+ \sum_{i=1}^{p} A^{-1}\Psi_i(\psi_2)_i/ST \\ &+ \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}(A^{-1}\Psi_{ij} - A^{-1}A_jA^{-1}\Psi_i - A^{-1}A_iA^{-1}\Psi_j)\times \\ &(\psi_1 + \sqrt{\frac{S}{T}}B(\alpha,T))_i(\psi_1 + \sqrt{\frac{S}{T}}B(\alpha,T))_j/ST + R/(ST)^{3/2}\end{aligned}$$

where $R = O_p(1)$. Since the $\Psi_i, \Psi_{ij}, \ldots$ are odd functions of V and $\widehat{\alpha}$ is an even function of V, it follows that

$$Bias(\widehat{\beta}_1(\widehat{\alpha})) = 0.$$

Using the independence obtained in Step 1, the higher order variance is

$$\begin{aligned}Var(\sqrt{ST}(\widehat{\beta}_1(\widehat{\alpha}) - \beta_1)) &= A^{-1} \\ &+ \mathrm{E}[(\sum_{i=1}^{p} A^{-1}\Psi_i(\psi_1 + \sqrt{\frac{S}{T}}B(\alpha,T))_i/\sqrt{ST})(\sum_{i=1}^{p} A^{-1}\Psi_i(\psi_1 + \sqrt{\frac{S}{T}}B(\alpha,T))_i/\sqrt{ST})'].\end{aligned}$$

Following Rothenberg (1984), note that since $\widehat{\alpha}$ is an even function of V and S_i is an odd function of V, the ψ's and Ψ_i are uncorrelated. Hence,

$$\begin{aligned}&\mathrm{E}[(\sum_{i=1}^{p} A^{-1}\Psi_i(\psi_1 + \sqrt{\frac{S}{T}}B(\alpha,T))_i/\sqrt{ST})(\sum_{i=1}^{p} A^{-1}\Psi_i(\psi_1 + \sqrt{\frac{S}{T}}B(\alpha,T))_i/\sqrt{ST})'] \\ &= \frac{1}{ST}\sum_{i=1}^{p}\sum_{j=1}^{p}\left(\mathrm{E}[\Psi_i\Psi_j']\mathrm{E}[(\psi_1)_i(\psi_1)_j] + \frac{S}{T}B(\alpha,T)_iB(\alpha,T)_j\right).\end{aligned}$$

Lemmas C.6 and C.7 and Assumption 4 imply that $\mathrm{E}[\psi_1\psi_1'] = \frac{ST}{S(T-p)}(\Gamma_p^{-1} + \frac{1}{T}\Gamma_p^{-1}\Theta\Gamma_p^{-1}) = \frac{ST}{S(T-p)}\Gamma_p^{-1} + O(1/T)$. Then defining $\zeta_{ij} = \mathrm{E}[\Psi_i\Psi_j']$ and

$$\Upsilon = \sum_{i=1}^{p}\sum_{j=1}^{p}\zeta_{ij}(\frac{ST}{S(T-p)}\Gamma_p^{-1})_{ij} \tag{29}$$

yields

$$Var(\sqrt{ST}(\widehat{\beta}_1(\widehat{\alpha}) - \beta_1)) = A^{-1} + \Upsilon/ST + \frac{1}{ST}\sum_{i=1}^{p}\sum_{j=1}^{p}\zeta_{ij}\frac{S}{T}B(\alpha,T)_iB(\alpha,T)_j + O(1/ST^2). \tag{30}$$

(ii) The derivation with $\widehat{\alpha}$ replaced with $\widehat{\alpha}^{(\infty)}$ is similar. In particular, the expansion for $\widehat{\beta}_1(\widehat{\alpha}^{(\infty)} - \beta_1)$ is

$$\sqrt{ST}(\widehat{\beta}_1(\widehat{\alpha}^{(\infty)}) - \beta_1) = \sqrt{ST}(\widehat{\beta}_1(\alpha) - \beta_1) + \sum_{i=1}^{p} A^{-1}\Psi_i(\psi_1)_i/\sqrt{ST} + \sum_{i=1}^{p} A^{-1}\Psi_i(\psi_2^{(\infty)})_i/ST$$
$$+ \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}(A^{-1}\Psi_{ij} - A^{-1}A_jA^{-1}\Psi_i - A^{-1}A_iA^{-1}\Psi_j)(\psi_1)_i(\psi_1)_j/ST$$
$$+ R/(ST)^{3/2}.$$

It then follows as in (i) that

$$Bias(\widehat{\beta}_1(\widehat{\alpha}^{(\infty)}) = 0$$

and that

$$Var(\sqrt{ST}(\widehat{\beta}_1(\widehat{\alpha}^{(\infty)}) - \beta_1)) = A^{-1} + \Upsilon/ST + O(1/ST^2). \tag{31}$$

Finally, the $O(1/ST^2)$ term in (30) and (31) is the same, so

$$Var(\sqrt{ST}(\widehat{\beta}_1(\widehat{\alpha}) - \beta_1)) - Var(\sqrt{ST}(\widehat{\beta}_1(\widehat{\alpha}^{(\infty)}) - \beta_1)) = \frac{1}{ST}\sum_{i=1}^{p}\sum_{j=1}^{p}\zeta_{ij}\frac{S}{T}B(\alpha,T)_iB(\alpha,T)_j$$

is positive semidefinite since

$$\frac{1}{ST}\sum_{i=1}^{p}\sum_{j=1}^{p}\zeta_{ij}\frac{S}{T}B(\alpha,T)_iB(\alpha,T)_j = \frac{1}{ST}\frac{S}{T}\mathrm{E}[(\sum_{i=1}^{p}B(\alpha,T)_i\Psi_i)(\sum_{j=1}^{p}B(\alpha,T)_j\Psi_j')].$$

■

References

Billingsley, P. (1995): *Probability and Measure.* New York: Wiley, third edn.

Doukhan, P. (1994): *Mixing: Properties and Examples*, vol. 85 of *Lecture Notes in Statistics (Springer-Verlag).* New York: Springer-Verlag.

Fitzpatrick, P. M. (1996): *Advanced Calculus: A Course in Mathematical Analysis.* PWS Publishing Company: Boston, Massachussetts.

Lehmann, E. L. (1983): *Theory of Point Estimation.* New York: Wiley.

Phillips, P. C. B., and H. R. Moon (1999): "Linear Regression Limit Theory for Nonstationary Panel Data," *Econometrica*, 67(5), 1057–1111.

Rothenberg, T. J. (1984): "Approximate Normality of Generalized Least Squares Estimates," *Econometrica*, 52(4), 811–825.

White, H. (2001): *Asymptotic Theory for Econometricians.* San Diego: Academic Press, revised edn.