# The relationship between the Beveridge-Nelson decomposition and other permanant-transitory decompositions that are popular in economics

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**Abstract**

The Beveridge-Nelson (BN) decomposition is a model-based method for decomposing time series into permanent and transitory components. When constructed from an ARIMA model, it is closely related to decompositions based on unobserved components (UC) models with random walk trends and covariance stationary cycles. The decomposition when extended to $I(2)$ models can also be related to non-model based signal extraction filters such as the HP filter. We show that the BN decomposition provides information on the correlation between the permanent and transitory shocks in a certain class of UC models. The correlation between components is known to determine the smoothed estimates of components from UC models. The BN decomposition can also be used to evaluate the efficacy of alternative methods. We also demonstrate, contrary to popular belief, that the BN decomposition can produce smooth cycles if the reduced form forecasting model is appropriately specified.

*Keywords: Permanent-transitory decomposition; Trend-cycle decomposition; Unobserved components time series model; State space models; Kalman filter.*

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# 1 Introduction

The Beveridge-Nelson (BN) decomposition is a model-based method for decomposing a univariate or multivariate time series into permanent and transitory (PT) components. It defines the stochastic trend as the limiting forecast of the level of the series minus any deterministic components given the current information set. The permanent component is a pure random walk while the remaining movements in the series are the $I(0)$ transitory component. Other than the random walk trend, the BN decomposition does not make assumptions about the structure of the components and the correlations between them. However, it is closely related to decompositions based on unobserved components (UC) models with random walk trends and covariance stationary cycles. The BN decomposition can also be related to non-model based signal extraction filters such as the Hodrick-Prescott (HP) filter and other Butterworth lowpass filters considered by Gomez (2001). These latter methods are indirectly related to the BN decomposition through their relationships with UC models.

In this paper, we study the BN decomposition when ARIMA models are used as the forecast function. Our contribution is to clarify the relationship between the BN decomposition and other univariate detrending methods popular in economics. In particular, for certain $I(1)$ and $I(2)$ models we describe the relationship between the BN decomposition and UC models with correlated permanent and transitory shocks. We clarify under what conditions the correlation between shocks is identified. For our application to U.S. real GDP, the correlation between components is identified up to a set within the parameter space. If the value of the correlation is within this set, the real-time or filtered estimates of trend and cycle from a UC model will be equivalent to the BN decomposition. We also demonstrate how the BN decomposition can be used as a benchmark to test the over-identifying restrictions that are commonly made in applied macroeconomics research. Examining the over-identifying restrictions can help applied researchers understand the implications that these assumptions have on the decomposition. Many of these results have been stated previously in the literature and part of our contribution is to present these results in a consistent manner.

The BN decomposition holds less relevance for researchers who believe that the trend is not a pure random walk. Consequently, our analysis is limited to models with random walk trend components. There also exist other types of PT decompositions in which the permanent component is an integrated series but not a pure random walk. These include the canonical decomposition of Hillmer and Tiao (1982) and the general PT decompositions of Quah (1992), but these are outside the scope of this paper. As emphasized by Quah (1992), the random walk trend implicit in the BN decomposition maximizes the importance of the permanent component.

This should always be recognized when interpreting the results of the BN decomposition.

In the following, we begin by presenting the BN decomposition of an $I(1)$ process. Then we discuss the relationship between the BN decomposition for ARIMA(p,1,q) models and different UC models. We extend this to $I(2)$ processes in the next section. We conclude with an application to U.S. real GDP which emphasizes key points from our previous analysis.

## 2 The BN Decomposition of an $I(1)$ Process

Assume that the univariate time series $y_t$ is an $I(1)$ process with Wold representation given by

$$\Delta y_t = \mu + \psi(L)\epsilon_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}, \tag{1}$$

where $\Delta = 1 - L$, $\psi(0) = 1, \psi(1) \neq 0$, $\sum_{j=0}^{\infty} j^{1/2}|\psi_j| < \infty$, and $\epsilon_t$ are iid $(0, \sigma^2)$ one-step-ahead forecast errors. The permanent component or trend $\tau_t$ of the BN decomposition of $y_t$ is defined as the limiting forecast minus any deterministic components

$$\tau_t^{BN} = \lim_{J\to\infty} E[y_{t+J} - J\mu|\Omega_t], \tag{2}$$

where $\Omega_t$ represents conditioning information available at time $t$. Writing $y_{t+J} = y_t + \Delta y_{t+1} + \cdots + \Delta y_{t+J}$ and using $E[\Delta y_{t+j}|\Omega_t]$ $(j = 1, \ldots, J)$ based on (1) allows for the analytic evaluation of $\tau_t^{BN}$ as

$$\tau_t^{BN} = \mu + \tau_{t-1}^{BN} + \psi(1)\epsilon_t. \tag{3}$$

Hence, the BN trend is a pure random walk with drift $\mu$ and has innovation variance $\sigma^2\psi(1)^2$. The transitory component or cycle, $c_t^{BN}$, is defined as the difference between $y_t$ and the BN trend

$$c_t^{BN} = y_t - \tau_t^{BN} = \tilde{\psi}(L)\epsilon_t, \tag{4}$$

where $\tilde{\psi}(L) = \sum_{j=0}^{\infty} \tilde{\psi}_j L^j$ and $\tilde{\psi}_j = -\sum_{k=j+1}^{\infty} \psi_k$. Solo (1989) showed that the $\frac{1}{2}$-summability of $\psi(L)$ and the uniqueness of the Wold decomposition guarantees the existence and uniqueness of the BN decomposition. From (3) and (4) it is clear that the BN decomposition produces real-time or one-sided estimates of the permanent and transitory components at time $t$.

An alternative derivation of the BN decomposition follows directly from the factorization $\psi(L) = \psi(1) + (1 - L)\tilde{\psi}(L)$. Then (1) may be rewritten as

$$\Delta y_t = \mu + \psi(1)\epsilon_t + (1 - L)\tilde{\psi}(L)\epsilon_t, \tag{5}$$

which identifies $(\mu + \psi(1)\epsilon_t)/(1 - L)$ as the permanent component and $\tilde{\psi}(L)\epsilon_t$ as the transitory component.

In practice, the BN decomposition can be computed in a number of ways. Typically, it is assumed that $\Delta y_t$ follows an ARMA$(p, q)$ process so that $\psi(L) = \theta(L)/\phi(L)$ where the orders of $\phi(L)$ and $\theta(L)$ are $p$ and $q$, respectively, and the roots of $\phi(L) = 0$ and $\theta(L) = 0$ are assumed to lie outside the complex unit circle. A brute force approach is based on estimating an ARMA$(p, q)$ model for $\Delta y_t$, using these estimates to compute an estimate of $\psi(1) = \theta(1)/\phi(1)$, and then forming estimates of the components using (2) and (4) with the ARMA residuals in place of $\epsilon_t$. Cuddington and Winters (1987), Miller (1988) and Newbold (1990) provided improvements to this brute force method. These methods are valid if the forecasting model for $\Delta y_t$ is a univariate ARMA$(p, q)$ model. Ariño and Newbold (1998) extended the algorithm of Newbold (1990) to multivariate forecasting models for $\Delta y_t$. Evans and Reichlin (1994) also discussed the BN decomposition for multivariate models. Recently, Morley (2002) provided a very simple state-space approach for calculating the BN decomposition that is valid for any forecasting model for $\Delta y_t$ that can be cast into state-space form. In particular, suppose $\Delta y_t - \mu$ is a linear combination of the elements of the $m \times 1$ state vector $\alpha_t$

$$\Delta y_t - \mu = \mathbf{z}'\alpha_t,$$

where $\mathbf{z}$ is an $m \times 1$ vector with fixed elements. Suppose further that

$$\alpha_t = \mathbf{T}\alpha_{t-1} + \eta_t,\ \ \eta_t \sim \text{iid } N(\mathbf{0}, \mathbf{Q}), \tag{6}$$

such that all of the eigenvalues of $\mathbf{T}$ have modulus less than unity, and $\mathbf{I}_m - \mathbf{T}$ is invertible. Then, Morley (2002) showed that

$$\begin{aligned} \tau_t^{BN} &= y_t + \mathbf{z}'\mathbf{T}(\mathbf{I}_m - \mathbf{T})^{-1}\alpha_{t|t}, \\ c_t^{BN} &= y_t - \tau_t^{BN} = -\mathbf{z}'\mathbf{T}(\mathbf{I}_m - \mathbf{T})^{-1}\alpha_{t|t}, \end{aligned} \tag{7}$$

where $\alpha_{t|t} = E[\alpha_t|\Omega_t]$ denotes the filtered or real-time estimate of $\alpha_t$ from the Kalman filter.[1] An important advantage of Morley's approach is its generality. It works the same way for univariate and multivariate forecasting models for $\Delta y_t$.

Disadvantages of the methods described above to compute the BN decomposition are that they lose the first observation due to differencing the data, and that they do not provide standard error bands for the extracted trend and cycle estimates. However, as discussed by Morley, Nelson, and Zivot (2003) (MNZ) and Anderson, Low, and Snyder (2006) and shown in the next section, the BN decomposition may also be computed directly using the Kalman filter from certain UC models. This allows for the use of all the data and for the calculation of standard error bands

[1] Throughout the paper we refer to filtered estimates as real-time estimates based on information only available at time $t$, and smoothed estimates as final estimates based on all available sample information.

for the extracted trend and cycle. It also allows for the extraction of trend and cycle estimates at time $t$ using information in the full sample, $\Omega_T$.

# 3 The BN Decomposition and Unobserved Components Models

The BN decomposition produces a decomposition into permanent and transitory components with minimal assumptions about the structure of the components. The definition of the BN trend (2) identifies the permanent component as a pure random walk, and this result can be used to link the BN decomposition with traditional UC models with random walk trends. The following subsections describe the class of UC models that are consistent with the BN decomposition. Throughout, we assume that $\Delta y_t$ has a reduced form covariance stationary and invertible ARMA$(p,q)$ representation such that $\psi(L) = \theta(L)/\phi(L)$ in (1).

## 3.1 Single Source of Error Model

The definitions of the BN permanent and transitory components in (3) and (4) suggest the following single-source-of-error (SSOE) state space representation[2]

$$\begin{aligned} y_t &= \tau_t + c_t, \\ (1-L)\tau_t &= \mu + \psi(1)\ \epsilon_t, \\ c_t &= \tilde{\psi}(L)\ \epsilon_t, \end{aligned} \tag{8}$$

where $\tilde{\psi}(L)\epsilon_t \sim$ ARMA$(p,n)$ with $n = \max(q-1,0)$. It is clear from (8) that the innovations to the permanent and transitory components are perfectly correlated

$$\rho = \frac{\text{cov}(\psi(1)\epsilon_t, \tilde{\psi}(0)\epsilon_t)}{\sqrt{\text{var}(\psi(1)\epsilon_t)\ \text{var}(\tilde{\psi}(0)\epsilon_t)}} = \frac{\psi(1)\ \tilde{\psi}(0)}{|\psi(1)\ \tilde{\psi}(0)|} = -1 \ \text{ or } \ 1,$$

where the sign of $\rho$ depends on the sign of $\tilde{\psi}(0)$. Hence, there always exists a UC representation with perfectly correlated shocks that is consistent with the BN decomposition. However, as discussed by MNZ, equation (8) is not the only UC representation that is consistent with the BN decomposition. We note that Ord, Koehler, and Snyder (1997) advocated the use of SSOE UC models because they do not impose the complicated restrictions on the reduced form that result from UC models with orthogonal components.

More generally, a SSOE UC model representation may be deduced from the Kalman filter equations for any UC model of the form

$$\begin{aligned} y_t &= \mathbf{z}'\alpha_t + \varepsilon_t, \\ \alpha_t &= \mathbf{T}\alpha_{t-1} + \eta_t, \end{aligned}$$

[2]Anderson, Low, and Snyder (2006) gave a slightly different, but equivalent, formulation of the SSOE model that includes $\epsilon_t$ in the measurement equation.

where the vector $\eta_t$ may contain several structural shocks that may be independent of each other and also independent of the observation errors $\varepsilon_t$. From the updating and prediction steps of the Kalman filter, we may deduce the following SSOE representation

$$\begin{aligned} y_t &= \mathbf{z}'\alpha_{t|t-1} + v_t, \\ \alpha_{t+1|t} &= \mathbf{T}\alpha_{t|t-1} + \mathbf{K}_t v_t, \end{aligned}$$

where $\mathbf{K}_t$ is the Kalman gain and $v_t = y_t - y_{t|t-1}$ is the prediction error. This result shows that a SSOE UC model cannot be ruled out a priori. However, there is no reason to think that such a model is more natural than the original UC model.[3]

Anderson, Low, and Snyder (2006) used the SSOE representation of the BN permanent and transitory components (8) to compute the BN decomposition directly from the Kalman filter. The filtered estimates $\tau_{t|t} = E[\tau_t|\Omega_t]$ and $c_{t|t} = E[c_t|\Omega_t]$ produced by the Kalman filter correspond to the BN permanent and transitory components (3) and (4), respectively. This approach allows for the direct calculation of $\psi(1)$ from an ARMA$(p, n)$ model for $\tilde{\psi}(L)$ which is one of its main advantages. Watson (1986), Harvey and Koopman (2000), and Morley (2007), emphasized the components $\tau_{t|t}$ and $c_{t|t}$ in the SSOE model are estimated with zero mean squared error because they are an exact function of past observations. As a result, the standard errors for $\tau_{t|t}$ and $c_{t|t}$ computed from the Kalman filter will be equal to zero and the two-sided or smoothed estimates will be equivalent. Morley (2007) used this result to argue against interpreting the SSOE representation as a structural model.

### 3.2 Two Source of Error Model

The perfect correlation between shocks to the components in (8) is due to the single disturbance term $\epsilon_t$, which represents the forecast error in the reduced form ARMA$(p, q)$ model for $\Delta y_t$. Shapiro and Watson (1988) and others have argued that the economic forces underlying movements in real output imply multiple sources of shocks. For example, suppose that $\psi(L)\epsilon_t$ is the sum of two independent processes $\psi_1(L)\epsilon_{1t}$ and $\psi_2(L)\epsilon_{2t}$, where $\epsilon_{1t} \sim$ iid $(0, \sigma^2_{\epsilon 1})$, $\epsilon_{2t} \sim$ iid $(0, \sigma^2_{\epsilon 2})$, and $\text{cov}(\epsilon_{1t}, \epsilon_{2t}) = 0$. Then equation (5) becomes

$$\begin{aligned} (1-L)y_t &= \mu + (\psi_1(1)\ \epsilon_{1t} + \psi_2(1)\ \epsilon_{2t}) + (1-L)(\tilde{\psi}_1(L)\epsilon_{1t} + \tilde{\psi}_2(L)\epsilon_{2t}), \\ &= \mu + \psi(1)\ \epsilon_t + (1-L)\left\{\tilde{\psi}_1(L)\epsilon_{1t} + \tilde{\psi}_2(L)\epsilon_{2t}\right\}. \end{aligned} \tag{9}$$

The permanent innovation is still $\psi(1)\epsilon_t$ but the transitory innovation is now $\tilde{\psi}_1(0)\epsilon_{1t} + \tilde{\psi}_2(0)\epsilon_{2t}$. If $\tilde{\psi}_1(L)$ is zero,[4] then the correlation between permanent and transitory innovations will be

[3] We thank a referee for this result. See Chapter 12 of Brockwell and Davis (1991) for more details.

[4] As a simple example, let $\psi_1(L)\epsilon_t$ be a white noise process. Then, $\psi(1) = 1$ and $\tilde{\psi}_1(L) = 0$.

between -1 and 1

$$\rho = \frac{\psi_2(1)\ \tilde{\psi}_2(0)\sigma^2_{\epsilon_2}}{\sqrt{\psi_1(1)^2\ \tilde{\psi}_2(0)^2\ \sigma^2_{\epsilon_1}\sigma^2_{\epsilon_2} + \psi_2(1)^2\ \tilde{\psi}_2(0)^2\sigma^4_{\epsilon_2}}}, \neq -1 \text{ nor } 1.$$

If $\text{cov}(\epsilon_{1t}, \epsilon_{2t}) \neq 0$, then the correlation between the permanent and transitory shocks will be between -1 and 1 even when $\tilde{\psi}_1(L)$ is not zero.

In equation (9), the permanent component $\psi(1)\epsilon_t$ does not depend on the individual shocks. This means that the variance of the permanent component is always identified and is exactly the same as the variance of the permanent component from the SSOE model (8). In contrast, the parameters of the cycle and the correlation between the permanent and transitory shocks are generally not identified without further assumptions on the parametric form of the UC model.

To understand the general relationship between the permanent and transitory components defined by the BN decomposition from an ARMA$(p, q)$ reduced form model for $\Delta y_t$ and those defined in an unobserved components model, consider a typical UC model with two sources of shocks

$$\begin{aligned} y_t &= \tau_t + c_t, \qquad (10)\\ \tau_t &= \tau_{t-1} + d + w_t,\ w_t, \sim \text{iid } (0, \sigma^2_w),\\ \phi(L)c_t &= \theta^*(L)\ v_t,\ v_t \sim \text{iid } (0, \sigma^2_v),\\ \text{cov}(w_t, v_t), &= \sigma_{wv}, \end{aligned}$$

where the order of $\phi(L)$ is $p$, the order of $\theta^*(L)$ is $q^*$, and the roots of $\phi(z) = 0$ and $\theta^*(z) = 0$ lie outside the complex unit circle. We call (10) a UC-ARMA$(p, q^*)$ model. In (10), $w_t$ is the permanent shock and $v_t$ is the transitory shock.

The reduced form of (10) is an ARMA$(p, q)$ model for $\Delta y_t$

$$\phi(L)(1 - L)y_t = \phi(1)d + \phi(L)w_t + (1 - L)\theta^*(L)v_t. \qquad (11)$$

The right-hand side of (11) is the sum of MA$(p)$ and MA$(q^* + 1)$ processes. To be consistent with the UC-ARMA$(p, q^*)$ model, the reduced form ARMA$(p, q)$ model must have MA order $q = \max(p, q^* + 1)$. As discussed in Harvey (1989), identification of the UC model parameters requires solving for these parameters uniquely from knowledge of the reduced form ARMA parameters. The AR polynomial in (11) and in the reduced form ARMA$(p, q)$ model are the same. The remaining parameters of the UC model to be identified are the $q^*$ MA parameters in $\theta^*(L)$ and the 3 covariance matrix parameters $\sigma^2_w, \sigma^2_v$, and $\sigma_{wv}$. From the MA portion of (11), the number of moments that can be calculated is $q + 1$. Therefore, the order condition for exact identification is $q+1 = q^*+3$ meaning the reduced form must have $q^*+2$ MA parameters. For example, when $p$ equals 2 and $q^*$ equals 0 the reduced form must have $q = 2$. Of course, the resulting UC model parameters must also satisfy certain necessary conditions such as positive definitiveness of the covariance matrix and invertibility of the MA coefficients. MNZ used this result to estimate $\sigma_{wv}$. If $q+1 < q^*+3$, the UC model (10) is under-identified and there is no unique UC model that matches the moments of the reduced form ARIMA model. For example, when $p = 2$ and $q^* = 1$ the reduced form has $q = 2$ and there are not enough moments to uniquely identify all of the UC model parameters.

As long as a UC model with random walk trend does not restrict the parameter space which matches the moments of the observed data, the UC model produces the same filtered estimates as the BN decomposition from an unrestricted ARIMA model. For these admissible UC models, the value of the correlation between the permanent and transitory components, $\rho_{wv}$, does not impact the point estimates of the trend $\tau_{t|t}$ or cycle $c_{t|t}$ computed by the Kalman filter. However, as noted by Harvey (1989), Harvey and Koopman (2000), Proietti (2006), and Morley (2007), the value of $\rho_{wv}$ does impact the precision of $\tau_{t|t}$ and $c_{t|t}$ if $|\rho_{wv}| \neq 1$. In this case, the trend and cycle, $\tau_{t|t}$ and $c_{t|t}$, are estimated with non-zero mean squared error because there are now two distinct sources of error. As a result, standard error bands for the extracted components computed by the Kalman filter will be positive and smoothed estimates of the trend and cycle will be different than the filtered estimates.

### 3.3 ARIMA(2,1,2) Model

To illustrate the relationship between a particular reduced form ARIMA model for $y_t$ and the class of observationally equivalent UC-ARMA models with correlated shocks, consider the following reduced form ARIMA(2,1,2) model for $y_t$ that was studied by MNZ

$$(1-\phi_1 L - \phi_2 L^2)(1-L)y_t = \mu + (1+\theta_1 L + \theta_2 L^2)\epsilon_t. \tag{12}$$

As shown in Proietti (2006), (12) is the unrestricted reduced form associated with a UC-ARMA(2,1) model with correlated shocks of the form (11). To determine which parameters can be identified, consider the moments of the MA part of (11)

$$\begin{aligned}
\gamma_0 &= (1+\phi_1^2+\phi_2^2)\sigma_w^2 + \left(1+(\theta_v-1)^2+\theta_v^2\right)\sigma_v^2 + 2(1-\phi_1(\theta_v-1)+\phi_2\theta_v)\sigma_{wv}, \qquad (13)\\
\gamma_1 &= (-\phi_1+\phi_1\phi_2)\sigma_w^2 + ((\theta_v-1)-\theta_v(\theta_v-1))\sigma_v^2 + (-\phi_1-\phi_2(\theta_v-1),\\
&\quad +(\theta_v-1)+\theta_v\phi_1)\sigma_{wv},\\
\gamma_2 &= -\phi_2\sigma_w^2 - \theta_v\sigma_v^2 + (-\phi_2-\theta_v)\sigma_{wv}.
\end{aligned}$$

Notice that $\sigma_w^2$ does not depend on the covariance

$$\sigma_w^2 = \frac{\gamma_0 + 2(\gamma_1 + \gamma_2)}{1 - 2\phi_1 - 2\phi_2 + 2\phi_1\phi_2 + \phi_1^2 + \phi_2^2}, \tag{14}$$

and is identified from the reduced form moments and parameters. The remaining parameters are not identified without further restrictions. If $\sigma_{wv}$ (or $\rho_{wv}$) is given arbitrarily, then the remaining two unknowns $(\sigma_v^2, \theta_v)$ may be determined. Similarly, if $\sigma_v^2$ or $\theta_v$ is given, then $(\sigma_{wv}, \theta_v)$ or $(\sigma_{wv}, \sigma_v^2)$ may be determined. The system (13) is nonlinear in the parameters, however, so there may exist multiple solutions. Admissible solutions must satisfy the covariance stationarity and invertibility conditions as well as the positive definiteness of the covariance matrix of $(w_t, v_t)'$. Equations (13) and (14) are an extension of MNZ to the case where $\theta_v$ is not equal to zero.

Assuming normal errors, we define the 'admissible' solutions as the set of parameter values with the same likelihood value as the unrestricted reduced form ARIMA(2,1,2) model. Once within this set, the filtered permanent and transitory components from the UC model will be equivalent to the BN decomposition.[5] In other words, there exists a set of UC-ARIMA(2,1) models (each with different correlations) that are observationally equivalent to the unrestricted reduced form ARIMA(2,1,2) model. The reduced form ARIMA model used to compute the BN decomposition may not uniquely identify the correlation between components but it does provide a set that is identified.

The SSOE model has $\rho_{wv} = \pm 1$ in (10). Evaluating $\psi(1)$ and $\psi(L)$ in (8) gives the resulting UC-ARMA(2,1) model parameters in terms of the reduced form ARIMA(2,1,2) model parameters, see Proietti (2006). He shows that $\rho_{wv} = -1$ if $\psi(1) > 1$ and $\rho_{wv} = 1$ if $\psi(1) < 1$.

The UC-ARMA(2,1) model with correlated components considered by MNZ imposes the restriction $\theta_v = 0$ in (11). As a result, the order condition for identification is satisfied which allows $\sigma_w^2$, $\sigma_v^2$ and $\sigma_{wv}$ to be recovered from (13). See MNZ for details. Using this result, MNZ estimated $\rho_{wv}$ to be $-0.9062$ for postwar quarterly real GDP. However, this is just one such UC-ARMA(2,1) model that is consistent with the reduced form. Setting $\theta_v = \bar{\theta}_v \neq 0$, where $\bar{\theta}_v$ denotes any permissable value, also gives a linear system that can be inverted to obtain a solution for $\sigma_w^2$, $\sigma_v^2$, and $\sigma_{wv}$. Hence, the deduced correlation is a function of $\bar{\theta}_v$ and different values of $\bar{\theta}_v$ will produce different correlations. Therefore, to draw conclusions about the correlation between shocks in the UC-ARMA(2,1) model one must derive the set of all admissible correlation values as a function of $\bar{\theta}_v$. We do this for U.S. postwar quarterly real GDP in the empirical section below.

[5] Importantly, two-sided or smoothed estimates are not equivalent for different values of the correlation within the identified set despite equivalent likelihood values. Consequently, smoothed estimates are not identified unless a unique correlation is chosen by the researcher.

Proietti (2006) considered the UC-ARMA(2,1) with $\rho_{wv} = 0$ and $\theta_v$ free to see if expanding the dynamics of the cycle could give a UC model with uncorrelated components that matched the moments of the US real GDP data used by MNZ. Unfortunately, the system of equations (13) relating the autocovariances to the remaining UC model parameters is nonlinear and it is not straightforward to determine admissibility. Proietti showed that any UC-ARMA(2,1) model with $\rho_{wv} > 0$ and $\theta_v = 0$ is equivalent to a model with $\rho_{wv} = 0$ and $\theta_v < 0$, and a model with $\rho_{wv} < 0$ and $\theta_v = 0$ is equivalent to one with $\rho_{wv} = 0$ and $\theta_v > 0$ provided $\sigma_w/\sigma_v$ is sufficiently small.

The UC model with uncorrelated components used by Watson (1986) imposes the restrictions $\theta_v = \sigma_{wv} = 0$. As discussed in Harvey (1989), the zero correlation assumption is often made when modeling components with distinct properties. From the order condition, the UC-ARMA(2,1) model is now overidentified and imposes complicated nonlinear restrictions on the reduced form ARIMA(2,1,2) model parameters. These restrictions can be checked for admissibility using the moment conditions (13). If they are not admissible, then the resulting estimated trend and cycles will be different from the BN decomposition. As noted by MNZ, the validity of the restrictions $\theta_v = \sigma_{wv} = 0$ can be tested using a likelihood ratio statistic.

# 4 The Relationship between the BN Decomposition and Unobserved Components Models for $I(2)$ Processes

In the previous section, we discussed the relationship between the BN decomposition and UC models with correlated shocks for $I(1)$ processes represented by an ARIMA model. Many empirical implementations of UC models allow the slope of the random walk trend to also evolve as a random walk, see e.g., Harvey (1985), Clark (1987) and Harvey and Jaeger (1993). This UC model allows for a more flexible trend that can pick up smooth structural breaks. In this specification, the time series $y_t$ follows an $I(2)$ process.

For $I(1)$ processes, we stressed how the BN decomposition could be compared to UC models by comparing their reduced-form ARIMA representations. It is possible to extend this comparison for $I(2)$ models to include other popular non-model based trend/cycle decompositions. For example, Gomez (2001) demonstrated that a class of two-sided Butterworth lowpass filters and bandpass filters (built from the lowpass filters) are equivalent to UC models and consequently ARIMA models.[6] This class of nonparametric filters is based in the frequency domain

[6] Not all of the nonparametric filters considered by Gomez (2001) will have random walk trends, in particular the bandpass filters he considers. They consequently may not always be equivalent to the BN decomposition. Harvey and Trimbur (2003) have extended his work by building UC models with higher-order stochastic cycles that include random walk trends and slopes. These models extract cycles within a predefined range of frequencies and consequently have bandpass filter properties. Using the results in Trimbur (2006), this class of models can be

and includes the Hodrick-Prescott (HP) filter as a special case. To apply these filters, a user chooses the tuning parameters of the gain function. These tuning parameters implicitly determine the underlying ARIMA model and its parameter values. Consequently, one can test the over-identifying restrictions imposed by the nonparametric filters by comparing their ARIMA representations to an unrestricted ARIMA model.

### 4.1 The BN Decomposition for an $I(2)$ Process

Assume that $y_t$ is an $I(2)$ process with a Wold representation given by

$$(1-L)^2 y_t = \psi(L)\epsilon_t, \tag{15}$$

where $\psi(L)$ and $\epsilon_t$ are defined as in (1). Using the BN factorization of $\psi(L)$, we can rewrite (15) as

$$(1-L)^2 y_t = \psi(1)\ \epsilon_t + (1-L)\tilde{\psi}(L)\epsilon_t.$$

Dividing both sides by $(1-L)$ and applying the BN factorization to $\tilde{\psi}(L)$, we obtain

$$(1-L)y_t = \frac{\psi(1)}{(1-L)}\ \epsilon_t + \tilde{\psi}(1)\epsilon_t + (1-L)\tilde{\tilde{\psi}}(L)\epsilon_t. \tag{16}$$

Splitting the integrated parts from the stationary part in (16), the permanent and transitory components in the BN decomposition for an $I(2)$ process may be defined as

$$(1-L)\tau_t = \frac{\psi(1)}{(1-L)}\ \epsilon_t + \tilde{\psi}(1)\epsilon_t, \tag{17}$$

$$c_t = \tilde{\tilde{\psi}}(L)\epsilon_t. \tag{18}$$

Defining the double integrated part of (17) as a random drift term $d_t$, the BN decomposition has the following SSOE UC model representation

$$\begin{aligned} y_t &= \tau_t + c_t, \\ (1-L)\tau_t &= d_{t-1} + [\psi(1) + \tilde{\psi}(1)]\epsilon_t, \\ (1-L)d_t &= \psi(1)\epsilon_t, \\ c_t &= \tilde{\tilde{\psi}}(L)\epsilon_t. \end{aligned} \tag{19}$$

Notice that in the $I(2)$ case there is an overall trend, $\tau_t$, which follows a double random walk, a drift term, $d_t$, that follows a random walk, and a residual cycle component, $c_t$.

When $\psi(L) = \theta(L)/\phi(L)$, Newbold and Vougas (1996) derived a computationally efficient algorithm for computing the components of (19). Alternatively, the components may be computed using the Kalman filter applied directly to the SSOE model (19). Oh and Zivot (2006)

shown to have reduced-form ARIMA representations and will be equivalent to the BN decomposition. Therefore, it is possible to test any over-identifying restrictions that are present, although we do not consider that here.

extended the method of Morley (2002) for cases in which $\Delta^2 y_t = \mathbf{z}'\alpha_t$, where $\mathbf{z}$ is an $m \times 1$ vector with fixed elements and the $m \times 1$ state vector $\alpha_t$ follows the transition equation (6). They showed that the components may be computed using

$$\begin{aligned} \tau_t^{BN} &= y_t - \mathbf{z}'\mathbf{T}^2(\mathbf{I}_m - \mathbf{T})^{-2}\boldsymbol{\alpha}_{t|t}, \\ d_t^{BN} &= \Delta y_t + \mathbf{z}'\mathbf{T}(\mathbf{I}_m - \mathbf{T})^{-1}\boldsymbol{\alpha}_{t|t}, \\ c_t^{BN} &= \mathbf{z}'\mathbf{T}^2(\mathbf{I}_m - \mathbf{T})^{-2}\boldsymbol{\alpha}_{t|t}. \end{aligned} \tag{20}$$

### 4.2 The Relationship between the BN Decomposition and UC Models

Assume that $\psi(L) = \theta(L)/\phi(L)$ in (15), and consider (19) rewritten as a typical UC model with three shocks

$$\begin{aligned} y_t &= \tau_t + c_t, \\ \tau_t &= \tau_{t-1} + d_{t-1} + w_t, \;\; w_t \sim \text{iid } (0, \sigma_w^2), \\ d_t &= d_{t-1} + u_t, \;\; u_t \sim \text{iid } (0, \sigma_u^2), \\ \phi(L)c_t &= \theta^*(L)\ v_t, \;\; v_t \sim \text{iid } (0, \sigma_v^2), \\ \text{cov}(w_t, u_t) &= \sigma_{wu}, \;\; \text{cov}(w_t, v_t) = \sigma_{wv}, \;\; \text{cov}(u_t, v_t) = \sigma_{uv}, \end{aligned} \tag{21}$$

where the order of $\theta^*(L)$ is $q^*$. The reduced form of (21) is an ARMA$(p, q)$ model

$$\phi(L)(1-L)^2 y_t = \phi(L)u_{t-1} + \phi(L)(1-L)w_t + (1-L)^2\theta^*(L)v_t. \tag{22}$$

The MA polynomial has order $q = \max(p+1, q^*+2)$ with respect to $L$. From the MA portion, $q+1$ moments may be calculated. Excluding the AR parameters, the unknown parameters in the UC model (21) are the $q^*$ MA parameters and the 6 covariance matrix parameters $\sigma_w^2, \sigma_u^2, \sigma_v^2, \sigma_{wu}, \sigma_{wv}$ and $\sigma_{uv}$. If $q^* + 6 > q + 1$ the model (21) is under-identified. In this case, we can match the moments of the reduced form and the moments of the corresponding ARIMA model given different choices for the correlations. Admissible choices must satisfy certain necessary conditions such as positive definitiveness of the covariance matrix and invertibility of the MA coefficients.

### 4.3 ARIMA(0,2,2) Model

Connections between the BN decomposition, UC models, and some commonly used signal extraction filters can be illustrated using the following ARIMA(0,2,2) reduced form model for $y_t$[7]

$$(1-L)^2 y_t = (1 + \theta_1 L + \theta_2 L^2)\epsilon_t. \tag{23}$$

[7] While we focus on the application to U.S. output, we note that Nelson and Schwert (1977), Pearce (1979) and Stock and Watson (2007) have used similar ARIMA(0,2,1) models to describe the dynamic behavior of the U.S. price level.

The ARIMA(0,2,2) model (23) is the unrestricted reduced form associated with the following UC-ARMA(0,0) model

$$\begin{aligned} y_t &= \tau_t + c_t, \\ \tau_t &= \tau_{t-1} + d_t + w_t,\ w_t \sim \text{iid } (0, \sigma_w^2), \\ d_t &= d_{t-1} + u_t,\ u_t \sim \text{iid } (0, \sigma_u^2), \\ c_t &= v_t,\ v_t \sim \text{iid } (0, \sigma_v^2), \\ \text{cov}(w_t, u_t) &= \sigma_{wu},\ \text{cov}(w_t, v_t) = \sigma_{wv},\ \text{cov}(u_t, v_t) = \sigma_{uv}. \end{aligned} \tag{24}$$

The reduced form of (24) is

$$(1-L)^2 y_t = u_{t-1} + (1-L)w_t + (1-L)^2 v_t, \tag{25}$$

which implies an ARIMA(0,2,2) model. Not all of the parameters of (25) are identified since $q^* + 6 = 6 > q + 1 = 3$. To determine which parameters may be identified, consider the moments of (23)

$$\begin{aligned} \gamma_0 &= 2\sigma_w^2 + \sigma_u^2 + 6\sigma_v^2 + 2\sigma_{wu} + 6\sigma_{wv} + 2\sigma_{uv}, \\ \gamma_1 &= -\sigma_w^2 - 4\sigma_v^2 - \sigma_{wu} - 4\sigma_{wv} - 2\sigma_{uv}, \\ \gamma_2 &= \sigma_v^2 + \sigma_{wv} + \sigma_{uv}. \end{aligned} \tag{26}$$

Notice that the variance of the slope shock is identified since it only depends on the reduced form moments, $\sigma_u^2 = \gamma_0 + 2\gamma_1 + 2\gamma_2$. However, the remaining parameters are not identified without further restrictions. All admissible UC models will satisfy the moment conditions (26), the invertibility of the MA polynomial in (23), the positive definiteness of the covariance matrix of $(w_t, v_t, u_t)'$, and will admit the same trend-cycle decomposition as the BN decomposition based on (23).

The SSOE model imposes three restrictions $|\rho_{wu}| = |\rho_{wv}| = |\rho_{uv}| = 1$, which exactly identifies the remaining parameters. Evaluating $\psi(1)$, $\tilde{\psi}(1)$ and $\tilde{\tilde{\psi}}(L)$ in (19) gives the resulting UC-ARMA(0,0) parameters in terms of the reduced form ARIMA(0,2,2) parameters. It can be shown that the signs of the correlations depend on the signs of $\theta_1$ and $\theta_2$.

The traditional local linear trend model (e.g., Harvey (1989)) sets $\rho_{wu} = \rho_{wv} = \rho_{uv} = 0$. With these restrictions, the remaining variance parameters may be recovered using

$$\begin{bmatrix} \sigma_w^2 \\ \sigma_u^2 \\ \sigma_v^2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 6 \\ -1 & 0 & -4 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{bmatrix}$$

This model will be admissible provided all of the variances are positive. However, as discussed by Harvey (1989), the parameter space for the ARIMA(0,2,2) that supports the local linear trend model is quite restrictive. As a result, the filtered estimates of trend and cycle from the local linear trend model are likely to be different from those computed from the BN decomposition.

As shown by Harvey and Jaeger (1993) and Gomez (1999), the HP filter (e.g., Hodrick and Prescott (1997)) results from a restricted version of the local linear trend model. HP defined the permanent component as the solution to

$$\min_{\tau_1,\ldots,\tau_T} \sum_{t=1}^{T}(y_t - \tau_t)^2 + \lambda \sum_{t=3}^{T}((1-L)^2\tau_t)^2, \tag{27}$$

where $\lambda$ is a smoothness parameter such that large values of $\lambda$ produce smooth trends. For quarterly data HP recommended using $\lambda = 1600$. This solution is equivalent to the smoothed estimate of $\tau_t$ from the local linear trend model with the additional restrictions $\sigma_w^2 = 0$ and $\sigma_v^2/\sigma_u^2 = \lambda$. The resulting cycle from the HP detrended data is equivalent to the smoothed estimate of $c_t$ from the restricted local linear trend model. If $\lambda$ is fixed then there are no parameters to be estimated, and the HP filter imposes two overidentifying restrictions which may be tested against the unrestricted reduced form.

Gomez (2001) showed that certain Butterworth or bandpass filters have more desirable properties than the HP filter and also have UC model representations. For example, consider the Butterworth filters based on the sine and tangent functions as described in Gomez (2001) and denoted BFS and BFT, respectively. These filters depend on a differencing parameter $d$ and a frequency parameter $x_c$. For the BFS, Gomez (2001) showed that $\lambda = [2\sin(x_c/2)]^{-2d}$ where $\lambda$ is the smoothness parameter for the HP filter in (27). For example, $\lambda = 1600$ is equivalent to $x_c = 0.1583$, or a period of 9.2 years. Using the results in the Appendix of Gomez (2001), it can be shown that the parameters of the BFS and BFT when $d = 2$ can be mapped into the parameters of the ARIMA(0,2,2) reduced form. For the BFS, the procedure is as follows. Set $\lambda = [2\sin(x_c/2)]^{-4}$, compute $C = \sin(x_c/2)^2$, $D = 1 - 2C\cos(\pi/2) + C^2$ and $E = \sqrt{(C+\sqrt{D})^2 - 1}$. Then the ARIMA(0,2,2) parameters are determined using

$$\theta_1 = \frac{2(C-\sqrt{D})}{C+\sqrt{D}+E},\ \theta_2 = \frac{C+\sqrt{D}-E}{C+\sqrt{D}+E},\ \sigma_\epsilon^2 = \lambda(C+\sqrt{D}+E)^2.$$

For the BFT, set $\lambda = 1/\tan(x_c/2)^4$, compute $C = \tan(x_c/2)$, $D = \cos(\pi/2)$, and $E = C\sqrt{2(1-D)}$. Then the ARIMA(0,2,2) parameters are determined using

$$\theta_1 = \frac{2(C^2-1)}{C^2+1+E},\ \theta_2 = \frac{C^2+1-E}{C+1+E},\ \sigma_\epsilon^2 = \lambda(C^2+1+E)^2.$$

As a result, the appropriateness of these filters can be tested against the reduced form ARIMA(0,2,2) model.

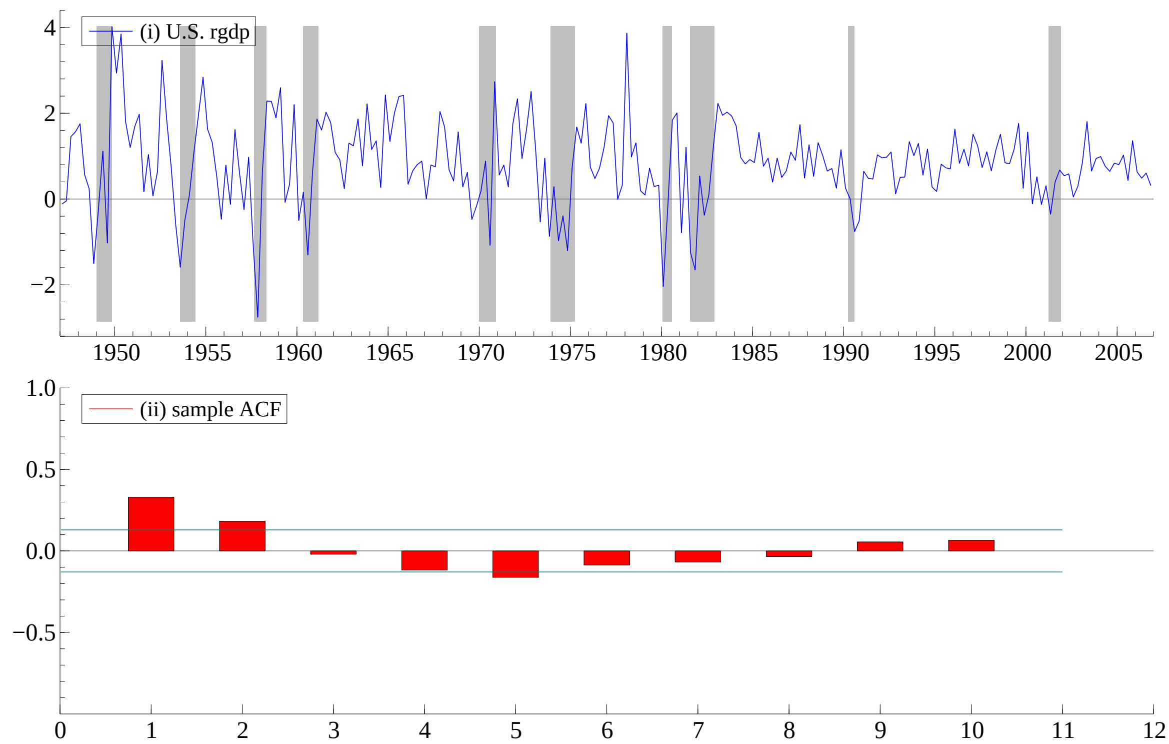


Figure 1: *(i) growth rate of U.S. real GDP from 1947 Q1 to 2007 Q1. Shaded regions represent the NBER recession dates. (ii) sample autocorrelation function.*

# 5 Empirical Illustration using U.S. Real GDP

In this section we illustrate the relationship between the BN decomposition and various UC models using U.S. postwar quarterly real GDP data from 1947:I through 2007:I. The data were obtained from the FRED database at the Federal Reserve Bank of St. Louis. We first consider $I(1)$ models and then $I(2)$ models.

## 5.1 $I(1)$ Models

Figure 1 shows the log quarterly growth rate in percent along with the first 10 sample autocorrelations. The first two autocorrelations are clearly non-zero and there appears to be a cyclical pattern in the higher order autocorrelations. Determining the most appropriate ARIMA(p,1,q) forecasting model for real GDP growth to compute the BN decomposition is a difficult task (e.g., see Campbell and Mankiw (1987)). Standard model selection criteria (e.g., AIC and BIC) tend to select low order ARIMA$(p, 1, q)$ models and the resulting BN cycles tend to be noisy and lack business cycle features. We recommend using the ARIMA$(p, 1, q)$ model that is the unrestricted reduced form associated with the most general UC model to be considered. This

| | ARIMA(2,1,2) | UC models | | | |
|---|---|---|---|---|---|
| | Estimate | SSOE | MNZ | UC0 | Proietti |
| $\mu$ | 0.3453 (0.0718) | - | - | - | - |
| $\phi_1$ | 1.3649 (0.1452) | 1.3649 (0.1386) | 1.3649 (0.1454) | 1.4971 (0.1027) | 1.4851 (0.1150) |
| $\phi_2$ | -0.7819 (0.1747) | -0.7819 (0.1640) | -0.7819 (0.1737) | -0.5687 (0.1067) | -0.5636 (0.1121) |
| $\theta_1$ | -1.1100 (0.2176) | - | - | - | - |
| $\theta_2$ | 0.6225 (0.2268) | - | - | - | - |
| $\sigma_\epsilon$ | 0.9049 (0.0413) | - | - | - | - |
| $d$ | - | 0.8279 (0.0718) | 0.8279 (0.0718) | 0.8309 (0.0405) | 0.8311 (0.0448) |
| $\sigma_\omega$ | - | 1.1118 (0.1283) | 1.1118 (0.1284) | 0.5998 (0.1018) | 0.6710 (0.3051) |
| $\sigma_\upsilon$ | - | 0.5486 (0.2802) | 0.5541 (0.3043) | 0.6293 (0.1141) | 0.5178 (0.6123) |
| $\theta_\upsilon$ | - | 0.0646 (0.2213) | 0 | 0 | 0.2076 (1.3344) |
| $\rho_{\omega\upsilon}$ | - | -1 | -0.9487 (0.1841) | 0 | 0 |
| $\gamma_0$ | 2.1449 | 2.1449 | 2.1449 | | |
| $\gamma_1$ | -1.4747 | -1.4747 | -1.4747 | | |
| $\gamma_2$ | 0.5097 | 0.5097 | 0.5097 | | |
| AR roots | $0.8727 \pm 0.7192i$ | | | | |
| MA roots | $0.8916 \pm 0.9008i$ | | | | |
| log-likelihood | -317.6529 | -317.6529 | -317.6529 | -319.2908 | -319.2572 |

Table 1: *Estimates from the ARIMA(2,1,2) model and the corresponding UC models for U.S. real GDP. Estimated standard errors are in parentheses.*

allows the reduced form model to potentially capture the dynamics implied by the UC models. In addition, any restrictions imposed by the UC models can be directly evaluated by comparing likelihood values. We consider the UC-ARIMA(2,1) model as the most general model and use the unrestricted ARIMA(2,1,2) model to compute the benchmark BN decomposition. Our ensuing analysis for $I(1)$ models focuses on the ARIMA(2,1,2) model because it was studied either implicitly or explicitly by previous authors; see e.g., Watson (1986), Morley, Nelson, and Zivot (2003), and Proietti (2006). To support this choice, we also fit all ARIMA$(p, 1, q)$ models with $p, q \leq 3$ and found that the ARIMA(2,1,2) is preferred by the AIC and the ARIMA(1,1,0) is preferred by the BIC.

Table 1 presents maximum likelihood (ML) estimates, under the assumption of normally

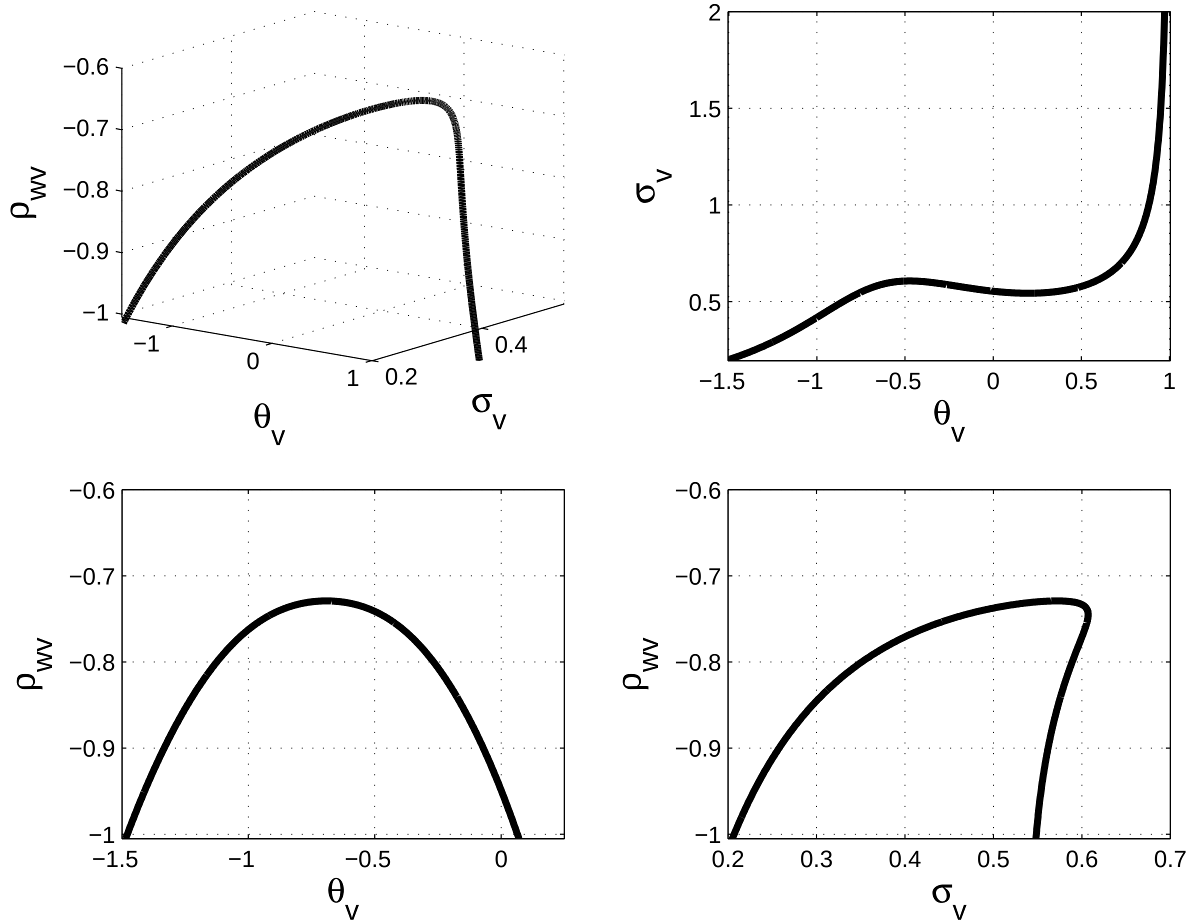


Figure 2: *Combinations of $\theta_v$, $\sigma_v$ and $\rho_{wv}$ that satisfy the moment equations (13) from an ARIMA(2,1,2) model for U.S. real GDP.*

distributed errors, of the ARIMA(2,1,2) model, of the identified UC-ARMA(2,1) models discussed in subsection 3.3, as well as the estimated moments $\gamma_0$, $\gamma_1$ and $\gamma_2$ derived from the MA portion of the estimated reduced form ARIMA(2,1,2) model. All of the estimated parameters in the ARIMA(2,1,2) are significantly different from zero, and the estimated persistence is $\hat{\psi}(1) = \hat{\theta}(1)/\hat{\phi}(1) = 1.229$. The SSOE and MNZ ($\rho_{wv}$ free and $\theta_v = 0$) models have the same likelihood value as the unrestricted reduced form and so are admissible models, as defined above in Section 3.3. In the SSOE model, $\rho_{wv} = -1$ (since $\hat{\psi}(1) > 0$) and $\hat{\theta}_v = -1.4789$ which implies a non-invertible model for the cycle. In the MNZ model, $\hat{\rho}_{wv} = -0.9487$. Although the model from Proietti (2006) ($\theta_v$ free and $\rho_{wv} = 0$) is just identified by the order condition, the lower likelihood value indicates that restrictions imposed by the model are potentially inconsistent with the unrestricted reduced form. The UC0 model ($\rho_{wv} = \theta_v = 0$) has the lowest likelihood and imposes one overidentifying restriction. The likelihood ratio statistic for testing the single overidentifying restriction from the UC0 model is 3.2758, with a $p$-value of 0.0703.

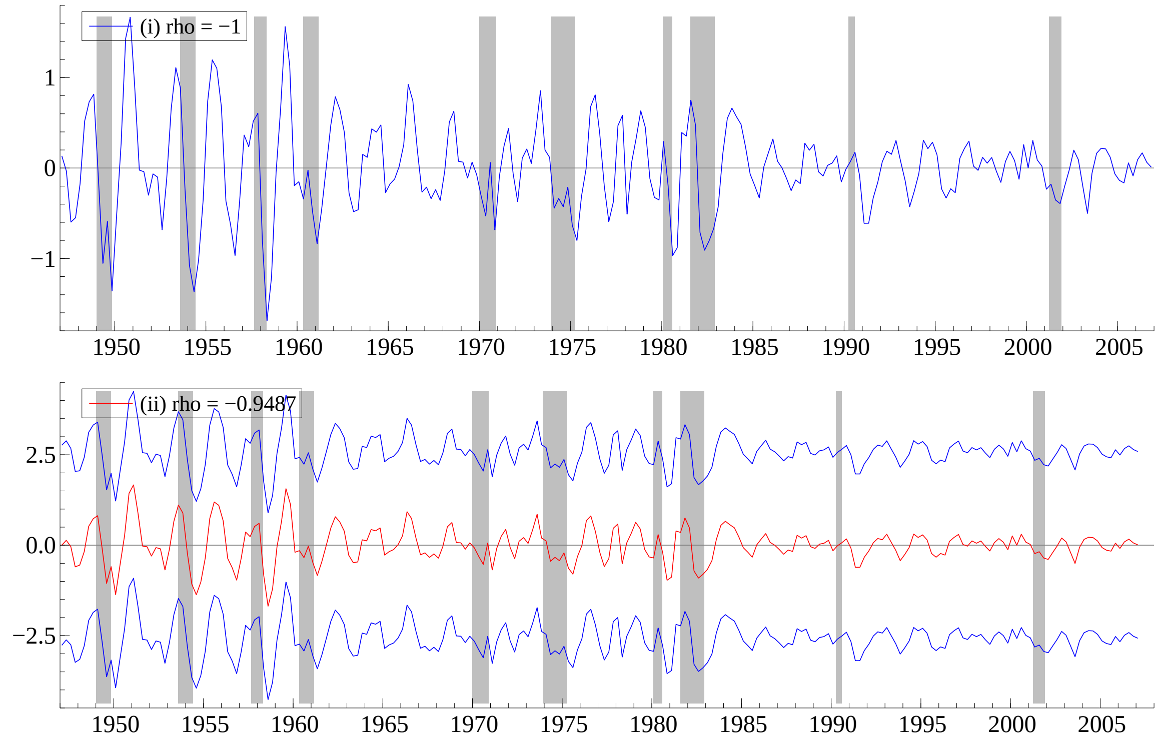


Figure 3: *BN decomposition for U.S. real GDP. Shaded regions represent the NBER recession dates. (i)* $c_t^{BN}$ *from ARIMA(2,1,2) model and* $c_{t|t}$ *from SSOE model with* $\rho_{wv} = -1$. *(ii)* $c_{t|t} \pm 2$ *SE from the MNZ model with* $\hat{\rho}_{wv} = -0.9487$.

The full set of admissible UC-ARMA(2,1) models is illustrated in Figure 2. This set is constructed by finding all of the values of $\rho_{wv}$, $\theta_v$ and $\sigma_v$, with $\sigma_w^2$ fixed according to (14) and the AR parameters fixed at the ARIMA(2,1,2) values, such that the moment conditions (13) are satisfied. The figure shows that the permissible range of correlation values, $\rho_{wv}$, is between about $-0.76$ and $-1$. For these values of $\rho_{wv}$, the signal-to-noise ratio $\sigma_w/\sigma_v$ varies from about 1.82 to 5.5. The set also shows that there is an invertible SSOE model with $\rho_{wv} = -1$ and $\theta_v \approx 0.06$, and this model is shown in Table 1.

The range of permissible values for $\rho_{wv}$ depends on the assumption of a UC-ARMA(2,1) model. Assuming different dynamics for the cycle will generally imply a different range for $\rho_{wv}$. However, some general results can be established. The sign of $\rho_{wv}$ is related to $\psi(1)^2$ computed from the reduced form ARIMA model. For example, Lippi and Reichlin (1992) showed that if $\rho_{wv} = 0$ then it must be the case that $\psi(1)^2 < 1$ , i.e. if the series is persistent then the series cannot be decomposed into uncorrelated components. Recently, Nagakura and Zivot (2006) showed that $\psi(1)^2 \geq 1$ implies $\rho_{wv} < 0$ and that $\rho_{wv} < -\sqrt{1 - V^{-1}}$ provided $V > 1$ where $V = \psi(1)^2\sigma_\epsilon^2/\text{var}(\Delta y_t)$.

The cyclical component $c_t^{BN}$ from the BN decomposition computed from the ARIMA(2,1,2) and the filtered cyclical estimate $c_{t|t}$ computed from the SSOE ($\rho_{wv} = -1$) and MNZ ($\hat{\rho}_{wv} = -0.9487$) models are reported in Figure 3. The BN decomposition is computed from (7) with

$$\alpha_t = \begin{bmatrix} \Delta^2 y_t \\ \phi_2 \Delta^2 y_{t-1} + \theta_1 \epsilon_t + \theta_2 \epsilon_{t-1} + \theta_3 \epsilon_{t-2} \\ \theta_2 \epsilon_t + \theta_3 \epsilon_{t-1} \\ \theta_3 \epsilon_t \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} \phi_1 & 1 & 0 & 0 \\ \phi_2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and $\mathbf{z}' = (1, 0, 0, 0)'$. The BN cycle is identical to the filtered cycles except for the first observation. The key difference between the decompositions is that in the MNZ model the cycle is estimated with error. Accordingly, $c_{t|t}$ is plotted with $\pm 2 \times SE$ bands computed from the output of the Kalman filter which shows considerable uncertainty about the estimated real-time cycle. These confidence bands do not take into account sampling uncertainty associated with the estimated model parameters. Bayesian methods could be used to produce highest posterior density intervals for the filtered cycles that incorporate parameter uncertainty; see the recent contribution by Harvey, Trimbur, and van Dijk (2007).

All of the UC models that match the moments of the data have the same filtered estimates, but they have different smoothed estimates. To illustrate this point, Figure 4 presents smoothed cycle estimates $c_{t|T}$ from UC models with $\rho_{wv} = -1$, $-0.9$, and $-0.8$ that are consistent with the estimated ARIMA(2,1,2) model as well as $c_{t|T}$ values computed from the fitted UC0 and Proietti models given in Table 1. We also computed the smoothed estimates from a second order stochastic trend/cycle model described in Harvey and Trimbur (2003).[8] The smoothed estimates from the SSOE model are identical to the filtered estimates since the filtered estimates are computed with zero mean squared error. The smoothed estimates for models with $\rho = -0.9$ and $\rho = -0.8$ are substantially different from the filtered estimates and attribute much more variability to the cycle. Harvey and Koopman (2000) show that this behavior is due to the asymmetric nature of the weights in the smoothing algorithm from UC models. Negatively correlated shocks will put more weight on future observations than on past observations. The smoothed estimates from the UC0, Proietti and Harvey-Trimbur models, which impose $\rho_{wv} = 0$, are substantially different and show more typical business cycle behavior.

The BN decomposition is often criticized because it is thought that the reduced form ARIMA model is not well suited for capturing the subtle dynamics that may exist in the data. Often low order ARIMA models (e.g., ARIMA(0,1,1) or ARIMA(1,1,0)) are found to be the best fitting models by traditional model selection criteria, and these models produce simplistic cycles by

[8]In particular, we follow Harvey and Trimbur (2003) and fit the balanced form of the second order stochastic trend cycle model fixing the frequency parameter $\lambda_c$ at the estimate from the first order stochastic cycle model. Details of the estimation are given in a technical appendix available from the authors upon request.

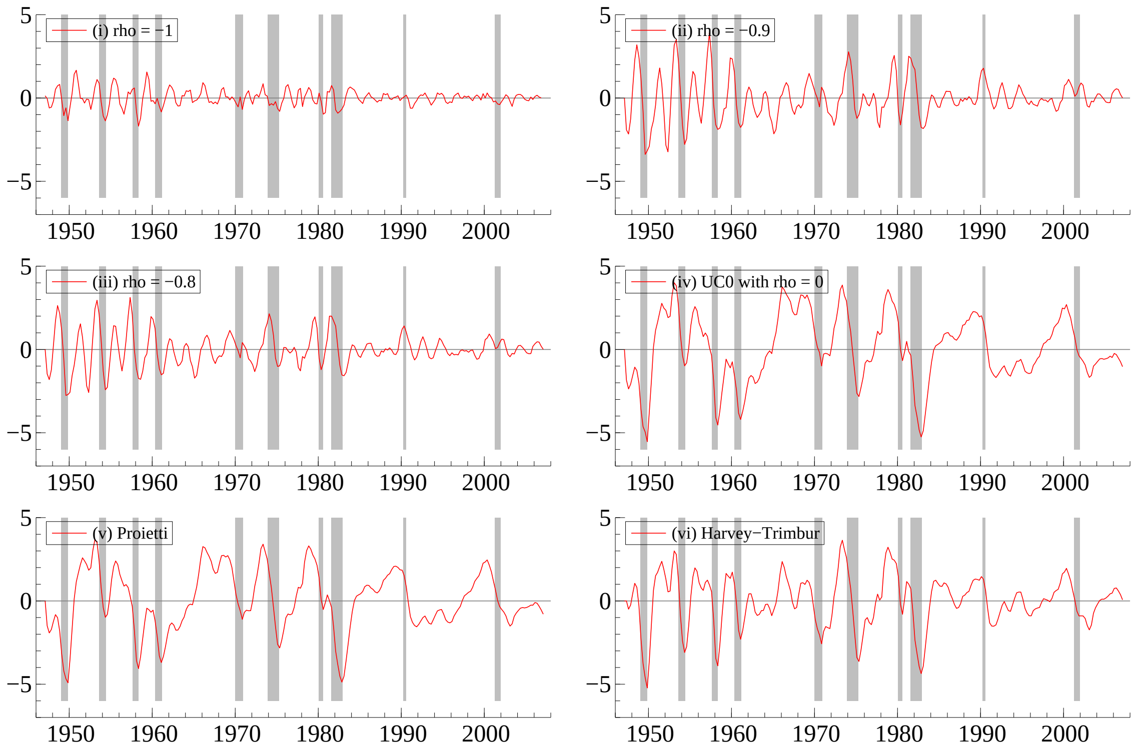


Figure 4: *Smoothed cycles from UC models. The models with* $\rho_{wv} = -1, -0.9$ *and* $-0.8$ *all have the same likelihood value. The UC0 model imposes* $\rho_{wv} = 0$ *and* $\theta_v = 0$*. The Proietti model imposes* $\rho_{wv} = 0$*. The Harvey-Trimbur model has trend and cycle order* $(m, n) = (2, 2)$ *and imposes* $\rho_{wv} = 0$*. (i)* $\rho_{wv} = -1$*; (ii)* $\rho_{wv} = -0.9$*; (iii)* $\rho_{wv} = -0.8$*; (iv) UC0 model; (v) Proietti model; (vi) Harvey-Trimbur model.*

construction. It is argued (e.g., Harvey and Jaeger (1993)) that structural UC models with orthogonal components can be tailored to capture business cycle dynamics better than reduced form ARIMA models. However, an appropriate reduced form ARIMA model can capture the same type of dynamic behavior as a structural UC-ARMA model. To illustrate this point, we simulated data from the fitted UC0 model given in column five of Table 1. We then fit the UC0, MNZ, ARIMA(2,1,2) models, and then computed the filtered cycles from each model. The estimation results are given in Table 2, and the extracted cycles are compared in Figure 5. In terms of estimation, the AR parameters from all models are similar and the estimated correlation from the MNZ model is small and positive. The LR statistic for testing $H_0 : \rho_{wv} = 0$ in the UC model is 0.0336 with a $p$-value of 0.8546.[9] The top panel of Figure 5 shows that $c_t^{BN}$

[9]To evaluate the asymptotic chi-square approximation to the finite sample distribution of the LR statistic, we repeated the simulation from the UC0 model $1,000$ times and found that the empirical rejection frequencies of the 1% and 5% tests were 2.8% and 10.8%, respectively. We also found that the empirical distribution of the estimated correlation, $\hat{\rho}_{wv}$, from the MNZ model had a bimodal distribution with one mode near zero and another mode near unity with little mass near minus one.

| | | ARIMA(2,1,2) | | UC models | |
|---|---|---|---|---|---|
| | True value | Estimate | Implied UC Est | MNZ | UC0 |
| $\mu$ | - | 0.0411 (0.0474) | - | - | - |
| $\phi_1$ | 1.4971 | 1.6042 (0.1395) | 1.6042 | 1.6042 (0.1363) | 1.5960 (0.1348) |
| $\phi_2$ | -0.5687 | -0.6551 (0.1358) | -0.6551 | -0.6551 (0.1344) | -0.6487 (0.1337) |
| $\theta_1$ | - | -1.3623 (0.1701) | - | - | - |
| $\theta_2$ | - | 0.3990 (0.1659) | - | - | - |
| $\sigma_\epsilon$ | - | 0.9721 (0.0444) | - | - | - |
| $d$ | 0.8309 | - | 0.8090 | 0.8090 (0.0474) | 0.8095 (0.0505) |
| $\sigma_\omega$ | 0.5998 | - | 0.7030 | 0.7030 (0.2305) | 0.7359 (0.0946) |
| $\sigma_\upsilon$ | 0.6293 | - | 0.3151 | 0.4727 (0.2681) | 0.5266 (0.1454) |
| $\theta_\upsilon$ | 0 | - | 1.6235 | 0 | 0 |
| $\rho_{\omega\upsilon}$ | 0 | - | -1 | 0.2449 (1.1709) | 0 |
| log-likelihood | | -334.8434 | - | -334.8434 | -334.8602 |

Table 2: *Estimates from the ARIMA(2,1,2) model and the corresponding UC models on simulated data from the UC0 model. Estimated standard errors are in parentheses.*

from the ARIMA(2,1,2) is very close to $c_{t|t}$ from the UC0 model. The bottom panel of Figure 5 compares $c_t^{BN}$ from an underspecified ARIMA(1,1,0) to $c_{t|t}$ and illustrates the typical empirical result that the BN decomposition produces small noisy cycle estimates. Several points are worth noting here. First, the fact that the filtered cycle from the UC0 model is not roughly equal to the BN cycle when applied to U.S. real GDP indicates that either the restrictions are incorrect or that the UC0 model is misspecified. If the model was correct, they should be roughly the same. If the restrictions were valid, the BN decomposition would provide the cycle. Finally, contrary to popular belief, the BN decomposition can produce smooth cycles. It does not produce smooth cycles, however, for the U.S. real GDP data as illustrated by Figure 3.

## 5.2 $I(2)$ Models

We focus initially on the ARIMA(0,2,2) model because it is the reduced form associated with the local linear trend model, the HP filter, and some of the BFS and BFT filters. Table 3 shows MLEs of the ARIMA(0,2,2) model, an identified UC-ARMA(0,0) model with $\rho_{wv} = -0.9$ and

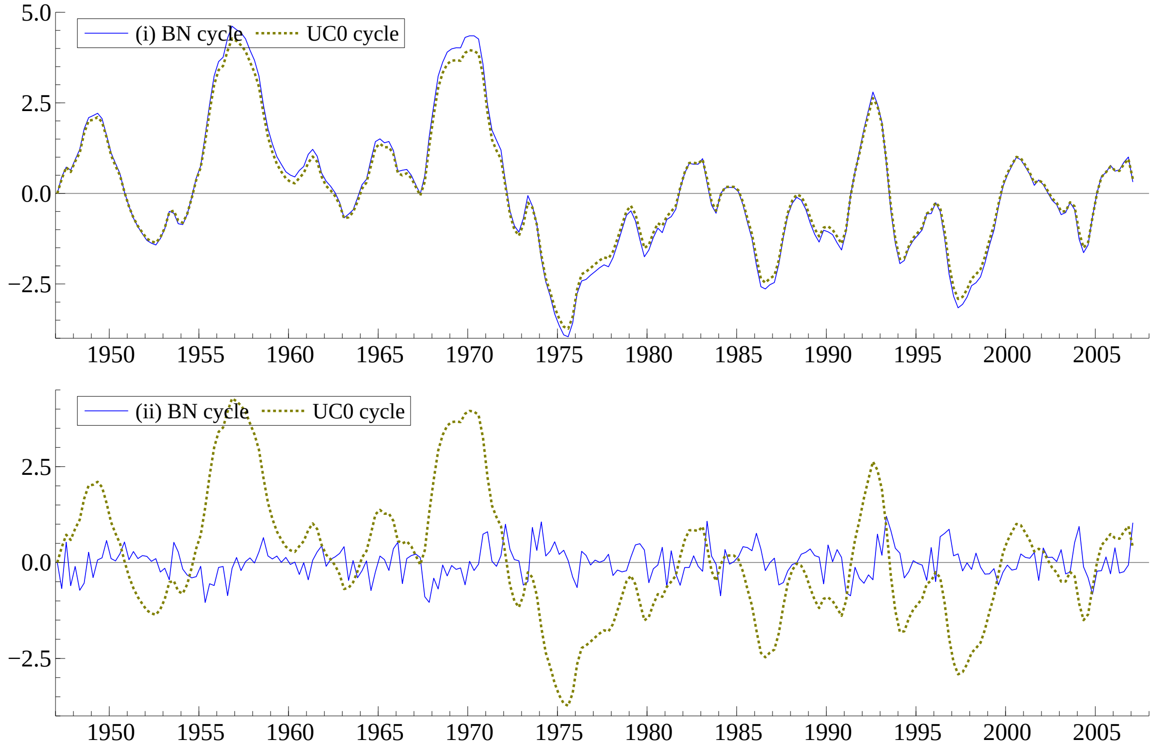


Figure 5: *(i) Estimated BN cycle,* $c_t^{BN}$*, and UC0 filtered cycle,* $c_{t|t}$*, from models fit to simulated data from the UC0 model. (ii) Estimated BN cycle,* $c_t^{BN}$*, from a misspecified ARIMA(1,1,0) model compared to the correctly specified UC0 filtered cycle,* $c_{t|t}$*.*

$\rho_{wu} = \rho_{uv} = 0$, the local linear trend model, and implied estimates for some other models. One of the MA roots in the ARIMA(0,2,2) is almost unity which indicates potential overdifferencing of the data. This is equivalently reflected by the near zero estimate of $\sigma_u$ in the UC-ARMA(0,0) models. Figure 5.2 shows the combinations of $\rho_{wv}$, $\rho_{wu}$ and $\rho_{uv}$ in the UC-ARMA(0,0) model that produce the same likelihood value as the ARIMA(0,2,2) model. The figure shows that the correlation between the trend and cycle shocks, $\rho_{wv}$, cannot take values higher than about -0.8, whereas $\rho_{wu}$ and $\rho_{uv}$ appear not to be restricted. This result explains the lower likelihood value for the local linear trend model which imposes zero correlation between all pairs of shocks. Column four in Table 3 gives the ARIMA(0,2,2) model estimates implied by the HP filter restrictions on the local linear trend model. The restrictions implied by the HP filter are clearly rejected by the data. Figure 7 shows the BN cycle computed from (20) with

$$\alpha_t = \begin{bmatrix} \Delta y_t \\ \epsilon_t \\ \epsilon_{t-1} \end{bmatrix}, \ \mathbf{T} = \begin{bmatrix} 0 & \theta_1 & \theta_2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

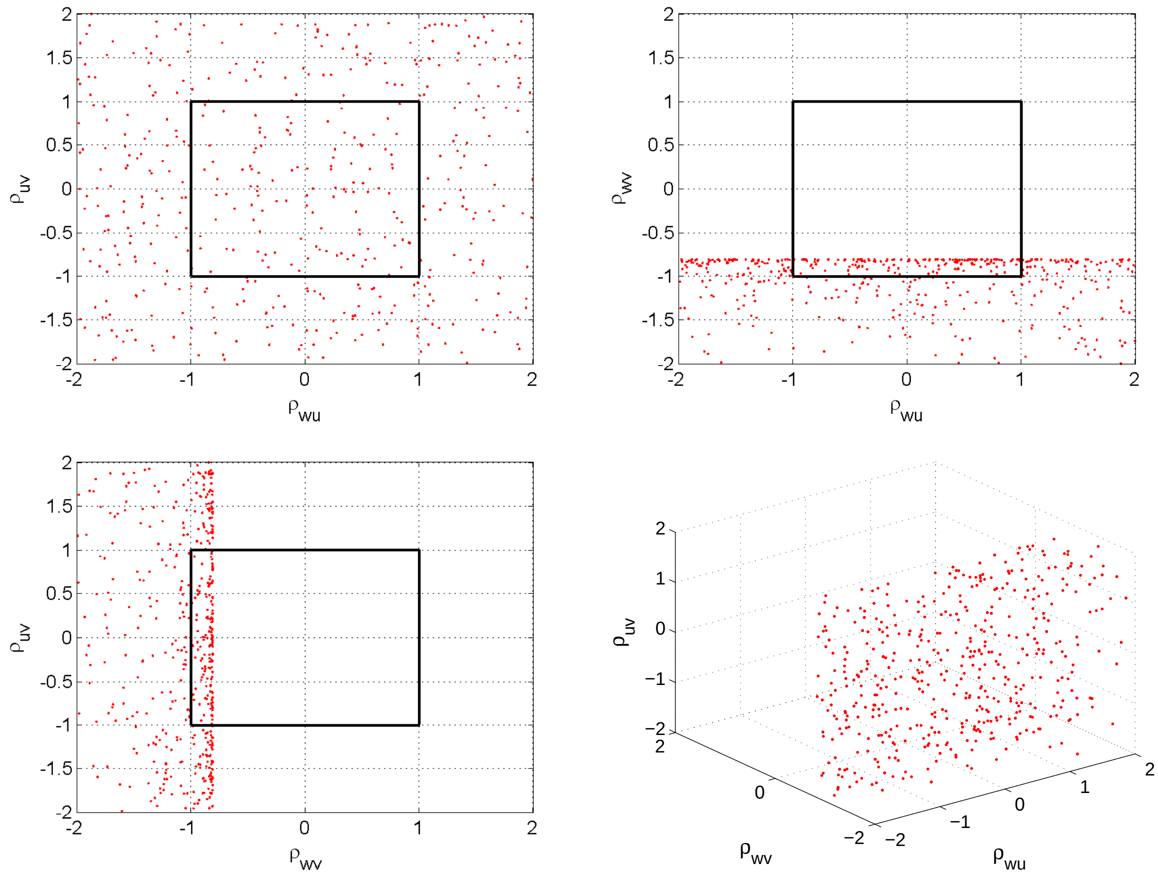


Figure 6: *Combinations of $\rho_{wv}$, $\rho_{wu}$ and $\rho_{uv}$ that satisfy the moment equations (26) from an ARIMA(0,2,2) model for U.S. real GDP.*

where $z = (1, 0, 0, 0)'$. The BN cycle is smaller in magnitude and noisier than the corresponding cycle from the ARIMA(2,1,2) model.

A more realistic model is the ARIMA(2,2,3), which is the reduced form associated with the stochastic slope UC-ARMA(2,0) and UC-ARMA(2,1) models used by Harvey (1985), Clark (1987) and Harvey and Jaeger (1993), respectively. In the UC-ARMA(2,1) model, not all of the parameters are identified since $q^* + 6 = 7 > q + 1 = 4$. Identification within the Clark model is discussed in Oh and Zivot (2006), who map the parameters of the UC model to its reduced-form ARIMA representation.[10] They show that after identifying the trend shock and choosing the covariances the remaining parameters of the model are determined. In our empirical analysis, we set the MA parameter of the cycle to zero in the UC-ARMA(2,1) model so that one of the correlations may be estimated when the other two correlations are restricted. For example, if in addition we impose $\sigma_{wu} = \sigma_{uv} = 0$, then the remaining parameters $(\sigma_v^2, \sigma_w^2, \sigma_u^2, \sigma_{wv})'$ can be calculated using the results from Oh and Zivot (2006). Table 4 reports the MLEs for the

[10]These results are summarized in a technical appendix that is available from the authors upon request.

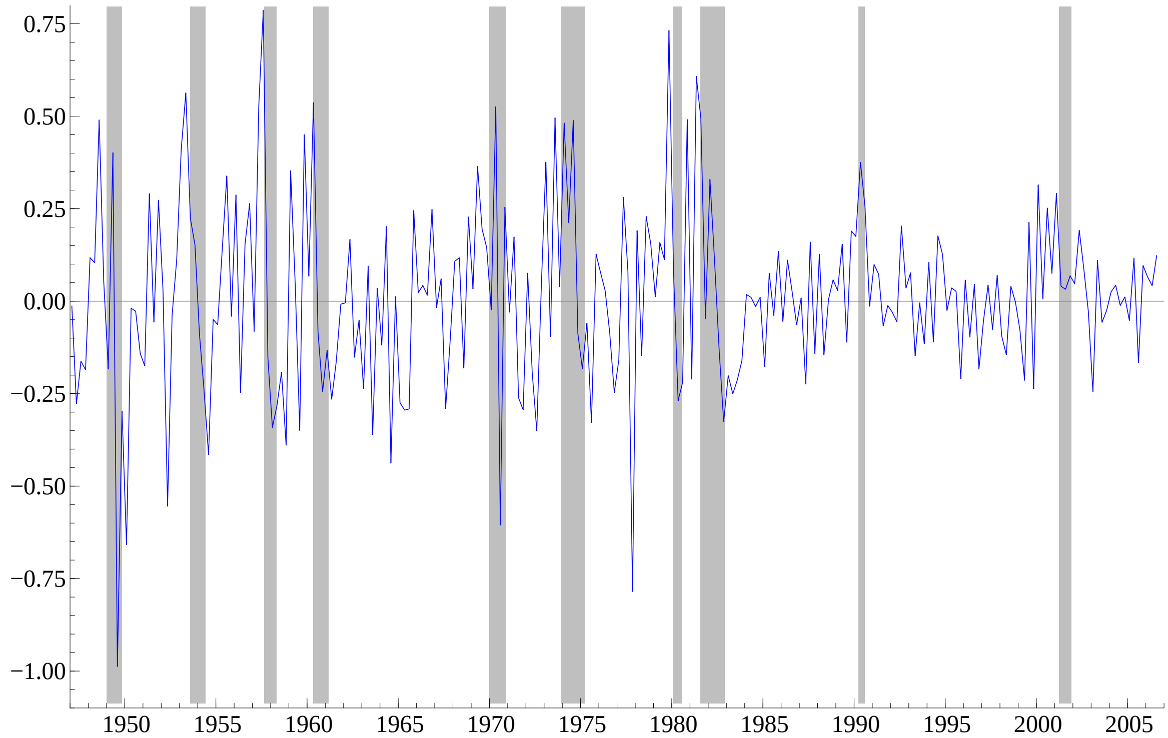


Figure 7: *BN decomposition from the ARIMA(0,2,2) model for U.S. real GDP.*

ARIMA(2,2,3). As with the ARIMA(0,2,2) model, there is a near unit moving average root in the ARIMA(2,2,3) model. The MLEs of the stochastic slope UC-ARMA(2,0) reported in Table 4 show $\hat{\sigma}_u \approx 0$ and $\hat{\rho}_{wv} = -0.9455$ and are very close to the MLEs of the MNZ model reported in Table 1. More importantly, the BN cycle and filtered cycles from the admissible UC-ARMA(2,1) models are essentially identical to those from the ARIMA(2,1,2) model.

## 6 Conclusion

The purpose of this paper was to clarify the relationship between the BN decomposition derived from an ARIMA model and trend/cycle decompositions computed from UC-ARMA models with correlated shocks and to understand what information the BN decomposition can provide. The ARIMA model used to compute the BN decomposition does convey some information about the correlation between permanent and transitory shocks in certain UC-ARMA models. It also imparts information on the fit of certain UC-ARMA models and non-model based filters, whose trends are random walks.

We emphasize that the BN and related permanent-transitory decompositions are confronted by more subtle technical issues than has been appreciated in the literature. Researchers must be

| | ARIMA(0,2,2) | | UC models | | |
|---|---|---|---|---|---|
| | Estimate | HP-ARIMA | SSOE | Two Source | Two Source |
| $\theta_1$ | -0.7396 (0.0592) | -1.777 | - | - | |
| $\theta_2$ | -0.2604 (0.0534) | 0.7994 | - | - | |
| $\sigma_\epsilon$ | 0.9391 (0.0447) | 44.7258 | - | - | |
| $\sigma_\omega$ | - | - | 1.1836 (0.0736) | 1.1836 (0.0736) | 0.9810 (0.0451) |
| $\sigma_u$ | - | - | 0.0000 (0.0005) | 0.0000 (0.0007) | 0.0066 (0.0177) |
| $\sigma_\upsilon$ | - | - | 0.2445 (0.0510) | 0.3000 (0.0729) | 0.0000 (0.0015) |
| $\rho_{\omega u}$ | - | - | 1 | 0 | 0 |
| $\rho_{\omega\upsilon}$ | - | - | -1 | -0.9 | 0 |
| $\rho_{u\upsilon}$ | - | - | -1 | 0 | 0 |
| $\gamma_0$ | 1.4241 | 9595.93 | 1.4241 | 1.4241 | |
| $\gamma_1$ | -0.4824 | -6396.62 | -0.4824 | -1.4747 | |
| $\gamma_2$ | -0.2296 | 1599.16 | -0.2296 | -0.2296 | |
| MA roots | $1.00 + 0i$ $-3.84 + 0i$ | $1.111\pm$ $0.12467i$ | | | |
| log-likelihood | -328.488 | -1132.94 | -328.488 | -328.488 | -339.311 |

Table 3: *Estimates from the ARIMA(0,2,2) model, the ARIMA(0,2,2) model implied by the HP filter with $\lambda = 1600$, and three UC models with the correlation fixed at different values. All models are estimated on U.S. real GDP. Estimated standard errors are in parentheses.*

careful about drawing strong conclusions about the nature of business cycles extracted by UC model decompositions because commonly used identifying assumptions can have large impacts on estimated trends and cycles. We believe that the BN decomposition serves as a useful benchmark for permanent-transitory decompositions in which the permanent component is a pure random walk. However, when comparing the BN decomposition to permanent-transitory decompositions from UC models with correlated shocks, the ARIMA model used to compute the BN decomposition should correspond with the unrestricted reduced form associated with the UC model. In this way both types of decompositions are based on the same underlying forecasting model and any overidentifying restrictions imposed by the UC model can be formally tested.

Univariate models have a limited ability to construct good forecasting models to obtain the BN cycle. Moving to multivariate forecasting models is one way to produce more realistic decompositions using the BN decomposition. Results in Evans and Reichlin (1994) indicate that better forecasting models will reduce the variability of the forecast error of the trend. This reduces the variability of the BN trend and consequently produces smoother cycles. We believe

| | ARIMA(2,2,3) | | UC models | |
|---|---|---|---|---|
| | Estimate | (standard error) | SSOE | Multi Sources |
| $\phi_1$ | 1.3535 | (0.1490) | 1.3535 | 1.3535 |
| $\phi_2$ | -0.7677 | (0.1714) | -0.7677 | -0.7677 |
| $\theta_1$ | -2.0915 | (0.2149) | - | - |
| $\theta_2$ | 1.6985 | (0.4208) | - | - |
| $\theta_3$ | -0.6069 | (0.2144) | - | - |
| $\sigma_\epsilon$ | 0.9069 | (0.0415) | - | - |
| $\sigma_w$ | - | - | 1.1280 | 1.1281 |
| $\sigma_u$ | - | - | 0.0001 | 0.0001 |
| $\sigma_v$ | - | - | 0.2213 | 0.5836 |
| $\theta_v$ | - | - | -1.4263 | - |
| $\rho_{wu}$ | - | - | 1 | 0 |
| $\rho_{wv}$ | - | - | -1 | -0.9455 |
| $\rho_{uv}$ | - | - | -1 | 0 |
| AR roots | 0.8815 ± 0.7249i | | | |
| MA roots | 1.0001, 0.8993 ± 0.9159i | | | |
| Log likelihood | -318.4413 | | -318.4413 | -318.4413 |

Table 4: *Estimates from the ARIMA(2,2,3) model and the corresponding UC models for U. S. real GDP.*

that researchers who are interested in using the BN decomposition should focus on developing better forecasting models using multivariate series that contain more information.

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