Bayesian persuasion and information design

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Abstract
A school may improve its students’ job outcomes if it issues only coarse grades. Google can reduce congestion on roads by giving drivers noisy information about the state of traffic. A social planner might raise everyone’s welfare by providing only partial information about solvency of banks. All of this can happen even when everyone is fully rational and understands the data generating process. Each of these examples raises the question of what is the (socially or privately) optimal information that should be revealed. In this article, I review the literature that answers such questions.

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1 What is Bayesian persuasion?

An implicit premise in most of economics is that behavior is driven by three factors: preferences, technology, and information. Consequently, if we wish to influence economic outcomes, there are three broad ways of doing so. The most straightforward one is to change the (induced) preferences over actions via incentives, e.g., contingent payments, threat of violence, or supply of complementary goods. A second way to engender an outcome is to make it easier for a decision maker to achieve it, i.e., to improve the relevant technology.\(^1\) This article is about the third path – persuasion — which we can define as influencing behavior via provision of information (Kamenica and Gentzkow 2011).\(^2\) Throughout, the focus will be on standard decision makers who understand how information is generated and react to information in a rational (Bayesian) manner; hence the moniker “Bayesian persuasion.” Bayesian persuasion is also referred to as information design,\(^3\) and a comparison with mechanism design is instructive (Bergemann and Morris 2016a, Taneva 2016). In mechanism design problems, who knows what is given, and the designer influences the outcome by selecting the game that the agents will play. In information design problems, the game that the agents play is given, and the designer influences the outcome by specifying who gets to knows what.

Bayesian persuasion can alternatively be seen as a communication protocol, in the tradition of cheap talk (Crawford and Sobel 1982), verifiable message (Grossman 1981; Milgrom 1981), and signaling games (Spence 1973). Relative to these other models of communication, Bayesian persuasion endows the sender with more commitment power. In the most common formulation, Bayesian persuasion allows the sender to commit to send any distribution of messages as a function of the state of the world. This “full commitment” formulation, however, yields equilibrium outcomes that are identical to those that arise in an alternative model where the sender publicly chooses how

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\(^1\)In the pages of this journal (Kamenica 2012), I have argued that some of the methods used in nudging and choice architecture (Thaler and Sunstein 2008) can be seen as technological interventions. For instance, teaching drivers to open their car door with their right hand when they are about to exit the car (which forces them to swivel and see whether there is a bicycle coming) is a technological innovation that allows drivers way to reduce the chance they cause an accident.

\(^2\)Of course, there is no reason why these three methods of influencing behavior must be used in isolation. A promising area for future research might be to explore how to best combine incentives, choice architecture, and information provision. Li (2017) adds transfers to the simplest two-action two-state model of persuasion. Lewis and Sappington (1994) study a monopolist who chooses both what price to charge and what information to provide to consumers about their valuations for the firm’s product.

\(^3\)I use the two terms interchangeably, but the former is probably used more when the designer is one of the players in the game and there is a single receiver while the latter is used more when the designer is a social planner or there are multiple, interacting receivers.
much information she will privately observe, and then strategically decides how much of this private information to reveal via verifiable messages (Gentzkow and Kamenica 2017a). This equivalence result may not be especially important, however, since in most applications, the full commitment formulation corresponds more closely to the real-world institution being analyzed.

In this review, I focus exclusively on environments where the information designer is motivated by the desire to influence the actions of those who observe the signal realization. A separate line of research, surveyed in this volume by Bergemann and Bonatti (2018), studies markets where a seller designs information in order to sell it. Bergemann et al. (2018), for example, consider a seller who offers a menu of signals to a buyer with unknown private information, while Kastl et al. (2018) analyze sale of information to competitive firms about their suppliers’ marginal costs.

Bayesian persuasion is also closely related to the literature on Bayes Correlated Equilibria (Bergemann and Morris 2013). Bayes correlated equilibria take as given a basic game (a set of players, a set of feasible actions for each player, and players’ payoffs as a function of the state of the world and the actions taken) and describe the set of all possible outcomes that could arise (as Bayes Nash equilibria) regardless of what each player knows (about the state and about what the other players know). One benefit of deriving this set is that it provides a prediction about the outcome of the basic game that is robust to the uncertainty about what the players engaged in the game know, but the set of Bayes Correlated equilibria, by definition, also coincides with the set of outcomes that can be attained through information design. Thus, a Bayesian persuasion problem is equivalent to a problem of selecting an optimal Bayes correlated equilibrium given an objective function. The relationship between these two literatures is also discussed in a complementary survey by Bergemann and Morris (2018).

The research on Bayesian persuasion has developed in two main directions. One strand of research, which is the primary focus of this article, is more abstract and seeks to extend the basic model in various dimensions and/or develop new approaches for solving the designer’s optimization problem. The other strand is more applied and aims to understand or improve real-world institutions via information design. Research in this second strand includes applications to (in no particular order) financial sector stress tests (Goldstein and Leitner 2018, Inostroza and Pavan 2018, Orlov et al. 2018a), grading in schools (Ostrovsky and Schwarz 2010, Boleslavsky and Cotton
2015), employee feedback (Smolin 2017, Habibi 2018), law enforcement deployment (Lazear 2006, Rabinovich et al. 2015, Hernandez and Neeman 2017), censorship (Gehlbach and Sonin 2014), entertainment (Ely et al. 2015), financial over-the-counter markets (Duffie et al. 2017), voter coalition formation (Alonso and Camara 2016a), research procurement (Yoder 2018), contests (Feng and Lu 2016, Zhang and Zhou 2016), medical testing (Schweizer and Szech 2018), medical research (Kolotilin 2015), matching platforms (Romanyuk and Smolin 2018), price discrimination (Bergemann, Brooks, and Morris 2013), financing (Szydlowski 2016), insurance (Garcia and Tsur 2018), transparency in organizations (Jehiel 2015), and routing software (Kremer et al. 2014, Das et al. 2017). The breadth of these topics reveals the wide applicability of Bayesian persuasion models.

2 The model and its interpretations

2.1 The basic model

The basic Bayesian persuasion model (of the static variety with a single sender and a single receiver) takes the following form. A player called Receiver (she) has a utility function \( u(a, \omega) \) that depends on her action \( a \in A \) and the state of the world \( \omega \in \Omega \). Another player, Sender, aka the information designer, (he), has a utility function \( v(a, \omega) \) that depends on Receiver’s action and the state of the world. In some applications, we might think of Sender not as a player but rather as the social planner with social welfare function \( v \). Sender and Receiver share a common prior \( \mu_0 \) on \( \Omega \).

The key object in Bayesian persuasion models is the thing that Sender chooses, which goes by many names, including signal, signal structure, information structure, experiment, Blackwell experiment, or the data generating process. I will call it signal. Let \( S \) be some “sufficiently large” set of signal realizations.\(^4\) In the basic model we are considering here, it suffices to assume that \( |S| \geq \min\{|A|, |\Omega|\} \), i.e., that the signal realization is not smaller than both the state space and the action space.\(^5\) The notation I will use here further assumes that \( S \) is finite. A signal is a map

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\(^4\)An alternative to taking \( S \) as given is to let Sender choose it in the process of selecting his signal. Doing so, however, creates an economically irrelevant if philosophically delightful complication that prevents us, because of Russell’s paradox, from saying that Sender optimizes over the “set of all signals.”

\(^5\)With multiple receivers, things are a little more tricky. Suppose there are \( n \) receivers, each with some action space \( A_i \). If we consider the private signals environment (cf: the discussion in Section 4.1) and equilibrium selection is resolved in favor of the designer, then it suffices to set \( |S| \geq |A_1 \times \ldots \times A_n| \) since, by the revelation-principle-type argument, we can restrict attention to signal realizations that are action recommendations. But, analogously to the fact that the
from the state to the distribution over signal realizations, \( \pi : \Omega \to \Delta (S) \).\(^6\) In other words, a signal specifies the statistical relationship between truth \( (\omega \in \Omega) \) and data \( (s \in S) \). Another, equivalent, way to define a signal is as a joint distribution over states and signal realizations, \( \pi \in \Delta (\Omega \times S) \), with the requirement that the marginal distribution over \( \Omega \) coincides with the prior. Let \( \Pi \) denote the set of all signals. The timing is as follows.

1. Sender chooses a signal \( \pi \).
2. Receiver observes which signal was chosen.
3. Nature chooses \( \omega \) according to \( \mu_0 \).
4. Nature chooses \( s \) according to \( \pi (\omega) \).
5. Receiver observes the realized \( s \).
6. Receiver takes action \( a \).

Receiver’s behavior is mechanical. Given her knowledge of \( \pi \), she uses Bayes’ rule to update her belief from the prior \( \mu_0 \) to the posterior \( \mu_\pi (\omega \mid s) = \frac{\pi(s \mid \omega)\mu_0(\omega)}{\sum_{\omega'} \pi(s \mid \omega')\mu_0(\omega')} \), and then she simply selects an action \( a^* (\mu_\pi (\cdot \mid s)) \) that maximizes \( E_{\omega \sim \mu_\pi (\cdot \mid s)} u(a, \omega) \).\(^7\) Given this behavior by Receiver, Sender solves

\[
\max_{\pi \in \Pi} E_{\omega \sim \mu_0} E_{s \sim \pi(\omega)} v(a^* (\mu_\pi (\cdot \mid s)), \omega).
\] (1)

The optimization problem in Equation (1) looks somewhat daunting for a couple of reasons. First, \( \Pi \) is a pretty large set. Second, the choice of \( \pi \) influences Sender’s payoff both by changing the distribution of signal realizations and by changing the action induced by a given signal realization.

\(^6\)Given a set \( X \), \( \Delta (X) \) denotes the set of all probability distributions on \( X \).

\(^7\)If Receiver does not have “duplicate” actions that yield the same utility in each state, she can only be indifferent across multiple actions at a set of beliefs that has lower dimensionality than \( \Delta (\Omega) \). In any equilibrium that leads her to hold such beliefs with positive probability, it will necessarily be the case that she breaks her indifference in Sender’s favor.
Much of the progress in the Bayesian persuasion literature has relied on recasting Sender’s optimization problem in a more approachable way. I discuss those reformulations and environments where they are applicable in Section 3.

2.2 Interpretations

The motivating example in Kamenica and Gentzkow (2011) considers a courtroom setting where Sender is a prosecutor and Receiver is a judge. The state of the world is the guilt of the defendant. We can think of the choice of the signal as consisting of forensic tests, questions asked to witnesses, etc. The prosecutor can ask for a DNA test, but does not have to; he can call an expert witness, but does not have to; etc. The assumption that the judge necessarily sees all of the evidence uncovered by the prosecution might seem problematic, but anything unfavorable to the accused the prosecutor will willingly share (the prosecutor prefers conviction) and any exculpatory evidence he is required to reveal by law. (In Brady v. Maryland,\textsuperscript{8} the Supreme Court of the United States ruled that a prosecutor violates the Due Process Clause of the Fourteenth Amendment when he fails to disclose evidence favorable to the accused.) The courtroom example might not seem to match the model because an actual prosecutor can conduct additional investigations (i.e., generate another signal) once he observes the realization of the initial signal. It turns out, however, that allowing for such additional signals does not change the game since any contingent information-gathering can be fully baked into the choice of the initial signal. Finally, it is implausible that a prosecutor could generate arbitrarily informative signals, i.e., discover the whole truth. This concern again does not diminish the applicability of the model, however, since we can redefine the state to be the realization of the most informative signal that Sender can generate; then the assumption that a fully informative signal is feasible becomes vacuous. Thus, the Bayesian persuasion approach can be useful even when the environment does not match the literal description of the model very well.

The assumptions underlying the commitment assumption vary substantially across applications. Consider grades in schools. There is a large population of students with varying ability. We denote the ability of a particular student by $\omega$. The distribution of ability in the student population, $\mu_0$, is known to everyone. Now, it may be the case – in contrast with the literal description of the model

\textsuperscript{8}373 US 83, 87 (1963).
that each student’s ability is directly observed by the school, perhaps even before the school chooses its grading policy (i.e., its signal). This will not matter. What is important is that the way in which the school assigns grades (the school’s grading policy) is publicly known. A grading policy maps each ability level \( \omega \) into a (potentially randomized) grade. The idea that the mapping from ability to grades can be stochastic might reflect a policy that takes attributes orthogonal to ability, e.g., attendance, into consideration when assigning grades. The set of possible grades here is \( S \) and \( \pi(s|\omega) \) is the probability that a student of ability \( \omega \) obtains grade \( s \). Receiver is the labor market at large, and the placement of a student whose ability is perceived to be \( \mu \) is some \( a^*(\mu) \). The school values placement \( a \) of a student of ability \( \omega \) at \( v(a, \omega) \); Equation (1) then implies that the school chooses its grading policy to maximize its average valuation of the placement of its students.

For yet another interpretation of what constitutes a signal, consider deployment of law enforcement. Lazear (2006) introduces the following model. There are \( Z \) miles of road. A driver can either speed or obey the speed limit on each mile. Speeding generates utility \( V \) per mile and the fine for speeding if caught is \( K > V \). There are \( G < Z \) police and each policeman can patrol one mile of road. The police wish to minimize the number of miles over which drivers speed. To map this environment to the Bayesian persuasion model we let \( \omega \in \Omega = \{0, 1\} \) denote the presence of a policeman on a given mile and hence we have \( \mu_0 = \frac{G}{Z} \). The set of signal realizations corresponds to the miles of the road: \( S = \{1, \ldots, Z\} \). The police is Sender and the driver is Receiver. A signal here represents the consistency or predictability of the patrolling strategy. A patrolling strategy induces a joint distribution over \( \Omega \) and \( S \), i.e., over the presence of a policeman and a mile of the road. The case where police randomly choose where to set up their speed traps each day corresponds to the completely uninformative signal \( \pi \) (with \( \omega \) and \( s \) uncorrelated) and \( \mu_\pi(\cdot|s) = \frac{G}{Z} \) on every mile of the road \( s \). This policy induces drivers to speed everywhere if \( V > \frac{GK}{Z} \) and speed nowhere otherwise. The case where the police always patrol the exact same locations corresponds to the

\[\text{If the grading policy is chosen after seeing the abilities, it is important that the student population is large, so there is no uncertainty about the realized distribution of ability, } \mu_0.\]

\[\text{If } \Omega \text{ is uncountable and } A \text{ is finite, Sender is not harmed by restricting } \Pi \text{ to deterministic, i.e., partitional, signals, } \pi : \Omega \to S.\]

\[\text{Of course, the impact of grades on labor market is more complex – placement of a student with a given apparent ability might depend on the grading policy at other schools, schools might differ in the quality of their education, etc. Papers that focus on applications of information design to grading develop extensions that take these issues into account (e.g., Ostrovsky and Schwarz 2010, Boleslavsky and Cotton 2015).}\]
completely informative signal $\bar{\pi}$ (with $\omega$ and $s$ perfectly correlated) and $\mu_{\bar{\pi}} (\cdot|s)$ is either zero or one: one on the $G_Z$ share of miles that are consistently patrolled and zero on the remainder of the road. When speeding is sufficiently appealing (i.e., $V > G_K$), neither of these policies will be optimal – a partially informative signal induced by an imperfect consistency in the location of the speed traps will be the best.

These three disparate examples are meant to illustrate the variety of situations where some party has control over the information that another party will observe. Other applications involve yet other interpretations of how a signal is generated. Best and Quigley (2017), for instance, identify the circumstances under which reputation-building motives (e.g., of a long-run Sender facing a sequence of short-run receiver) allow Sender to generate arbitrary signals.

3 Sender’s optimization problem

3.1 Concavification

Let us return to the optimization problem in Equation (1). We aim to reformulate this problem in a way that will, at least in some circumstances, be more manageable than brute force optimization over all possible signals. Our reformulation relies on two steps.

First, note that whatever signal Sender chose, his expected payoff is fully determined by Receiver’s posterior. In particular, if Receiver holds belief $\mu$, Sender’s expected utility is

$$\hat{v} (\mu) = \mathbb{E}_{\omega \sim \mu} v (a^* (\mu) ; \omega).$$

Note that $\mu$ enters $\hat{v}$ in two ways – it both affects Receiver’s action and impacts Sender’s expected utility from that action. The latter channel relies on the fact that Receiver’s beliefs, being formed by Bayes’ rule, are expected to be well calibrated: if we look at all of the cases when Receiver had some belief $\mu$, we should expect to find that the state was $\omega$ in $\mu (\omega)$ share of those cases.

The second step draws on the relationship between signals and distributions of beliefs. When Sender chooses some signal $\pi$, each signal realization $s$ leads to some posterior $\mu_{\pi} (\cdot|s)$. From the ex-ante perspective, however, before the realization of the signal, we can think of the choice of $\pi$ as inducing a distribution of posteriors. We use notation $\tau = \langle \pi \rangle$ to indicate that a distribution of
posteriors $\tau$ is induced by signal $\pi$.\textsuperscript{12} We say a distribution of posteriors $\tau$ is Bayes-plausible if it equals the prior in expectation, i.e., $\mathbb{E}_{\mu \sim \tau} \mu = \mu_0$. By the law of iterated expectations, we know that every distribution of posteriors induced by a signal is Bayes-plausible. Moreover, Bayes-plausibility is the only restriction on induced distributions of posteriors: for every Bayes-plausible $\tau$ there is a $\pi \in \Pi$ such that $\tau = \langle \pi \rangle$ (Kamenica and Gentzkow 2011). The proof of this observation is constructive and allows to easily find a signal that induces any given Bayes-plausible $\tau$.\textsuperscript{13}

Combining these two observations allows us to reformulate Sender’s problem as

$$
\max_{\tau} \mathbb{E}_{\mu \sim \tau} \hat{v}(\mu) \tag{2}
$$

$$
s.t. \ E_{\mu \sim \tau} \mu = \mu_0
$$

This formulation has a nice geometric interpretation. Consider Figure 1. Sender’s payoff from a signal that induces some beliefs $\{\mu_l, \mu_h\}$ is the height of the point at the intersection of the line segment connecting $\hat{v}(\mu_l)$ and $\hat{v}(\mu_h)$ and the vertical line from $\mu_0$. This observation makes it clear that the optimal binary signal is the one that induces the distribution of posteriors with the support on $\{\mu_l^*, \mu_h^*\}$. Moreover, it is easy to see that a signal with more than two realizations cannot improve Sender’s payoff.

We can generalize this line of reasoning with the notion of concavification. A concavification of $\hat{v}$ is the smallest concave function everywhere greater than $\hat{v}$. It is indicated as the thick red line in Figure 1. The concavification of $\hat{v}$ evaluated at $\mu_0$ equals $\max \{z| (\mu_0, z) \in \co(\hat{v})\}$ where $\co(\hat{v})$ denotes the convex hull of the graph of $\hat{v}$. Therefore, since the set of Sender’s payoffs across all signals is $\{z| (\mu_0, z) \in \co(\hat{v})\}$,\textsuperscript{14} Sender’s payoff under the optimal signal is precisely the concavification of $\hat{v}$ evaluated at the prior.

When there are only two or three states of the world, we can plot $\hat{v}$, and then the concavification approach allows us to simply “read off” the optimal distribution of posteriors. It is then straightforward to identify a signal that induces this distribution of posteriors. When the state space is larger, the concavification approach can still be used to derive some qualitative features.

\textsuperscript{12}Algebraically, a signal $\pi$ induces a distribution of posteriors $\tau$ if $\tau(\mu) = \sum_{s: \pi(s|\omega) = \mu} \sum_{\omega' \in \Omega} \pi(\omega'|s) \mu_0(\omega')$.

\textsuperscript{13}Specifically, associate every $\mu$ in the support of $\tau$ with some signal realization $s$, and let $\tau(s|\omega) = \frac{\mu(s|\omega) \tau(\mu)}{\mu_0(\omega)}$.

\textsuperscript{14}Note that $\{z| (\mu_0, z) \in \co(\hat{v})\} = \{\mathbb{E}_{\mu \sim \tau} \hat{v}(\mu) | \mathbb{E}_{\mu \sim \tau} \mu = \mu_0\}$. 

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of the optimal signal, but it does not immediately deliver the solution to Sender’s problem. When I discuss the various extensions of the basic model in Section 4, I will note the cases where the concavification approach remains useful.

A few words on the intellectual history of information design and the concavification approach might be in order. At the height of the cold war, from 1966 to 1968, a group of economists was retained by the United States Arms Control and Disarmament Agency to study game-theoretic aspects of arms control and disarmament. This group included, among others, Robert Aumann, Michael Maschler, and Richard Stearns. These three authors wrote a number of reports that focused on a particular concern, namely “that the negotiating strategy used by the Americans in a series of arms control conferences might implicitly send signals to the Russians about the nature of the US arsenal.”\textsuperscript{15} To study this issue, Aumann and Maschler (1966) consider the following model. There are two players, called (I)nformed and (U)ninformed. There are two zero-sum games, $G_A$ or $G_B$ with identical action spaces. With probability $\mu_0$, the players will repeatedly play $G_A$ ad infinitum, and with the complementary probability, they will repeatedly play $G_B$ ad infinitum. Before they start playing, player I learns which game they will be playing. After each period, player U observes

\textsuperscript{15}This quote is from the Preface (p. xiii) of the book that collected and reorganized the original technical reports (Aumann and Maschler 1995).
the action of player I, but she does not observe her payoff nor which game they had played (and will continue to play). Both players seek to maximize their undiscounted average payoff. While this situation might seem quite distinct from the Bayesian persuasion model, an important step in characterizing the equilibria of these repeated games of incomplete information involves solving an optimization problem that is analogous to the one in Equation (2). Aumann and Maschler (1966) developed the concavification approach for solving such problems. For over half a century, however, Aumann and Maschler’s contribution played a part in the analysis of repeated games but did not spur the development of of the applications mentioned in Section 1. Formulating a model with an explicit information design step seems to have been important for stimulating further research on the topic.

Another early, though more recent, contribution to information design is Brocas and Carrillo (2007). They analyze an environment that fits squarely within the framework of the model in Section 2.1, with one exception: Sender, rather than choosing any signal whatsoever from Π, selects how many i.i.d. draws of a fixed (binary) signal to generate. Even though it might seem easier to optimize over this one-dimensional set, the fact that the concavification approach is only applicable when Sender can choose from all possible signals has led to the unrestricted model being more widely used.

### 3.2 An important special case

As mentioned earlier, the concavification approach delivers a “visual solution” to the information design problem only when the state space is small, with two or three elements. Another special case that has received much attention is the case where the state space is large, in fact uncountable, but Sender’s payoff depends only on the mean of Receiver’s posterior.

Suppose that \( \Omega = [0, 1] \) and that \( a^*(\mu) = f(\mathbb{E}_\mu \omega) \) for some function \( f \).\(^{17}\) Then, there exists a function \( \tilde{v} \) that captures Sender’s payoff as a function of Receiver’s posterior mean, i.e., \( \tilde{v}(\mathbb{E}_\mu \omega) = \tilde{v}(\mu) \). Even though \( \mu \) is infinite-dimensional, \( \tilde{v} \) can be plotted on a piece of paper. We now might hope that the concavification of \( \tilde{v} \) yields the solution to Sender’s optimization problem, but

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\(^{16}\)In work contemporaneous with Kamenica and Gentzkow (2011), Rayo and Segal (2010) analyze a specific case of the baseline model from Section 2.1 with an added twist that Receiver has some private information about her preferences.

\(^{17}\)It only matters that \( \Omega \) is a compact subset of \( \mathbb{R} \). I assume it is a unit interval to simplify notation.
unfortunately this turns out not to be the case. The problem is that, even though we can induce every distribution of posteriors whose average value is the prior, this is not the case for every distribution of posterior means whose average value is the prior mean.

So, we cannot solve these cases via concavification, but some progress has been made using other routes. Gentzkow and Kamenica (2016) derive a way to solve these problems when the action space is small. The first step is to transform each signal into a convex function. Let $G_\pi$ denote the cumulative distribution function of the posterior means induced by $\pi$. Then, let $c_\pi$ be the integral of $G_\pi$, i.e., $c_\pi(x) = \int_0^x G_\pi(t) \, dt$. Note that $c_\pi$ is an increasing convex function. Let me illustrate the construction of $c_\pi$ with some examples. Suppose the prior $\mu_0$ is uniform on $[0,1]$. First, consider a completely uninformative signal $\pi$. This signal induced a degenerate distribution of posterior means always equal to $\frac{1}{2}$, so $G_\pi$ is a step function equal to 0 below $\frac{1}{2}$ and equal to 1 above. Hence, $c_\pi$ is flat on $[0,\frac{1}{2}]$ and then linearly increasing on $[\frac{1}{2},1]$ with a slope of 1. At the other extreme, consider a fully informative signal $\bar{\pi}$. Under this signal, the posterior mean equals the true state, so the distribution of posterior means is uniform, $G_{\bar{\pi}}$ is linear, and $c_{\bar{\pi}}$ is quadratic: $c_{\bar{\pi}}(x) = \frac{x^2}{2}$. Finally, consider a partitional signal $P$ that gives a distinct signal realization depending on whether the state is above or below $\frac{1}{2}$. Then, $G_P$ is a step function and $c_P$ is piecewise-linear. Figure 2 depicts these functions. Note that signal $\pi$ is more informative than $\pi'$ if and only if $c_\pi \geq c_{\pi'}$ (Blackwell 1951, Blackwell and Girschick 1954). That implies that for any signal $\pi$, we have $c_{\bar{\pi}} \geq c_\pi \geq c_{\pi'}$.

The concavification is not entirely useless here. If the concavification of $\tilde{v}$ is strictly above $\tilde{v}$ at the prior mean, we can at least conclude that providing no information is strictly suboptimal (Kamenica and Gentzkow 2011).

Throughout this article, when I say more informative, less informative, or comparable, I mean in terms of Blackwell order.
In fact, every convex function that is “sandwiched” between $c_{\pi}$ and $c_{\bar{\pi}}$ is induced by some signal (Gentzkow and Kamenica 2016). Hence, we can represent the set of all signals $\Pi$ as the set of convex functions in the shaded region of Figure 3. Then, if Receiver’s action space is finite (so that $\tilde{v}$ is a step function), we can solve for the optimal signal by working inside this new space. For a simplest example, suppose that prior is uniform and $\tilde{v}$ is a step function equal to 0 below some cutoff $\gamma$ and 1 above it.\(^{20}\) Then, Sender’s payoff from $\pi$ is the likelihood that Receiver’s induced posterior mean is $\gamma$ or above, which corresponds to $1 - G_\pi(\gamma)$ or $1 - \bar{c}_\pi(\gamma^-)$.\(^{21}\) So, Sender wants to minimize the left derivative of $c_\pi$ at $\gamma$, which is attained by $c_\pi^*$ indicated in Figure 3.

The special case where Sender’s payoff depends only on the mean of Receiver’s posterior is also studied by Ivanov (2015), Kolotilin (2018), and Dworczak and Martini (2018).\(^{22}\) Ivanov (2015) considers an extension of the basic model that allows for Sender’s payoff to also depend on the rank of the realized posterior mean among the posterior means that might be generated. Kolotilin (2018) and Dworczak and Martini (2018) draw on linear programming methods and represent Sender’s problem as a consumer-choice situation where Sender purchases posterior means using the prior as

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\(^{20}\)This extremely simple case can also be solved directly, but I use it rather than a case that cannot be solved using other methods for the simplest illustration of how this method works.

\(^{21}\)Expression $c_\pi'(\gamma^-)$ denotes the left derivative of $c_\pi$ at $\gamma$.

\(^{22}\)Ostrovsky and Schwarz (2010) similarly consider this special case, but in a setting with multiple senders discussed in Section 4.2.
her endowment.\textsuperscript{23} Kolotilin (2018) assumes the action space is binary, but allows for Receiver to have private information about his preferences. Dworczak and Martini (2018) allow for a general action space and thus for a $\tilde{v}$ that can take any shape and derive a simple way to verify whether a given signal is optimal.\textsuperscript{24}

### 3.3 Computational methods

Despite the aforementioned progress in techniques for deriving optimal signals in various settings, many information design problems of applied interest may not be amenable to analytic solutions. This naturally leads to a question of whether we might be able to employ computational methods instead. A recent literature in computer science and algorithmic game theory delivers a number of results on this question. Dughmi (2017) provides an excellent survey.

Dughmi and Xu (2016) analyze algorithmic approaches to the Bayesian persuasion problem with a single Receiver we have been discussing thus far. They deliver two positive results and one negative one. The first positive result concerns environments with strong symmetry across actions and states. In particular, given any action $a \in A$, we can think of both $u(a, \omega)$ and $v(a, \omega)$ as real-valued random variables. Now, suppose that for any pair of distinct actions $a$ and $a'$ we have that $u(a, \omega)$ and $u(a', \omega)$ are i.i.d. and $v(a, \omega)$ and $v(a', \omega)$ are i.i.d. In this case, there is a polynomial time algorithm for computing the optimal signal. The second positive result concerns approximately optimal signals and approximately rational behavior by Receiver. If we only require that $\mathbb{E}_\mu u(a^*(\mu), \omega) \geq \mathbb{E}_\mu u(a', \omega) - \epsilon$ for all $a' \in A$, then we can find a signal that delivers Sender a payoff within $\epsilon$ of the maximal one using an algorithm that is polynomial in $|A|$ and $\frac{1}{\epsilon}$. The negative result states that, without the aforementioned simplifications, the general problem of computing Sender’s maximal payoff is $\#P$-hard (which basically means it is very hard).\textsuperscript{25} The literature has

\textsuperscript{23}Moreover, Dworczak and Martini (2018) show that this reformulation does not require the assumption that Sender’s preferences only depend on the posterior mean.

\textsuperscript{24}These and other papers also deliver results on circumstances under which optimal signals fall in a particular class. Ivanov (2015) and Dworczak and Martini (2018) provide necessary and sufficient conditions for optimality of monotone partitional signals. Kolotilin (2018) provides necessary and sufficient condition for optimality of a signal that reveals moderate types and hides extreme types. Mensch (2018) establishes necessary and sufficient condition for optimality of monotone partitions without assuming that Sender’s payoff depends only on the mean of Receiver’s posterior. Guo and Shmaya (2017) show that when the action space is binary and Receiver has private information about the state, the optimal signal has a particular structure they term a nested interval.

\textsuperscript{25}I suspect economists are likely to have encountered complexity classes $P$ and $NP$, but might be less familiar with $\#P$. Roughly, $\#P$ is the set of counting problems associated with decision problems in $NP$. For example, asking whether a traveling salesman can visit a set of cities and return home while traversing less than $X$ miles is in class
also explored computation of Bayesian persuasion problems with multiple receivers, but I postpone discussion of those until Section 4.1.

4 Extensions

There are three main extensions of the basic model: (i) multiple receivers, (ii) multiple senders, and (iii) dynamic environments. Before discussing each of these three, I will briefly mention a few other generalizations that have been considered thus far.

A natural and easy extension is to allow for the possibility that Receiver might have some private information. This information might be about her own preferences (e.g., Rayo and Segal 2010, Kolotilin 2018) or about the state of the world (e.g., Guo and Shmaya 2017). In both cases, the literature has also examined the possibility that Sender elicits information from Receiver prior to releasing the signal (Kolotilin et al. 2017, Li and Shi 2017).

Matyskova (2018) analyzes situations where Receiver has no private information at the outset, but can gather additional costly information after observing the realization of Sender’s signal. She shows that it is without loss of generality to focus on cases where Receiver never actually gathers information on the equilibrium path. The threat of additional information gathering of course weakly harms Sender and can be

$NP$, so asking how many distinct paths allow the salesman to visit a set of of cities and return home while traversing less than $X$ miles is in class $\#P$. Every problem in $\#P$ is at least as hard as the corresponding problem in $NP$.

It is perhaps incoherent at first glance. Since Receiver’s utility is $u(a, \omega)$, uncertainty about the state is the same thing as uncertainty about preferences over actions. What we mean when we say that Receiver has private information about her preferences is that the actual state space is some $\Theta = \Omega \times T$, the prior over $\Omega$ is independent of the prior over $T$, Receiver’s utility $u(a, \theta)$ depends on the full state $\theta \in \Theta$, but Sender’s signal is a function of $\omega \in \Omega$ only, i.e., it cannot depend on $t \in T$.

Both varieties of the models allow a reformulation to the concavification-friendly form of Equation (2); we simply redefine $\hat{v}$ by integrating private information out (cf: Kamenica and Gentzkow 2011).

Kolotilin et al. (2017) consider the case where Receiver’s private information is about her preferences and Sender observes Receiver’s reported type. Li and Shi (2017) consider the case where Receiver (a buyer) has private information about the state, while Sender (a seller) conditions her signal on Receiver’s report, but does not observe the report prior to setting the price. Kolotilin et al. (2017) show that Sender does not benefit from the ability to elicit Receiver’s private information. Li and Shi (2017) show that discriminatory disclosure (sending different signals for different reported types) dominates full disclosure.
beneficial or harmful for Receiver.

Alonso and Camara (2016b) analyze Bayesian persuasion in situations where Sender and Receiver have heterogeneous priors. They establish a striking result that, as long as Receiver’s action depends only on her posterior mean, Sender generically benefits from his ability to generate a signal. This result is particularly surprising because we might think that, in the case where Receiver’s prior differs from Sender’s in the direction that benefits Sender, the last thing that Sender would want to do is to generate information and thus sober up Receiver who is about to (mistakenly from Sender’s perspective) take an action that Sender likes. This intuition indeed helps us understand why Sender will not want to generate a fully informative signal, but II is rich enough that even when the audience is mistaken in a favorable direction, Sender can benefit from some manipulation of beliefs.

Gentzkow and Kamenica (2014) extend the basic model by making the signals potentially costly for Sender; if Sender generates signal π, his overall payoff if v(a, ω) − c(π) for some cost function π. This model of costly persuasion provides a bridge between the literatures on Bayesian persuasion and rational inattention (e.g., Sims 2003). Both of those literatures consider special cases of costly persuasion: Bayesian persuasion assumes c(π) = 0 while rational inattention assumes u = v. The assumption that c(π) has a posterior-separable form, i.e., that it can be expressed as c(π) = E[µ∼⟨π⟩][H(µ0) − H(µ)] for some function H, is nearly-universal in the literature on rational

32Their model admits the concavification approach since there is a function that maps Receiver’s posterior into Sender’s posterior that does not depend on the signal or the signal realization that caused the updating of the beliefs.

33Kosterina (2018) considers a model with heterogeneous priors and an ambiguity averse Sender. Specifically, Sender is uncertain about Receiver’s prior and thinks that, whatever signal he chooses, Receiver’s prior will turn out to be the one that minimizes his payoff. She characterizes the optimal signal and shows it has features that are qualitative different from the optimum in the standard set-up. In particular, under ambiguity aversion, it is no longer the case that when Receiver takes an action that Sender dislikes, she is certain of the state (cf: Proposition 4 in Kamenica and Gentzkow 2011). The ambiguity averse model can also be reinterpreted as a model where Sender is trying to convince a group of receivers with heterogeneous priors to unanimously approve his proposal (cf: Section 4.1).

34Note that this is a weaker condition that the environment analyzed in Section 3.2 since Sender’s preferences area allowed to depend on the state.

35We can also think of situations where Sender’s choice of signals is restricted to some strict subset of II (e.g., Brocas and Carrillo 2007) as special cases of the costly model, but then we typically lose the natural monotonicity of the cost function, i.e., the property that if π is more informative than π’, then c(π) ≥ c(π’). Ichihashi (2017) explores how restrictions of Sender’s signal impact Receiver’s welfare. For the case of binary actions, he fully characterizes which Sender’s choice set maximizes Receiver’s utility. Tsakas and Tsakas (2018) explore limitations on signals that arise from exogenous noise in signal realizations. Perez-Richet and Skreta (2018) analyze an environment with a given signal π where Sender chooses a manipulation strategy, i.e., the probability with which in state ω signal realizations will be generated according to π(·|ω’) rather than π(·|ω). The manipulation strategy is observable to Receiver so this model is equivalent to a modification where Sender has access to a smaller set of signals; Sender cannot generate every signal that is less informative than π, however, so the set of available signals has a non-trivial structure.
inattention (where $H$ is typically assumed to be entropy)\textsuperscript{36} and has important implications for the tractability of the costly persuasion model.\textsuperscript{37} When $c$ is posterior-separable, Sender’s optimization problem, which is difficult if expressed analogously to Equation (1):

$$\max_{\pi \in \Pi} \left( \mathbb{E}_{\omega \sim \mu_0} \mathbb{E}_{s \sim \pi(\omega)} v \left( a^* \left( \mu_\pi \left( \cdot | s \right) \right), \omega \right) \right) - c \left( \pi \right)$$

can be reformulated\textsuperscript{38} analogously to Equation (2)

$$\max_{\tau} \mathbb{E}_{\mu \sim \tau} \left[ \tilde{v} \left( \mu \right) + H \left( \mu \right) \right]$$

s.t. $\mathbb{E}_{\mu \sim \tau} \mu = \mu_0$.

Thus, the concavification approach can be used to derive the optimal signal. When $c$ is not posterior-separable, such reformulation is not possible.

Finally,\textsuperscript{39} an important class of extensions try to weaken the assumption of Sender’s commitment.\textsuperscript{40} Min (2017), Frechette et al. (2017), and Lipnowski et al. (2018) all consider a model where, after publicly choosing the signal, with some probability Sender can change the signal realization.\textsuperscript{41} Lipnowski et al. (2018) provide the strongest results for this setting.\textsuperscript{42} They show that, as long as Sender’s preferences are state-independent, a geometric approach similar to concavification can be used to derive the equilibrium.\textsuperscript{43} They observe that Receiver might be better off when Sender

\textsuperscript{36}See the discussion in Caplin et al. (2017) and Frankel and Kamenica (2018).

\textsuperscript{37}Function $H$ is referred to as a measure of uncertainty, so posterior-separable functions assume the cost of a signal is the expected reduction in uncertainty. Gentzkow and Kamenica (2014) show that the baseline level of uncertainty need not be $H \left( \mu_0 \right)$ but can be taken to be the uncertainty at any benchmark belief. Posterior-separable function satisfy monotonicity (cf. Footnote 35) as long as $H$ is concave. All measures of uncertainty with a decision-theoretic foundation are indeed concave.

\textsuperscript{38}I’ve suppressed the constant term $-H \left( \mu_0 \right)$ since it does not affect the optimization problem.

\textsuperscript{39}A curious pair of papers modifies the Bayesian persuasion setting by replacing the standard formulation of uncertainty with quantum uncertainty (Danilov and Lambert-Mogiliansky 2018a, 2018b). My knowledge of quantum mechanics is insufficient to understand these papers, but from what I can gather, the idea is not to expand Sender’s toolkit (say by allowing him to quantum-entangle some signal realizations), but rather to assume a different (as far as I can tell, an irrational) model of Receiver’s belief formation process based on an analogue to quantum systems.

\textsuperscript{40}Perez-Richet (2014) and Hedlund (2017) consider settings where Sender chooses the signal after observing some information about the state.

\textsuperscript{41}These models are related to the literature on lying costs (e.g., Kartik 2009 and Guo and Shmaya 2018).

\textsuperscript{42}Min (2017) shows that commitment helps both Sender and Receiver in Crawford and Sobel’s (1982) uniform-quadratic setting. Frechette et al. (2017) conduct laboratory experiments and analyze the extent to which subjects’ behavior corresponds to equilibrium predictions. Other explorations of Bayesian persuasion models in the lab include Nguyen (2017) and Au and Li (2018).

\textsuperscript{43}Lipnowski and Ravid (2017) show that in a model of pure cheap talk, if Sender’s preferences do not depend on the state, his maximal equilibrium payoff is determined by the quasiconcavification of $\tilde{v}$. (Recall that concavification is
becomes less credible (i.e., has a higher chance of altering the signal realization).\footnote{This echoes the aforementioned results by Ichihashi (2017) on the potential benefit to Receiver from restricting the set of signals available to Sender.} Sender is obviously always harmed by a reduction in his credibility. Moreover, there is generically a key level of credibility at which Sender’s payoff changes discontinuously. Under an additional assumption,\footnote{The assumption is that Sender does not need to rule out any states to obtain his preferred action.} however, full credibility is not such a threshold; this means that a small departure from the baseline model (i.e., a small chance that Sender can manipulate the signal realization) can only lead to a small reduction in Sender’s payoff.

\subsection{Multiple receivers}

By far the most important extension of the basic model is to allow for multiple receivers. Space constraints prevent me from giving this case as much attention as it deserves, but the consequences of this omission are lessened by the excellent survey by Bergemann and Morris (2018) that focuses on this case.

There are two classes of multiple receiver environments that are as easy to analyze as the single receiver case. One is when Sender can only send public signals observed by all receivers. In this case, we simply need to reinterpret \( a^* (\mu) \) as the vector of (potentially mixed) equilibrium actions when receivers share the posterior \( \mu \), and then the analysis proceeds as with a single receiver.\footnote{The fact that actions might be mixed means that we need to set \(|S| \geq |\Omega|\) (cf: discussion in Footnote 5). This does mean that when the state space is large, the problem can be difficult. Bhaskar et al. (2016) establish that if two receivers engage in a zero-sum game and Sender wishes to maximize a weighted sum of the receivers’ payoffs, computing the optimal public signal is \( NP \)-hard. On the more positive side, Cheng et al. (2015) introduce a class a games where approximately optimal signals can be computed in polynomial time.} The other case is when Sender can send separate signals to each receiver, each receiver cares only about her own action, and Sender’s utility is separable across receivers’ actions. Then, Sender can determine the optimal signal receiver-by-receiver and faces a set of independent problems of single-receiver variety. If Sender can send separate signals to each receiver and (i) a receiver’s optimal action depends on what other receivers do, or (ii) Sender’s utility is not separable across receiver’s actions,\footnote{Babichenko and Barman (2016) provide results on the computational difficulty of the model where receiver’s optimal action does not depend on actions of other receivers, but Sender’s payoff is not separable across receiver’s} the problem becomes significantly more difficult and cannot be expressed in the form

\( \text{the smallest concave function above } \hat{v}. \text{ Quasiconcavification is the smallest quasiconcave function above } \hat{v}. \text{) Lipnowski et al. (2018) extend this result to Bayesian persuasion with limited commitment by showing that Sender’s payoff can be characterized by an object that combines the concavification and the quasiconcavification. (Alas, it is not a simple convex combination of the two.)} \)

\( \text{NP} \)-hard. On the more positive side, Cheng et al. (2015) introduce a class a games where approximately optimal signals can be computed in polynomial time.
equivalent to the single receiver case. Of course, the most general case allows for both (i) and (ii).

One approach to Sender’s optimization problem in the general version takes a two-step approach (Bergemann and Morris 2016a, Taneva 2017). The first step is to characterize the set of all outcomes (joint distributions over the state and receivers’ actions) that can be attained by some signal. This set of outcomes is referred to as the set of Bayes correlated equilibria (Bergemann and Morris 2013, 2016b). Identifying the set of Bayes correlated equilibria is often of interest even in the absence of the second, optimization stage. This set describes the outcomes we should think might arise in a game if we are agnostic about the information obtained by the players. Moreover, it pinpoints the worst-case scenario for a given game and thus aids informationally-robust mechanism design (cf: Bergemann et al. 2017b, 2018; Brooks and Du 2018).

To turn the analysis of Bayes correlated equilibria into information design, we simply select the best equilibrium given some objective function. A different approach to the general class of information design problems with multiple receivers is provided by Mathevet et al. (2018). They propose a procedure where we first identify the optimal purely private signal for every prior $\mu$. Denoting the payoff from such a signal by $\hat{v}(\mu)$ if the common prior were $\mu$, we can compute the optimal public signal via the concavification of $\hat{v}$ as in Equation (2). Combining the optimal public signal with the optimal private signals (which are contingent on the realization of the public signal) then yields the overall optimum. Unlike the approach via Bayes correlated equilibria, the method proposed by Mathevet actions. Suppose that $\Omega = \{0, 1\}$, each receiver’s action space $A_i = \{0, 1\}$, and Sender’s payoff $V(Q)$ depends on the set of receivers $Q$ who take action $a_i = 1$. This set-up includes voting models such as Alonso and Camara (2016a). Babichenko and Barman (2016) show that if Sender’s utility is submodular (in the sense that $V(Q \cup \{i\}) - V(Q) \geq V(Q' \cup \{i\}) - V(Q')$ for every $Q' \subset Q$ and every receiver $i$), then computing the optimal signal is $NP$-hard, but computing a signal that yields a payoff at least $(1 - \frac{1}{e})$ times the maximal one can be done in polynomial time.

48By the logic analogous to the revelation principle, this involves deriving a set of linear constraints, each of which requires that a receiver prefers an action recommended by her signal realization over all other actions.

49The literature on Bayes correlated equilibria also allows for receivers to obtain some exogenous information about the state that is not private, i.e., that is observable by Sender.

50Bergemann et al. (2015) establish a striking, beautiful result that characterizes what combinations of firm profits and consumer surplus might arise as we span all possible signals that the firm might observe about the consumer’s valuation. Bergemann et al. (2017a) characterize combinations of revenue and bidder surplus that might arise in a first-price auction as we span all possible signals that each bidder might observe about her and other bidders’ valuations.

51By informationally-robust mechanism design, we mean choosing the mechanism that yields the best possible outcome under the worst-case signal.

52This decomposition requires the assumption that equilibrium selection works in Sender’s favor. Otherwise, not every Bayes correlated equilibrium is attainable.

53Computing this signal requires optimizing over the set of minimal consistent distributions over belief hierarchies. A consistent distribution over hierarchies is one where receivers’ belief arise from a common prior. A consistent distribution $h$ is minimal if there is no consistent distribution $h'$ such that the support of $h'$ is a strict subset of the support of $h$. 
et al. (2018) is applicable even when Sender is concerned that equilibrium selection may not be in his favor.

Much of the research on information design with multiple receivers has focused on specific applications.\textsuperscript{54} For instance, a number of papers analyze the problem of persuading voters. Alonso and Camara (2016a) and Kosterina (2018) restrict Sender to public signals.\textsuperscript{55} Wang (2015) restricts Sender to i.i.d signals. Arieli and Babichenko (2016), Bardhi and Guo (2018), and Chan et al. (2018) allow for arbitrary private signals.\textsuperscript{56} While each of these papers makes a nice contribution on its own, taken together they deliver a laundry list of results rather than a coherent picture of how to persuade voters. One of the challenges facing the Bayesian persuasion literature going forward will be to synthesize existing results rather than just add to the list of cases that have been considered.\textsuperscript{57}

### 4.2 Multiple senders

Bayesian persuasion models with multiple senders have proved especially useful in engaging an important topic that far predates the idea of information design.\textsuperscript{58} A long tradition in political and legal thought places special emphasis on competition in information provision. A widely held view that competition increases information revelation has motivated protection of freedom of speech and freedom of the press, media ownership regulation, the adversarial judicial system, and many other policies (Gentzkow and Shapiro 2008, Gentzkow and Kamenica 2017c).

A number of papers in information economics provide support for this view (e.g., Milgrom and Roberts 1986, Shin 1998, Battaglini 2002), but Bayesian persuasion-style models have proved

\textsuperscript{54}Das et al. (2017) apply information design to reducing congestion. They provide a simple example where a suitably designed i.i.d. signal about the state of traffic can fully eliminate congestion externalities. Congestion games seem like particularly fertile ground for future work on information design. If crucial data about traffic is user generated, network externalities might lead to the rise of a monopolist provider of routing software who could afford to withhold some information from its users without the fear of being displaced by a competitor. Such a monopolist could raise everyone’s welfare by providing drivers with suitably designed partial information about the route that is optimal for them. This information design problem is unlikely to be amenable to analytic solutions, but development and deployment of welfare-maximizing, dynamic information-provision algorithms might greatly reduce time wasted in traffic.

\textsuperscript{55}Alonso and Camara (2016a) assume voters vary in their preferences. Kosterina (2018) assumes they vary in their priors.

\textsuperscript{56}A receiver’s payoff is assumed to depend on her own action (Arieli and Babichenko 2016), the collective action (Bardhi and Guo 2018), or both (Chan et al. 2018).

\textsuperscript{57}This article fails in this respect as it has turned out to be more of a survey and less of a synthesis than one might have hoped.

\textsuperscript{58}I limit my discussion to papers that focus on the impact of competition on information revelation. Papers with multiple senders that focus on other topics include Ostrovsky and Schwarz (2010), Brocas, Carrillo, and Palfrey (2012), Gul and Pesendorfer (2012), and Boleslavsky et al. (2017)
especially tractable for exploring the issue. Gentzkow and Kamenica (2017b) consider a model that closely follows the basic model, with the only change being that instead of a Sender with utility $v(a, \omega)$, we have a set of senders indexed by $i$, each with utility $v_i(a, \omega)$. All senders simultaneously choose a signal.\footnote{Li and Norman (2018a, 2018b) analyze the impact of competition under rich signal spaces when senders move sequentially. They show that adding an additional sender who moves first cannot result in a strictly less informative equilibrium, but adding a sender to the middle or the end of the line up can. They also show that an equilibrium of the sequential move game can never be strictly less informative than the equilibrium of the simultaneous move game with the same set of senders.} Receiver observes the signal realizations from all senders prior to choosing her action. This description of the model, however, leaves an important issue unspecified: if sender $i$ chooses some signal $\pi$ and sender $j$ also chooses $\pi$, how much information does Receiver obtain? In particular, are the two signals redundant or does Receiver gets two i.i.d. draws from $\pi$? To specify a Bayesian persuasion model with multiple senders, we need to reformulate our definition of the set of all signals in a way that will eliminate such ambiguity. The following formulation does the job.\footnote{This definition of a signal was proposed by Green and Stokey (1978). It was employed to study Bayesian persuasion with multiple senders by Gentzkow and Kamenica (2017b).}

The set of signal realizations $S$ is the set of (Lebesgue-measurable)\footnote{A set $X \subset [0, 1]$ is Lebesgue-measurable if it is meaningful to ask what is the probability that a random variable uniformly distributed on $[0, 1]$ will fall inside $X$. A set $s \subset \Omega \times [0, 1]$ consists of a collection of subsets of $[0, 1]$, one for each state. The set $s$ is Lebesgue-measurable if each element of this collection is Lebesgue-measurable.} subsets of $\Omega \times [0, 1]$. Thus, a signal realization $s \in S$ is a subset of $\Omega \times [0, 1]$. A signal $\pi$ is a partition of $\Omega \times [0, 1]$, with each element of the partition in $S$. The interpretation is that a random variable drawn uniformly from $[0, 1]$ determines the signal realization conditional on the state; the probability of observing $s \in \pi$ when the state is $\omega$ is the probability that this uniform random variable lands inside $\{x \in [0, 1] \mid (\omega, x) \in s\}$. Figure 4 illustrates this formalism. In this example, $\Omega = \{L, R\}$ and

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{Signals as partitions of $\Omega \times [0, 1]$}
\end{figure}
\( \pi_1 = \{l, r\} \) where \( l = (L, [0.07]) \cup (R, [0, 0.3]) \) and \( r = (L, [0.7, 1]) \cup (R, [0.3, 1]) \). Signal \( \pi_1 \) is a partition of \( \Omega \times [0, 1] \) and the state-specific likelihood of signal realizations is \( Pr(l|L) = Pr(r|R) = 0.7 \). This formulation immediately yields the joint informational content of multiple signals. The set of partitions has a lattice structure and we can define the \( \lor \) operator that denotes the coarsest common refinement of two signals. In Figure 4, \( \pi_1 \lor \pi_2 \) is the signal that yields the same information as observing both signal \( \pi_1 \) and signal \( \pi_2 \). In this sense, the operator \( \lor \) “adds” signals together.

Given a vector of signals \( \pi = (\pi_1, \ldots, \pi_n) \), we write \( \lor \pi \) for \( \pi_1 \lor \pi_2 \ldots \lor \pi_n \). Note that the set of all signals is quite rich in the sense that every sender can choose any signal whatsoever, including one whose realizations are arbitrarily correlated with signal realizations of other senders. This richness assumption is in the spirit of the basic model that did not impose any restrictions on signals that could be generated.\(^{62}\)

We now have an easy way of describing how each sender’s payoff depends on the vector of signals chosen by the senders. As before, we can denote sender \( i \)’s payoff when Receiver’s posterior is \( \mu \) by \( \hat{v}_i(\mu) \), and we can denote the distribution of posteriors induced by signal \( \pi \) by \( \langle \pi \rangle \). Hence, if senders’ strategy profile is \( \pi = (\pi_1, \ldots, \pi_n) \), sender \( i \)’s payoff is \( \mathbb{E}_{\mu \sim \langle \lor \pi \rangle} \hat{v}_i(\mu) \). This means that a profile \( \pi \) is a (pure strategy)\(^{63}\) equilibrium if \( \mathbb{E}_{\mu \sim \langle \lor \pi \rangle} \hat{v}_i(\mu) \geq \mathbb{E}_{\mu \sim \langle \pi'_i \lor \pi_{-i} \rangle} \hat{v}_i(\mu) \) for all signals \( \pi'_i \). If \( \pi \) is an equilibrium, we say \( \tau = \langle \pi \rangle \) is an equilibrium outcome.

Characterizing the set of equilibrium outcomes turns out to be surprisingly easy. The approach again draws on concavification. As long as there are multiple senders, a Bayes-plausible distribution of posteriors \( \tau \) is an equilibrium outcome if and only if for every belief \( \mu \) in the support of \( \tau \), for every sender \( i \), \( \hat{v}_i \) coincides with its concavification at \( \mu \) (Gentzkow and Kamenica 2017b). Figure 5 illustrates how this result can be used in practice. Suppose there are two senders \( A \) and \( B \) with value functions \( \hat{v}_A \) and \( \hat{v}_B \). Then, if we denote by \( M_i \) the set of beliefs where \( \hat{v}_i \) coincides with its concavification, we know that a distribution of posteriors \( \tau \) is an equilibrium outcome if and only if its support lies in \( M = M_A \cap M_B \).

Now let us return to the question from the beginning of this subsection, namely: does compe-

\(^{62}\)Despite the richness, it is not clear whether under this formalism, if we have \( n \) receivers, we can induce any coherent hierarchy of beliefs (cf: Mertens and Zamir 1985, Brandenburger and Dekel 1993) for the receivers simply by having each of them observe a realization of some signal. This question is not relevant for our analysis here, since we have a single Receiver who observes all signals, but the question seems interesting in its own right.

\(^{63}\)For now, I focus exclusively on pure strategy equilibrium. This is a substantive restriction for reasons I will discuss later.
(a) \( \hat{\psi}_A(\mu) \) and its concavification for sender \( A \)

(b) \( \hat{\psi}_B(\mu) \) and its concavification for sender \( B \)

(c) Sets of belief where \( \hat{\psi} \) coincides with its concavification

Figure 5: Characterizing equilibrium outcomes
tition increase the amount of information revealed? There are a few things we might mean by that question, but one natural interpretation would be to compare a collusive outcome that would arise if senders’ jointly maximized their welfare with the competitive, equilibrium outcome. Specifically, suppose that $\tau^c$ is the unique distribution of posteriors that maximizes the sum of sender’s utilities, i.e., $\tau^c = \argmax_{\tau} E_{\mu \sim \tau} \sum_i \hat{v}_i(\mu)$. Now, suppose that $\tau^*$ is some equilibrium outcome. Can we conclude that $\tau^*$ is more informative than $\tau^c$? This, strong version of the conjecture that competition increases information revelation, turns out not to be true: it could happen that $\tau^c$ and $\tau^*$ are not comparable, so that some utility functions $u$, Receiver is better off under collusion and for other utility functions, she is better off under competition. We can establish a somewhat weaker form of the conjecture, however. In particular, if $\tau^c$ and $\tau^*$ are comparable, it must be the case that $\tau^*$ is more informative than $\tau^c$. In other words, no matter what preferences senders have, it can never be the case that the collusive outcome is strictly more informative than an equilibrium outcome. This analysis thus provides some qualified support for the common view about competition and information that has played such an important role in shaping public policy.

Of course, the rich signal space assumed in the analysis above is rather special, so we would like to know whether the aforementioned conclusions generalize. Suppose that each sender $i$ has access to some set of signals $\Pi_i$. We refer to $\Pi = (\Pi_1, ..., \Pi_n)$ as the informational environment. Other than relaxing the assumption that each $\Pi_i$ is the rich set of signals, we keep the model the same as before. Can we still conclude that the collusive outcome can never be strictly more informative than any equilibrium? The answer turns out to be: it depends.

Without the rich signal space, we no longer have the $\vee$ operator, but we can still talk about the distribution of posteriors induced by any set (or vector) of signals. We denote by $\tau = \langle \cup \pi_i \rangle$ the distribution of posteriors induced by the set $\{\pi_1, ..., \pi_n\}$. We say an outcome $\tau$ is feasible if there exist $(\pi_1, ..., \pi_n) \in \Pi$ such that $\tau = \langle \cup \pi_i \rangle$. We say the informational environment is Blackwell-connected if for any feasible $\tau$, any sender $i$, and any $\pi_{-i} \in \Pi_{-i}$ such that $\tau$ is more informative.

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64Gentzkow and Kamenica (2017b, 2017c) also analyze the impact of introducing additional senders or increasing the misalignment of senders’ preferences.

65This argmax will generically be unique. If we allow for multiple collusive outcomes, analogous results can be stated using orders on sets (cf: Gentzkow and Kamenica 2017b, 2017c).

66The Blackwell order is usually defined as an order over signals, not distribution of posteriors. We abuse the terminology somewhat and say that $\tau$ is more informative than $\tau^*$ if a signal that induces $\tau$ is Blackwell more informative than a signal that induces $\tau^*$, i.e., if $\tau$ is a mean-preserving spread of $\tau^*$. 

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than $\langle \cup_{j\neq i} \pi_j \rangle$, there exists a $\pi \in \Pi_i$ such that $\tau = \langle \pi \cup (\cup_{j\neq i} \pi_j) \rangle$. In other words, an information environment is Blackwell-connected if, given any strategy profile, each sender can unilaterally deviate to induce any feasible outcome that is more informative. Importantly, if each $\Pi_i$ is the rich set of signals (as in the model we considered earlier), the environment is Blackwell-connected. This turns out to be the key feature of the rich signal space that led to our conclusions. Gentzkow and Kamenica (2017c) show that the collusive outcome cannot be strictly more informative than an equilibrium (regardless of senders’ preferences) if and only if the information environment is Blackwell-connected.\footnote{Gentzkow and Kamenica (2017c) establish this result in a slightly more general setting where senders can have an arbitrary utility over induced distribution of posteriors, not necessarily one that takes the form of $E_{\mu \sim \tau} \hat{v}_i (\mu)$. Their proof of the “only if” direction utilizes preferences that do not conform to the $E_{\mu \sim \tau} \hat{v}_i (\mu)$ formulation. It is an open question whether there is a weaker condition on the information environment that suffices for the result to hold when each sender’s utility is some $E_{\mu \sim \tau} \hat{v}_i (\mu)$.} This result further sharpens our understanding of the circumstances under which we can safely assume that competition cannot reduce the amount of information revealed.\footnote{This result also explains why it was important to focus on pure strategy equilibria. Once senders use mixed strategies, the environment immediately fails to be Blackwell-connected. Consequently, even with the rich signal space, it is possible to have a collusive outcome that is strictly more informative than a mixed strategy equilibrium.}

Board and Lu (2017) analyze a closely related question in a search setting. Senders are sellers who offer the same product.\footnote{Board and Lu (2017) consider a more general setting. I focus on a special case of their model.} There is a binary state of the world that determines whether the buyer is better off buying the product or not; price is taken to be exogenous. At the prior, buyer prefers not to buy. Seller wish to sell the product regardless of the state. The buyer approaches sellers one by one, paying a search cost $c > 0$ to visit each additional seller. When a buyer visits a particular seller, the seller chooses a signal about the state to reveal to the buyer. The buyer then decides whether to purchase the product from this seller, visit another seller, or exit the market. Board and Lu (2017) show that if each seller has access to the rich signal space (and can thus correlate with signal realizations of other sellers) and buyer’s belief is observable when she visit a seller,\footnote{One interpretation of this would be that we are in an online setting and cookies in the buyer’s browser reveal all of the information she has received thus far.} then competition will not have any bite: the unique equilibrium is one where each seller generates the same signal he would generate if he were the only seller in town. On the other hand, if sellers cannot perfectly correlate their signal realizations and buyer’s belief is not observable, the unique equilibrium as $c$ converges to zero leads to full revelation of the state. It is quite interesting that in the static setting, senders’ access to the rich signal space suffices to ensure that competition
induces information revelation whereas in the search setting, senders’ ability to correlate signals reduces the impact of competition.\textsuperscript{71}

4.3 Dynamics

Now let us add time to the basic model. If the state of the world and the set of available actions do not change, and the players are patient (so the payoffs do not depend on when the action is taken), dynamics do not matter. Any sequence of signals (including those where signal sent in period \( t \) depends on the signal realization in period \( t - 1 \)) in the end induces some Bayes-plausible distribution of posteriors \( \tau \), so Sender might as well just induce his preferred \( \tau \) at the outset.

Dynamics become important\textsuperscript{72} when state evolves over time, past behavior influences current opportunities, and/or Sender and Receiver have conflicting preferences on the optimal timing of Receiver’s action.\textsuperscript{73} Ely (2017) derives the optimal information policy (a map from the history and the current state to a distribution over signal realizations) in environments where the state evolves through a Markov process and Receiver is myopic (in each period, she chooses an action that maximizes her contemporaneous utility given her current belief).\textsuperscript{74} For example, suppose that Receiver is an employee who is either ready for promotion (\( \omega = 1 \)) or not (\( \omega = 0 \)) and can either request the promotion (\( a = 1 \)) or not (\( a = 0 \)). Time is continuous. It is common knowledge that \( \omega = 0 \) at the outset and that the state transitions to \( \omega = 1 \) at Poisson rate \( \lambda \). It is optimal for the worker to ask for the promotion if and only if her belief that she is ready is above some cutoff \( p \). Sender is the firm that always prefers employees not to ask for promotion. Via its design of employee evaluations, feedback procedures, and transparency, the firm can implement any information policy. If the firm provides no information, the employee’s belief that she is ready will drift upward; at time \( t \), she will think the probability she is ready is \( 1 - e^{-\lambda t} \). Consequently, regardless of the true

\textsuperscript{71}Levy et al. (2018) consider a Receiver who suffers from correlation neglect. A monopolist owner of multiple news outlets can, by suitably correlating signal realizations of the various outlets, manipulate Receiver’s belief in a way that violates Bayes-plausibility. Competition is assumed to reduce the ability to correlate signal realizations and thus benefits Receiver.

\textsuperscript{72}Space constraints prevent me from covering the entire literature on dynamic Bayesian persuasion. In addition to the papers discussed below, important contributions to this literature include Horner and Skrzypacz (2016) and Henry and Ottaviani (2018).

\textsuperscript{73}Ely et al. (2015) put belief dynamics directly into the players’ utility function. They postulate that entertainment utility stems from suspense (variance of next period’s beliefs) and surprise (realized movement of beliefs) and analyze how to entertain a Bayesian audience. They apply this analysis to the design of mystery novels, engaging political primaries, casino gambling, game shows, charity auctions, and sports.

\textsuperscript{74}Renault et al. (2017) also analyze a case of this model.
state, she will ask for promotion at \( t = \frac{-\ln(1-p)}{\lambda} \). If the firm is fully transparent, the employee will ask for promotion exactly when she is ready for it, which on average happens at time \( t = \frac{1}{\lambda} \). This means that when the employee is very eager (\( p \) is low), it is better to be fully transparent; otherwise, providing no information is better. It turns out, however, that neither of these policies is ever optimal. The optimal policy is to reveal to the employee that she has become ready with some deterministic delay.\(^{75}\) It is rather remarkable that the optimal policy is so simple.

Ely and Szydlowski (2018) examine a setting where Receiver’s past decisions influence her current opportunities.\(^{76}\) Suppose Receiver is again an employee concerned about her promotion, but this time promotion is as function of employee’s effort, not her type. Specifically, time is continuous, at every moment the employee chooses whether to continue working or not (incurring a constant cost of effort), and she is eligible for promotion once the time she spent working exceeds some unknown state \( \omega \). Sender is a firm that designs its information policy to maximize employee’s effort (reaping benefits even if the employee continues working beyond the level required for promotion). Here the state is assumed to remain constant over time, but dynamics still matter for two reasons. First, the firm can use future information provision as an incentive to get the worker to exert current effort.\(^{77}\) Second, after the employee has sunk some effort, provision of additional information can induce her to continue working; thus, the employee might exert more overall effort than she would have been willing to do at the outset. Ely and Szydlowski (2018) characterize the optimal policy and demonstrate that (for the two aforementioned reasons) dynamic information provision improves upon what could be achieved with static information design.

The model in Ely and Szydlowski (2018) can be seen as a special case of a broader class of environments.\(^{78}\) Suppose in period \( t = 0 \), Sender chooses a signal \( \pi_0 \), and Receiver, after observing its realization, selects an element of some partition \( P_0 \) of the action space \( A \). Denote Receiver’s choice by \( A_1 \). The interpretation is that \( A_1 \) is the set of actions that will remain available to Receiver. In period, \( t = 1 \), sender chooses a signal \( \pi_1 \), and Receiver, after observing

\(^{75}\) As long as the delay is shorter than \( \frac{-\ln(1-p)}{\lambda} \), the employee will not ask for promotion until she has been informed that she is ready. Consequently, setting the delay to \( \frac{-\ln(1-p)}{\lambda} \) maximizes the firm’s payoff.

\(^{76}\) Using a similar modeling approach, Smolin (2017) derives optimal feedback that employees should be given about their past performance.

\(^{77}\) The firm can fully commit to her information policy. Orlov et al. (2018b) and Bizzotto et al. (2018) analyze settings where Sender lacks commitment in the sense that he must choose a sequentially rational signal.

\(^{78}\) These observations come from a discussion with Andy Skrzypacz and Ilan Kremer.
its realization, selects an element of some partition $P_1$ of $A_1$; and so on. In the (discrete time version of) Ely and Szydlowski (2018), the action set $A$ is the total number of periods worked and $P_t = \{t, \{t + 1, t + 2, \ldots\}\}$ until the employee quits. In their context, the nature of sunk costs determined this particular partitional structure, but in other applications one could imagine that specifying the partitions in each period is part of the design problem. For example, a college can control both how much information students get about their aptitude and the rules about when students have to commit to their major. Or, the legal system can determine whether a prosecutor can provide additional information after the conviction but before the sentencing. Such combination of designing potential paths of narrowing options alongside designing information seems like potentially fertile ground for future research.
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