An Improved Bound for the Shapley-Folkman Theorem*

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Abstract

Abstract: We provide an improvement on the error bound for the Shapley-Folkman theorem.

1. Introduction

Budish (2011) considers the indivisible-goods combinatorial assignment problem in the context of a Fisher economy wherein agents are endowed with fiat money. Budish establishes the existence of an approximate competitive equilibrium in which the agents have nearly equal money endowments and in which markets clear up to an error that tends to zero as the number of agents tends to infinity. That money endowments can be made nearly equal is important for establishing a number of results on the fairness of the final allocation.

A careful look at Budish's proof reveals that it contains a proof of the Shapley-Folkman theorem with an improved error bound.[1] The purpose of this note is to provide an explicit statement and proof of this result.

2. The Shapley-Folkman Theorem

Throughout, $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^m$, and, for any subset X of $\mathbb{R}^m$, coX denotes its convex hull.

For any nonempty subset S of $\mathbb{R}^m$, define the *diameter* of S by

$$diam(S) := \sup_{x,y \in S} \|x - y\|,$$

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[1] Reny (2017) extends Budish's (2011) result to settings with both divisible and indivisible goods.

define the *radius* of S by

$$rad(S) := \inf_{y\in\mathbb{R}^m} \sup_{x\in S} \|x-y\|,$$

define the *inner diameter* of S by

$$indiam(S) := \sup_{y\in coS} \inf_{\{T\subseteq S: y\in coT\}} diam(T),$$

and define the *inner radius* of S by

$$inrad(S) := \sup_{y\in coS} \inf_{\{T\subseteq S: y\in coT\}} rad(T).$$

When S is compact, all of the infima and suprema above are attained.

Because T can always be chosen to be equal to S, it is clear that $indiam(S) \leq diam(S)$ and $inrad(S) \leq rad(S)$. It is also not difficult to show that,[2]

$$\frac{diam(S)}{2} \leq rad(S), \tag{2.1}$$

from which it follows that,[3]

$$\frac{indiam(S)}{2} \leq inrad(S). \tag{2.2}$$

If S is a sphere or if $\#S = 2$, then (2.1) and (2.2) are equalities. However, the inequalities can be strict, e.g., if $S = \{(1,0,0),(0,1,0),(0,0,1)\}$, then $diam(S)/2 = \sqrt{2}/2 < \sqrt{6}/3 = rad(S)$. Our improved bound exploits inequalities (2.1) and (2.2).

The version of the Shapley-Folkman theorem that has been most useful in the literature is as follows (see Starr 2008).

Theorem 2.1. *(Shapley-Folkman) Suppose that $n \geq m$. If $S_1, ..., S_n$ are compact subsets of $\mathbb{R}^m$, if $y \in co(S_1 + ... + S_n)$, and if $R = \max(rad(S_1), ..., rad(S_n))$, then there exist $x_i \in S_i$ and $y_i \in coS_i$, $i = 1, ..., n$, such that $y = \sum_i y_i$, $y_i = x_i$ for all but at most m indices i, and*

$$\left\| y - \sum_{i=1}^{n} x_i \right\| \leq R\sqrt{m}.$$

By using the inner radii of the sets S_i rather than their radii, Starr (1969) obtains the following result with an improved error bound.

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[2] Suppose $rad(S) = \sup_{x\in S} \|x - y^*\|$ and $diam(S) = \|\hat{x} - \hat{y}\|$ for some $y^* \in \mathbb{R}^m$ and $\hat{x}, \hat{y} \in S$. Then, by the triangle inequality, $\|\hat{x} - \hat{y}\| \leq \|\hat{x} - y^*\| + \|y^* - \hat{y}\|$, and so $rad(S) \geq \max(\|\hat{x} - y^*\|, \|y^* - \hat{y}\|) \geq \|\hat{x} - \hat{y}\|/2$.

[3] Indeed, if $diam/2 \leq rad$, then the number, $diam(T)$, that appears on the right-hand side of the definition of $indiam(S)$, is less or equal to $2rad(T)$, implying that that right-hand side is less or equal to $2inrad(S)$.

Theorem 2.2. *(Starr) Suppose that $n \geq m$. If $S_1, ..., S_n$ are compact subsets of $\mathbb{R}^m$, if $y \in co(S_1 + ... + S_n)$, and if $r = \max(inrad(S_1), ..., inrad(S_n))$, then there exist $x_i \in S_i$ and $y_i \in coS_i$, $i = 1, ..., n$, such that $y = \sum_i y_i$, $y_i = x_i$ for all but at most m indices i, and*

$$\left\| y - \sum_{i=1}^{n} x_i \right\| \leq r\sqrt{m}.$$

Remark 1. *The Shapley-Folkman and Starr Theorems 2.1 and 2.2 each have more refined versions (see Starr 1969). The statement of the more refined Shapley-Folkman theorem is as follows: If $S_1, ..., S_n$ are compact subsets of $\mathbb{R}^m$ and if $y \in co(S_1 + ... + S_n)$, then there exist $x_i \in S_i$ and $y_i \in coS_i$, $i = 1, ..., n$, such that $y = \sum_i y_i$, $y_i = x_i$ for all but at most $\min(m, n)$ indices i, and*

$$\left\| y - \sum_{i=1}^{n} x_i \right\|^2 \leq \sum rad^2(S_i),$$

where the sum on the right-hand side is over the $\min(m, n)$ highest among the n numbers $rad^2(S_1), ..., rad^2(S_n)$. The statement of the more refined Starr theorem is the same except that $inrad(S_i)$ everywhere replaces $rad(S_i)$.

3. An Improved Bound

We can improve on the error bounds in Theorems 2.1 and 2.2 by using the diameters of the S_i instead of their radii, and by using the inner diameters of the S_i instead of their inner radii. Our results are as follows.

Theorem 3.1. *Suppose that $n \geq m$. If $S_1, ..., S_n$ are compact subsets of $\mathbb{R}^m$, if $y \in co(S_1 + ... + S_n)$, and if $D = \max(diam(S_1), ..., diam(S_n))$, then there exist $x_i \in S_i$ and $y_i \in coS_i$, $i = 1, ..., n$, such that $y = \sum_i y_i$, $y_i = x_i$ for all but at most m indices i, and*

$$\left\| y - \sum_{i=1}^{n} x_i \right\| \leq D\sqrt{m}/2.$$

Theorem 3.2. *Suppose that $n \geq m$. If $S_1, ..., S_n$ are compact subsets of $\mathbb{R}^m$, if $y \in co(S_1 + ... + S_n)$, and if $d = \max(indiam(S_1), ..., indiam(S_n))$, then there exist $x_i \in S_i$ and $y_i \in coS_i$, $i = 1, ..., n$, such that $y = \sum_i y_i$, $y_i = x_i$ for all but at most m indices i, and*

$$\left\| y - \sum_{i=1}^{n} x_i \right\| \leq d\sqrt{m}/2.$$

Remark 2. *That the inequality bounds in Theorems 2.1, 2.2, 3.1 and 3.2 can all be satis-*

fied with equality for any given dimension m is established by the following example. Let $\hat{z} = \left(\frac{-1}{2(m-1)}, ..., \frac{-1}{2(m-1)}\right) \in \mathbb{R}^m$ and for each $i = 1, ..., m$, let $S_i = \{(0, \hat{z}_{-i}), (1, \hat{z}_{-i})\}$. Then $diam(S_i) = indiam(S_i) = 1$ and $rad(S_i) = inrad(S_i) = 1/2$ for every $i = 1, ..., m$. Moreover, $0 = \frac{1}{2}\left(\sum_i (0, \hat{z}_{-i})\right) + \frac{1}{2}\left(\sum_i (1, \hat{z}_{-i})\right) \in co\,(S_1 + ... + S_m)$, and, for every $z \in S_1 + ... + S_m$, the ith coordinate of z is $\pm\frac{1}{2}$ for every $i = 1, ..., m$. Hence, $\|0 - z\| = \|z\| = \sqrt{m}/2 = R\sqrt{m} = r\sqrt{m} = D\sqrt{m}/2 = d\sqrt{m}/2$, where R, r, D, and d are as in Theorems 2.1, 2.2, 3.1 and 3.2.

3.1. How Much Better?

Our next result indicates that the bounds offered in Theorems 3.1 and 3.2, while always at least as small as the bounds in 2.1 and 2.2, respectively, are never more than a factor of $\sqrt{2}/2$ ($\approx .707$) smaller, uniformly in the dimension of the ambient space. Given the dimension, m, of the ambient space, it is posible to make the following somewhat stronger statement.

Lemma 3.3. *For any $m \geq 1$ and for any nonempty compact subset S of $\mathbb{R}^m$,*

$$diam(S) \geq \sqrt{2(1 + 1/m)}rad(S).$$

Moreover, for any $m \geq 1$, there are subsets S of $\mathbb{R}^m$ for which equality holds.

Remark 3. *As a consequence of Lemma 3.3, if R, r, D, and d are as in Theorems 2.1, 2.2, 3.1, and 3.2, then*

$$\frac{\sqrt{2(1 + 1/m)}}{2}R \leq \frac{D}{2} \leq R,$$

and

$$\frac{\sqrt{2(1 + 1/m)}}{2}r \leq \frac{d}{2} \leq r,$$

where the right-hand inequalities in the two displays follow from (2.1) and (2.2), respectively.

Hence, the bounds that we obtain by using D and d are smaller than those obtained by using R and r by a factor of at most $\sqrt{2(1 + 1/m)}/2$, which decreases to $\sqrt{2}/2$ as $m \to \infty$.

To see that, for each m, there are subsets S of $\mathbb{R}^m$ for which the best possible reduction of $\sqrt{2(1 + 1/m)}/2$ is achieved, it is convenient to consider $\mathbb{R}^m$ as a subset of $\mathbb{R}^{m+1}$, specifically, as the (isometrically isomorphic) subset $L_m = \{(x_0, ..., x_m) \in \mathbb{R}^{m+1} : x_0 + ... + x_m = 1\}$ of $\mathbb{R}^{m+1}$. Letting $e_i = (0, ..., 0, 1, 0, ..., 0)$ denote the i-th unit vector in L_m, it is straightforward to verify that the subset $S = \{e_1, ..., e_{m+1}\}$ of L_m has both diameter and inner diameter equal to $\sqrt{2}$, and has both radius and inner radius equal to $\sqrt{m/(m+1)}$, which is the distance of any e_i to the barycenter, $(1/(m+1), ..., 1/(m+1))$, of S. Hence, the ratio of the diameter to the radius (and of the inner diameter to the inner radius) is $\sqrt{2(1 + 1/m)}$, as desired.

4. Proofs.

Proof of Theorem 3.1.

Since $y \in co(S_1 + ... + S_n)$ and because $co(S_1 + ... + S_n) = coS_1 + ... + coS_n$, we have $y = \sum_{i=1}^{n} \sum_k \alpha_{ik} x_{ik}$, for some finitely many $\alpha_{ik} \geq 0$ such that $\sum_k \alpha_{ik} = 1$ for every i and for some $x_{ik} \in S_i$ for every i and k.

For each $i = 1, ..., n$, let e_i denote the i-th unit vector $(0, ..., 0, 1, 0, ..., 0)$, and, for any $z \in \mathbb{R}^m$, let $(e_i, z) \in \mathbb{R}^{n+m}$ denote the concatenation of e_i and z. Then $\sum_{i,k}(\alpha_{ik}/n)(e_i, x_{ik}) = (1, ..., 1, y)/n$ and therefore, because $\sum_{i,k}(\alpha_{ik}/n) = 1$,

$$\frac{1}{n}(1, ..., 1, y) \in co\left(\cup_{i=1}^{n}(\{e_i\} \times S_i)\right) \subseteq \Delta_n \times \mathbb{R}^m, \tag{4.1}$$

where Δ_n denotes the $n-1$ dimensional unit simplex.

By Caratheodory's theorem (Rockafellar 1970) $(1, ..., 1, y)/n$ can therefore be written as a convex combination of $n+m$ or fewer points belonging to $\cup_{i=1}^{n}(\{e_i\} \times S_i)$. Thus, for some positive integer K we may write

$$\frac{1}{n}(1, ..., 1, y) = \sum_{i=1}^{n} \sum_{k=1}^{K} \lambda_{ik}(e_i, x_i^k), \tag{4.2}$$

where the λ_{ik}'s are nonnegative and sum to one, and, at most $n+m$ of the λ_{ik} are strictly positive and $\lambda_{ik} > 0$ implies that x_i^k is in S_i.[4]

For each $i = 1, ..., n$, let $S_i^+ = \{x_i^k : \lambda_{ik} > 0\}$. Since the first n coordinates of the vector on the left-hand side of (4.2) are positive, each S_i^+ contains at least one element. Reindexing if necessary, let $S_1^+, ..., S_j^+$ denote those S_i^+ that contain two or more elements. So $S_{j+1}^+, ..., S_I^+$ are singletons, and, since at most $n+m$ of the λ_{ik} are strictly positive, the union of $S_1^+, ..., S_j^+$ contains no more than $m+j$ elements.

Since S_i^+ is a finite subset of S_i, the distance between any point in S_i^+ and the simple average of all of the points in S_i^+ is no greater than $diam(S_i)(\#S_i^+ - 1)/(\#S_i^+)$. Hence

$$rad(S_i^+) \leq diam(S_i)(\#S_i^+ - 1)/(\#S_i^+), i = 1, ..., j. \tag{4.3}$$

The equality in (4.2) for the first n coordinates implies that $\sum_{k=1}^{K} n\lambda_{ik} = 1$ for each i, and the equality for the last m coordinates then implies that y is contained in the sum of the convex hulls of the sets $S_1^+, ..., S_n^+$. Hence, y is contained in the convex hull of $S_1^+ + ... + S_n^+$.[5]

[4] Zhou (1993) makes essentially the same use of Caratheodory's theorem. Zhou does not consider the implications for the error bound.

[5] Using once again that the sum of the convex hulls of any finite number of sets is equal to the convex hull of their sum.

Consequently, by (4.3) and because $rad(S_i^+) = 0$ for $i > j$, the Shapley-Folkman theorem (see Starr 1969 or Remark 1 in Section 2) implies that there exist $x_i \in S_i^+$ and $y_i \in coS_i^+$, $i = 1, ..., n$, such that $y = \sum_{i=1}^n y_i$, $y_i = x_i$ for all but at most m indices i, and,

$$\left\| y - \sum_{i=1}^n x_i \right\|^2 \leq \sum_{i=1}^j \left(\frac{(\#S_i^+ - 1) diam(S_i)}{\#S_i^+} \right)^2.$$

Since $\#S_i^+ \geq 2$ for every $i = 1, ..., j$, we have $\big((\#S_i^+ - 1)/(\#S_i^+)\big)^2 \leq (\#S_i^+ - 1)/4$ for every $i = 1, ..., j$. Hence,

$$\left\| y - \sum_{i=1}^n x_i \right\|^2 \leq \sum_{i=1}^j \frac{(\#S_i^+ - 1) diam^2(S_i)}{4} \leq D^2 m/4,$$

where the second inequality follows because the union of the sets $S_1^+, ..., S_j^+$ contains no more than $m + j$ elements and so $\sum_{i=1}^j (\#S_i^+ - 1) \leq m$. Hence, we may conclude that

$$\left\| y - \sum_{i=1}^n x_i \right\| \leq D\sqrt{m}/2.$$

Q.E.D.

Proof of Theorem 3.2. Since $y \in co(S_1 + ... + S_n) = coS_1 + ... + coS_n$, there exist $z_i \in coS_i$, $i = 1, ..., n$, such that $y = z_1 + ... + z_n$.

Fix any $\varepsilon > 0$. By the definition of the inner diameter, for each $i = 1, ..., n$, there is $T_i \subseteq S_i$ such that $z_i \in coT_i$ and $diam(T_i) \leq indiam(S_i) + \varepsilon$.

Hence, $y = z_1 + ... + z_n \in coT_1 + ... + coT_n = co(T_1 + ... + T_n)$ and so, letting $D = \max(diam(T_1), ...diam(T_n))$, Theorem 3.1 implies that there exist $x_i \in T_i \subseteq S_i$ and $y_i \in coT_i \subseteq coS_i$, $i = 1, ..., n$, such that $y = \sum_i y_i$, $y_i = x_i$ for all but at most m indices i, and

$$\left\| y - \sum_{i=1}^n x_i \right\| \leq D\sqrt{m}/2.$$

Setting $d = \max(indiam(S_1), ..., indiam(S_n))$, we have $D \leq d + \varepsilon$ and so,

$$\left\| y - \sum_{i=1}^n x_i \right\| \leq (d + \varepsilon)\sqrt{m}/2.$$

Letting $\varepsilon \to 0$ and taking convergent subsequences of the $x_i \in S_i$ and the $y_i \in coS_i$ completes the proof. Q.E.D.

Proof of Lemma 3.3. We first prove the lemma for finite subsets S of $\mathbb{R}^m$. So, let S be any nonempty finite subset of $\mathbb{R}^m$ and suppose that $rad(S) = \rho$. By the definition of $rad(S)$ and because S is finite, there exists $y^* \in \mathbb{R}^m$ such that $\max_{x \in S} \|y^* - x\| = \rho$. Let $\{x_0, ..., x_k\} = \{x \in S : \|y^* - x\| = \rho\}$.

We claim that $y^* \in co\{x_0, ..., x_k\}$. To prove this claim, let us suppose not. Then, by the separating hyperplane theorem, there is $p \in \mathbb{R}^m$ such that $py^* < px_j$ for every $j = 0, 1, ..., k$. But then

$$\left.\frac{d}{dt}\right|_{t=0} \|y^* + tp - x_j\|^2 = 2(py^* - px_j) < 0, \text{for every } j = 0, 1, ..., k,$$

from which we can conclude, since S is finite, that there is a small $t^* > 0$ such that $\max_{x \in S} \|y^* + t^*p - x\| < \rho$ for every $x \in S$. But this contradicts the definition of y^* and proves the claim.

Let $S^* = \{x_0, ..., x_k\}$ and let $\hat{S} = (1/\rho)(S^* - \{y^*\})$. Then $0 \in co\hat{S}$ and $\|x\| = 1$ for every $x \in \hat{S}$. Consequently, by Lemma 4.1 below, $diam(\hat{S}) \geq \sqrt{2(1 + 1/m)}$. Moreover, since $diam(\hat{S}) = diam((1/\rho)(S^* - \{y^*\})) = (1/\rho)diam(S^*)$, this implies that $diam(S^*) \geq \sqrt{2(1 + 1/m)}\rho = \sqrt{2(1 + 1/m)}rad(S)$. Finally, since $S \supseteq S^*$ implies that $diam(S) \geq diam(S^*)$, we obtain $diam(S) \geq \sqrt{2(1 + 1/m)}rad(S)$, proving the lemma for finite S.

To prove the lemma for nonempty compact S, choose any $\varepsilon > 0$ and let S' be a finite ε-dense subset of S. Let us first show that $rad(S) \leq rad(S') + \varepsilon$.

By the definition of $rad(S')$, there is $y' \in \mathbb{R}^m$ such that $\sup_{x \in S'} \|y' - x\| = rad(S')$. Therefore, by the definition of the radius of S, $rad(S) \leq \sup_{x \in S} \|y' - x\|$. By the compactness of S, there is $x^* \in S$ such that $\sup_{x \in S} \|y' - x\| = \|y' - x^*\|$. And since S' is ε-dense in S, there is $x' \in S'$ such that $\|x' - x^*\| < \varepsilon$. Hence,

$$\begin{aligned} rad(S) &\leq \sup_{x \in S} \|y' - x\| = \|y' - x^*\| \\ &\leq \|y' - x'\| + \|x' - x^*\| \\ &\leq \sup_{x \in S'} \|y' - x\| + \varepsilon = rad(S') + \varepsilon, \end{aligned}$$

where the second inequality is the triangle inequality, and the third inequality follows because $x' \in S'$.

By what have already shown for finite sets, $diam(S') \geq \sqrt{2(1 + 1/m)}rad(S')$. Hence, $diam(S) \geq diam(S') \geq \sqrt{2(1 + 1/m)}rad(S') \geq \sqrt{2(1 + 1/m)}(rad(S) - \varepsilon)$, where the first inequality follows because $S \supseteq S'$. Taking the limit as $\varepsilon \to 0$ gives the desired result.

That the inequality in Lemma 3.3 can be achieved as an equality for any m has already been demonstrated in the main text (see the end of Section 3.1). Q.E.D.

Lemma 4.1. *If S is any nonempty subset of $\mathbb{R}^m$ whose convex hull contains the origin, and $\|x\| = 1$ for every $x \in S$, then $diam(S) \geq \sqrt{2(1+1/m)}$.*

Proof of Lemma 4.1. Since $0 \in coS$, Caratheodory's theorem implies that there are $m+1$ not necessarily distinct points, $x_0, x_1, ..., x_m$, in S and there are nonnegative $\lambda_0, \lambda_1, ..., \lambda_m$ that sum to 1 such that $\sum_{i=0}^{m} \lambda_i x_i = 0$. Let $X = \{x_0, ..., x_m\}$. Since $S \supseteq X$, we have $diam(S) \geq diam(X)$. Hence, it suffices to show that $diam(X) \geq \sqrt{2(1+1/m)}$.

Without loss of generality, we may assume that $\lambda_0 \geq \lambda_i$ for every $i = 0, 1, ..., m$. In particular, $\lambda_0 > 0$.

For any $x_i \in X$, since $\|x_i\| = \|x_0\| = 1$ we have,

$$\|x_i - x_0\|^2 = \|x_i\|^2 - 2x_i x_0 + \|x_0\|^2 = 2 - 2x_i x_0.$$

Since $\sum_{i=0}^{m} \lambda_i x_i = 0$, we have

$$\sum_{i=1}^{m} \lambda_i x_i x_0 = -\lambda_0 \|x_0\|^2 = -\lambda_0.$$

Hence there is $j \in \{1, ..., m\}$ such that $\lambda_j x_j x_0 \leq -\lambda_0/m < 0$, which implies that $-x_j x_0 \geq (\lambda_0/\lambda_j)(1/m) \geq 1/m$, since $\lambda_0 \geq \lambda_j$. Hence,

$$\begin{aligned} diam^2(X) &\geq \|x_j - x_0\|^2 = 2 - 2x_j x_0 \\ &\geq 2 + 2/m. \end{aligned}$$

Q.E.D.

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