

The Combinatorial Assignment Problem: Approximate Competitive Equilibrium from Equal Incomes

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The Combinatorial Assignment Problem

General question: How can we divide a set of indivisible objects amongst a set of agents without using monetary transfers, in a way that is efficient, incentive compatible, and fair?

Specific instance: Course Allocation at Universities

- ▶ The indivisible objects are seats in courses
- ▶ Each student requires a bundle of courses
- ▶ Exogenous restriction against monetary transfers (even at Chicago!)

Other examples: assigning interchangeable workers to tasks or shifts; leads to salespeople; takeoff and landing slots to airlines; shared scientific resources amongst scientists; players to teams

Relation to the Literature

Combinatorial assignment is one feature removed from canonical market design problems that have received considerable attention and have compelling solutions

- ▶ No restriction on money → Combinatorial Auction Problem
 - ▶ Theory: Vickrey 1961 ...
 - ▶ Applications: Spectrum Auctions, Power Auctions, Adwords Auctions ... (e.g., Milgrom 2000, 2004)
- ▶ Single-Unit Demand → School/House Assignment Problem
 - ▶ Theory: Shapley and Scarf 1974 ...
 - ▶ Applications: Redesign of School Choice procedures in New York, Boston, San Francisco ... (e.g., Abdulkadiroglu et al 2005a, 2005b, 2009)
- ▶ Two-Sided Preferences → Matching Problem
 - ▶ Theory: Gale and Shapley 1961 ...
 - ▶ Applications: National Resident Matching Program ... (e.g., Roth and Peranson 1999)

Yet, Mostly Negative Results

1. Dictatorship Theorems. The only mechanisms that are ex-post Pareto efficient and strategyproof are dictatorships (Klaus and Miyagawa, 2001; Papai 2001; Ehlers and Klaus, 2003; Hatfield 2009)
 - ▶ Dictatorship: for any two agents, one makes *all* her choices before the other makes *any*
2. Impossibility of ex-ante efficiency and strategyproofness even in the single-unit case (Zhou 1990)
3. Other impossibility results: specific criteria that are compatible for single-unit assignment are not compatible for multi-unit assignment (Sönmez, 1999; Konishi, Quint and Wako, 2001; Klaus and Miyagawa, 2001; Manea, 2007; Kojima, forthcoming)

Takeaway: there is no "perfect" mechanism. Any solution will involve compromise.

N.B. Mechanisms found in the field practice have severe fairness and incentives problems (Sonmez and Unver forth., Budish and Cantillon 2009)

This Paper: A New Mechanism

This paper proposes a new mechanism inspired by the old general-equilibrium theory idea of Competitive Equilibrium from Equal Incomes (Foley 1967, Varian 1974)

There are two basic challenges in adapting CEEI to the problem of combinatorial assignment

1. CEEI prices need not exist
 - ▶ Either indivisibilities or complementarities alone complicate existence. Our economy features both.
2. The fairness criteria at the heart of the argument for CEEI are either undefined or unrealistic in our environment
 - ▶ For instance, Envy-free allocations need not exist

This Paper: A New Mechanism

Goal: develop the *Approximate CEEI Mechanism* and show that it satisfies attractive criteria of efficiency, fairness and incentives:

1. Existence theorem for *Approximate CEEI*
2. New criteria of outcome fairness, tailored to the case of indivisible goods: *maximin-share guarantee* and *envy bounded by a single good*
3. Fairness theorems
4. Incentives: *strategyproof in the large*
5. Comparison to alternative mechanisms from theory and practice

A Simple Example: Two Diamonds, Two Rocks

- ▶ Two agents. Four objects: two valuable Diamonds (Big, Small) and two ordinary Rocks (Pretty, Ugly). At most two objects per agent.

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 - ▶ Randomly assign budgets of 1 and $1 + \beta$, for $\beta \gtrsim 0$
 - ▶ Set the price of the Big Diamond strictly greater than 1
 - ▶ Set other prices such that the poorer agent can afford {Small Diamond, Pretty Rock}, wealthier agent gets {Big Diamond, Ugly Rock}

A Simple Example: Two Diamonds, Two Rocks

1. Is the allocation $\{\text{Big Diamond, Ugly Rock}\}$, $\{\text{Small Diamond, Pretty Rock}\}$ fair?
 - ▶ Indivisibilities create a certain irreducible unfairness: only one Big Diamond
 - ▶ Criteria will formalize the sense in which this allocation is fair
2. It is critical for fairness that budget inequality is small
 - ▶ Else, there may exist prices at which the wealthier agent can afford both Diamonds while the poorer agent can afford neither
3. It is also critical for fairness that we use item prices, not bundle prices
 - ▶ Else, price the bundle $\{\text{Big Diamond, Small Diamond}\}$ at $1 + \beta$ without either Diamond being affordable to the poorer agent
4. In this example, an arbitrarily small amount of budget inequality enables perfect market clearing.
 - ▶ In general, my existence result allows for a "small" amount of market-clearing error
 - ▶ Error on real preference data from HBS is 6 course seats per semester, versus 4500 allocated

Environment

- ▶ Set of N students \mathcal{S} (s_i)
- ▶ Set of M courses \mathcal{C} (c_j) with integral capacities $\mathbf{q} = (q_1, \dots, q_M)$. No other goods in the economy.
- ▶ Each student s_i has a set of permissible schedules $\Psi_i \subseteq 2^{\mathcal{C}}$, and a vNM utility function $u_i : 2^{\mathcal{C}} \rightarrow \mathbb{R}_+$
 - ▶ Impermissible schedules have utility of zero. Otherwise ordinal preferences over bundles are strict.
 - ▶ Complementarities, Substitutabilities are allowed
 - ▶ No peer effects. No uncertainty about preferences.
 - ▶ Maximum number of courses in a permissible schedule is k
- ▶ An allocation $\mathbf{x} = (x_i)_{i=1}^N$ is feasible if each $x_i \in 2^{\mathcal{C}}$ and $\sum_{i=1}^N x_{ij} \leq q_j$ for each j
- ▶ An economy is a tuple $(\mathcal{S}, \mathcal{C}, \mathbf{q}, \Psi, (u_i)_{i=1}^N)$.

N.B. I often use "students" and "courses" rather than "agents" and "objects"

Competitive Equilibrium from Equal Incomes

What would CEEI mean in our environment?

1. Agents report preferences over bundles
2. Agents are given equal budgets b^* of an artificial currency
3. We find an item price vector \mathbf{p}^* such that, when each agent is allocated his favorite affordable bundle, the market clears
4. We allocate each agent their demand at \mathbf{p}^*

It is easy to see that existence is problematic with indivisibilities. Consider the case in which agents have identical preferences.

Approximate CEEI

Definition. An allocation \mathbf{x}^* , budget vector \mathbf{b}^* and price vector \mathbf{p}^* constitute an (α, β) -**approximate competitive equilibrium from equal incomes** (*Approximate CEEI*) if:

- (i) Each agent i is allocated her most-preferred bundle in her budget set $\{x \in 2^C : \mathbf{p}^* \cdot x \leq b_i^*\}$
- (ii) Euclidean distance of market-clearing error at \mathbf{p}^* is $\leq \alpha$
market-clearing error $_j$ = demand $_j$ - supply $_j$ if $p_j > 0$
market-clearing error $_j$ = $\max(\text{demand}_j - \text{supply}_j, 0)$ if $p_j = 0$
- (iii) The ratio of the max to the min budget in \mathbf{b}^* is $\leq 1 + \beta$

Exact CEEI: $\alpha = \beta = 0$

Theorem 1

Existence of Approximate CE from Approximate EI

Theorem 1. Let k be the maximum number of courses in any permissible schedule. Define $\sigma = \min(2k, M)$

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2. Moreover, for any budget vector \mathbf{b}' with inequality ratio $\leq 1 + \beta$, and any $\epsilon > 0$, there exists a $(\frac{\sqrt{\sigma M}}{2}, \beta)$ -Approximate CEEI with budgets \mathbf{b}^* that are pointwise within ϵ of \mathbf{b}'

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- ▶ If we seek exact market clearing ($\alpha = 0$) may require arbitrarily large budget inequality (Dictatorship β)
 - ▶ If we seek exactly equal budgets ($\beta = 0$) may require arbitrarily large market clearing error (Identical prefs α)
 - ▶ Theorem 1 indicates that "a little budget inequality goes a long way"

Discussion of Market-Clearing Error

Approximate Efficiency: $\frac{\sqrt{\sigma M}}{2}$ is small in two senses

1. $\frac{\sqrt{\sigma M}}{2}$ does not grow with N (number of agents) or \mathbf{q} (number of copies of each good). As $N, \mathbf{q} \rightarrow \infty$, error goes to zero as a fraction of the endowment (e.g., Starr 1969)

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3. I also show that the bound is tight.

Discussion of Market-Clearing Error

In course allocation, a small amount of market-clearing error likely is not too costly in practice

1. Envelope theorem argument: adding / removing a small number of students close to the optimum
2. Secondary market can correct error in the primary market ("add drop period")

In other contexts, market-clearing error is intolerable.

- ▶ In the paper I describe two variants of the proposed mechanism that have perfect market clearing
- ▶ Of course there are tradeoffs in terms of other properties

Relationship of Theorem 1 to Prior Work on GE w Non-Convexities

Starr (1969)

- ▶ Divisible goods exchange economy
- ▶ Continuous but non-convex preferences
- ▶ In our context, bound would be $\frac{M}{2} \geq \frac{\sqrt{\sigma M}}{2}$ (strict if $k < \frac{M}{2}$)

Dierker (1971)

- ▶ Indivisible goods exchange economy
- ▶ In our context, bound would be $(M - 1)\sqrt{M} \gg \frac{\sqrt{\sigma M}}{2}$

The substantive reason why the Starr and Dierker results cannot apply here is that approximately equal incomes need not be well defined in exchange economies with indivisibilities

That is why I use a Fisher economy in which agents are directly endowed with budgets

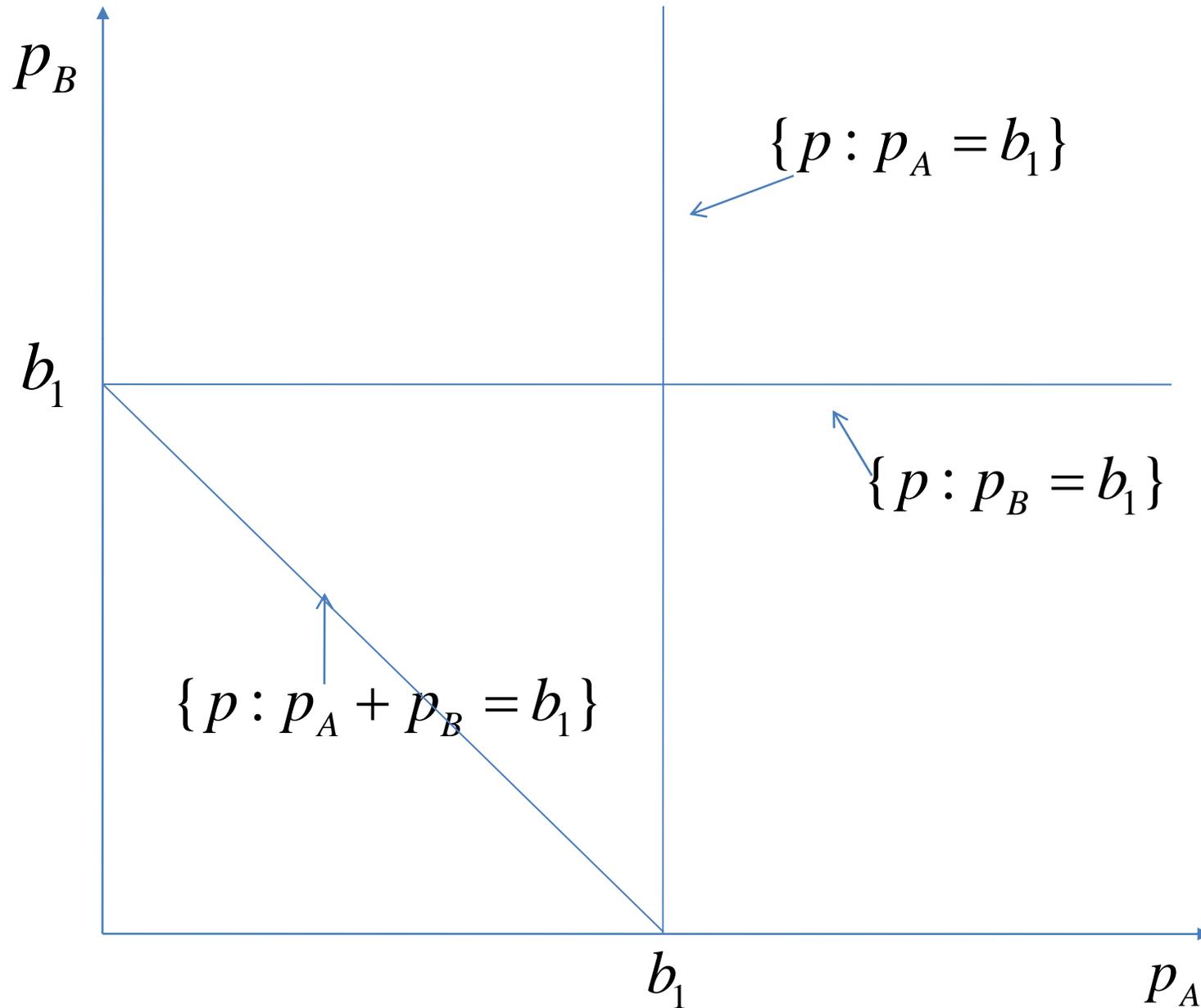
Proof of Theorem 1: Overview

Consider a tâtonnement price-adjustment function of the form

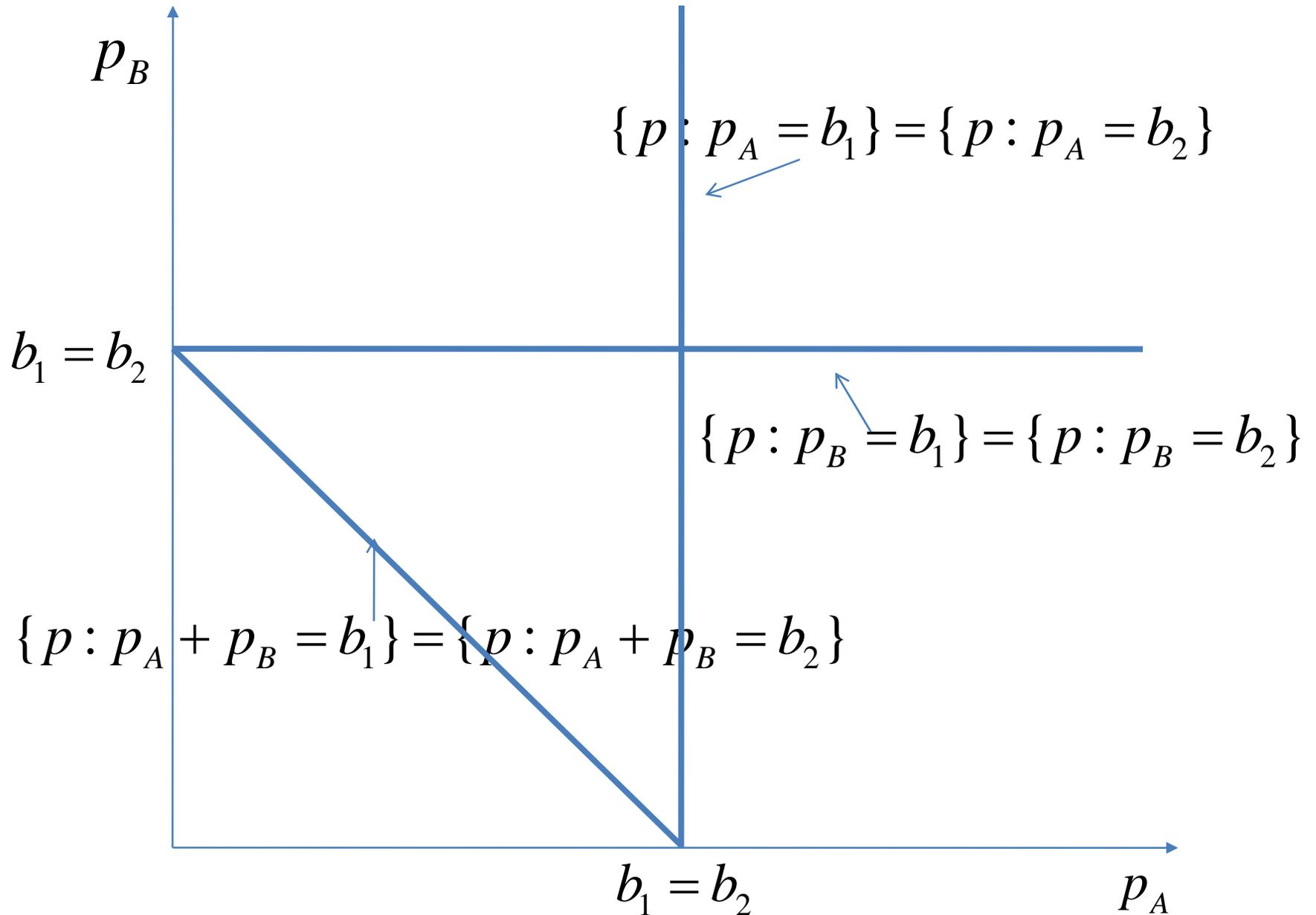
$$f(\mathbf{p}) = \mathbf{p} + z(\mathbf{p})$$

1. Mitigate discontinuities in $f(\cdot)$ using budget perturbations

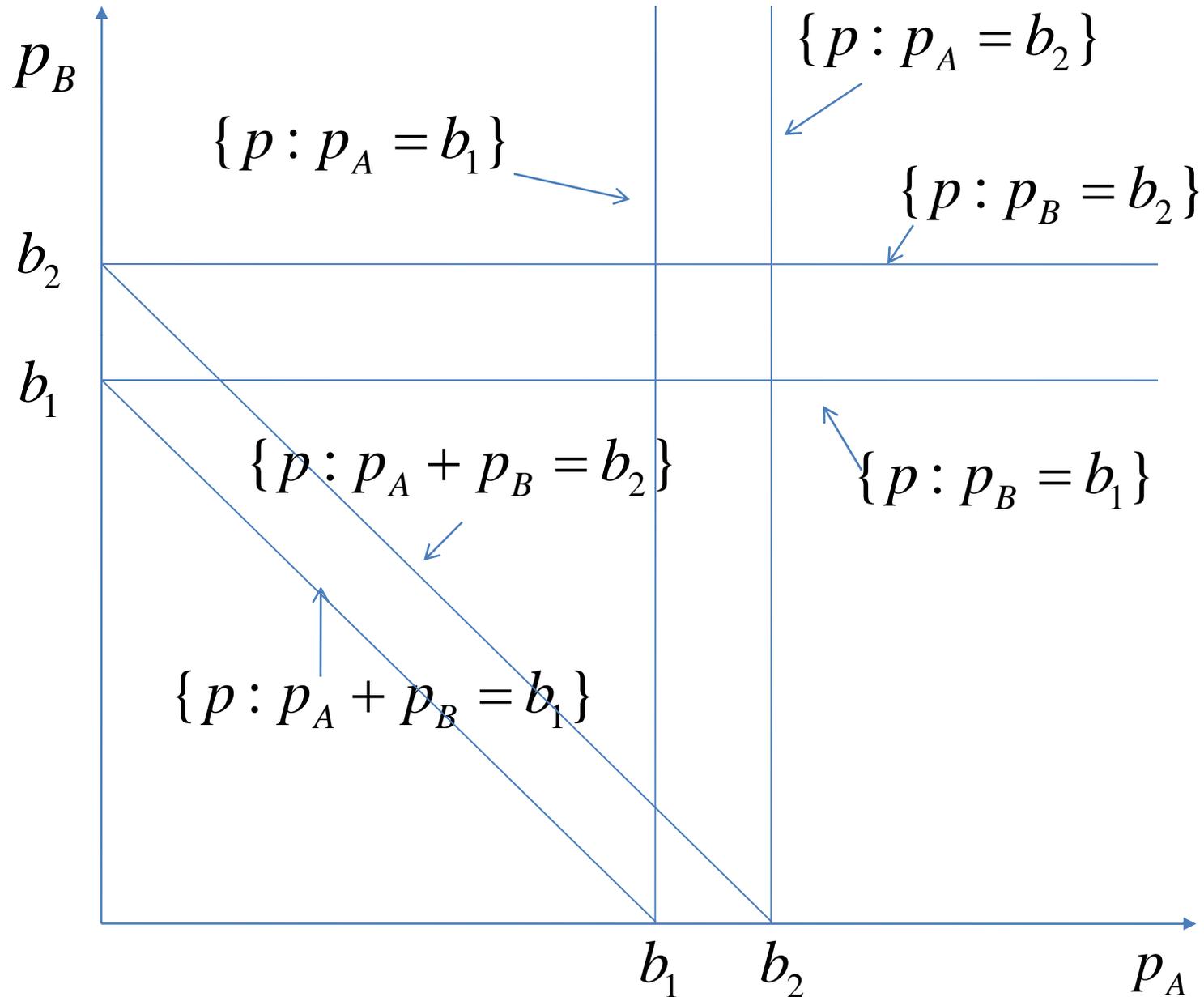
The Role of Budget Inequality: Budget-Constraint Hyperplanes



The Role of Budget Inequality: What if $b_1 = b_2$?



The Role of Budget Inequality: “A Little Inequality Goes A Long Way”



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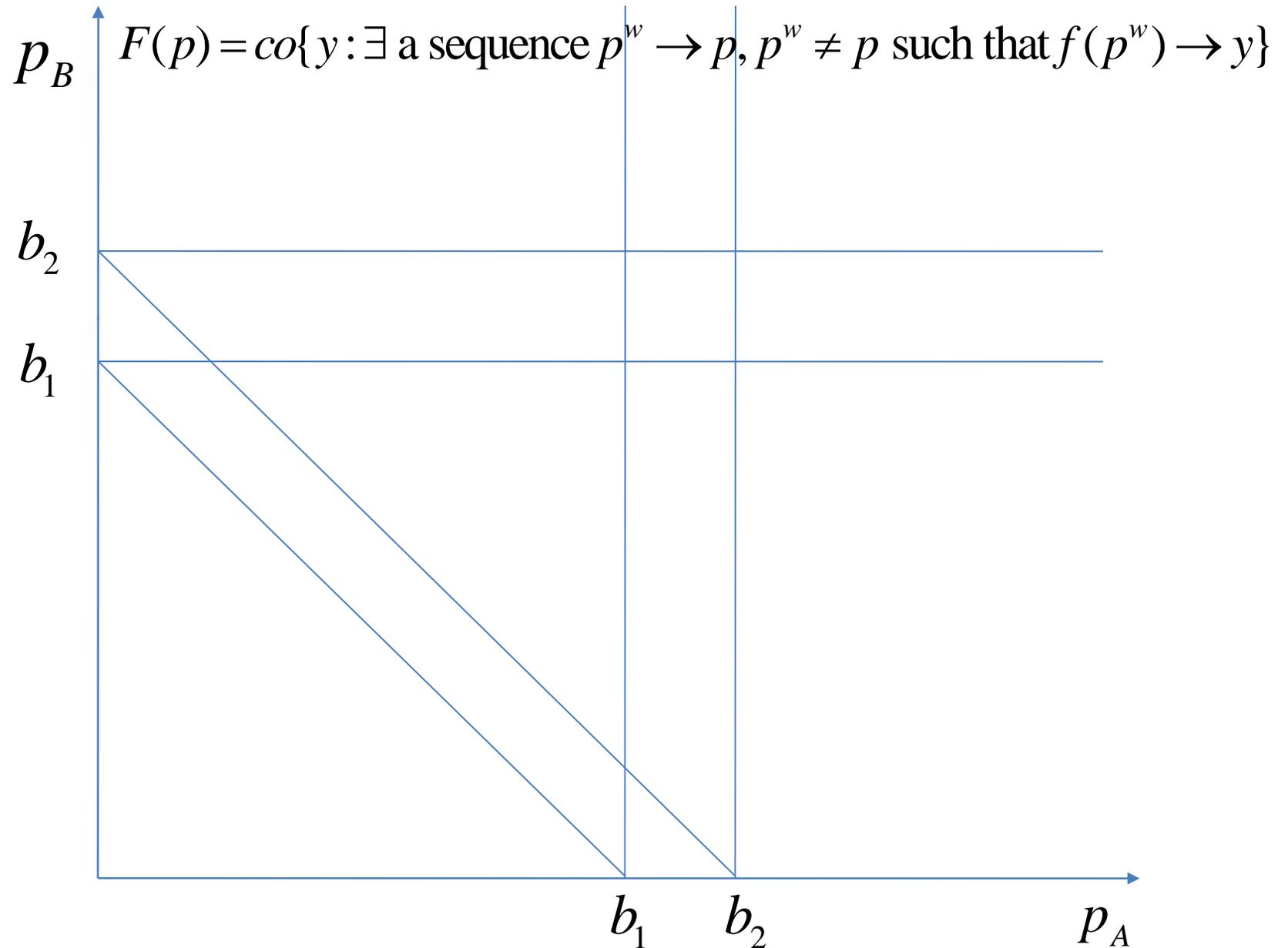
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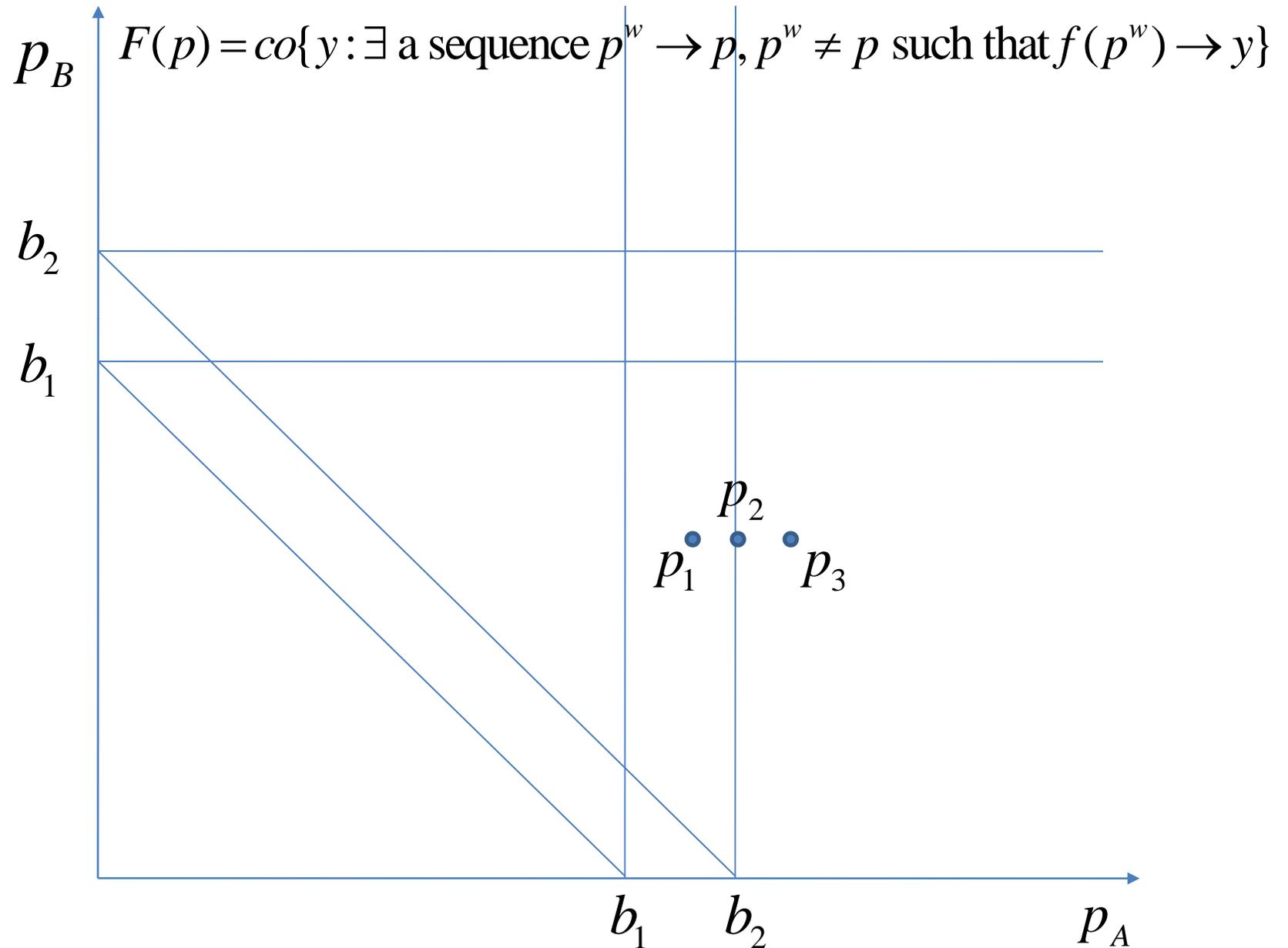
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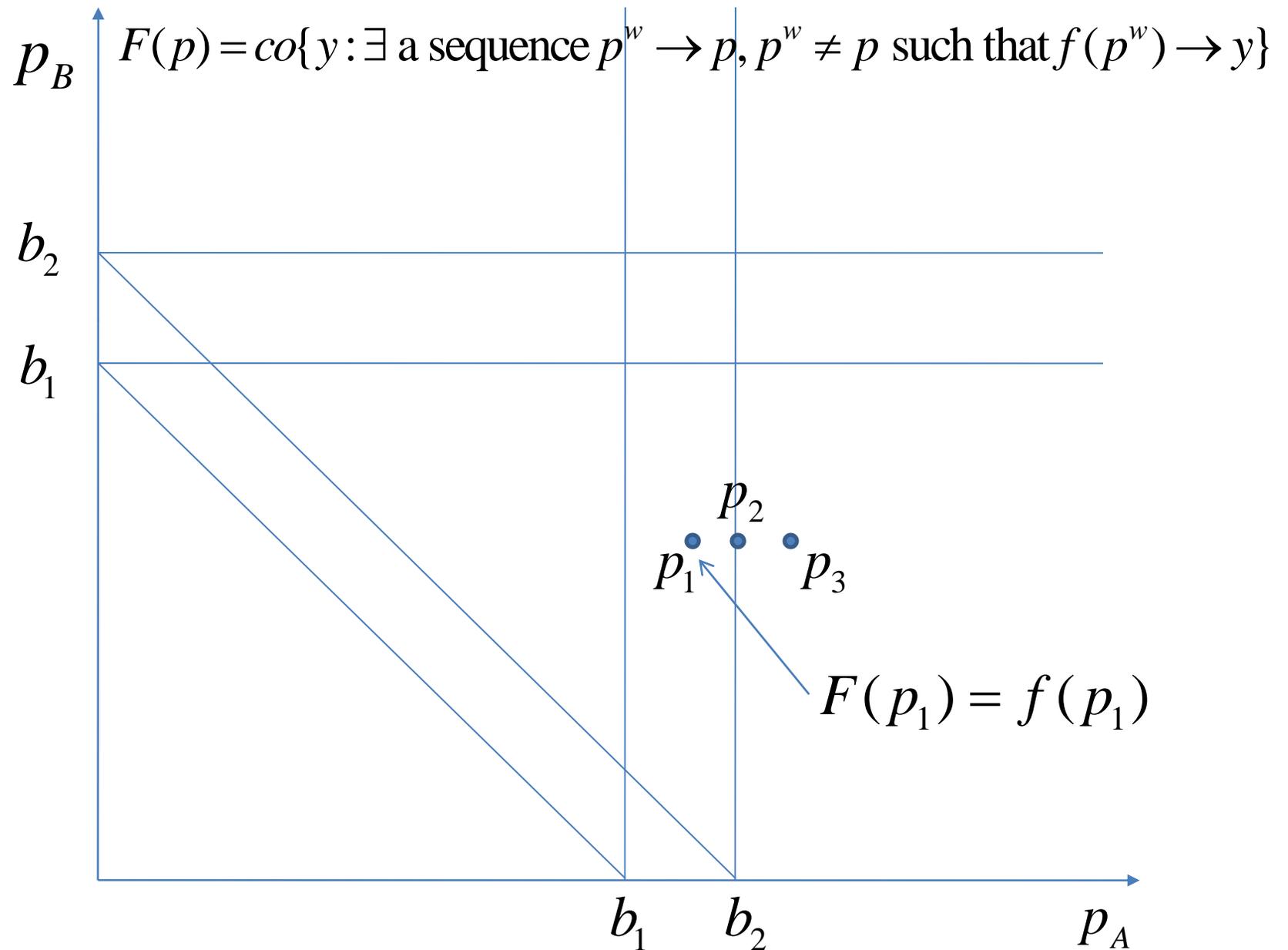
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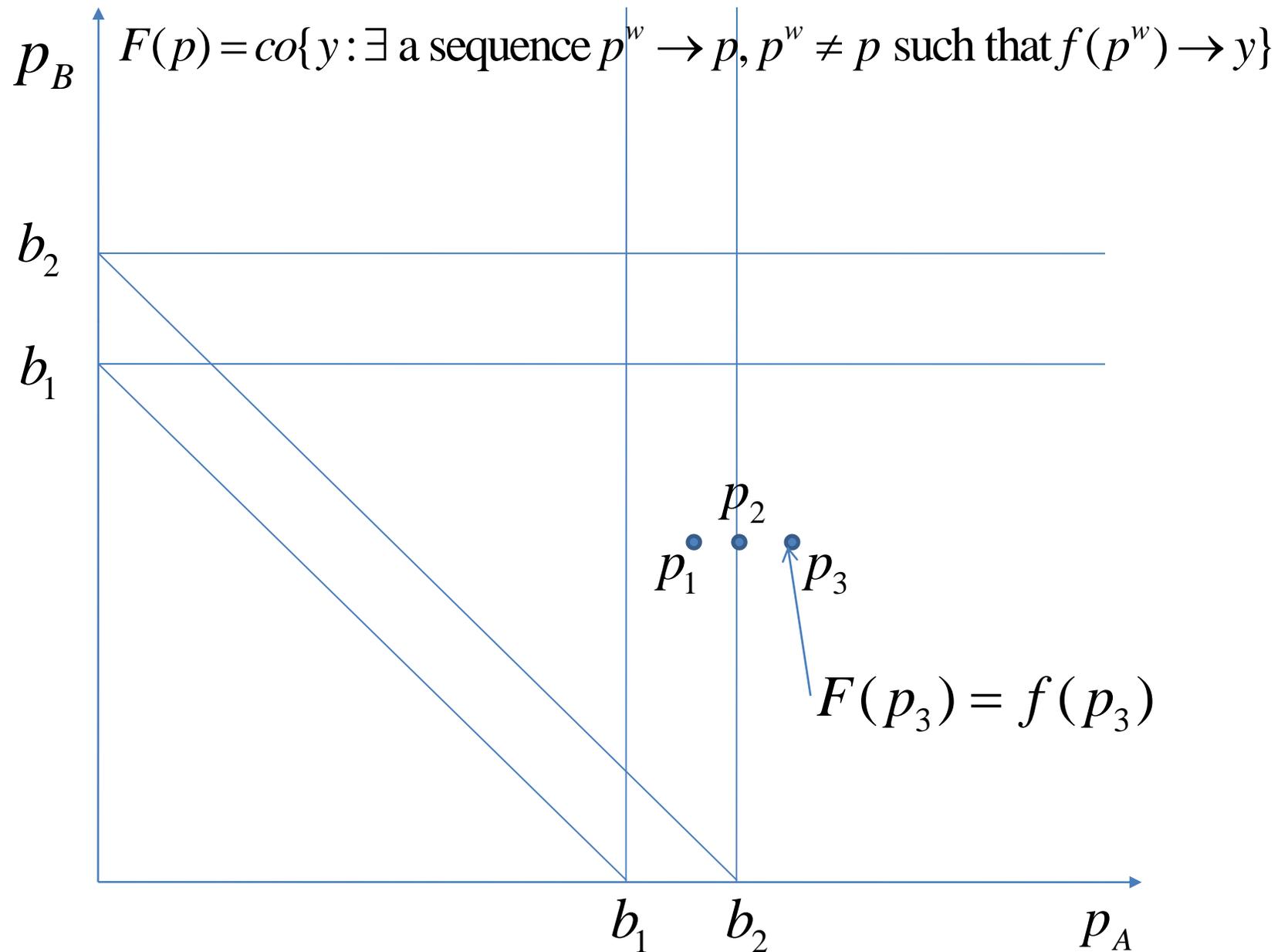
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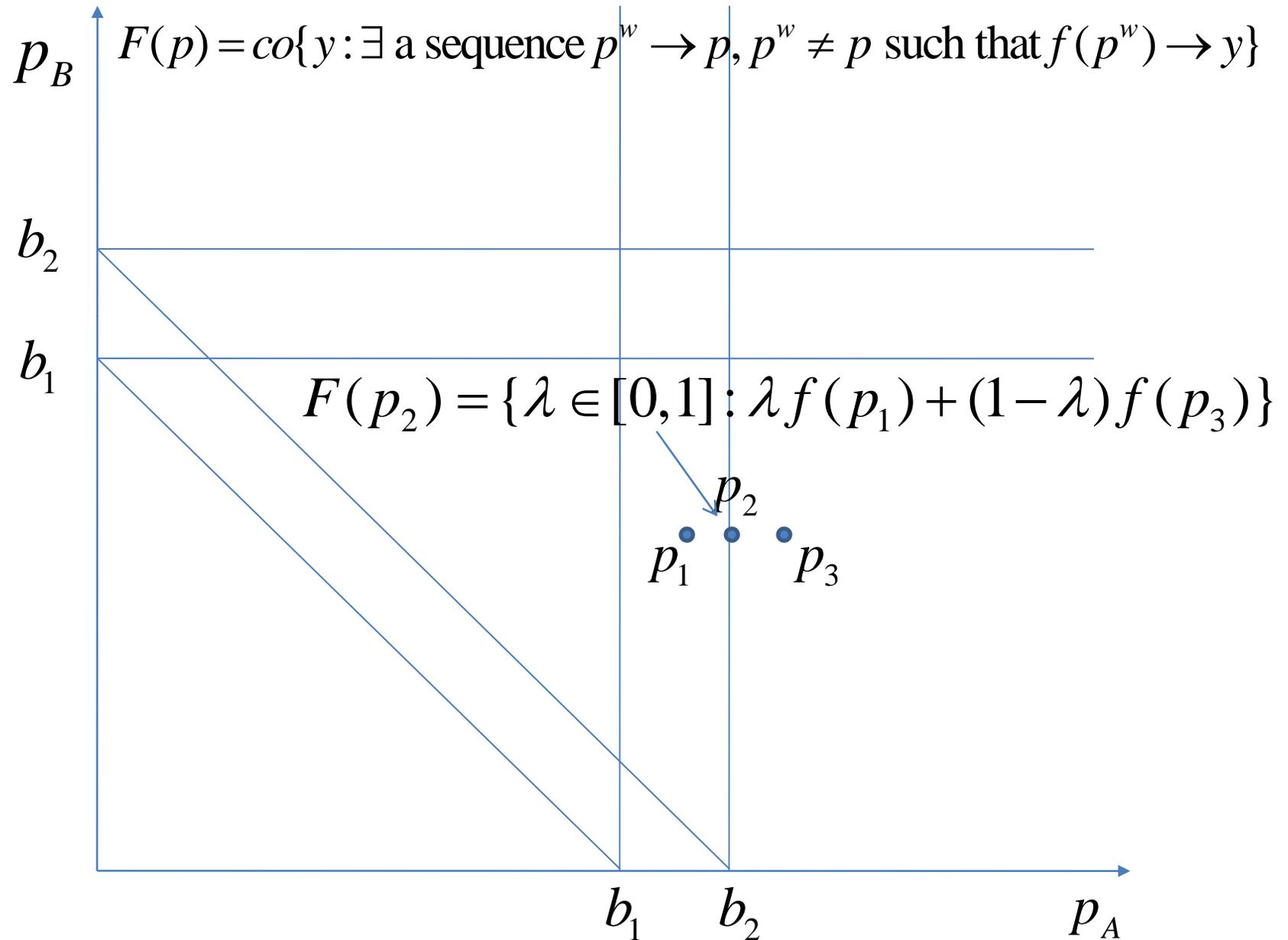
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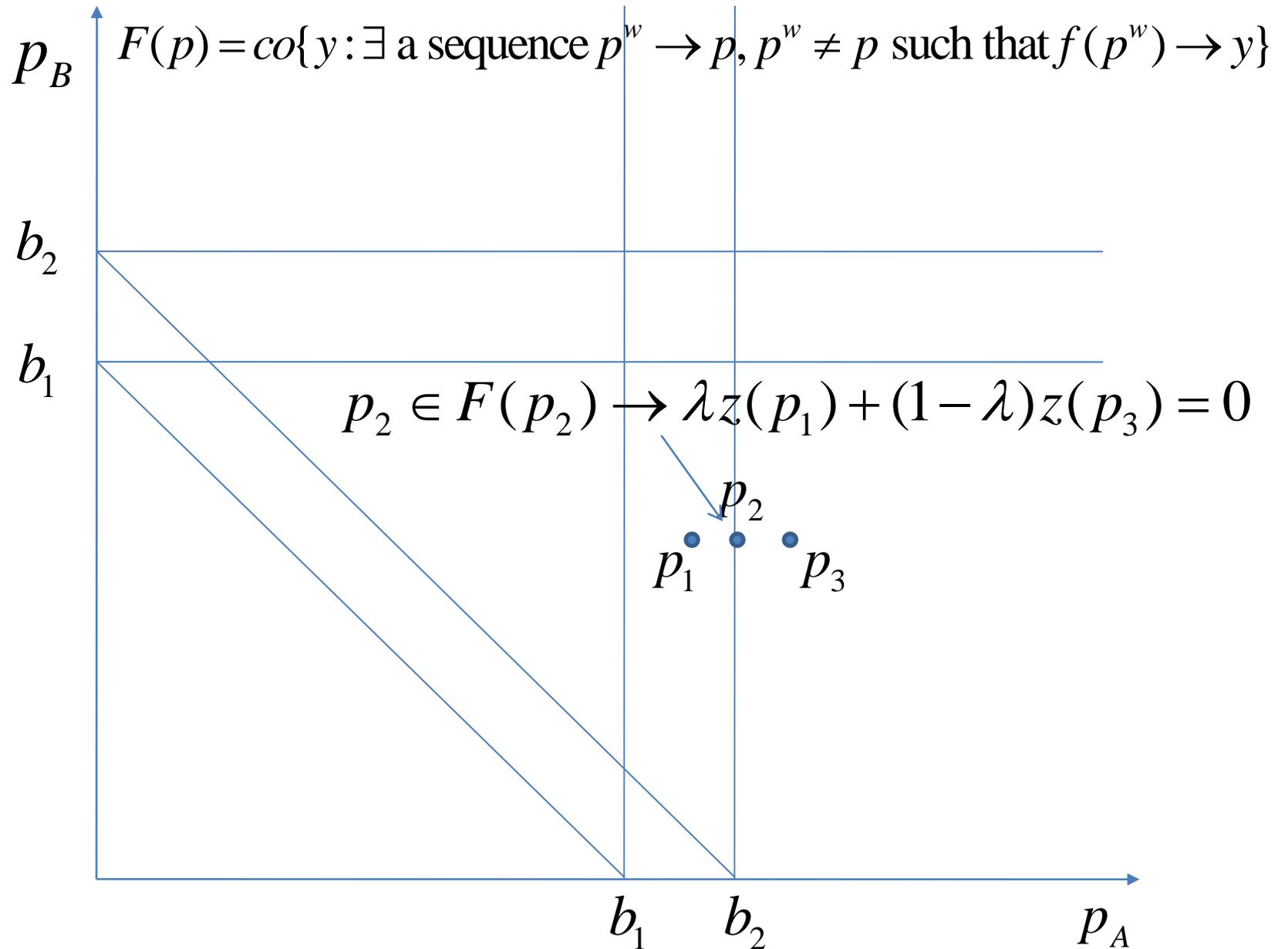
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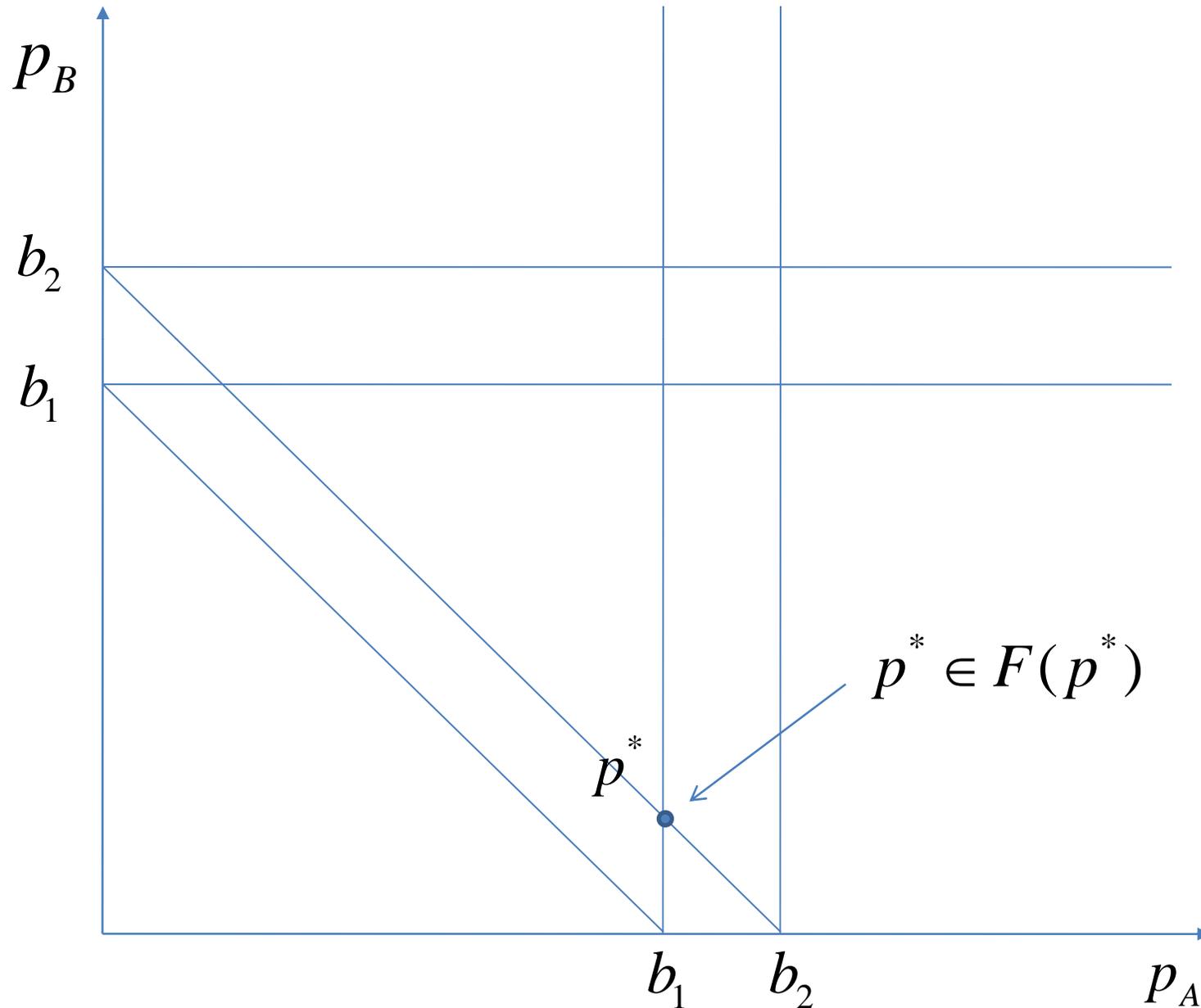
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 - ▶ Key idea: a little inequality goes a long way
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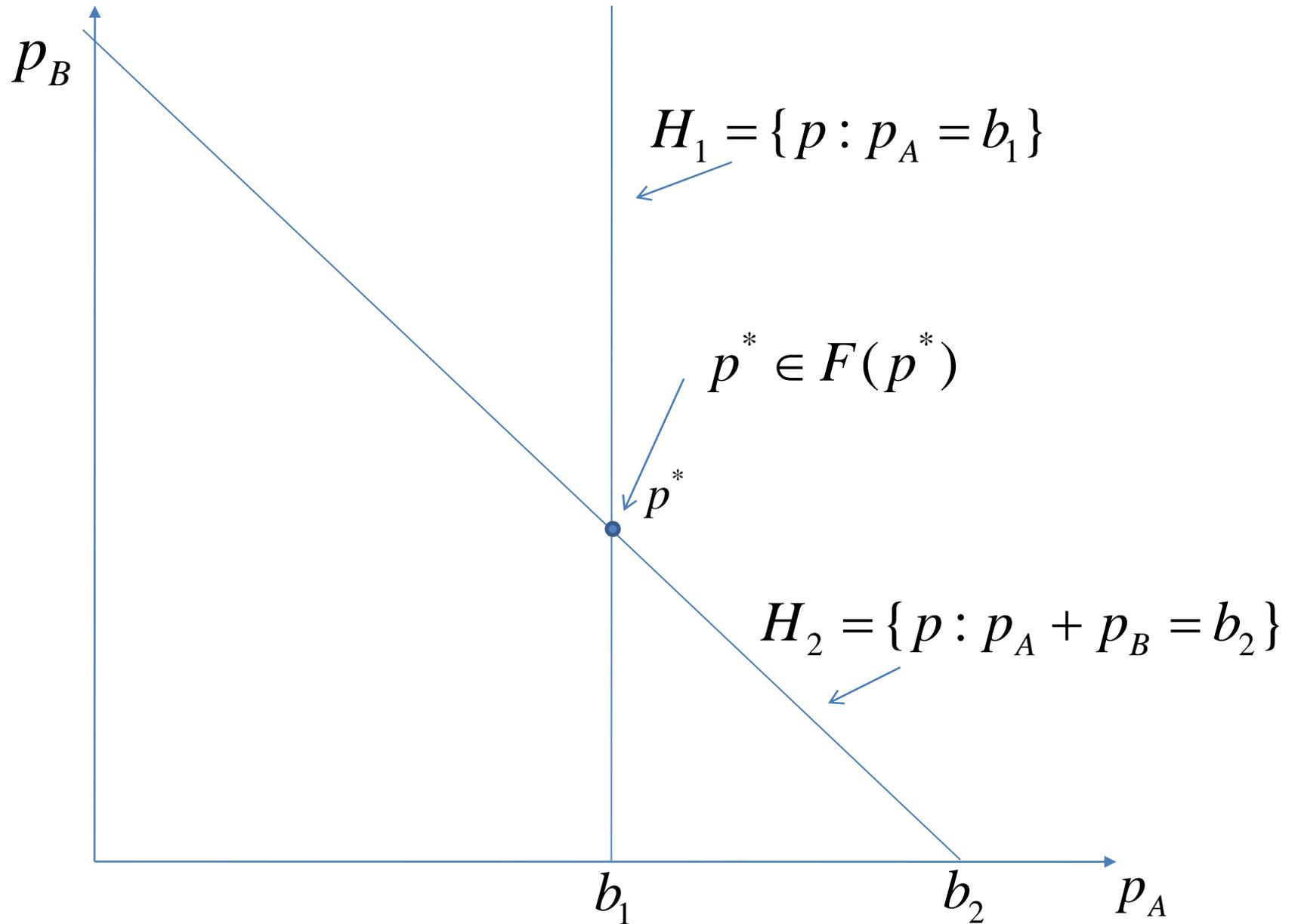
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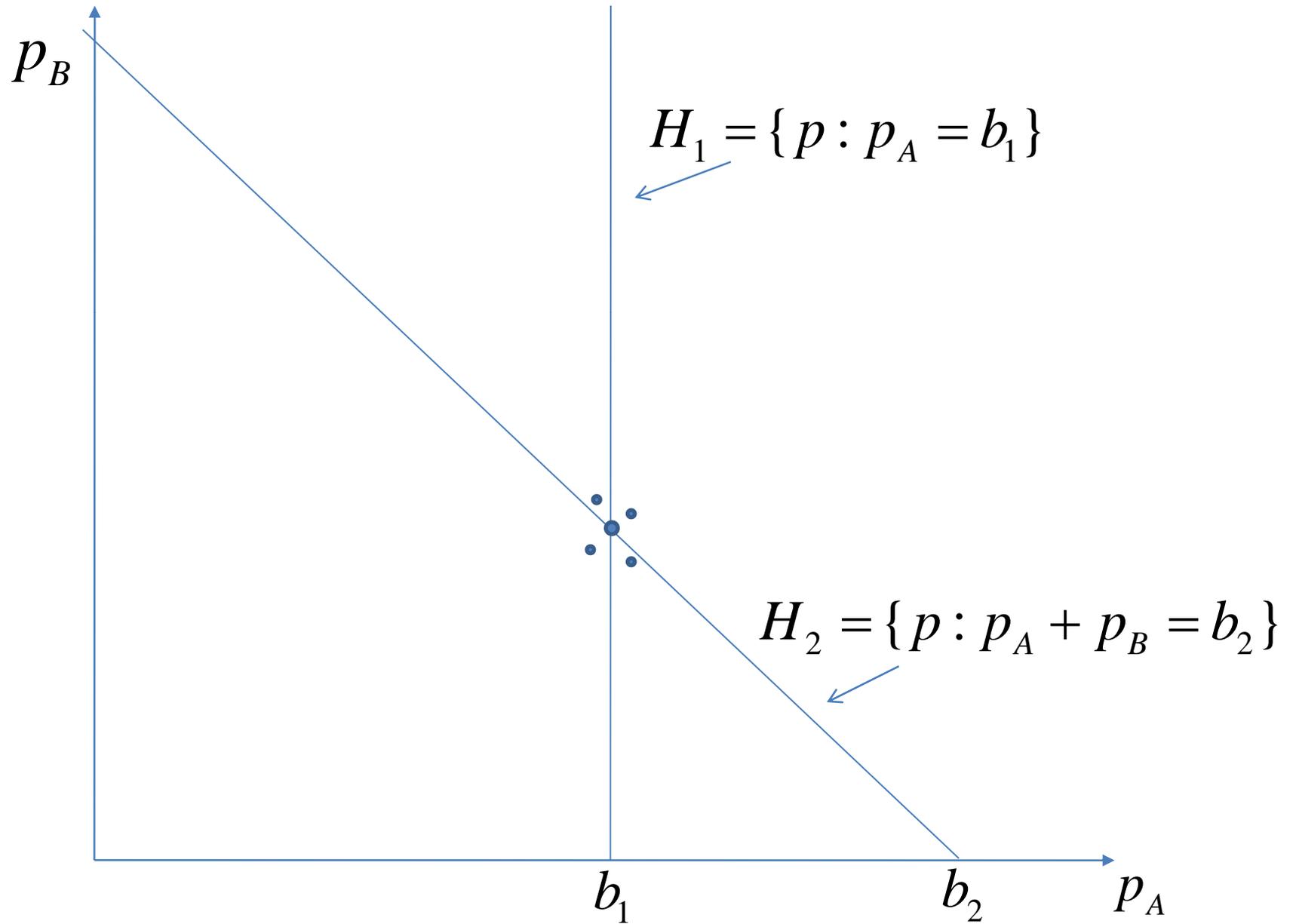
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 - ▶ We can describe demand at these 2^M points using at most M individual-agent change-in-demand vectors

Convexification of $f(p)$ into correspondence $F(p)$:
 $F(p)$ has a fixed point

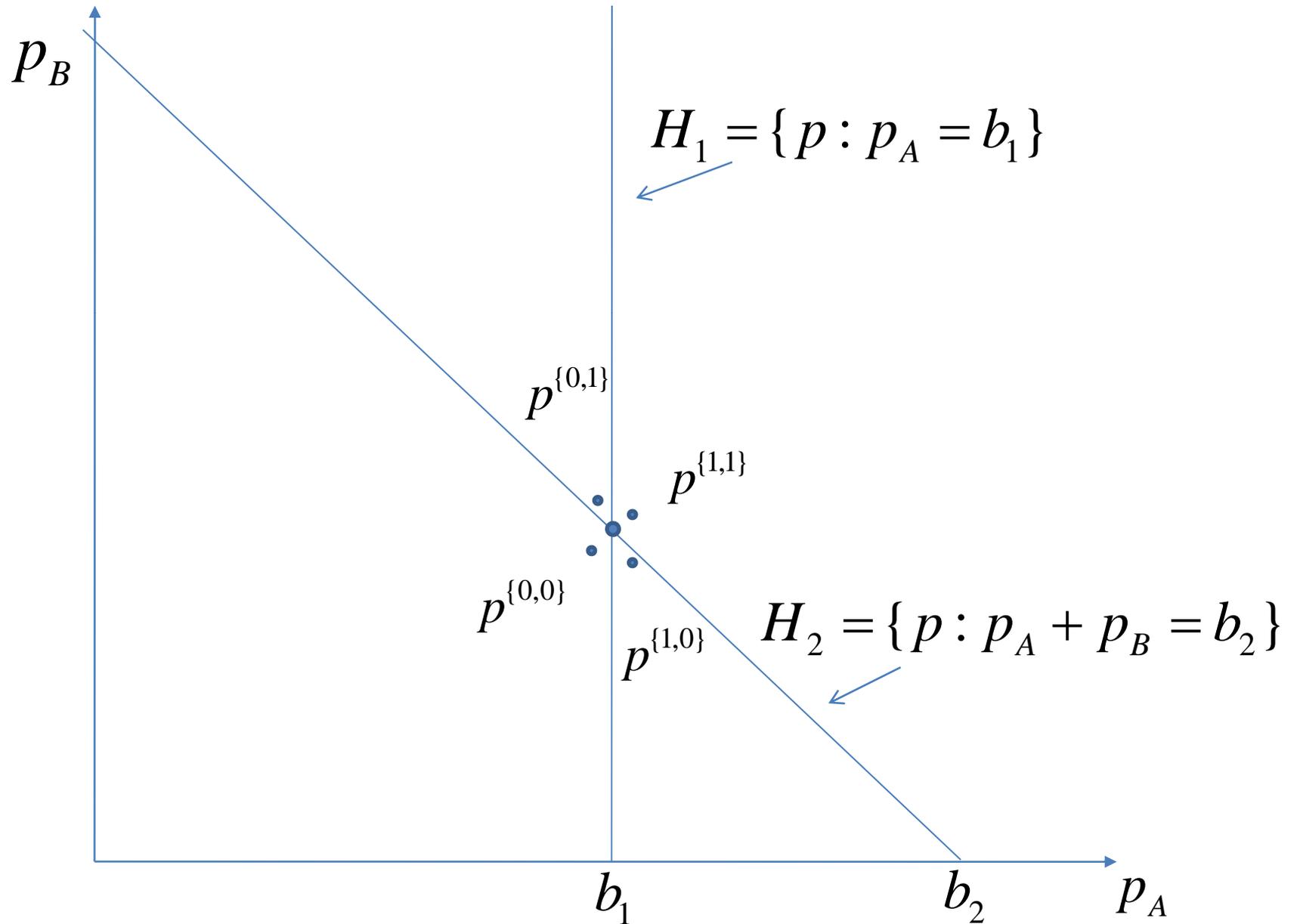


Map from Price Space to Demand Space I

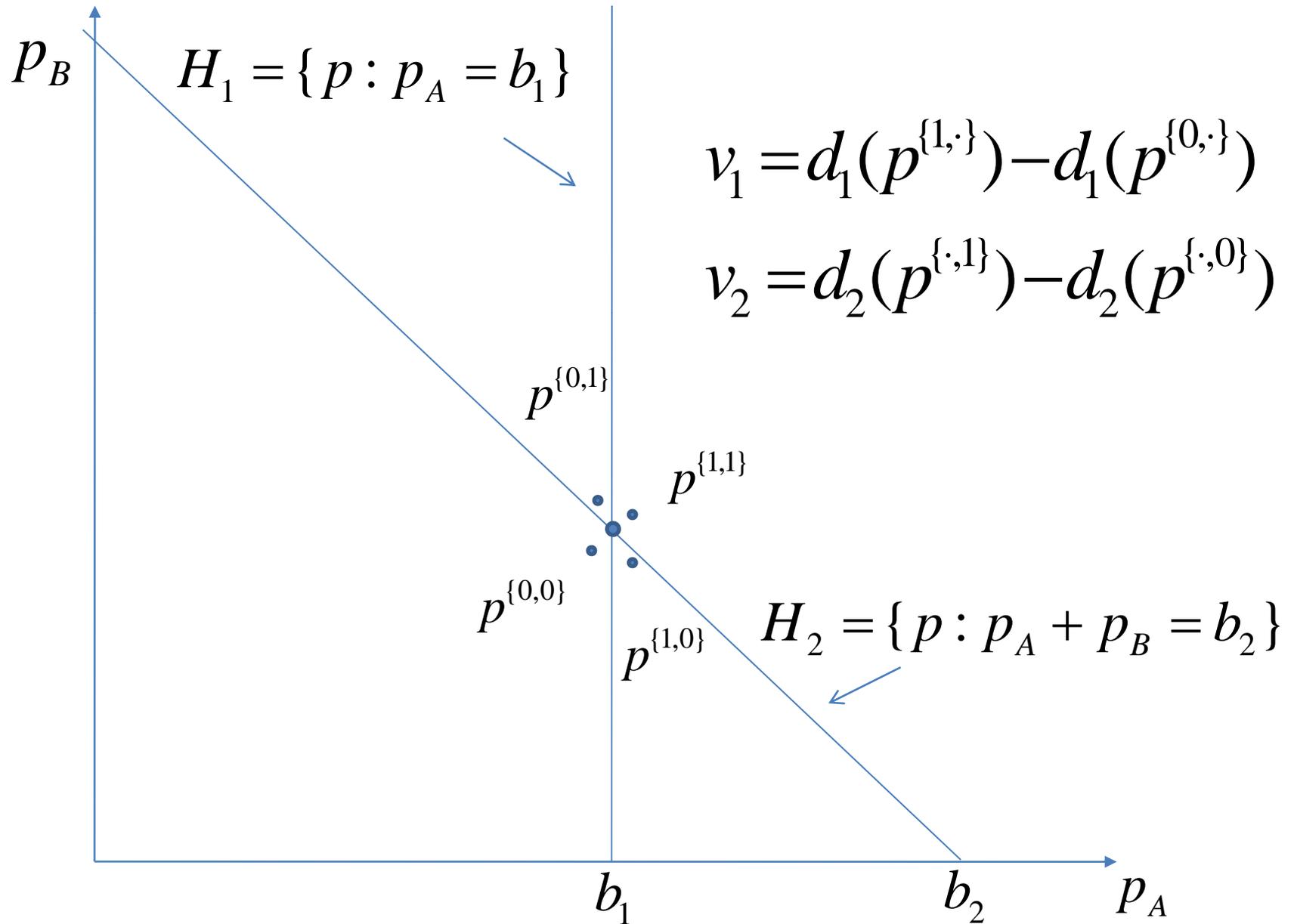
Ball around p^*



Map from Price Space to Demand Space: Ball around p^*



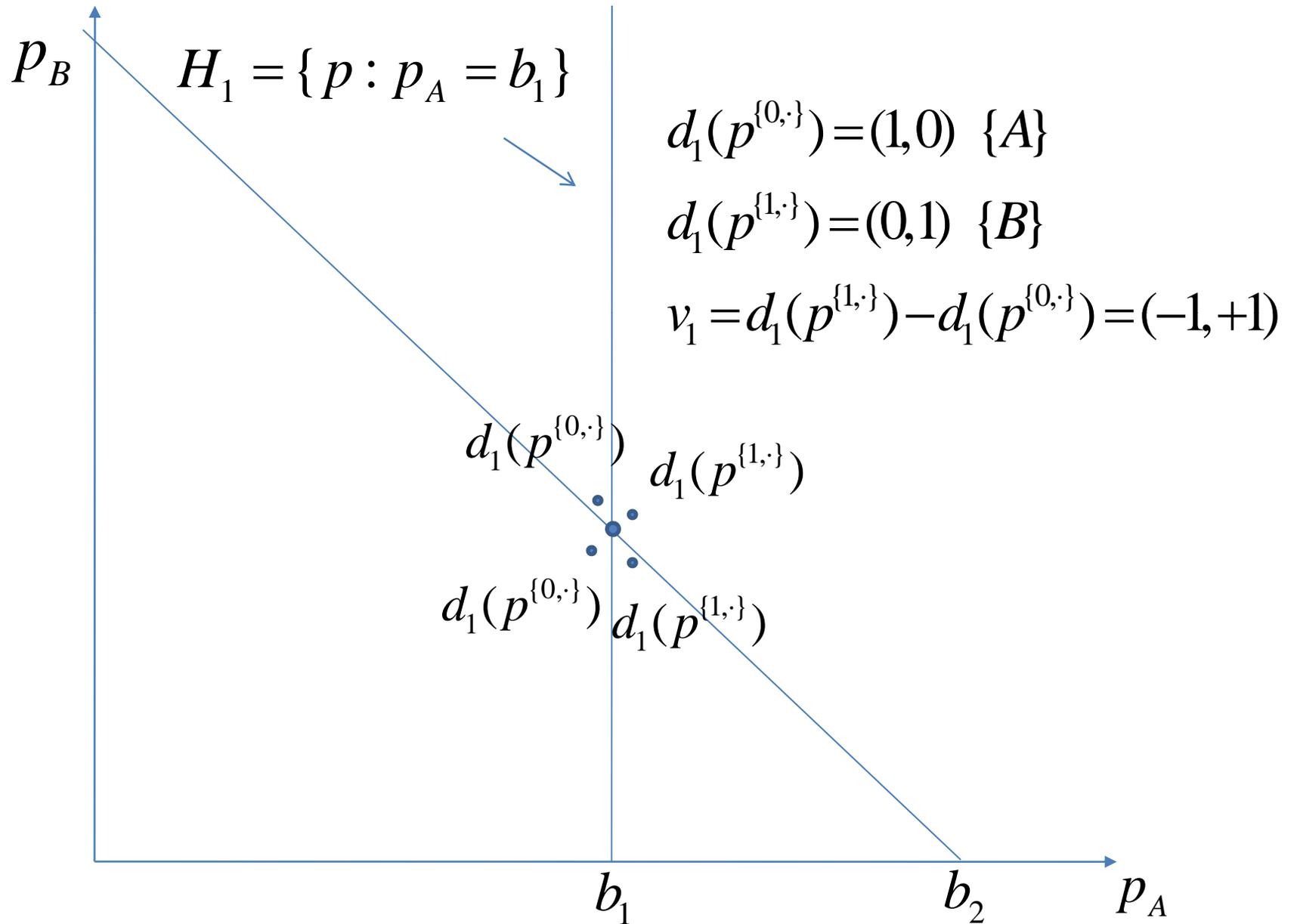
Map from Price Space to Demand Space: “Change-in-Demand” vectors near p^*



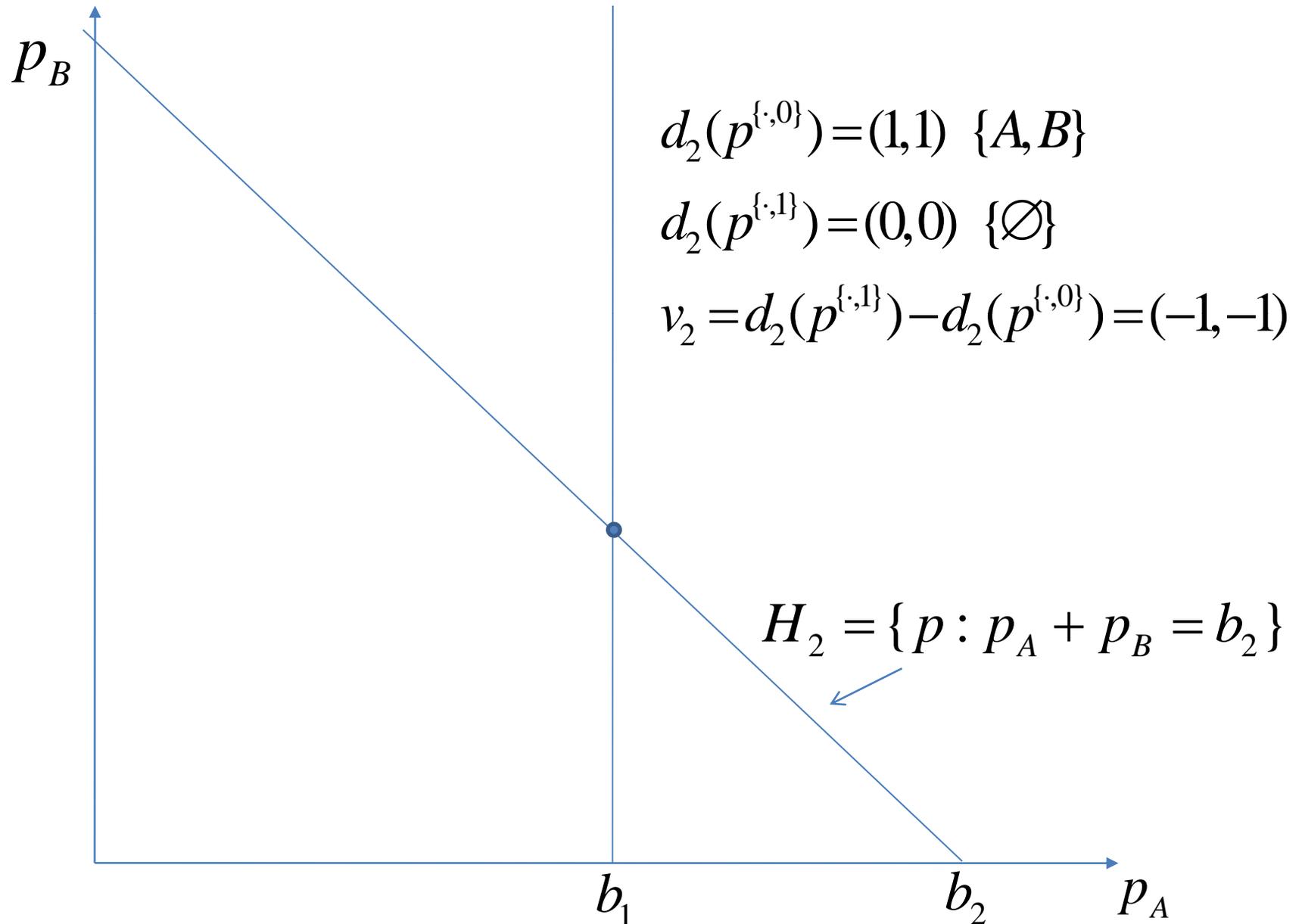
$$v_1 = d_1(p^{\{1,\cdot\}}) - d_1(p^{\{0,\cdot\}})$$

$$v_2 = d_2(p^{\{\cdot,1\}}) - d_2(p^{\{\cdot,0\}})$$

Map from Price Space to Demand Space: Agent 1's "Change-in-Demand" vector near p^*



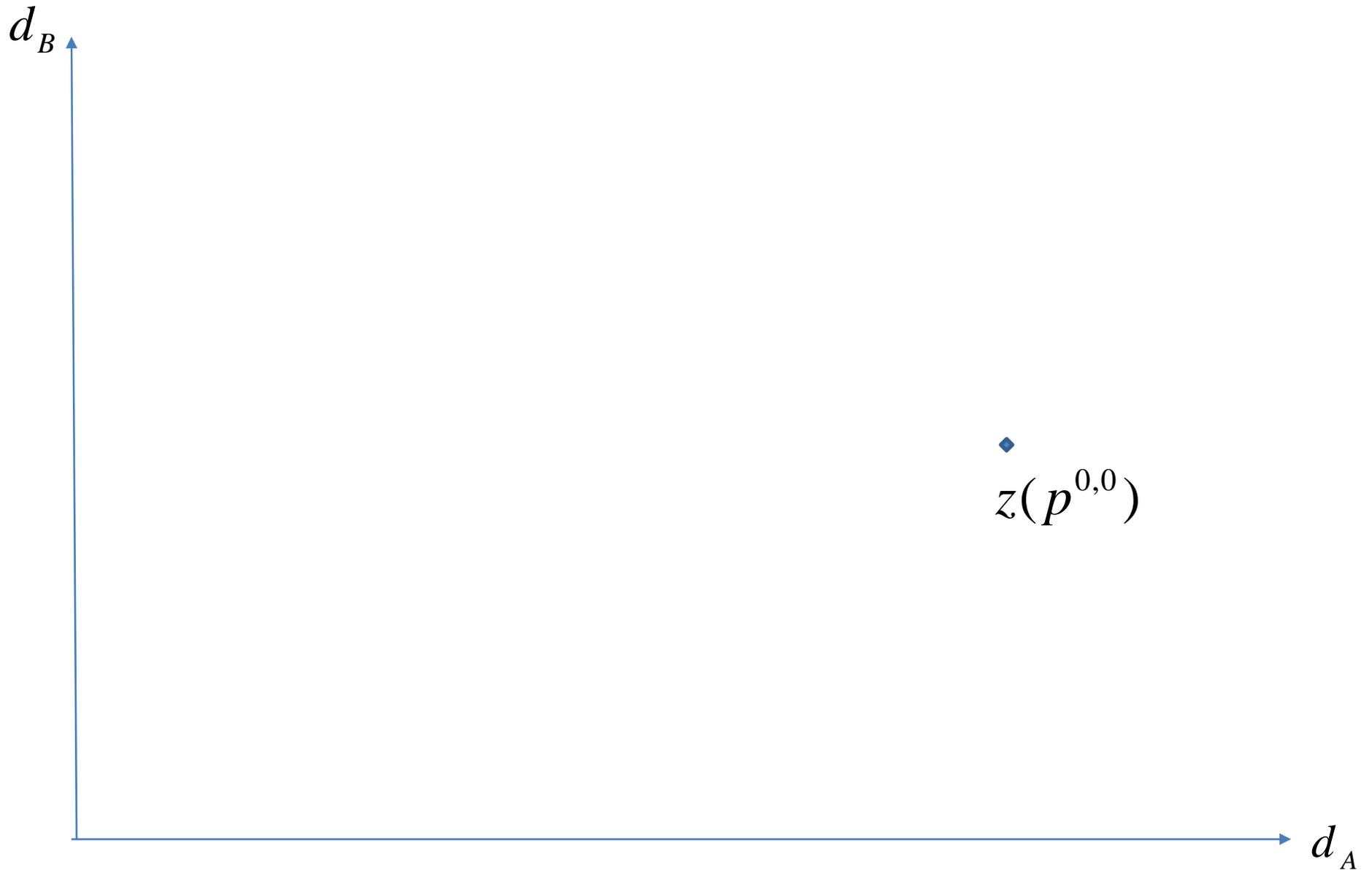
Map from Price Space to Demand Space: Agent 2's "Change-in-Demand" vector near p^*



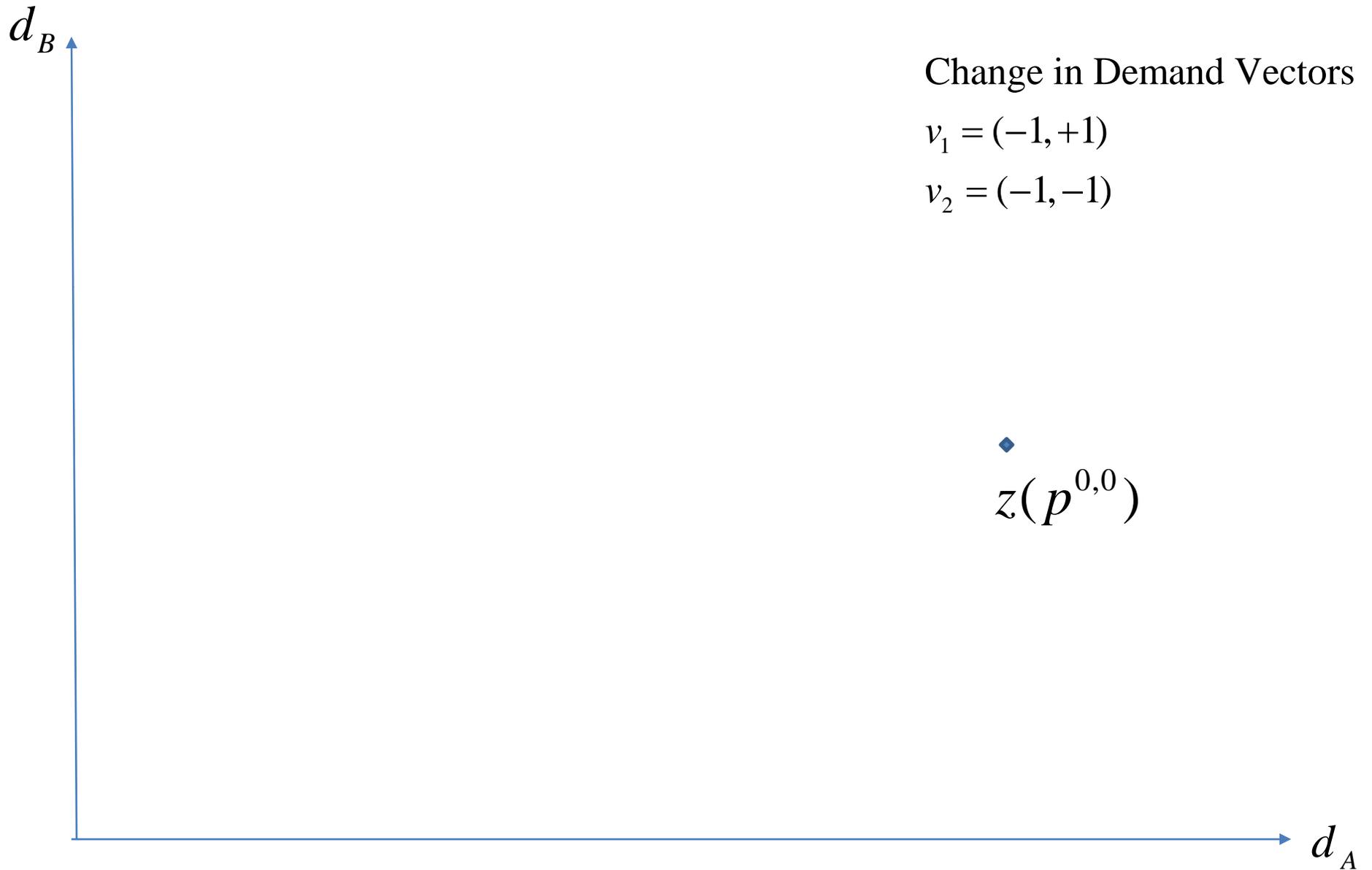
Proof of Theorem 1: Overview

4. Bound market-clearing error, using the structure of demand discontinuities near to \mathbf{p}^* . Use an exact fixed point of $F(\cdot)$ to find an approximate fixed point of $f(\cdot)$

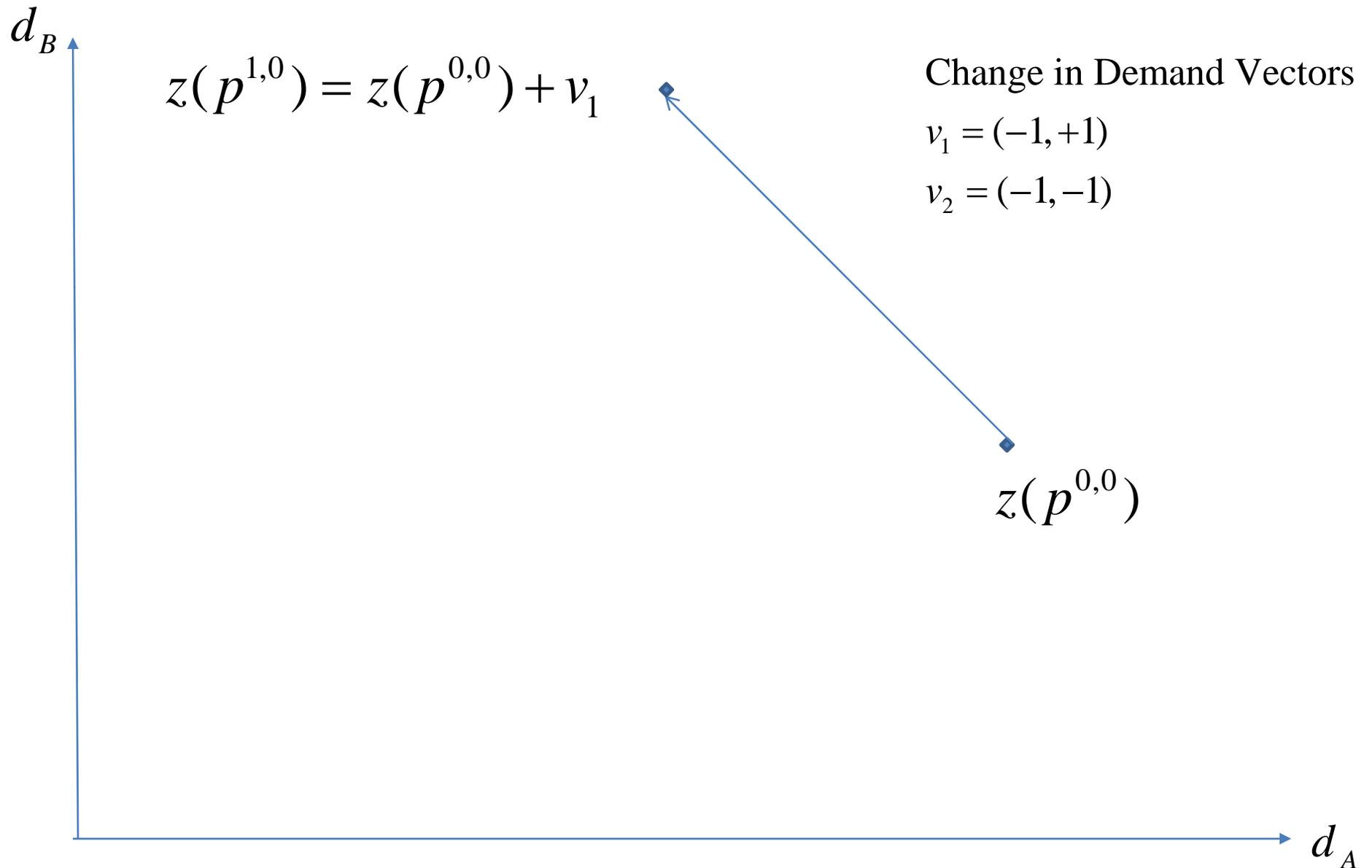
Map from Price Space to Demand Space: Demands near p^* form a zonotope



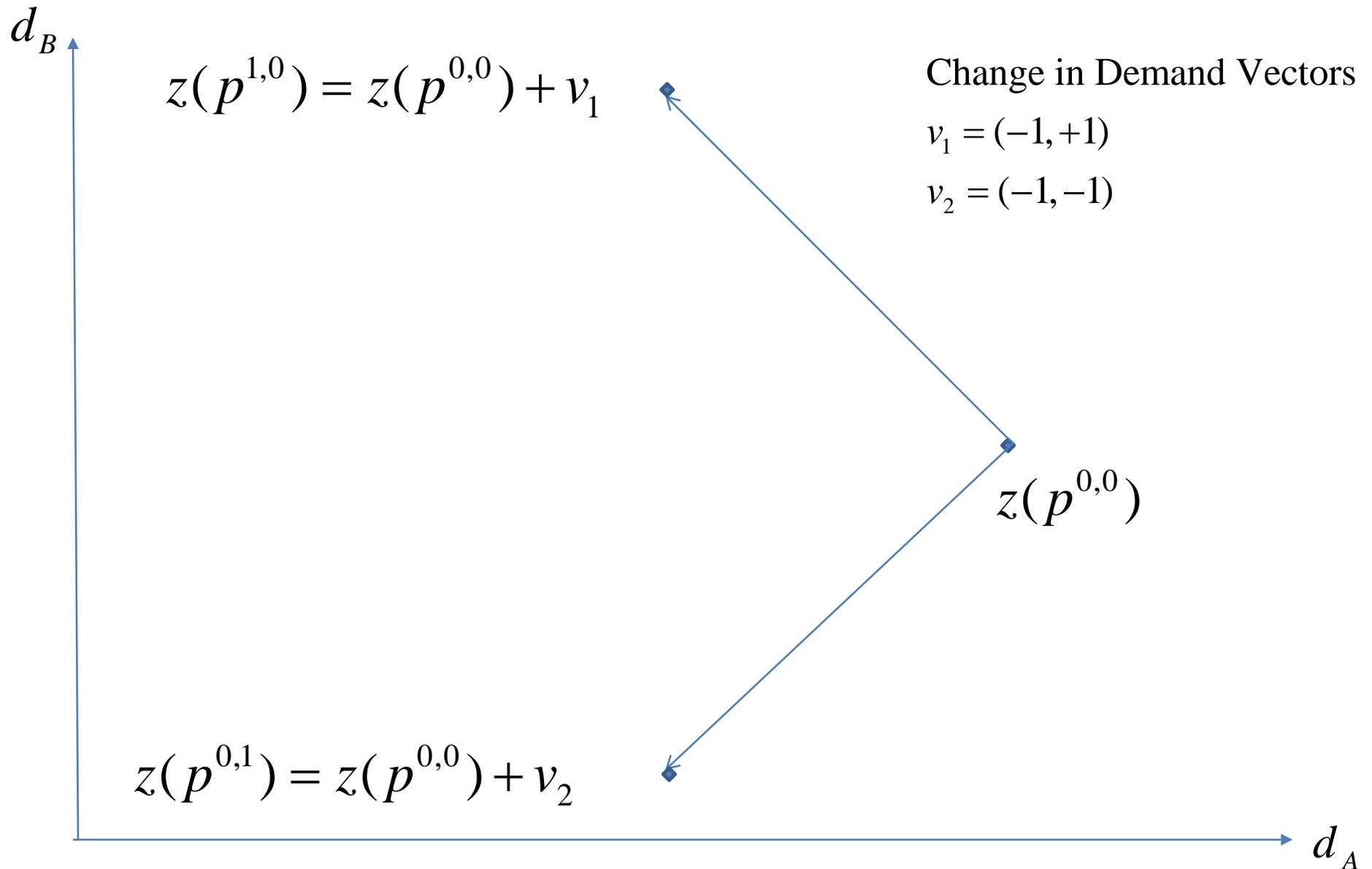
Map from Price Space to Demand Space: Demands near p^* form a zonotope



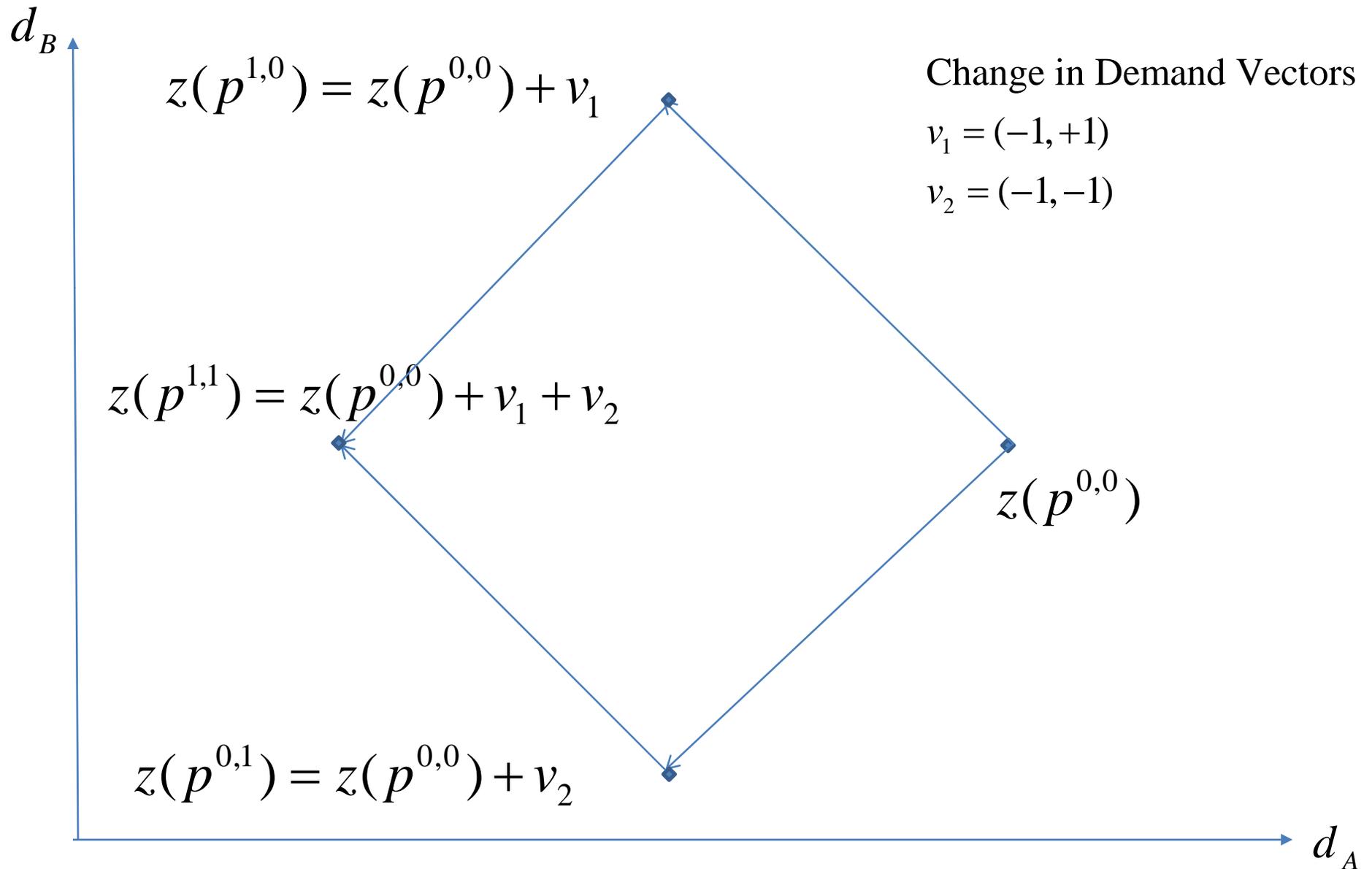
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 - ▶ Bound is half the diagonal of this cube: $\frac{\sqrt{\sigma M}}{2}$

Remarks on Theorem 1

1. Bound is only meaningful if $\sqrt{\sigma}$ is small relative to the endowment.
2. We achieved approximate existence using M item prices, not 2^C bundle prices
3. The monotone price path techniques that have been successfully applied in auction contexts cannot be applied here, due to complementarities
 - ▶ Complementarities are intrinsic to allocation problems with indivisible goods and budget constraints, be they of fake money or real money
 - ▶ In the simple example, Big Diamond and Ugly Rock are complements

Criteria of Outcome Fairness

"In fair division, the two most important tests of equity are 'fair share guaranteed' and 'no envy'" (Moulin, 1995)

*Suppose the goods in the economy, \mathbf{q} , are perfectly divisible. An allocation \mathbf{x} satisfies the **fair-share guarantee** if $u_i(x_i) \geq u_i(\frac{\mathbf{q}}{N})$ for all i*

*An allocation \mathbf{x} is **envy free** if $u_i(x_i) \geq u_i(x_j)$ for all i, j*

In divisible-goods economies, CEEI satisfies both criteria. But indivisibilities complicate fair division:

- ▶ Fair share is not well defined with indivisibilities - what is $\frac{1}{N}$ of the endowment?
- ▶ Envy freeness will be impossible to guarantee with indivisibilities. What if there is just a single "big diamond"?

Previous Approaches to Outcome Fairness with Indivisibilities

There have been several previous approaches to defining outcome fairness in the presence of indivisibilities:

1. Allow for monetary transfers (Alkan et al, 1991)
2. Assume that indivisible goods are actually divisible if needed (Brams and Taylor, 1999)
3. Assess criteria of outcome fairness at an interim stage (Hylland and Zeckhauser, 1979; Bogomolnaia and Moulin, 2001; Pratt, 2007)

Common thread in previous approaches:

- ▶ Modify either the problem, or the time at which fairness is assessed.
- ▶ Then apply traditional criteria.

My approach:

- ▶ Keep my problem as is, but weaken the criteria to accommodate indivisibilities in a realistic way

The Maximin Share Guarantee

I explicitly accept that indivisibilities complicate fair division and propose weaker criteria

Definition. *An allocation*

$$\begin{aligned} \mu_i \in \arg \max_{(x_l)_{k=1}^N} & [\min(u_i(x_1), \dots, u_i(x_N))] \\ \text{s.t. } & (x_l)_{k=1}^N \text{ is feasible} \end{aligned}$$

*is said to be i 's maximin-share split. Agent i 's **maximin share** is any least-preferred bundle in μ_i . A mechanism satisfies the **maximin-share guarantee** if each agent always gets a bundle they weakly prefer to their maximin share.*

- ▶ Divide-and-choose interpretation
- ▶ Rawlsian guarantee from what Moulin (1991) calls a "thin veil of ignorance"
- ▶ Coincides with fair share if goods divisible, prefs convex and monotonic

Envy Bounded by a Single Good

I explicitly accept that indivisibilities complicate fair division and propose weaker criteria

Definition 2. *An allocation x satisfies **envy bounded by a single good** if*

For any two agents i, i' either:

$$(i) \quad u_i(x_i) \geq u_i(x_{i'}) \text{ or}$$

$$(ii) \quad u_i(x_i) \geq u_i(x_{i'} \setminus \{j\}) \text{ for some } j \in x_{i'}$$

- ▶ In words: if student i envies i' , the envy is bounded: by removing some single good from i' 's bundle we could eliminate i' 's envy
- ▶ Coincides with envy-freeness in a limit as consumption bundles become perfectly divisible

Diamonds and Rocks Revisited

Two agents. Four objects: two Diamonds (Big, Small) and two Rocks (Pretty, Ugly). At most two objects per agent.

$$\begin{aligned}\text{Maximin Share} &= \min[u(\{\text{Big Diamond, Ugly Rock}\}), \\ &\quad u(\{\text{Small Diamond, Pretty Rock}\})] \\ &= u(\{\text{Small Diamond, Pretty Rock}\})\end{aligned}$$

- ▶ So the A-CEEI allocation in which one agent obtains {Small Diamond, Pretty Rock} and the other obtains {Big Diamond, Ugly Rock} gives each agent at least their maximin share
- ▶ This allocation also satisfies envy bounded by a single good: striking the Big Diamond from the wealthier agent's bundle would eliminate the other agent's envy

Dictatorships and Fairness

- ▶ Dictatorships are procedurally fair if the choosing order is uniform random
- ▶ However, dictatorships fail the outcome fairness criteria: whichever student chooses first gets both diamonds
- ▶ The criteria thus help to formalize why dictatorships are unfair in multi-unit assignment. By contrast:

Remark 1: In single-unit assignment (e.g., one diamond, one rock), dictatorships satisfy the maximin-share guarantee and envy bounded by a single good.

- ▶ Dictatorships are frequently used in practice for single-unit assignment problems (school choice, housing assignment)
- ▶ The fairness properties help us to make sense of the empirical patterns of dictatorship usage. Useful external validity check.

Fairness Properties of Approximate CEEI

To what extent do approximately equal budgets guarantee that students will receive fair outcomes ex-post? We might worry for several reasons

- ▶ In single-unit demand case, cardinal budget information is meaningless; all that matters is the order of the budgets
 - ▶ e.g., two students and two objects, no difference between budgets of $(1000, 999)$ and $(1000, 1)$. In either case, the budget of 1000 gets his favorite object.
- ▶ More generally, since goods are indivisible, students' optimal consumption bundles might not exhaust their budgets.
 - ▶ e.g., a student whose favorite bundle costs 1000 and whose second favorite bundle costs 1 doesn't care if her budget is 999 or 1.

Theorem 2: Approximate CEEI Guarantees Approximate Maximin Shares

Theorem 2: if $\beta < \frac{1}{N}$ then \mathbf{x}^* guarantees each agent their $N + 1$ -maximin share (maximin share in a hypothetical economy with one additional agent)

Intuition for proof:

1. If $\beta < \frac{1}{N} \Rightarrow$ even poorest student has $> \frac{1}{N+1}$ of the income endowment
2. If \mathbf{p}^* is an exact c.e. \Rightarrow goods endowment costs weakly less than the income endowment.
3. So if \mathbf{p}^* is an exact c.e., each student must be able to afford some bundle in any $N + 1$ -way split.
4. Hence, each student must be able to afford some bundle weakly preferred to her $N + 1$ -maximin share.

The full argument is a bit messier because \mathbf{p}^* might be an approximate c.e.

Theorem 3: Approximate CEEI Guarantees that Envy is Bounded by a Single Good

Theorem 3: if $\beta < \frac{1}{k-1}$ then \mathbf{x}^* satisfies envy bounded by a single good

Sketch of proof:

- ▶ Suppose i envies j . Then

$$1 \leq b_i^* < \mathbf{p}^* \cdot \mathbf{x}_j^* \leq b_j^* \leq \frac{k}{k-1}$$

- ▶ Since \mathbf{x}_j^* contains at most k goods, one of them must cost at least $\frac{1}{k-1}$. i can afford the bundle formed by removing this good from \mathbf{x}_j^*
- ▶ By revealed preference, i must weakly prefer her own bundle to the bundle formed by removing this single good from \mathbf{x}_j^* , so her envy is bounded.

Notice that budget inequality plays slightly different roles in the two proofs.

The Approximate CEEI Mechanism (A-CEEI)

1. Agents report their preferences
2. Agents are given **approximately** equal budgets of an artificial currency (uniform draws from $[1, 1 + \beta]$ for β suitably small)
3. We find an item price vector \mathbf{p}^* such that, when each agent i is allocated his favorite bundle in his budget set $\{x \in \Psi_i : \mathbf{p}^* \cdot x \leq b_i^*\}$ the market **approximately** clears (market-clearing error as small as possible, and certainly no larger than $\frac{\sqrt{\sigma M}}{2}$)
4. We allocate each agent their demand at \mathbf{p}^*

Note 1: choosing budgets and prices uniform randomly ensures that the procedure is Strategyproof in the Large. There are other such tie-breaking rules.

Note 2: we can add a step in which we first seek an exact CEEI.

Incentives

- ▶ A-CEEI is not strategyproof in finite markets, but instead is only SP in a limit economy in which agents are price takers (Theorem 4)
- ▶ I call this "Strategyproof in the Large"
- ▶ This seems like a very mild criterion of approximate IC. However
 1. It has bite in the design of A-CEEI: if we ignored incentives we could execute Pareto-improving trades ex-post and correct market-clearing error
 2. It has bite in practice: all course-allocation mechanisms currently found in practice are manipulable even by price takers
 - ▶ Budish and Cantillon (2009): empirically, this manipulability has welfare consequences

Manipulability of A-CEEI in Finite Markets

Even in small markets it is not obvious how to manipulate A-CEEI

- ▶ The usual way to manipulate a competitive equilibrium mechanism is to withhold some portion of one's demand for a good: get less of the good, but at a sufficiently lower price
- ▶ Here, demand is 0-1. So demand reduction does not work.
- ▶ A student can certainly lower the price of some star professor's course by pretending not to demand it, but this is not a useful manipulation.

Example in which A-CEEI is Manipulable

Two agents $\{i, j\}$, and four objects, $\{a, b, c, d\}$

$u_i : \{d, a\}, \{d, b\}, \{d, c\}, \dots$

$u_j : \{a, b\}, \{a, c\}, \{a, d\}, \dots$

There are two exact CEEIs: \mathbf{x}^* in which i gets $\{d, b\}$ and \mathbf{x}^{**} in which i gets $\{d, c\}$. The mechanism will randomize between the two. Suppose i misreports his preferences as

$\hat{u}_i : \{a, b\}, \{d, a\}, \{d, b\}, \{d, c\}, \dots$

That is, i feigns envy for j 's allocation under \mathbf{x}^{**} . This kills \mathbf{x}^{**} being a CEEI, leaving only \mathbf{x}^* .

Manipulability of A-CEEI in Finite Markets

Notice that i 's manipulation is informationally demanding and potentially risky

- ▶ i has to know that by feigning envy for $\{a, b\}$ he
 - ▶ will kill \mathbf{x}^{**} being a CEEI
 - ▶ will not actually get allocated the bundle he is pretending to like
- ▶ Simulation evidence suggests that such manipulations are not profitable in markets that are larger and in which agents have some uncertainty about others' preferences
- ▶ A formal convergence result is beyond the scope of this paper (the relationship between agents' reports and prices is too non-constructive)

Properties of the Approximate CEEI Mechanism

Efficiency

- *Ex-post efficient, but for small error*

Fairness

- *Symmetric*
- *$N+1$ Maximin Share Guaranteed*
- *Envy Bounded by a Single Good*

Incentives

- *Strategyproof in the Large*

Comparison to Other Mechanisms

Because the Approximate CEEI Mechanism constitutes a compromise of first-best criteria, it is useful to compare the proposed mechanism to alternatives.

Table 2 in the paper lists the properties of every mechanism I am aware of from both theory and practice

- ▶ Every other mechanism is severely unfair ex-post or manipulable even in large markets

I then compare A-CEEI to three important mechanisms in more detail:

1. Random Serial Dictatorship
2. Multi-Unit Hylland and Zeckhauser (1979)
3. Bidding Points Mechanism

Relationship to Random Serial Dictatorship

Single-Unit Demand

- ▶ The Approximate CEEI Mechanism coincides with Random Serial Dictatorship
- ▶ Both satisfy maximin-share guarantee and envy bounded by a single good
- ▶ Dictatorships frequently used in practice (school choice, housing assignment)

Multi-Unit Demand

- ▶ The mechanisms are importantly different.
- ▶ Suppose students require at most k objects. RSD corresponds to an exact competitive equilibrium ($\alpha = 0$) from budgets of

$$\mathbf{b}^{RSD} = (1, k + 1, (k + 1)^2, (k + 1)^3, \dots, (k + 1)^{N-1})$$

- ▶ Dictatorships not frequently observed in practice

Comparison to Multi-Unit Hylland and Zeckhauser (1979)

In a seminal paper, HZ propose CEEI in "probability shares" as a solution to the single-unit assignment problem. Recent work by Pratt (2007) and Budish, Che, Kojima and Milgrom (2010) enable an extension of HZ to multi-unit demand under the following conditions:

1. Each agent's vNM preferences are additive-separable over objects (risk neutral, no super/sub-additivities)
2. Permissible schedule sets satisfy a technical condition called "hierarchy"

Under these conditions

- ▶ Efficiency: multi-unit HZ is exactly ex-ante efficient, which is more attractive than approximate ex-post efficiency.
- ▶ Fairness: multi-unit HZ violates the outcome fairness criteria of this paper. A student may spend her entire budget on a < 1 chance to take a star professor's class, only then not to get it.

Comparison to the Bidding Points Mechanism

Or: Isn't CEEI Already Used in Practice?

"Bidding Points Mechanisms" are used at Berkeley, UChicago, Columbia, Kellogg, Michigan, MIT, NYU, Princeton, Wharton, Yale, etc. Here is roughly how they work:

1. Each student is given an equal budget of artificial currency, say 10000 points.
2. Students express preferences by bidding for individual classes, the sum of their bids not to exceed 10000
3. For a course with q seats, the q highest bidders get a seat (modulo some quota issues)
 - ▶ Schools describe the q^{th} highest bid as the "price", and the procedure as a "market".
 - ▶ To the casual observer, this procedure looks like CEEI ... which we know need not exist

The Bidding Points Mechanism is not a CEEI

- ▶ Two mistakes: wrong prices, wrong demands
- ▶ Conceptual error: the market treats fake money as if it were real money that enters the utility function.
- ▶ Correct "fake money" demand

$$x_i^* = \arg \max_{x \in 2^C} (u_i(x) : \mathbf{p}^* \cdot x \leq b_i)$$

- ▶ Incorrect "real money" demand

$$x_i^* = \arg \max_{x \in 2^C} (u_i(x) - \mathbf{p}^* \cdot x)$$

- ▶ Some virtues: prices always exist, easy to compute ...

What Goes Wrong in the BPM: Incentives

Incentives to misreport are easy to see.

- ▶ Three courses, $\{A, B, C\}$
- ▶ Budgets are 10000 points
- ▶ Suppose $u_{Alice} = (7000, 2000, 1000)$ and $\mathbf{p}^* = (8000, 3000, 1500)$
- ▶ Bid truthfully \rightarrow get zero courses
- ▶ BR: bid $\hat{u}_{Alice} = (8001, 0, 1501)$

What's so bad about this?

- ▶ Alice simply tricked a "real money" demand function into behaving like a "fake money" demand function

What Goes Wrong in the BPM: Fairness

Answer: Betty!

- ▶ Alice's bid of 8001 for A displaces Betty who bid 8000
- ▶ Betty now wastes 8000 of her points; at best, gets correct demand given a budget of 2000.

Proposition 10: Suppose an exact CEEI actually exists

- ▶ Truthful play $\not\Rightarrow$ CEEI
- ▶ Eqm play $\not\Rightarrow$ CEEI

Proposition 11:

- ▶ Truthful play \Rightarrow Some students get ex-post utility of zero
- ▶ Eqm play \Rightarrow Some students get ex-post utility of zero

By contrast: A-CEEI yields an exact CEEI whenever one exists, and the Fairness Theorems prevent highly unfair outcomes

What Goes Wrong in the BPM: Fairness

The University of Chicago's Booth School of Business adopted a BPM in 2008.

- ▶ In the past four quarters, the number of students allocated zero courses in the main round of bidding has been 17, 64, 37, and 53.
- ▶ Some examples from full-time MBAs graduating in Spring 2010:
 - ▶ Bid (5466, 5000, 1500, 1) for courses that then had prices of (5741, 5104, 2023, 721)
 - ▶ Bid (11354, 3, 3, 3, 2) for courses that then had prices of (13266, 2023, 1502, 1300, 103)
- ▶ Another implication of Proposition 11, and more broadly of the treatment of fake money as if it were real money, is that students will graduate with large leftover budgets
 - ▶ On average, full-time MBA students graduate with 7500 leftover points (roughly a full term's worth)
 - ▶ 10% of students graduate with >17000 leftover points

Ex-Ante Welfare Performance of A-CEEI

- ▶ The Approximate CEEI Mechanism has an element of randomness: the budgets.
- ▶ Efficiency ideally should be assessed ex-ante, not ex-post
 - ▶ A necessary but not sufficient condition for a lottery over allocations to be ex-ante Pareto efficient is that all its realizations are ex-post Pareto efficient
 - ▶ Impossibility theorems are even more severe (Zhou, 1990)
 - ▶ Jointly resolving lotteries over bundles is not possible in general (Budish, Che, Kojima and Milgrom 2010)
- ▶ In this paper, I assess ex-ante efficiency empirically in a specific course-allocation environment
- ▶ Specifically, compare A-CEEI to the HBS Draft Mechanism studied by Budish and Cantillon (2009)

N.B. with fairness, ex-post is actually the more stringent perspective

Computational Analysis - Algorithm

Theorem 1 is non constructive, and implementing the Approximate CEEI Mechanism is non-trivial. There are two key challenges:

1. Calculating excess demand at a particular price ($z(\mathbf{p})$) is NP Hard – each agent must solve a set-packing problem
2. Price space is large. So even if $z(\mathbf{p})$ were easy to compute, finding an approximate zero is a difficult search problem

Othman, Budish and Sandholm (2010) develop a computational procedure that overcomes these challenges in life-size problems.

1. Demands are calculated using an integer program solver, CPLEX
2. We use a method called "Tabu Search" to find an approximate zero. Departure point is the Tatonnement process $\mathbf{p}^{t+1} = \mathbf{p}^t + z(\mathbf{p}^t)$

The algorithm can currently handle "semester-sized" economies in which students consume 5 courses. Each run takes 1 hour.

Computational Analysis - Data and Key Assumptions

- ▶ HBS data: preferences are ordinal over individual courses.
- ▶ To convert into utilities over bundles I assume average-rank preferences
 - ▶ E.g. prefer 2nd+3rd favorite to 1st+5th favorite
 - ▶ Theory can handle more complex preferences but this seems reasonable given data incompleteness
- ▶ I also assume students report their preferences truthfully under Approximate CEEI
 - ▶ 916 students seems large but I am unable to empirically validate whether students have exact incentives
 - ▶ Space of possible deviations is too large to meaningfully search

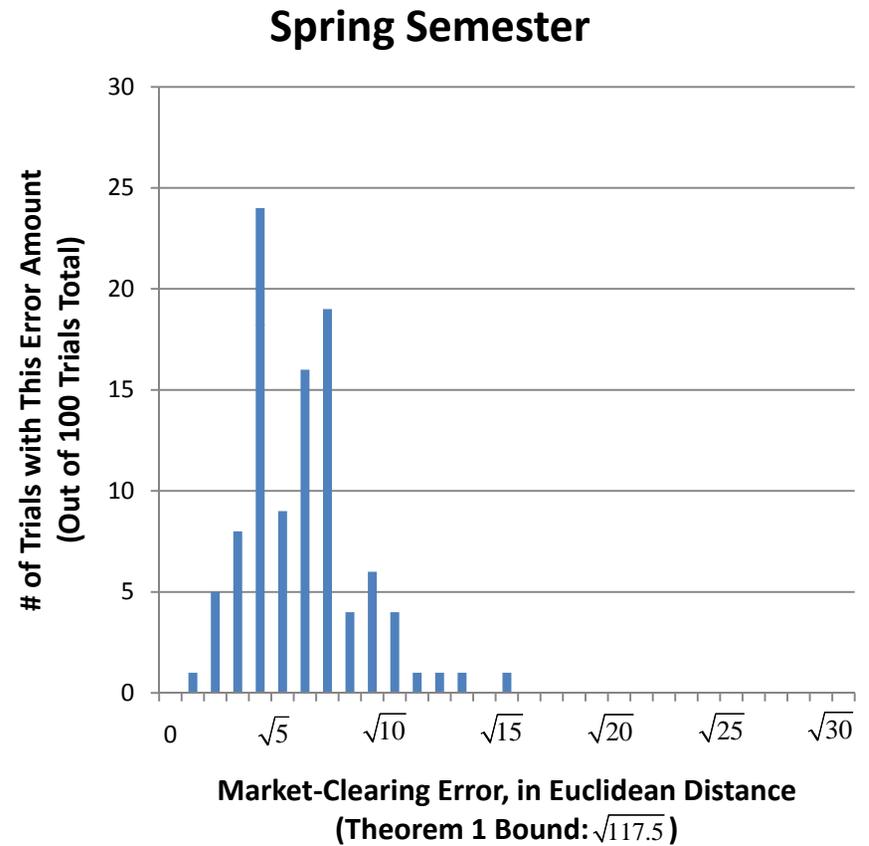
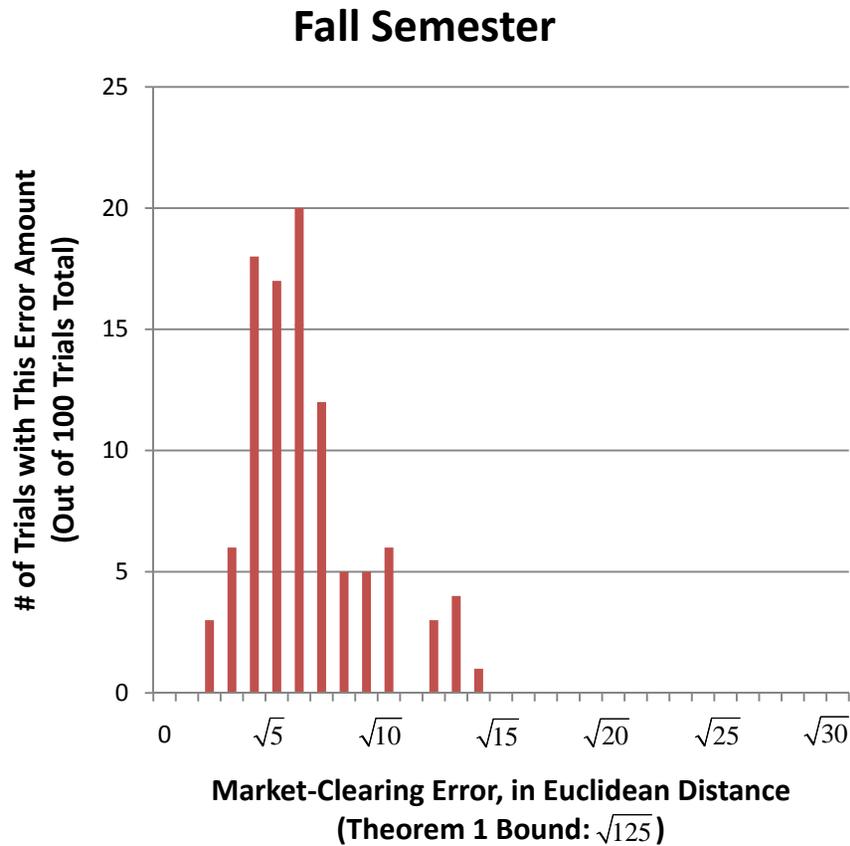
Ex-Ante Welfare Performance of Approximate CEEI

Summary of Findings

1. Market-clearing error is small

Figure 1: Ex-Post Inefficiency

Distribution of Market-Clearing Error



Description: The Othman, Budish and Sandholm (2010) Approximate CEEI algorithm is run 100 times for each semester of the Harvard Business School course allocation data (456 students, ~50 courses, 5 courses per student). Each run uses randomly generated budgets. This table reports the distribution of the amount of market-clearing error per trial, measured in Euclidean Distance (square-root of sum of squares). Both excess demand and excess supply count as error (except that courses priced at zero are allowed to be in excess supply without counting as error).

Ex-Ante Welfare Performance of Approximate CEEI

Summary of Findings

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 - ▶ Implication: ex-post inefficiency is small

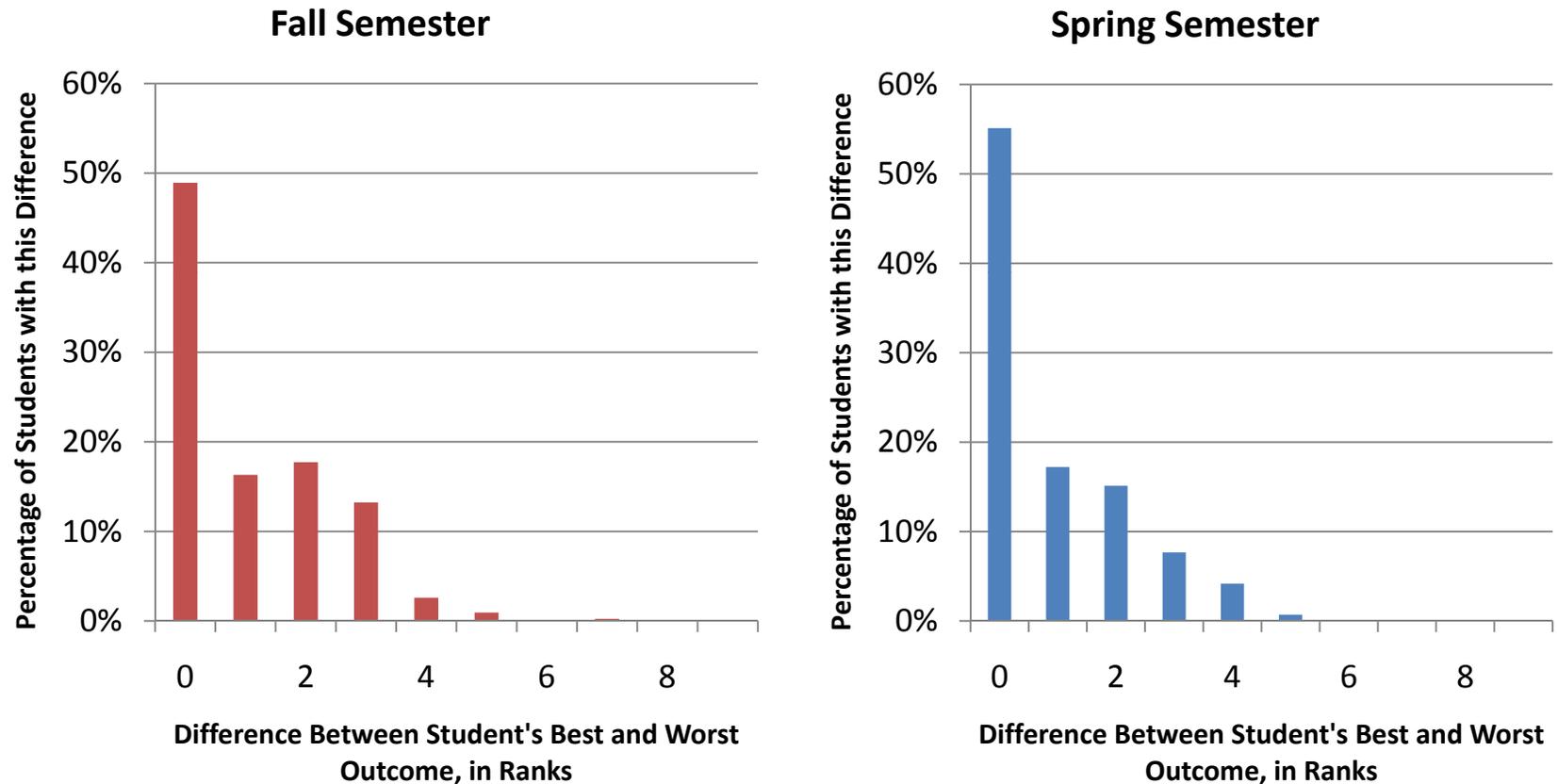
Ex-Ante Welfare Performance of Approximate CEEI

Summary of Findings

1. Market-clearing error is small
 - ▶ Implication: ex-post inefficiency is small
2. Individual students' outcomes seem not to vary much with the random budgets

Figure 2: Relationship of Ex-Post to Ex-Ante Efficiency

Distribution of Difference Between Best and Worst Outcomes



Description: The Othman, Budish and Sandholm (2010) Approximate CEEI algorithm is run 100 times for each semester of the Harvard Business School course allocation data (456 students, ~50 courses, 5 courses per student). Each run uses randomly generated budgets. This table reports the distribution of the difference between a student's single best and single worst outcome over the 100 trials, in ranks. Here is an example calculation: a student whose best received bundle consists of his 1,2,3,4 and 5th favorite courses, and worst bundle consists of his 2,3,4,6 and 7th favorite courses has a difference of $(2+3+4+6+7) - (1+2+3+4+5) = 7$

Ex-Ante Welfare Performance of Approximate CEEI

Summary of Findings

1. Market-clearing error is small
 - ▶ Implication: ex-post inefficiency is small
2. Individual students' outcomes seem not to vary much with the random budgets
 - ▶ Implication: ex-post efficiency is a reasonable proxy for ex-ante efficiency (unlike for Random Serial Dictatorship)

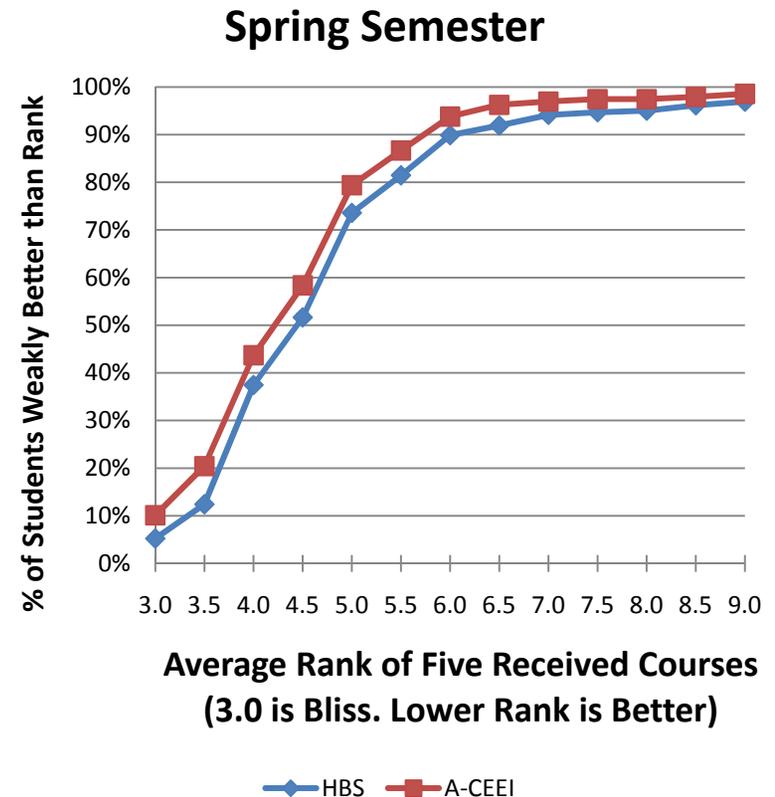
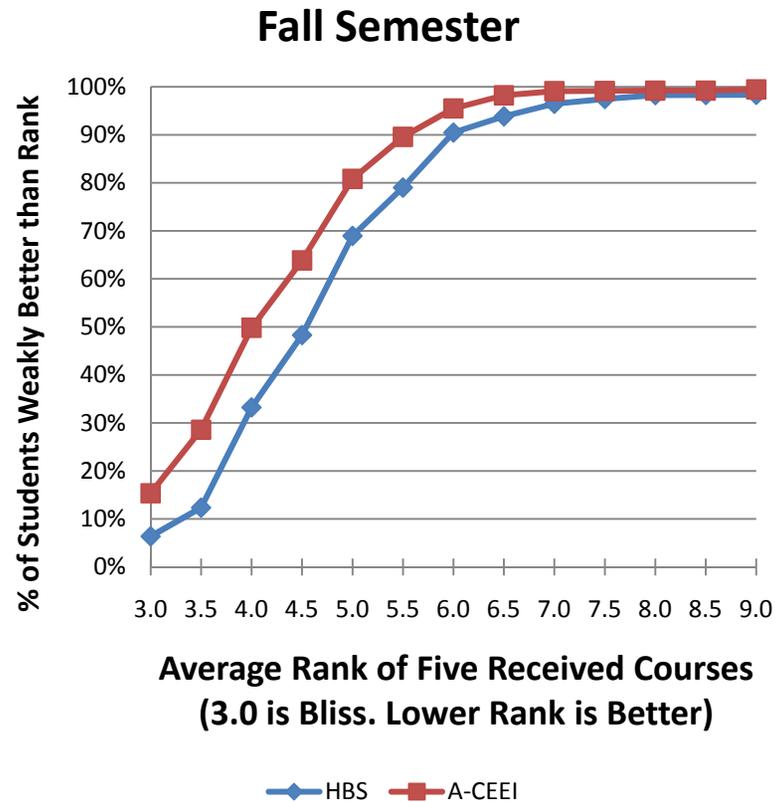
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3. Distribution of utilities f.o.s.d.'s that from HBS's own mechanism, and s.o.s.d.'s that from RSD.

Figure 3: Ex-Ante Efficiency Comparison

Approximate CEEI Mechanism vs. HBS Draft Mechanism



Description: The Othman, Budish and Sandholm (2010) Approximate CEEI algorithm is run 100 times for each semester of the Harvard Business School course allocation data (456 students, ~50 courses, 5 courses per student). Each run uses randomly generated budgets. For each random budget ordering I also run the HBS Draft Mechanism, using the random budget order as the draft order. The HBS Draft Mechanism is run using students' actual strategic reports under that mechanism. The Approximate CEEI algorithm is run using students' truthful preferences. This table reports the cumulative distribution of outcomes, as measured by average rank, over the $456 \times 100 = 45,600$ student-trial pairs. Average rank is calculated based on the student's true preferences. For instance, a student who receives her 1,2,3,4 and 5th favorite courses has an average rank of $(1+2+3+4+5)/5 = 3$.

Ex-Ante Welfare Performance of Approximate CEEI

Summary of Findings

1. Market-clearing error is small
 - ▶ Implication: ex-post inefficiency is small
2. Individual students' outcomes seem not to vary much with the random budgets
 - ▶ Implication: ex-post efficiency is a reasonable proxy for ex-ante efficiency (unlike for Random Serial Dictatorship)
3. Distribution of utilities f.o.s.d.'s that from HBS's own mechanism, and s.o.s.d.'s that from RSD.
 - ▶ Implication: a utilitarian social planner should prefer Approximate CEEI to either of these alternatives

Conclusion

- ▶ Practical market design problems often prompt the development of new theory that enhances and extends old ideas
- ▶ The beautiful theory of CEEI is too simple for practice because it assumes perfect divisibility and well-behaved preferences
- ▶ This paper proposes a richer theory that accommodates indivisibilities and general preferences
- ▶ Indivisibilities complicate existence
 - ▶ Especially with equal incomes
 - ▶ But we can guarantee approximate market clearing by using an arbitrarily small amount of budget inequality
- ▶ Indivisibilities complicate fairness
 - ▶ Especially if there is just a single diamond
 - ▶ But we can reduce unfairness to that necessitated by the degree of indivisibility in the economy