I. INTRODUCTION

In Chapter 2 we considered the problems of optimal operating-financing decisions by firms in a world of perfect certainty and perfect capital markets. Two fundamental "separation principles" were established. First, it was shown that given its operating, that is, production-investment, decisions, the market value of a firm at any point in time is independent of its financing decisions; thus operating decisions need not be affected by financing decisions.1 Second, in making its operating decisions, an optimal policy for the firm is to maximize the market value of the holdings of its current owners, irrespective of the details of owner tastes. Finally, in Chapter 3 it was then shown that with perfect certainty and perfect capital

1 As in preceding chapters, the terms "production," "investment," and "operating decisions" are used interchangeably. Likewise we feel free to refer to an individual as a "consumer," an "investor," or a "consumer-investor."
markets, observed market interest rates provide the appropriate cutoff rates or "costs of capital" for the firm's production-investment decisions.

In this chapter we show that the two separation principles continue to hold when the perfect certainty assumption is dropped. The goal is to demonstrate that the validity of these propositions is a direct implication of the assumed existence of a perfect capital market. We find, however, that in a world of uncertainty the perfect capital market assumption is not sufficient to give real meaning to the notion of a cutoff rate or cost of capital for a firm's investment decisions.

The first separation principle is considered in Section III. We initially take a "market equilibrium" approach in which the effects of the capital structure or financing decisions of firms on the market values of their securities is discussed in the context of the entire process by which the equilibrium holdings and prices of holdings by all investors in all firms are determined in the market. Although most general and in fact quite simple, this market equilibrium approach may seem, at least to the beginning reader, a little overly abstract. Thus for purposes of illustration two other somewhat more concrete, but correspondingly more restrictive, approaches are also used to present the major propositions concerning the effects of financing decisions on market values. In these the method of analysis is to derive the market value of a firm's securities from the market values of other securities that provide identical investment positions.

In Section IV we consider the problem of determining criteria for optimal operating decisions by firms. We find that, as in a world of perfect certainty, the "market value rule," that is, maximize the market values of currently outstanding securities, is a criterion for investment decisions that is optimal in a perfect capital market. In a world of uncertainty, however, implementation of this criterion runs into difficulties, because firms can have more than one type of security outstanding—for example, bonds and common stock—and an investment decision that maximizes market value for one group of security holders need not do so for others.

Finally, neither the analysis of the effects of capital structure on market values nor the discussion of criteria for optimal investment decisions make use of the concept of a cost of capital. The chapter concludes by considering the somewhat limited role of this concept in a world of uncertainty.

The immediate order of business, however, is a description of the market setting—a discrete-time, two-period model—within which most of the analysis takes place.

II. MARKET SETTING

For convenience, the economic agents that carry on production activities are called firms. At the beginning of period 1 firms purchase the services of
inputs—labor, machinery, and so on—and use these to produce goods and services to be sold at the beginning of period 2, at which time all firms are disbanded. A firm finances its outlays for production in period 1 by issuing financial assets or securities that are claims against its total market value at period 2. In a discrete-time, two-period world the market value of a firm at period 2 is just the difference between cash revenues and costs at period 2, and this total market value is paid in full to the investors holding the firm’s securities from period 1.

At the beginning of period 1, investors are assumed to have given quantities of resources—labor, which will be sold to some firm, and portfolio assets, which are the securities of firms carried forward from previous periods—that must be allocated to consumption for period 1 and a portfolio investment whose market value at period 2 determines the investor’s period 2 consumption. For simplicity, we assume that the investor will be paid for his labor at period 1.\(^3\)

A given set of production decisions by firms at period 1 determines the number of firms actually producing as well as the set of probability distributions on market values of firms at period 2 that will be available in the capital market at period 1.\(^3\) These distributions are the basic objects that must be priced and cleared from the capital market at the beginning of period 1.

But the capital market also includes financial opportunities through which probability distributions on market value at period 2 can be fragmented into new distributions in different ways. We can distinguish two major types of fragmentations of the probability distribution on a firm’s market value: (1) a division into different types of securities or claims—for example, bonds and common stock—and (2) a division of a particular financial asset into equivalent units—for example, the common stock of a firm is subdivided into individual shares.

Some fragmentations of the distribution of a firm’s period 2 market value will be provided by the firm itself when it issues securities. Additional fragmentations may be carried out by investors by issuing claims against securities or portfolios of securities purchased from firms.

Equilibrium at the beginning of period 1 is assumed to be reached through a process of tâtonnement with recontracting; that is, investors come to market with their resources, tastes, and expectations on market values for period 2, and firms bring their production opportunity sets. Firms announce tentative production and financing decisions, investors offer their labor to

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\(^3\) As in earlier chapters, for brevity we commonly use the phrase “at period \(t\)” when we have in mind the longer phrase “at the beginning of period \(t\).”

\(^4\) A brief review of the elementary statistical concepts that are required background for the uncertainty section of this book is provided in the Appendix to Chap. 5.
firms and begin bidding for consumption goods and securities, and a
tentative set of prices for consumption goods, labor, and securities is
established. Prices and decisions are tentative, because it is agreed that no
decisions will be executed until an equilibrium set of prices, that is, a set of
prices at which all markets can clear at period 1, has been determined. Our
treatment of this model, however, concentrates on the nature of equilibrium
in the capital market.

III. CAPITAL STRUCTURE AND MARKET VALUES

The effects of the financing decisions of firms on the market values of
their securities is analyzed first in the context of the entire process by which
the equilibrium holdings and prices of holdings by all investors in all firms
are simultaneously determined in the capital market. The advantage of
this market equilibrium approach is its generality; major propositions about
the effects of capital structure can be presented without specifying much
about the details of either investor tastes or the types of claims against
firms that can and will be held when equilibrium is reached. By way of
contrast, and for purposes of illustration, we later discuss two common
and much more highly specified approaches, the “states of the world”
model of Arrow [6], Debreu [7], Hirschleifer [5], and others and the
original “risk class” model of Modigliani and Miller [1], in which the
effects of financing decisions on market values was first given rigorous
treatment. Most of the discussion in this section concentrates on the two-
period case. We later show that the major results are easily generalized to
a multiperiod context.

III.A. Two-Period Market Equilibrium Model

Suppose that the capital market and the tastes of individual investors
are characterized by the following general conditions, which we say define
a perfect capital market in the context of the two-period model:

1. The capital market is frictionless in the sense that all securities are
   infinitely divisible, information is costless and available to everybody, and
   there are no transactions costs or taxes.

2. Any financial arrangements available to firms are equally available to
   individuals; that is, any claims that a firm can issue at period 1 against its
   probability distribution of market value at period 2 can also be issued by
   any investor who holds an equivalent distribution. Thus, for example, it is
   assumed that, although his portfolio may include other probability dis-
   tributions, an investor can issue claims against a given distribution with his
   liability limited only to the period 2 market value obtained from this
distribution. In short, the limited liability of shareholders in firms applies
also to investors who issue securities on personal account. More generally, the possible ways in which a given probability distribution of period 2 market value can be fragmented and sold in the market at period 1 are independent of whether the distribution is presented to the market by an individual or a firm.

3. In choosing among available probability distributions on market value at period 2, investors are not concerned with who happens to issue a distribution; in particular, investors are not concerned with whether a distribution is issued by an individual or a firm.

4. Investors perceive that there are always perfect substitutes for any securities issued by an individual investor or firm, and individual investors and firms are "atomistic competitors" in the sense that their activities in the capital market have no effect on the prices of securities issued by other investors and firms. In short, as usual, we assume that investors and firms are price takers in the capital market.  

5. Finally, as in any perfect market, a certain amount of maximizing behavior on the part of investors is presumed. Specifically, investors are assumed to protect themselves against any sort of "financing decisions" by individuals or firms that have the effect of expropriating their positions without appropriate compensation. For example, suppose that at period 1 a firm initially has one bond outstanding, issued in some earlier period, which at this point is simply a promise to pay $R(2)$ at period 2. If the firm issues no new bonds at period 1, the outstanding bond is in effect a promise to pay the lesser of $R(2)$ or the market value of the firm $V(2)$ at period 2. But suppose that the firm issues one additional bond, which is again simply a promise to pay $R(2)$ at period 2. If there are no further priority arrangements, each bond now represents a claim to the lesser of $R(2)$ or $\frac{1}{2} V(2)$. If $V(2)$ turns out to be equal to or greater than $2R(2)$, the old bond receives $R(2)$, just as it would if no new bond were issued. But when $V(2)$ turns out less than $2R(2)$, the payoff on the old bond is strictly less than if no new bond were issued. In effect, by issuing a new bond with no priority arrangements, the firm has expropriated without compensation part of the distribution of $V(2)$ that would belong to the old bond if no

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4 A note for the more sophisticated: Because we have not presented a complete theory of market value determination under uncertainty, at this point we cannot really say exactly what constitutes a perfect substitute for a given security; that is, a model of market value determination would tell us what characteristics of a security's probability distribution of period 2 market value are looked at by the market when it prices the security at period 1. A perfect substitute for the security would then be defined in terms of these characteristics of return distributions that are relevant in the pricing process. Thus what we are saying is that, whatever the relevant theory of market value determination, there always are perfect substitutes for the securities issued by any investor or firm.
new bond were issued. But the old bond could easily have been protected against such infringements by a "me first" rule; that is, the contract on the old bond explicitly states that any new bonds issued can only have claim to observed market values greater than $R(2)$. Our concept of a perfect market assumes that bondholders protect themselves with such "me first" rules, and in fact, such priority arrangements are common practice.

Likewise, we also assume that a firm's common stockholders do not allow it to engage in capital structure changes that result in uncompensated shifts of holdings from them to the bondholders. For example, suppose that at period 1 the firm initially has two bonds of equal priority outstanding, each of which is a fixed claim to $R(2)$ at period 2. If $V(2) \geq 2R(2)$, the bondholders receive full payment at period 2, but if $V(2) < 2R(2)$, each bondholder receives only $\frac{1}{2}V(2)$. Now suppose that the firm uses retained earnings, which would otherwise be paid to the stockholders, or issues additional common stock at period 1 and uses the proceeds to "retire" one of the bonds. For any realized market value $V(2) < 2R(2)$ at period 2 the remaining bond receives a higher payoff than it would if the other were not retired. Thus, the claims of the remaining bond against the firm have implicitly been increased without compensation to the shareholders. The moral is clear: To avoid uncompensated shifts of holdings from shareholders to bondholders, the firm should either retire an entire bond issue, or in repurchasing part of an issue, it should allow the claims represented by the repurchased bonds technically to remain outstanding either by paying the bonds themselves as dividends to its stockholders or, equivalently, by holding the repurchased bonds as assets. Moreover, it is also easy to show that the firm should retire lower priority bonds before higher priority bonds; that is, the second mortgage should be retired before the first, because the reverse would lead to an uncompensated increase in the holdings of the second mortgagees.\footnote{It is clear, however, that stockholders and bondholders need only worry about protecting themselves from one another when the bonds of the firm are risky, that is, there is some chance that the market value of the firm at period 2 will be insufficient to cover the total promised payments to all bondholders. When all bonds are riskless, protective arrangements are unnecessary.}

The major result of capital structure theory is then the "first separation principle": For any given set of operating decisions by firms at period 1, when the capital market is perfect, the equilibrium total market value of any firm at period 1 is unaffected by its financing decisions. Moreover, the firm's financing decisions have no effects on either the wealths or the capital market opportunities of its security holders, so that these decisions are a matter of indifference to the security holders. It follows that optimal
operating decisions for the firm do not depend on its financing decisions; that is, operating and financing decisions are separable.\(^6\)

In essence, establishing these propositions involves showing that, given its operating decisions, the financial decisions of firms have no effect either on (1) the set of probability distributions on market value at period 2 that investors can bid for before equilibrium is established in the capital market at period 1 or on (2) the distributions that can be offered for sale by investors who come to market at period 1 with the securities of firms as part of their resources carried forward from period 0. In short, the financing decisions of firms have no effect on the opportunities facing investors. It then follows that the financing decisions of firms cannot affect either the ultimate equilibrium holdings of an individual investor in a given firm—that is, the fragmentation of the distribution of the firm’s total market value at period 2 held by the investor at period 1—or the prices of these holdings. Thus the financing decisions of firms are a matter of indifference to investors. And because the sum of the period 1 market values of all holdings by all investors in any given firm is the firm’s total market value, this also is unaffected by its financing decisions.

Let us expand. In a perfect market any types of claims that a firm can issue against its probability distribution of market value at period 2 can be issued by any investor who holds an equivalent distribution. It follows that a firm’s financial decisions cannot affect the set of fragmentations of the distribution of its period 2 market value that could possibly be bid for when investors come to market at period 1.\(^7\)

It is somewhat more tedious to show that the probability distributions that can be offered for sale in the market at period 1 by investors who hold the securities of a firm as part of their resources carried forward from period 0 are also unaffected by the firm’s financing decisions. It is convenient to consider separately two reasons that a firm might want to issue new securities at period 1. First, the firm may have to cover outlays for production. If the firm’s bondholders always use “me first” rules to protect

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\(^6\) In discussing these propositions in this section, we limit attention, for simplicity, and without departing too much from reality, to a world in which firms issue only the usual types of bonds, which provide for a stated maximum payoff at period 2, and common stock. Eventually the reader should be able to convince himself, however, that the analysis is not dependent on the types of securities that exist in the market and would, for example, also apply to the states of the world model, to be considered later, in which there are fixed payoff securities in the form of “contingent claims” that pay fixed sums but only if given states of the world are realized at period 2.

\(^7\) Note that the analysis never requires that investors have the same view of the probability distribution on a firm’s period 2 market value. Investors must simply be able to recognize when claims against a given distribution that are issued by investors are equivalent to claims against the same distribution that are issued by firms.
themselves against uncompensated expropriations of their holdings, any new securities issued to finance production have no effect on the distributions that can be offered to the market at period 1 by the holders of bonds outstanding from period 0. Thus any effects of the financing decision must fall on the shareholders. But in a perfect capital market the firm and its shareholders are atomistic competitors with respect to all types of securities. Thus if the firm chooses one configuration of new securities and its shareholders, individually or as a group, would prefer some other configuration, the shareholders can achieve their desired positions by repurchasing the securities issued by the firm— with the shareholder who owns the proportion \( \alpha \) of the firm’s common stock purchasing the proportion \( \alpha \) of the new securities—and issuing their desired configurations of securities on personal account. The probability distributions on market value at period 2 presented to the market are then exactly as if the firm itself had chosen to issue the configurations of new securities preferred by its shareholders. Moreover, in a perfect capital market the firm cannot provide any configuration of new securities that could not be offered by its shareholders on personal account. Thus new issues of securities by the firm to finance production have no effect on the distributions of period 2 market value that can be offered for sale at period 1 by investors who hold the firm’s common stock outstanding from period 0.

The second reason that the firm may issue new securities at period 1 is that it may simply wish to bring about some change in its capital structure. In this case it issues new claims—bonds, common stock, or retained earnings—and either pays the proceeds directly to the stockholders or uses them to repurchase old claims. But in a perfect capital market such shifts in capital structure amount to refragmentations of the distribution of the firm’s period 2 market value in which in any market equilibrium the firm’s security holders acquire holdings with a given market value in exchange for other holdings with exactly the same market value. Thus, given atomistic competition, any security holder can always reacquire his former position by simply reversing the exchange on personal account, using his new holdings to reacquire his old holdings. Moreover, in a perfect capital market the firm cannot carry out any refragmentation of security holder positions that could not be obtained by the security holder on personal account. Thus, like financing decisions made to finance production, simple changes of capital structure also have no effect on the distributions of period 2 market value.

*The term “securities” includes cash earnings obtained at period 1 from the production of period 0 that are retained to finance the production of period 1. In terms of their effects on the distributions of period 2 market values accruing to the old shareholders, such retained earnings are formally equivalent to a preemptive issue of new common stock, with each shareholder using dividends paid by the firm to purchase the new issue in proportion to his holdings of the shares outstanding from period 0.*
that can be offered for sale at period 1 by investors who hold the firm’s securities.8

In sum, because in a perfect market investors can issue the same sorts of claims as firms, the financial decisions of firms have no effect on the fragmentations of distributions on total market values of firms at period 2 that could be bid for by investors at period 1. Moreover, the distributions that can be offered to the market by investors who come to market at period 1 with the securities of a given firm as part of their resources are also unaffected by the financial decisions of the firm. Thus, a given set of financial decisions by firms neither provides investors with opportunities to trade probability distributions on market value at period 2 that are not available with any other set of financing decisions nor precludes opportunities that would be available with other financing decisions.10 Moreover, in a perfect market, investors are not concerned with whether a given distribution of period 2 market value is offered by an individual or a firm. It then follows that the financial decisions of firms cannot affect the ultimate equilibrium holdings of individual investors in firms, that is, the fragmentations of distributions of period 2 market values held by investors, or the equilibrium prices of these holdings. More formally, any equilibrium sets of holdings and prices of holdings that can be reached with one set of financing decisions

9 To give some concreteness to this analysis, consider a firm that has had no debt in its capital structure prior to period 1 and at period 1 simply wishes to bring about some change in its capital structure, with any proceeds from issuing debt going directly to the shareholders. In this case, all investors realize that regardless of what the equilibrium price of the firm’s debt turns out to be, the firm’s shareholders are always compensated precisely the amount needed by them to reacquire an unlevered position, that is, repurchase the bonds, if they so desire. Thus, before equilibrium is reached, the shareholders can proceed to undertake tentative trades exactly as if the firm will not lever their shares. And, of course, if the firm did not lever their shares, the shareholders could still offer to provide the levered positions to other investors by creating debt claims against the shares, exactly as would the firm, and then offering to sell the debt and the resulting levered shares separately.

10 These conclusions are, however, critically dependent on the assumption that bondholders and shareholders protect themselves against uncompensated expropriations of their positions. For example, we saw earlier that if the bondholders do not insist on “me first” rules, their holdings in the distribution of the firm’s period 2 market value can be reduced if the firm issues new bonds with no priority arrangements; thus the distributions of period 2 market value that can be offered for sale at period 1 by the holders of the old bonds can be affected by the firm’s financing decisions. Moreover, in this case the firm’s shareholders cannot similarly “dilute” the positions of its bondholders by issuing additional debt on personal account, so that their trading positions can also be affected by the firm’s financing decisions. On the other hand, we also saw earlier that if the firm retires part of a debt issue, it implicitly increases the claims against the firm of the remaining bonds in the issue, so that again the firm’s financing decisions can affect the distributions of period 2 market value that can be offered for sale at period 1 by its different security holders.
by firms can equally well be reached with any other set of financing decisions. Because a firm's total market value at period 1 is just the sum of the market values of all holdings by all investors, it follows that the total market values of firms are unaffected by their financial decisions. And because these results hold for any set of operating (production-investment) decisions by firms—that is, for any set of operating decisions by firms, the financing decisions of firms are a matter of indifference to their security holders—it follows that operating and financing decisions are separable.

III.B. Multi-period Model

In the two-period model to establish the proposition that the total market value of a firm at period 1 is unaffected by the firm's financing decisions, it is fairly clear and unobjectionable to specify that the objects of concern to investors in their portfolio decisions at period 1 are probability distributions on market value at period 2. Because the proposition is not concerned with what does determine market values, for its purposes we need not specify the nature of investor tastes or the process generating period 2 market values in more detail. The proposition is a consequence of a perfect market and in particular of the perfect market assumptions that (1) any claims that firms can issue against the future realizations of production decisions can also be issued by the firm's security holders against their holdings in these same realizations; (2) investors are concerned only with the "real characteristics" of a claim and not with who happens to issue it; and (3) there are always perfect substitutes for the securities issued by any investor or firm, and investors and firms are atomistic competitors in the capital market.

But the logic of the two-period model applies equally well to corresponding multi-period models. In a multi-period world, specifying the characteristics of an uncertain payoff stream that are relevant to investors is generally much more difficult than in a similar two-period model. The market value of a firm at any point in time depends in some way on the current and future production-investment opportunities of this firm and other firms, all of which are uncertain and must be specified in different ways, depending on the model in hand. Nevertheless, although it may be difficult to develop in detail models that tell what does determine the market value of a firm at any point in time, if investors and firms are atomistic competitors in the capital market, if investors are concerned only with the real characteristics of securities, somehow defined, and if any claims that firms can issue against the future realizations of production and investment decisions can also be issued by the firm's security holders against their shares in these realizations—in short, if the capital market is perfect—
then exactly the reasoning of the preceding section applies; that is, a firm’s financing decisions at any point in time are a matter of indifference to its security holders, so that operating and financing decisions are separable.

III.C. Two Partial Equilibrium Treatments

The preceding discussions analyze the effects of the capital structure or financing decisions of firms on their total market values in the context of the process by which the equilibrium holdings and prices of holdings by all investors in all firms are simultaneously determined in the market. The advantage of this market equilibrium approach is its generality; the important capital structure propositions were established without specifying much about the details of either investor tastes or the types of claims against firms that would be held when a market equilibrium is reached.

The two approaches to be discussed now are much more concrete but correspondingly less general. They can be characterized as partial equilibrium models in the sense that they take the financing decisions and prices of securities of all firms but one as given and then examine the effects of this firm’s financing decisions on its total market value. The mode of analysis is to derive the market value of the firm and its securities from the known market values of equivalent positions in other firms, that is, positions that provide payoff streams identical with those of the securities of the firm.

Two partial equilibrium models are discussed: (1) a two-period states of the world model very similar to that of Hirshleifer [5] and (2) a model based on the existence of risk classes more or less in the sense defined by Modigliani and Miller [1]. The primary purpose is to provide some contrast with and insight into the reasoning of the market equilibrium approach presented above.

III.C.1. Two-period states of the world model

In the two-period model the basic objects that must be priced in the capital market at period 1 are the probability distributions on the total market values of individual firms at period 2. The fragmentations of these distributions made possible by financial opportunities represent ways of subdividing the risks of any given distribution among investors. The types of fragmentations that are carried out by investors depend in part on their tastes and in part on the characteristics of the process generating distributions of period 2 market values. One especially simple specification of tastes and opportunities is the states of the world model of Arrow [6], Debreu [7], Hirshleifer [5], and others.
At period 1 suppose that investors agree that there are a finite number $S$ of mutually exclusive possible states of the world$^{11}$ at period 2 and that for a given set of production decisions by firms at period 1 the market value for certain of firm $j$ if state $s$ occurs is $V_j(2,s)$. All uncertainty attaches to the states of the world at period 2, and investors need not assign the same probabilities to states. Finally, in their portfolio decisions investors are assumed to be concerned only with the total number of dollars obtained in each state at period 2, and, other things equal, they always prefer more dollars in any given state to less.$^{12}$

A simple way in which such a market might be organized at period 1 would be to have $S$ separate contingent claims, where a given contingent claim is a promise to pay dollars at period 2 only if a given state of the world occurs. Firms might then fragment their distributions of total market value at period 2 into the values that occur in each state and then sell these separately in the market. The total dollars in each state made available by firms and the tastes of investors then determine $S$ prices for contingent claims $p(1), p(2), \ldots, p(S)$, where $p(s)$ is the price at period 1 of a dollar to be delivered at period 2 only if state $s$ occurs.

But suppose that firms do not issue contingent claims but rather issue only the more conventional bonds and common stock. If investors are still free to trade contingent claims among themselves, and if $v_k(2,s)$ is the market value of an arbitrary security $k$ in state $s$ at period 2, and if there are not to be arbitrage opportunities, the price $v_k(1)$ of security $k$ at period 1 must be

$$v_k(1) = \sum_{s=1}^{S} p(s) v_k(2,s).$$

That is, investors can always replicate the payoffs across states provided by any security by purchasing the appropriate numbers of contingent claims to dollars to be delivered in each possible future state. Thus, assuming that investors can both issue and purchase contingent claims, if there are not to be arbitrage opportunities, the market value of any security at period 1 must be just the sum of the market values of the contingent

$^{11}$ The complex problem of defining what is meant by a state has not been adequately treated in the literature, and fortunately it is not necessary to do so here.

$^{12}$ This is not inconsistent with the analyses in the preceding sections in which investors were viewed as choosing among probability distributions on market value at period 2. Given an assignment of probabilities to states, in choosing among possible arrays of total dollars to be received in each state, the investor is choosing among probability distributions on market value at period 2. The distributions are determined by the dollars to be received in each state and the distribution of probabilities across states. Of course no such specifications of either tastes or opportunities were required in the market equilibrium approach.
claims (the dollars to be delivered in each possible future state) that it implies.

It follows directly that, regardless of its financing decisions, the total market value at period 1 of any firm, that is, the market value of bonds plus common stock, is just

\[ V_j(1) = \sum_{s=1}^{s} p(s) V_j(2,s), \]

where \( V_j(2,s) \) is the market value of the firm in state \( s \) at period 2; that is, given the firm's production decisions at period 1, its period 1 total market value is independent of its financing decisions and is always just the sum of period 1 market values, computed by means of the prices of contingent claims, of the total dollars to be delivered in each possible future state.

For example, suppose that there are two possible states of the world at period 2, and that if state 1 occurs the value of firm \( j \) is 10, and that if state 2 occurs the value of firm \( j \) is 15. Suppose first that the firm has only common stock in its capital structure. Then the market value of the firm, and its stock, at period 1 is

\[ V_j(1) = p(1)10 + p(2)15. \]

Alternatively, suppose that the firm issues both debt and common stock, and the debt is in the form of a promise to pay 5 whichever state occurs at period 2. Then the period 1 market value \( B_j(1) \) of the firm's debt must be

\[ B_j(1) = p(1)5 + p(2)5. \]

The stockholders, as the residual claimants, get 5 if state 1 occurs and 10 if state 2 occurs. The total value of the stock is thus

\[ S_j(1) = p(1)5 + p(2)10, \]

so that

\[ V_j(1) = S_j(1) + B_j(1) = p(1)10 + p(2)15. \]

Finally, the firm might issue both debt and common stock, with the debt now being a promise to pay 12 in either state at period 2. But because the firm's value in state 1 is only 10, full payment on the debt occurs only in state 2, so that

\[ B_j(1) = p(1)10 + p(2)12, \]

\[ S_j(1) = p(1)0 + p(2)3, \]

\[ V_j(1) = B_j(1) + S_j(1) = p(1)10 + p(2)15. \]

Thus the period 1 market value of the firm is the same under all three different assumptions about financing decisions and in fact is always just
equal to the sum of the period 1 values of the total resources generated by
the firm in each possible future state.$^{13}$

To show that, given its operating decisions, a firm's financing decisions
have no effect on its total market value is not to show, however, that
operating and financing decisions are separable. The separation principle
requires that the firm's financing decisions do not affect the wealths, which
in this model determine the trading opportunities, of any of its security
holders. But at this point it is easy to see that in the states of the world
model a firm's financing decisions are indeed a matter of indifference to
its security holders. If, in accordance with the perfect market assumption,
security holders always protect themselves from one another, shifts in
capital structure by a firm always result in exchanges of positions among
security holders in which any security holder always receives a position with
market value identical with any position he gives up. Because the market
value of any security is just the sum of the market values of the contingent
claims that it implies, the security holder can always recover his initial
position, or any other position consistent with the total market value of
his resources, in the market for contingent claims.

Thus, for example, if, given its operating decisions, a firm shifts its
capital structure by issuing additional bonds, the shareholders lose claim to
some period 2 resources, but the market value of their losses is precisely
equal to the receipts that they obtain from issuing the bonds. On the other
hand, if a firm uses retained earnings to retire debt, the bondholders receive
the current market value of the claims to future resources that they give
up, so that they can purchase contingent claims in the open market and
regain precisely the positions that they give up. The shareholders, in turn,
give up current resources with market value identical with the claims to
future resources that they receive, so that their trading opportunities are
also unaffected by the firm's financing decision. Hence, given its operating
decisions, the firm's financing decisions are a matter of indifference to its
security holders, so that operating and financing decisions are separable.

The separation principle is also easily established in a multiperiod states
of the world model, and we leave this as an exercise for the reader. The
reader may also find it useful to illustrate the preceding discussion of the
irrelevance of capital structure decisions to a firm's security holders with a
numerical example similar to the one presented earlier in this section.

\textit{III.C.2. The market value of a firm in the
two-period risk class model}

In the states of the world model the market prices of a firm's securities
are derived by comparing them with the prices of perfect substitutes that

$^{13}$ The reader may find it helpful to show what would happen in this example if this
result did not hold. By this time the required arbitrage arguments should be so familiar
that this is an easy task.
are themselves just combinations of contingent claims. In the original treatment of capital structure problems by Modigliani and Miller [1] a different approach is taken. The market prices of a firm's securities are derived by comparing them with prices of identical positions in other firms from the same risk class.

We say that two firms \(i\) and \(j\) are in the same risk class if for all \(t\),

\[
X_i(t) = \lambda_i X_j(t) \quad \text{and} \quad I_i(t) = \lambda_i I_j(t),
\]

(4.1)

where \(X_i(t)\) and \(X_j(t)\) are the net cash earnings, before interest, of the firms at \(t\), \(I_i(t)\) and \(I_j(t)\) are cash outlays at \(t\) for investment, and \(\lambda_i\) is a proportionality factor, which, it should be noted, is the same for all \(t\) for both earnings and investment. In periods before \(t\), earnings and investments at \(t\) are uncertain; but for the two firms to be in the same risk class, investors must agree that whatever values earnings and investment outlays take in any period, for these two firms they are always proportional by the factor \(\lambda_i\) and hence perfectly correlated.\(^{14}\)

In our initial discussions of the effects of financing decisions in the context of the risk class model, we concentrate as in preceding sections, on the two-period case. The results are then generalized to the multiperiod case.

In the two-period model we presume as always that firms make production decisions at period 1 and that these production decisions yield probability distributions on net cash earnings to be received at period 2. Each firm is presumed to pay out its earnings in full to those who hold its securities from period 1, and these earnings are the only source of returns on these securities at period 2. The role of the capital market at period 1 is to establish market prices for the securities. Our goal initially is to use the risk class model to establish the by now familiar proposition that, given its production decision at period 1, or equivalently, the probability distribution of its total earnings at period 2, the total market value of a firm is inde-

\(^{14}\) This is not the definition of a risk class used by Modigliani and Miller in their initial paper [1], in which condition (4.1) is imposed only on the average earnings of firms in the same risk class. Thus, in their original definition only the average earnings, assumed to be uncertain, of firms in the same risk are assumed to be perfectly correlated, whereas in our definition this assumption is applied to the perceptions of investors concerning realized values of earnings. Their definition imposes fewer restrictions on the characteristics of the earnings streams of firms in the same risk class, but implicitly imposes more restrictions on the tastes of investors, because investors must be assumed to be concerned only with the average earnings of firms over time.

In their later empirical work, Modigliani and Miller apply (4.1) to realized earnings in defining a risk class [4, fn. 3], but (4.1) is not imposed on investment outlays. Some such condition on investment outlays is usually necessary, however, if the securities of firms in the same risk class are to be perfect substitutes. Intuitively, even if their earnings are proportional, the securities of the two firms need not be perfect substitutes if the earnings streams are obtained with nonproportional and uncertain investment outlays.
pendent of its financing decisions, that is, the way the distribution of period 2 earnings is fragmented and sold at period 1 to the different classes of security holders. Later we complete the proof of the separation principle by showing that a firm’s financing decisions are a matter of indifference for any of its security holders, so that operating and financing decisions are separable.

For simplicity, the argument centers on an example involving two firms from the same risk class, one of which, firm \( L \), is levered, that is, it has bonds as well as common stock in its capital structure at period 1, and the other, firm \( u \), is unlevered, that is, its capital structure is composed entirely of common stock. It is assumed that the two firms not only are in the same risk class but also are anticipated to have the same total net cash earnings at period 2; that is, whatever period 2 earnings turn out to be, investors anticipate at period 1 that they will be identical for the two firms:

\[
X_L(2) = X_u(2) = X(2).
\]

Consider first an investor who holds the fraction \( \alpha \) of the shares of the unlevered firm \( u \). The market value of his investment at period 1 is thus

\[
\alpha V_u(1) = \alpha S_u(1),
\]

and his return at period 2 is

\[
\alpha X_u(2) = \alpha X(2).
\]

The investor could obtain this same return, however, by buying the proportion \( \alpha \) of both the bonds and the common stock of the levered firm; that is, he could invest \( \alpha B_L(1) \) in the debt of the levered firm, which debt has total market value \( B_L(1) \), and invest \( \alpha S_L(1) \) in the firm’s shares, which have total market value \( S_L(1) \). If \( R_L(2) \) is the total payment to the firm's debtholders at period 2, his investment in the firm’s debt yields \( \alpha R_L(2) \), and his shares in the firm yield \( \alpha(X(2) - R_L(2)) \). Thus his total return is

\[
\alpha[X(2) - R_L(2)] + \alpha R_L(2) = \alpha X(2),
\]

which is indeed identical with the return that he gets from holding the proportion \( \alpha \) of the shares of the unlevered firm. His total investment in the levered firm is

\[
\alpha[V_L(1) - B_L(1)] + \alpha B_L(1) = \alpha V_L(1).
\]

Thus if \( V_L(1) \) were less than \( V_u(1) \), our investor, or any other investor, would not hold the shares of the unlevered firm, because the returns provided by these shares can be obtained at lower cost by buying both the

---

14 Previously we used \( R(2) \) to denote the promised payments to bondholders at period 2. In this section \( R(2) \) represent actual payments, which may, of course be uncertain at period 1.
bonds and the shares of the levered firm. It follows that the total value of the unlevered firm must be equal to or less than that of the levered firm.

In holding the shares of the unlevered firm \( u \), the investor has claim to an unlevered equity return from the particular risk class under consideration. When he buys equal proportions of the bonds and shares of the levered firm, he also has claim to an unlevered equity return from this class. In effect, by buying both the bonds and shares of the levered firm, he has unlevered the firm's capital structure, at least as far as his own portfolio is concerned. We now see that likewise an investor can use a combination of personal debt and the shares of the unlevered firm to obtain returns identical with those provided by the shares of the levered firm. In this case the investor lever the shares of the unlevered firm, at least as far as his own portfolio is concerned. And the availability of these types of transactions in a perfect market is shown to imply that the period 1 market values of the two firms must be identical.

Thus consider now an investor who owns the fraction \( \alpha \) of the shares of the levered firm \( L \). The market value of his shares at period 1 is

\[
\alpha S_L(1) = \alpha[V_L(1) - B_L(1)],
\]

and his return at period 2 is

\[
\alpha[X(2) - R_L(2)].
\]

Exactly the same return could be obtained by purchasing the fraction \( \alpha \) of the shares of the unlevered firm, financing the purchase in part by issuing \( \alpha B_L(1) \) of personal debt, using the shares of the unlevered firm as collateral. In a perfect capital market, if the debt of the levered firm, issued against its period 2 earnings \( X(2) \), has market value \( B_L(1) \) at period 1, the investor, borrowing against his ownership of \( \alpha X(2) \) of the period 2 earnings of the unlevered firm, must be able to obtain \( \alpha B_L(1) \) of personal debt at period 1 by promising to pay lenders \( \alpha \) times whatever turn out to be the bond payments of the levered firm at period 2. Thus the net period 1 cost of the position involving "homemade" leverage is

\[
\alpha V_u(1) - \alpha B_L(1) = \alpha[V_u(1) - B_L(1)],
\]

and the return at period 2 from this combination of personal debt and shares in the unlevered firm is

\[
\alpha X(2) - \alpha R_L(2) = \alpha[X(2) - R_L(2)],
\]

which is indeed identical with the return obtained by holding the fraction \( \alpha \) of the shares of the levered firm.

But the investment cost at period 1 of holding the shares of the levered firm is greater than the cost of the combination of personal debt and shares of firm 2 if \( V_L(1) > V_u(1) \). It follows that if the shares of the levered firm are to be held, \( V_L(1) \) must be equal to or less than \( V_u(1) \). Our first example
established, however, that \( V_u(1) \leq V_L(1) \). Thus we can conclude that in equilibrium the total period 1 market values of the two firms must be equal. In short, given their production decisions, or equivalently given the probability distributions of their period 2 earnings, the period 1 market values of the firms are unaffected by the differences in their capital structures.

III.C.3. The market value of a firm in a multiperiod risk class model

Extension of this result to a multiperiod risk class model is conceptually straightforward, although notationally the model becomes rather cumbersome. Thus let \( V_i(t) \) and \( V_j(t) \) be the market values at period \( t \) of firms \( i \) and \( j \). The two firms are assumed to be in the same risk class, so that

\[
X_i(t) = \lambda X_j(t) \quad \text{and} \quad I_i(t) = \lambda I_j(t). \tag{4.1}
\]

We want to establish the following proposition: If there is some \( t \) such that \( V_i(t) = \lambda V_j(t) \), then for all \( t \leq t \), \( V_i(t) = \lambda V_j(t) \), regardless of the financial decisions of the firms.\(^{16}\)

The total value at period \( t \) of the shares and bonds of firm \( j \) that were outstanding from \( t - 1 \) is

\[
[D_j(t) + S_{j,t-1}(t)] + [R_j(t) + B_{j,t-1}(t)], \tag{4.2}
\]

where \( S_{j,t-1}(t) \) and \( B_{j,t-1}(t) \) are respectively the market values at \( t \) of the common stocks and bonds outstanding from \( t - 1 \); \( D_j(t) \) are total dividends paid at \( t \), and it is assumed that any new stock issued at \( t \) does not share in these dividends; and \( R_j(t) \) are actual payments\(^{17}\) at \( t \) on bonds outstanding from \( t - 1 \). At period \( t - 1 \) the values of most period \( t \) variables are uncertain. Nevertheless, because cash inflows at \( t \) must equal outflows, we must have

\[
X_j(t) + b_j(t) + s_j(t) = I_j(t) + R_j(t) + D_j(t), \tag{4.3}
\]

where \( s_j(t) \) and \( b_j(t) \) are respectively the market values of the new stocks and bonds issued at \( t \). Thus first solving Equation (4.3) for \( D_j(t) \) and then noting that

\[
S_{j,t-1}(t) = V_j(t) - B_{j,t-1}(t) - s_j(t) - b_j(t),
\]

for the total wealth expression (4.2), we can obtain

\[
[D_j(t) + S_{j,t-1}(t)] + [R_j(t) + B_{j,t-1}(t)] = X_j(t) - I_j(t) + V_j(t). \tag{4.4}
\]

\(^{16}\) The assumed condition \( V_i(t) = \lambda V_j(t) \) is met, for example, if the life of the risk class is finite and terminates at period \( t \). Then \( V_i(t) = X_i(t) \) and \( V_j(t) = X_j(t) = \lambda X_j(t) = \lambda V_j(t) \).

\(^{17}\) Which are, of course, equal to or less than promised payments.
Similarly, the total value at \( t \) of the securities of firm \( i \) outstanding from \( t - 1 \) is
\[
[D_i(t) + S_{i, t-1}(t)] + [R_i(t) + B_{i, t-1}(t)] = X_i(t) - I_i(t) + V_i(t).
\]
(4.5)

Arguments similar to those advanced with the two-period model could be used now to show that equilibrium in the capital market implies \( V_i(t - 1) = \lambda_i V_j(t - 1) \). For example, an unlevered position in firm \( j \), that is, a direct ownership in the total return \( X_j(t) - I_j(t) + V_j(t) \), could be obtained at \( t - 1 \) by purchasing the proportion \( \alpha \) of both the bonds and shares of the firm. But precisely the same return at period \( t \) could be obtained by purchasing the proportion \( \alpha / \lambda_i \) of both the bonds and shares of firm \( i \), because
\[
X_i(t) - I_i(t) + V_i(t) = \lambda_i [X_j(t) - I_j(t) + V_j(t)],
\]
so that
\[
\frac{\alpha}{\lambda_i} [X_i(t) - I_i(t) + V_i(t)] = \alpha [X_j(t) - I_j(t) + V_j(t)].
\]
The costs of unlevered investments in the two firms are equal, however, only if \( V_i(t - 1) = \lambda_i V_j(t - 1) \).

Alternatively, using this type of argument, one could show that if \( V_i(t - 1) < \lambda_i V_j(t - 1) \), a return identical with that provided by the shares of firm \( j \) can be obtained from a combination of the securities of firm \( i \) with personal borrowing or lending, and the investment cost at \( t - 1 \) of this combination is less than the cost of the shares of firm \( j \). It follows that with \( V_i(t - 1) < \lambda_i V_j(t - 1) \), no investor is willing to hold the shares of firm \( j \). Likewise if \( V_i(t - 1) > \lambda_i V_j(t - 1) \), an investor could combine the securities of firm \( j \) with personal borrowing or lending at period \( t - 1 \) and obtain a return at \( t \) identical with that provided by the shares of firm \( i \) but at a cost lower than the cost of the shares of firm \( i \). It would then follow that if \( V_i(t - 1) > \lambda_i V_j(t - 1) \), no investors would hold the shares of firm \( i \) at \( t - 1 \). We could thus conclude that if the shares of both firms are to be held, that is, if we are to have a market equilibrium, it must be the case that \( V_i(t - 1) = \lambda_i V_j(t - 1) \). And the same arguments could then be applied to the values at \( t - 2, t - 3 \), and so on.

But because the approach is so similar to that of the preceding section, we leave its detailed development as an exercise for the reader\(^\text{a}\) and consider instead a different route to establishing the proposition, stated at the

\(^{\text{a}}\text{Hint: A simple way to go about this is as follows: First suppose that } V_i(t - 1) \neq \lambda_i V_j(t - 1). \text{ Then show that the capital structure of the "undervalued" firm can be unlevered by the investor on personal account, and then personal debt can be introduced to obtain a return at period } t \text{ identical with that to be obtained from the shares of the "overvalued" firm but at a cost less than the cost of the shares of the overvalued firm.}\)
beginning of this section, about the market values of firms in the same risk class. This alternative proof centers on showing that if firms $i$ and $j$ are in the same risk class and if $V_i(t) = \lambda_i V_j(t)$ but $V_j(t-1) \neq \lambda_i V_j(t-1)$, there are arbitrage opportunities, that is, opportunities for investors to earn sure profits without expending any of their own resources. Because this is inconsistent with market equilibrium, we must have $V_i(t-1) = \lambda_i V_j(t-1)$.

Suppose that $V_i(t-1) > \lambda_i V_j(t-1)$. Our hypothesis is that this would imply that firm $i$ is “overpriced” relative to firm $j$. We show that an investor could take advantage of this by issuing two claims at $t-1$ whose payoffs at $t$ are tied directly to those of the stocks and bonds of firm $i$ and then using the proceeds from his private issues to purchase an equivalent position in the securities of firm $j$. Specifically, on one of his private issues he promises to pay $\left[D_i(t) + S_{i,t-1}(t)\right]/V_i(t-1)$, whatever the (as of $t-1$) uncertain values of $D_i(t)$ and $S_{i,t-1}(t)$ turn out to be; and on the other he promises to pay $\left[R_i(t) + B_{i,t-1}(t)\right]/V_i(t-1)$. As long as he can deliver on these promises, and we see later that this is true as long as $V_i(t-1) \geq \lambda_i V_j(t-1)$, in a perfect capital market their market values at $t-1$ must be $S_i(t-1)/V_i(t-1)$ and $B_i(t-1)/V_i(t-1)$; that is, in a perfect market if the common stock of firm $i$ has market value $S_i(t-1)$ at $t-1$, the investor’s promise to pay $\left[D_i(t) + S_{i,t-1}(t)\right]/V_i(t-1)$ at $t$ must have market value $S_i(t-1)/V_i(t-1)$, and his promise to pay $\left[R_i(t) + B_{i,t-1}(t)\right]/V_i(t-1)$ must have market value $B_i(t-1)/V_i(t-1)$ at $t-1$. Thus the total proceeds from his two private issues are exactly $1$, and, making use of Equations (4.1) and (4.5) and the assumption that $V_i(t) = \lambda_i V_j(t)$, at period $t$ he pays

$$\frac{\left[D_i(t) + S_{i,t-1}(t)\right] + \left[R_i(t) + B_{i,t-1}(t)\right]}{V_i(t-1)} = \frac{X_i(t) - I_i(t) + V_i(t)}{V_i(t-1)} = \lambda \frac{X_j(t) - I_j(t) + V_j(t)}{V_i(t-1)}.$$

Just as he issues common stock and debt in exactly the proportions in which they are outstanding in the capital structure of firm $i$ at period $t-1$, the investor takes the $1$ proceeds from his private issues and invests

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18 Cf. Chap. 1, Sec. II.B.1.

19 Equivalently, at $t-1$ the investor “short-sells” the stocks and bonds of firm $i$ in the amounts $S_i(t-1)/V_i(t-1)$ and $B_i(t-1)/V_i(t-1)$; that is, he borrows these amounts of the stocks and bonds of firm $i$ from investors who hold them at $t-1$ and promises to return them at $t$ and to pay any dividends and interest declared by the firm at $t$. After borrowing the securities at $t-1$, he immediately sells them in the market and then proceeds to purchase the securities of firm $j$ in the manner to be described in the text.
\( S_j(t - 1)/V_j(t - 1) \) in the common stock of firm \( j \) at \( t - 1 \) and \( B_j(t - 1)/V_j(t - 1) \) in its bonds. From Equation (4.4) the dollar return at \( t \) from these investments is

\[
\frac{[D_j(t) + S_{j,t-1}(t)] + [R_j(t) + B_{j,t-1}(t)\]}{V_j(t - 1)} = \frac{X_j(t) - I_j(t) + V_j(t)}{V_j(t - 1)}.
\]

(4.7)

Comparing Equations (4.6) and (4.7), if \( V_i(t - 1) > \lambda_i V_j(t - 1) \), the investor’s dollar returns at \( t \) are greater than the claims he must pay, and this is true regardless of the realized values of \( V(t) \), \( X(t) \), and \( I(t) \) for the two firms.\(^2\) (Thus \( V_i(t - 1) > \lambda_i V_j(t - 1) \) does indeed imply that the investor can always deliver on his promises at \( t \).)

But remember that the investor realizes this sure profit at \( t \) without any expenditure of his own resources. Thus any investor could take advantage of such opportunities, and there is no reason for anyone to limit the extent to which he does so. In short, as long as \( V_i(t - 1) > \lambda_i V_j(t - 1) \), the existence of arbitrage opportunities implies that the market cannot clear.\(^2\)

On the other hand, if \( V_i(t - 1) < \lambda_i V_j(t - 1) \), there are likewise arbitrage (sure profit at no cost) opportunities in which investors issue claims equivalent to those of firm \( j \) and use the proceeds to purchase the securities of firm \( i \). And such opportunities are also inconsistent with equilibrium. Thus market equilibrium at \( t - 1 \) requires \( V_i(t - 1) = \lambda_i V_j(t - 1) \). But the same reasoning then applies to the market values of \( t - 2 \) and then to those of \( t - 3 \), and so on, so that the proposition stated at the beginning of this section is established by induction.

III.C.4. The effects of financing decisions on the firm's bondholders and shareholders

The preceding discussion of the risk class model has been concerned with showing that, given a firm’s operating (production-investment) decisions, its total market value at any point in time is independent of its financing

\(^2\) But this statement is true only if \( X_j(t) - I_j(t) + V_j(t) \geq 0 \). In fact this must always be the case. Making use of Equation (4.3),

\[ X_j(t) - I_j(t) + V_j(t) = R_j(t) + D_j(t) - [b_j(t) + s_j(t)] + V_j(t). \]

Thus \( X_j(t) - I_j(t) + V_j(t) < 0 \) implies \( b_j(t) + s_j(t) > V_j(t) + D_j(t) + R_j(t) \). But because \( D_j(t) \) and \( R_j(t) \) must be nonnegative, this last condition implies \( b_j(t) + s_j(t) > V_j(t) \); that is, the firm raises new capital in excess of its total market value. This is, of course, inconsistent with a perfect capital market, because it means that the new security holders suffer an immediate loss. Thus, necessarily, \( X_j(t) - I_j(t) + V_j(t) \geq 0 \).

\(^2\) Alternatively, if we think of investors as exercising the arbitrage opportunity by short-selling the securities of firm \( i \) and using the proceeds from sale of these securities at \( t - 1 \) to purchase the securities of firm \( j \), no investors will want to hold the securities of \( i \) and the market cannot clear.
decisions. It is well to reemphasize, however, that the firm's security holders are indifferent to its financing decisions, so that operating and financing decisions are separable, only if the financing decisions have no effect on their separate market values or trading positions. Thus to complete the treatment of the risk class model, we now show that this indifference proposition always holds. The analysis, however, is valid outside the context of the risk class model, and indeed one of its major advantages is in more closely relating the results of this chapter to those obtained with the perfect certainty model in Chapter 2.

Given that a firm is going to make an operating decision that requires outlays of $I(t) \text{ at period } t,$ we now examine the effects, if any, of different financing arrangements on the wealth at $t$

$$W_{S,t-1}(t) = D(t) + S_{t-1}(t)$$  \hspace{1cm} (4.8a)

of the common stock that is outstanding from $t - 1$ and on the wealth at $t$

$$W_{B,t-1}(t) = R(t) + B_{t-1}(t)$$  \hspace{1cm} (4.8b)

of the bonds outstanding from $t - 1$.

If the firm finances the outlay $I(t)$ with retained earnings, total dividends paid at $t$ are

$$D(t) = X(t) - I(t) - R(t);$$  \hspace{1cm} (4.9)

that is, the dividend is just net cash earnings $X(t)$ less the outlay $I(t)$, less the interest payments $R(t)$.

Likewise

$$S_{t-1}(t) = V(t) - B_{t-1}(t);$$  \hspace{1cm} (4.10)

that is, because no new securities are issued, the market value at $t$ of the shares outstanding from $t - 1$ is just the total market value of the firm less the market value at $t$ of the bonds outstanding from $t - 1$. Thus

$$W_{S,t-1}(t) = D(t) + S_{t-1}(t) = X(t) - I(t) + V(t) - [R(t) + B_{t-1}(t)].$$  \hspace{1cm} (4.11)

Suppose now that the firm finances all or part of the outlay $I(t)$ with new shares. If, as always, we assume that the new shares do not participate in dividends paid at $t$ and if the market value of the new shares is $s(t)$, we have

$$D(t) = X(t) - I(t) - R(t) + s(t),$$  \hspace{1cm} (4.12)

$$S_{t-1}(t) = V(t) - B_{t-1}(t) - s(t),$$  \hspace{1cm} (4.13)

so that

$$W_{S,t-1}(t) = X(t) - I(t) + V(t) - [R(t) + B_{t-1}(t)].$$  \hspace{1cm} (4.14)

---

\(^{\text{a}}\) Except that we now drop the subscript identifying the firm, the notation is the same as that for the multiperiod risk class model.

\(^{\text{b}}\) Equivalently, Equation (4.9) is obtained directly from Equation (4.3) by setting the value of new security issues, $b(t) + s(t)$, equal to zero.
In previous sections it has already been established that, given the firm's operating decisions, the value of the firm \( V(t) \) is independent of its financing decisions. Moreover, as long as the firm's bondholders and stockholders protect themselves from one another in the manner described in Section III.A, the wealth \([R(t) + B_{t-1}(t)]\) of the firm's bondholders depends only on the terms of the debt, that is, the payments promised, and on the firm's operating decisions, which are the sole determinant of the firm's future earnings prospects and thus of its "ability to pay." Thus because \( I(t) \) is taken as given and \( X(t) \) is the result of previous operating decisions, and is thus independent of current financing decisions, a comparison of Equations (4.14) and (4.11) shows that the wealth of the old shares at \( t \) is the same whether the outlay \( I(t) \) is financed with retained earnings or new shares.

And if we compare Equations (4.9) and (4.10) with (4.12) and (4.13), we find the reason, already familiar from Chapter 2, for this result. With new share financing the dividend \( D(t) \) is higher by the amount of the new issue \( s(t) \), but the old shareholders' part of the value of the firm is also lower by the same amount. In more familiar terms, new share financing allows the firm to pay a higher current dividend than when retained earnings financing is used, but there is an immediate equal and offsetting reduction in capital gains, so that the old shareholders are no better off with one form of financing than with another.

Finally consider now the case in which the firm perhaps issues both bonds, in the amount \( b(t) \), and shares, \( s(t) \), to cover all or part of its investment \( I(t) \). Then

\[
D(t) = X(t) - I(t) - R(t) + b(t) + s(t),
\]

\[
S_{t-1}(t) = V(t) - B_{t-1}(t) - b(t) - s(t),
\]

so that, as in the two previous cases,

\[
W_{S_{t-1}}(t) = X(t) - I(t) + V(t) - [R(t) + B_{t-1}(t)].
\]

This result is again easily explained when we compare Equations (4.15) and (4.16) with (4.9) and (4.10) and with (4.12) and (4.13). Like share financing, new bonds allow the firm to pay higher current dividends than when retained earnings are used, but the new bonds lead to an immediate equal and offsetting reduction in the old stockholders' share of \( V(t) \), the market value of the firm. The net result is that the wealth of the current shareholders is the same under all three methods of financing, as is the wealth of the current bondholders.

* That is, the bondholders protect themselves with "me first" rules, and the stockholders require the firm to avoid the types of financing decisions, that is, retiring the first mortgage before the second, that lead to increase in the positions of the bondholders at the expense of those of the shareholders.
Thus given the firm's operating decisions, its financing decisions are a matter of indifference to its security holders. It follows that operating and financing decisions are separable; that is, they can be made independently.

III.C.5. Summary

The partial equilibrium approaches to examining the effects of financing decisions are obviously quite restrictive. For example, the states of the world model requires that investors agree on the relevant states of the world at period 1; the risk class model requires the existence of meaningful risk classes, that is, containing more than one firm. But concentrating on the restrictiveness of any particular model obscures the important fact that the propositions concerning the “irrelevance of financing decisions” are direct consequences of a perfect capital market and not of restrictions imposed by more detailed specifications of the market context. This fundamental point comes through clearly in the market equilibrium approach, which is free of the restrictions of the partial equilibrium models.

III.D. Market Imperfections: The Effects of Tax Laws

In this section we consider briefly the effects of certain market imperfections on the propositions derived in previous sections concerning the irrelevance of financing decisions. Our attention concentrates on the effects of tax laws, and our goal is simply to show that the class of market imperfections that arise from existing provisions of tax laws can be treated with the same analytical apparatus used in the perfect market model.

III.D.1. The tax deductibility of corporate interest payments

We give most attention to those United States corporate tax laws by which a firm can deduct interest payments on its debt in computing its income for tax purposes, but other payments, that is, dividends, to security holders are not tax-deductible in this way. We find that this type of law should lead to higher market values for levered firms than for equivalent unlevered firms.

The analysis is carried out primarily in the context of the two-period risk class model. We assume again the existence of two firms identical in all respects except capital structure: one firm \(L\) is levered; the other firm \(u\) is not. The market anticipates that at period 2 earnings before interest and taxes are the same for the two firms,

\[
X_u(2) = X_L(2) = X(2),
\]

and the firms pay out all their period 2 posttax earnings to their security holders.

Because interest payments are deductible in computing corporate taxes, however, the posttax earnings available to security holders are not the same
for the two firms. In particular, if \( \tau \) is the tax rate, the posttax earnings of the unlevered firm are

\[
X(2) - \tau X(2) = X(2) (1 - \tau),
\]

and the posttax earnings available to the security holders of the levered firm are

\[
X(2) - \tau [X(2) - R_L(2)] = X(2) (1 - \tau) + \tau R_L(2).
\]

The expression on the left of the equality reflects the fact that the levered firm pays taxes only on earnings net of interest payments. In terms of the expression on the right of the equality, we see that this means that the posttax earnings available to the security holders of the levered firm are greater than those available to the security holders of the unlevered firm by the quantity \( \tau R_L(2) \), which represents the tax saving of the levered firm that arises from having debt in its capital structure. It is as if the government paid a subsidy of \( \tau R_L(2) \) to the levered firm for having debt in its capital structure.

We now show that, just as the posttax earnings of the levered firm at period 2 are greater than those of the unlevered firm by the amount of the tax saving \( \tau R_L(2) \), the market value of the levered firm at period 1, \( V_L(1) \), must be greater than the market value of the unlevered firm \( V_u(1) \) by the amount of the market value at period 1 of the period 2 tax saving. As in earlier sections, the method of proof is to compare investment positions in the two firms that yield identical returns to the investor at period 2 and then to show the relationship between the period 1 market values of the two firms that must hold if, as must be the case in a market equilibrium, the equivalent positions are to sell at the same investment cost.

First, consider a position in which at period 1 the investor holds the proportion \( \alpha \) of the common stocks of the levered firm. The return at period 2 on his holdings is

\[
\alpha [X(2) - \tau [X(2) - R_L(2)] - [R_L(2) + B_L(2)]]
\]

\[
= \alpha X(2) (1 - \tau) - \alpha [R_L(2) + B_L(2)] + \alpha \tau R_L(2)
\]

\[
= \alpha X(2) (1 - \tau) - \alpha (1 - \tau) R_L(2) - \alpha B_L(2);
\]

that is, the period 2 returns to the common stock of the levered firm are less than total pretax cash earnings \( X(2) \) by the sum of (1) total tax payments \( \tau [X(2) - R_L(2)] \) and (2) payments to bondholders of interest \( R_L(2) \) and principal \( B_L(2) \).

\# In our previous treatment of a world with no taxes we used \( R_L(2) \) to represent total payments—interest and principal—to bondholders at period 2. Now, however, the differential tax treatment makes it necessary to distinguish between interest and principal.
As usual, however, it is possible to replicate the cash resources at period 2 to be provided by the shares of the levered firm with a combination of personal debt and the shares of the unlevered firm. In particular, the investor purchases \( \alpha V_u(1) = \alpha S_u(1) \) of the shares of the unlevered firm and finances the purchase in part by issuing personal debt in the form of a promise to pay \( \alpha (1 - \tau) R_L(2) \) and \( \alpha B_L(2) \) at period 2. Thus the investor's cash return at period 2 is

\[
\alpha X(2)(1 - \tau) - \alpha (1 - \tau) R_L(2) - \alpha B_L(2),
\]

which is indeed identical with what he would get from simply holding the proportion \( \alpha \) of the shares of the levered firm.

Note that to get a cash return from the shares of the unlevered firm that is the same as that obtained from those of the levered firm, the investor does not simply replicate the debt of the levered firm on personal account; that is, rather than issuing a promise to pay \( \alpha [R_L(2) + B_L(2)] \), he promises to pay only \( \alpha [R_L(2)(1 - \tau) + B_L(2)] \). This reflects the fact that \( \tau R_L(2) \) of the levered firm's total debt payments are in effect "paid" by the government in terms of the tax saving, or government subsidy, on interest payments. Thus in issuing debt to finance a position in the unlevered firm, the investor replicates on personal account that part of the levered firm's total debt payments which is "paid" by the levered firm itself.

Now the total investment cost at period 1 of the combination of personal debt and the shares of the unlevered firm is

\[
\alpha V_u(1) - \alpha [B_L(1) - \tau v_{BL(1)}(1)];
\]

that is, in a perfect market if the period 2 debt payments \( R_L(2) + B_L(2) \) of the levered firm have market value \( B_L(1) \) at period 1, the investor's promise to pay \( \alpha (1 - \tau) R_L(2) \) and \( \alpha B_L(2) \) at period 2, or equivalently, his promise to pay \( \alpha [R_L(2) + B_L(2) - \tau R_L(2)] \), must have period 1 market value \( \alpha B_L(1) \) less \( \alpha v_{BL(1)}(1) \), where \( v_{BL(1)}(1) \) is the period 1 market value of the interest payment \( R_L(2) \). On the other hand, the period 1 investment cost that would be incurred by simply holding the proportion \( \alpha \) of the shares of the levered firm is

\[
\alpha S_L(1) = \alpha [V_L(1) - B_L(1)].
\]

Because the cash returns at period 2 from these shares of the levered firm are identical with those obtained from the combination of personal debt with the shares of the unlevered firm, in a market equilibrium the period 1 investment costs of the two positions must be equal; that is, we must have

\[
\alpha [V_L(1) - B_L(1)] = \alpha V_u(1) - \alpha [B_L(1) - \tau v_{BL(1)}(1)],
\]

which implies

\[
V_L(1) = V_u(1) + \tau v_{BL(1)}(1); \quad (4.18)
\]
that is, just as the posttax earnings available at period 2 to the security holders of the levered firm exceed those available to the security holders of the unlevered firm by the amount of the tax saving $rR_L(2)$, the market value of the levered firm at period 1 exceeds that of the unlevered firm by the amount of the period 1 market value of the tax saving.\footnote{More rigorously, we could obtain Equation (4.18) by first noting that if $V_L(1) > V_u(1) + rR_{LM}(1)$, no investors would hold the shares of the levered firm, because an equivalent period 2 cash return could be obtained with lower period 1 investment cost from a combination of personal debt with the shares of the unlevered firm. On the other hand, and we leave it to the reader to show that, if $V_L(1) < V_u(1) + rR_{LM}(1)$, then no investors would hold the shares of the unlevered firm, because the cash return that they provide at period 2 can be obtained with lower investment cost at period 1 by buying a combination of the debt and shares of the levered firm.}

Of course when the market values of the two firms are in the equilibrium given by Equation (4.18), there are no advantages or disadvantages to the investor who purchases the shares of the levered firm rather than the equivalent position involving personal debt and the shares of the unlevered firm. Likewise the reader should convince himself that there are no advantages or disadvantages in holding the shares of the unlevered firm rather than the equivalent unlevered position involving the bonds and shares of the levered firm.

But the shareholders of any firm are better off whenever the firm increases the amount of debt in its capital structure. Indeed the ideal situation would be to have a capital structure that is all debt. This would of course be a ruse, and one that almost surely would not fool the tax collector, because the debt in this case is common stock. And we must admit that at this point there is little in the way of convincing research, either theoretical or empirical, that explains the amounts of debt that firms do decide to have in their capital structure.

Finally, the result given by Equation (4.18) is easily generalized to the multiperiod case. Without going into details, which, at this point, the interested reader could surely provide, we simply state that in a market equilibrium the market value of a levered firm must be equal to the value of an equivalent unlevered firm from the same risk class plus the current market value of all anticipated future corporate tax savings, including those on debt to be issued in the future as well as those on currently outstanding debt, that result from the tax deductibility of corporate interest payments.\footnote{For the more sophisticated, we note that our analysis and conclusions with respect to the effects of corporate tax laws differ somewhat from those of Modigliani and Miller in Ref. 1 or 4. In particular, they argue that the market value of the tax saving of the levered firm is just $r$ times the market value of its debt; that is, $V_L(1) = V_u(1) + rB_L(1)$. This result holds under several special cases: (1) in the two-period model when all the}
III.D.2. The tax deductibility of personal interest payments

But we have concentrated so far on the effects of corporate tax laws. Suppose now that, in computing personal income taxes, investors have the same tax privileges as corporations; that is, as is the case with United States tax laws, interest payments on personal debt are deductible in computing taxable personal income just as interest payments on corporate debt are deductible in computing corporate taxable income. Does this reverse our preceding conclusions concerning the effects of corporate tax laws on market values; that is, can the individual investor replicate the tax advantages of corporate debt on personal account, so that in equilibrium the market value of a levered firm is once again equal to the market value of an equivalent unlevered firm from the same risk class?

These questions are easily answered, and in the usual way. Reverting again to the two-period model, we need only compare the cash returns at period 2 on the shares of a levered firm with those of an equivalent combination of personal debt with the shares of the unlevered firm. Now, however, the relevant comparison is in terms of cash returns net of both corporate and personal taxes.

Thus let \( \tau_c \) represent the corporate tax rate and \( \tau_p \) the personal tax rate, which for simplicity is assumed to be the same for all investors. Suppose now that the shareholders of the unlevered firm issue debt on personal account equivalent to the total debt of the levered firm; that is, at period 1 the shareholders of the unlevered firm issue a promise to pay a total of \( R_L(2) \) at period 2. (For simplicity we assume that the total debt payment \( R_L(2) \) is interest.) Hence their net cash returns, after both corporate and personal income taxes, are

\[
[X(2)(1 - \tau_c) - R_L(2)](1 - \tau_p)
= X(2)(1 - \tau_c)(1 - \tau_p) - R_L(2) + \tau_p R_L(2). \tag{4.19}
\]

The first term, \( X(2)(1 - \tau_c)(1 - \tau_p) \), represents the net posttax earnings of the unlevered firm; the second term, \( R_L(2) \), is the payment by the shareholders on their personal debt; and the last term, \( \tau_p R_L(2) \), is the savings in personal taxes that result from the tax-deductible debt payment \( R_L(2) \).

On the other hand, at period 2 the shareholders of the levered firm obtain

\[
[X(2) - \tau_c[X(2) - R_L(2)] - R_L(2)](1 - \tau_p)
= X(2)(1 - \tau_c)(1 - \tau_p) - R_L(2) + \tau_p R_L(2) + \tau_p R_L(2)(1 - \tau_p). \tag{4.20}
\]

firm's debt payments at period 2 are tax-deductible and (2) in the multiperiod model when all the firm's debt is in the form of a perpetuity and no additional debt is ever issued nor is any existing debt ever repurchased or, equivalently, when all debt is immediately replaced when it matures and there are never net new additions to or reductions from the amount of outstanding debt.
Thus comparing Equations (4.19) and (4.20), we see that by introducing personal debt identical with that of the levered firm, the shareholders of the unlevered firm were nevertheless unsuccessful in replicating the period 2 posttax cash return of the shareholders of the levered firm. The return to the shares of the levered firm is greater by the quantity \( \tau_c R_c(2)(1 - \tau_p) \), and this simply represents the fact that investors cannot reproduce on personal account the tax savings that arise at the level of the firm from having debt in its capital structure. We leave it to the reader to show that in fact the presence of personal taxes does not affect the relationship between the market values of the levered and unlevered firms as given by Equation (4.18).

### III.D.3. Some closing comments on market imperfections

We could extend a little further this analysis of the effects of the market imperfections that arise from tax laws. Rapidly, however, the conclusions that we could obtain would become more and more ambiguous, and the discussion would become more philosophical than analytical.

For example, we could use the techniques of the preceding sections to analyze the effects of those provisions of United States tax laws by which the dividends and interest received by individuals are treated as regular income for tax purposes but the tax rate on capital gains is at most half the rate on regular income. In itself this tax provision would destroy the equivalence of dividends and capital gains as sources of wealth in a perfect market, and it would lead to a preference for wealth received in the form of capital gains. An exact analysis of these effects, however, is complicated by the fact that the tax benefits of capital gains depend somewhat on the tax bracket of the investor, and indeed there are some large investors, primarily nonprofit institutions, that pay no taxes and so have no tax incentives to seek out capital gains rather than dividends.

Thus rather than speculate about the none too clear-cut effects of these and other market imperfections on the relationships between the financing decisions of firms and their market values, we leave the study of these effects to future research, both theoretical and empirical. The rest of this book is concerned with developing further the implications of a perfect capital market in a world of uncertainty. The justification for limiting the analysis in this way is threefold: (1) it is consistent with the goal of this book, which is to present those ideas in finance which have clear-cut rigorous foundations, and models based on a perfect market are the only ones that satisfy this criterion; (2) at the very least such models help us to organize our thinking about the ubiquitous “real world”; and finally (3) hopefully models based on seemingly unrealistic assumptions yield insights and hypotheses that help to explain real market data. And we argue in Chapter 8 that this is in fact the case.
IV. THE FIRM'S OBJECTIVE FUNCTION: THE MARKET VALUE RULE

Given perfect capital markets, the major result that we have derived so far for a world of uncertainty is what we call the first separation principle; that is, for given operating (production-investment) decisions, the financing decisions of a firm are a matter of indifference to its security holders, so that production and financing decisions are separable. This result, however, just tells us that operating decisions can be made independently of financing decisions; it does not provide a criterion, or decision rule, for optimal operating decisions. The goal of this section is to establish such a decision rule. In particular, we show that, given perfect capital markets, optimal operating decisions for a firm at any point in time involve maximizing the market value of those securities outstanding before the operating decision is made; that is, optimal operating decisions are independent of, or separable from, the details of security holder tastes and can be made according to the market value rule.

Having obtained this "second separation principle," we will have generalized to a world of uncertainty most of the major results concerning investment-financing decisions by firms obtained in the first part of this book under the assumption of perfect certainty. And this should suffice to show that the necessary ingredient for these results is the assumption of a perfect capital market.

We shall find, however, that implementation of the market value rule for a firm's operating decisions is much less straightforward in a world of uncertainty than in a world of certainty. First of all, a firm may have more than one type of security outstanding, and operating decisions that maximize the market value of one type of security need not do so for others. Second, the assumption of perfect capital markets does not in itself lead to a theory of how market values are determined. The assumption allows us to make the negative statement that financing decisions do not affect market values, but positive statements regarding how market values are determined require a more detailed specification of the market context. Thus to say that a firm should follow the market value rule in operating decisions is to say very little until a theory of how market values are determined has been presented.

IV.A. The Market Value Rule: Derivation

Because the arguments here are very similar, it is well to review the basis of the market value rule in a world of perfect certainty. With perfect certainty, the assumption of perfect capital markets is taken to mean (1) no transactions costs or other frictions in making portfolio adjustments, (2) equal access to capital markets by individuals and firms, and (3) firms and individuals are atomistic competitors in the capital market; that is, their individual actions in the market have no effect on the ruling one-

* The problems that arise on this score, however, were not inconceivable in a world of certainty. See, for example, Chap. 2, p. 71, footnote 4.
period market interest rates. It follows from this definition that an individual's consumption-investment opportunities can be determined from knowledge of his wealth—the current market value of all the resources he has or will obtain in time—and the sequence of one-period interest rates. Because it is assumed that the activities of any individual firm do not affect market interest rates, the only way that the firm can maximize its security holders' consumption-investment opportunities is by maximizing their wealths.

Given a similar specification of the capital market, precisely the same line of reasoning applies to a world of uncertainty. And we have in fact already assumed that the capital market is perfect in the sense that (1) there are no transactions costs, (2) there is equal access to financial opportunities by all individuals and firms, and (3) investors perceive that there are always close substitutes for any securities of a firm and the firm's decisions have no effect on the prices of the securities issued by other investors and firms.

What constitutes a close substitute for the securities of a firm is a simple matter in a world of perfect certainty. There it is always possible to replicate the (sure) payoffs through time to be provided by a firm's securities in terms of simple claims to resources to be obtained in future periods. (See, for example, Chapter 1, especially Section II.A.) Things are not so simple in a world of uncertainty, however, in which the future returns to be received on a firm's securities are now subject to probability distributions, and the notion of what constitutes a perfect substitute for a security depends both on investor tastes, that is, the characteristics of return distributions that are important to investors, and on the nature of the distributions, that is, the ways in which the distributions can be summarized. 30

30 Much of the rest of this book is concerned with developing such detailed specifications of investor tastes and the nature of distributions of investment returns and then with how these combine to produce a theory of market equilibrium, that is, a theory that says what is of positive importance in determining the market values of securities.

We might note, however, that one quite simple model of this sort has already been presented—the states of the world model in Section III.C.1. In the two-period version of this model, there are perfect substitutes for any probability distributions on period 2 market value generated by a firm as long as its production decisions do not affect prices of contingent claims for any state and the firm is an atomistic competitor in the capital market. In this model the payoffs in different states provided by any of a firm's securities can be replicated simply by purchasing the equivalent numbers of contingent claims to dollars to be delivered in each state.

On the other hand, the risk class model gave us a way of obtaining perfect substitutes for the securities of a firm that did not require a complete theory of how market values are determined; that is, when Equation (4.1) holds between two firms, the securities of one firm can be used to obtain perfect substitutes for the securities of the other, regardless of the model determining the market values of the two firms. But the conditions of Equation (4.1) are obviously quite strong, and in the model of later chapters we shall see that less stringent conditions are sufficient for firms to be in the same risk class.
Nevertheless, whatever model is used to obtain the definition of a perfect substitute, given a perfect capital market, including the assumed existence of perfect substitutes, the reasoning underlying the market value rule for a firm’s operating decisions is precisely the same in a world of uncertainty as in a world of perfect certainty; that is, given a security holder’s wealth, the firm’s operating decisions do not affect the consumption-investment opportunities that are available to the security holder in the market. Thus all that the firm can affect with its operating decisions is the wealth of its security holders, and here the optimal path is clear: more wealth is preferred to less.

IV.B. The Market Value Rule: Implementation

Translating the conclusion that in a perfect capital market the firm should maximize the wealth of its security holders into a concrete criterion for optimal production decisions is straightforward if there is a production-investment plan that maximizes the wealths of all security holders. Potential problems arise only when maximum wealth for one group does not imply maximum wealth for others.

Thus suppose that firms issue only the usual types of bonds and common stock. From Equation (4.5) we know that for a given firm the sum of \( W_{B,t-1}(t) \), the wealth at period \( t \) of the bonds outstanding from \( t - 1 \), and \( W_{S,t-1}(t) \), the wealth at period \( t \) of the stock outstanding from \( t - 1 \), is

\[
W_{B,t-1}(t) + W_{S,t-1}(t) = [R(t) + B_{t-1}(t)] + [D(t) + S_{t-1}(t)]
\]

\[
= V(t) - I(t) + X(t),
\]

(4.21)

where \( B_{t-1}(t) \) and \( S_{t-1}(t) \) are the market values at \( t \) of the bonds and stocks outstanding from \( t - 1 \), \( R(t) \) and \( D(t) \) are interest and dividend payments on these bonds and stocks, and, as always, \( V(t) \), \( I(t) \), and \( X(t) \) are, respectively, the market value of the firm, outlays for operating (production-investment) decisions made at period \( t \), and net cash earnings.

In these discrete-time models \( X(t) \) is assumed to be a result of production-investment decisions of previous periods and thus to be unaffected by the decisions taken at period \( t \). It is therefore clear from Equation (4.21) that a production-investment decision at \( t \) that maximizes \( V(t) - I(t) \) also maximizes the combined wealths of the bonds and shares outstanding from \( t - 1 \).

But why do we talk only about the bonds and shares outstanding from \( t - 1 \)? The reason is that there is nothing the firm can do to help, or hurt, those who buy any new securities issued at \( t \) to finance the operating decisions made at \( t \). In a perfect capital market these new securities are always sold at prices that “fully reflect” the operating decisions to be made at \( t \), and these new securities do not provide investors with any new types of investment opportunities that would not be available, and at the same prices, if the firm made some other operating decision at \( t \).

The situation here is the same as that obtained in a world of perfect certainty in which
There are some important circumstances in which the production-investment rule, maximize $V'(t) - I(t)$, maximizes the separate as well as the combined wealths of the firm's security holders. This is the case, in order of decreasing obviousness:

1. By chance, or perhaps as a general rule.
2. When the firm has only one type of security, common stock, in its capital structure.
3. When the firm has debt as well as common stock in its capital structure but the debt is riskless; that is, the firm is always able to make full payment on its debt promises, regardless of the operating decisions that it makes, so that its operating decision at $t$ does not affect the wealth of its debtholders.
4. The firm's bondholders and stockholders are free to compensate one another for the effects of operating decisions that increase the wealth of one group but not the other. In this case, because maximizing $V'(t) - I(t)$ maximizes the combined wealths of bondholders and shareholders, there may be some way, and indeed there may be many ways, that side payments between the bondholders and shareholders can be arranged, so that with the operating decision that maximizes $V'(t) - I(t)$, every security holder's wealth is at least as great as it would be with any other operating decision.

The rule of maximize $V'(t) - I(t)$ for the firm's operating decisions can fail to maximize the separate as well as the combined wealth of its bondholders and stockholders in the case in which the firm's debt is risky and side payments between the firm's bondholders and stockholders are ruled out. Then it is easy to construct examples in which a production plan that maximizes shareholder wealth does not maximize bondholder wealth, or vice versa.

Thus in the two-period states of the world model, consider a firm that has two mutually exclusive production decisions $a$ and $b$ available at period 1, and either can be carried out without additional expenditures of resources at period 1; that is, for both decisions $I(1) = 0$. There are also assumed to be two possible states of the world at period 2. The price $p(1)$ at period 1

---

any excess of value over cost generated by the firm's operating decision made at period $t$ goes to shares outstanding from previous periods. New shares issued to finance the operating decision of period $t$ are issued at a price that "fully reflects" the effects of the operating decision on cash flows. The difference between the certainty and uncertainty results, of course, is that under certainty the term "fully reflects" simply implies pricing of known cash flows at known market interest rates, whereas under uncertainty the pricing model must be somewhat more complicated.
of a contingent claim to $1 to be received only if state 1 occurs at period 2 is $0.5; and likewise the price $p(2)$ at period 1 of $1 to be received at period 2 if state 2 occurs is $0.5. At period 1, the firm is assumed to have bonds in its capital structure in the form of a promise to pay $5 at period 2, whichever state occurs.

For each of the production decisions, Table 1 shows the payoffs in the two states at period 2, along with the period 1 market values of the firm, its bonds, and its common stock. Thus if production plan $a$ is chosen, the period 1 market value of the firm's bonds is $B(1) = 5(0.5) + 5(0.5) = 5$, and the value of the shares is $S(1) = 2(0.5) + 2(0.5) = 2$. On the other hand, if plan $b$ is chosen, the firm is not able to deliver in full on its debt promise if state 1 occurs. Hence with this production plan the period 1 market value of the bonds is $B(1) = 1(0.5) + 5(0.5) = 3$, and the value of the shares is 2.5.

Thus the market value of the shares is higher with plan $b$, but the market value of the bonds is higher with plan $a$. If side payments between the bondholders and shareholders were possible, the bondholders could give the shareholders a subsidy of $0.5 to induce them to choose plan $a$; then the shareholders would have as much wealth as if plan $b$ were chosen, and the bondholders would have more. But if side payments are ruled out and the shareholders control the firm, plan $b$ is chosen. And it is well to note that, although this decision maximizes shareholder wealth, it does not maximize $V(1) - I(1)$, which in this case is just $V(1)$.

From a practical viewpoint, however, situations of potential conflict between bondholders and shareholders in the application of the market value rule are probably unimportant. In general, investment opportunities that increase a firm's market value by more than their cost both increase the value of the firm's shares and strengthen the firm's future ability to meet its current bond commitments. A much more important impediment to the application of the market value rule is the fact that we have not yet

\[\text{Indeed the bondholders would be willing to make side payments up to $2 to induce the adoption of plan $a$ rather than $b$.}\]
developed a theory of how market values are determined, and until we do, in later chapters, the market value rule remains somewhat empty as a criterion for optimal operating decisions.

But this leads us naturally to the last topic of this chapter.

V. THE COST OF CAPITAL AND THE RETURN ON A FIRM’S SHARES

Our analysis so far in this chapter has made no use of terms like “cost of capital,” “discount rate,” and “present value.” And indeed such concepts are unnecessary, and would be somewhat distracting, in the development of capital structure theory. This theory does not propose to say exactly how the market value of a firm and its securities is determined but only that market values, however determined, are unaffected by financing decisions, so that operating and financing decisions are separable.

But discount rates and costs of capital, interpreted as prices that determine current values for future payoffs, imply more positive statements about the determination of market values. Interpreting discount rates and costs of capital in this way requires a detailed theory of investor, firm, and market equilibrium that would specify, among other things, the characteristics of probability distributions on future payoffs that are important to investors and the way in which optimal consumption-investment decisions by all individuals and operating decisions by all firms combine to produce a market equilibrium in which discount rates are appropriately interpreted as prices that determine market values. The capital structure model of this chapter does not, of course, provide this much detail.

V.A. Discount Rates and the Cost of Capital

If one is willing to assign them a much less positive role, however, discount rates and cost of capital can be introduced into, or perhaps more accurately, imposed on, the model, and more or less in the way described by Modigliani and Miller [1]. Concentrating again on the two-period model, for any firm \( j \) we can always define a proportionality factor \( 1/(1 + \rho_j) \) relating \( V_j(1) \), the market value of the firm at period 1, to \( E[X_j(2)] \), its expected net cash earnings at period 2, as

\[
V_j(1) = \frac{1}{1 + \rho_j} E[X_j(2)]. \tag{4.22}
\]

It is then natural to think of \( \rho_j \) as the one-period rate of discount of the

\( ^{29} \) Although the states of the world model is such a theory of market value determination, in our view its usefulness at this time is mainly pedagogical, that is, in providing a concrete framework for illustrating basic notions in finance and other areas of economics. Because as yet the model has yielded little in the way of testable implications, we are reluctant to use it as the cornerstone in our treatment of uncertainty.
expected period 2 earnings. But we could just as well have defined a proportionality factor \(1/(1 + \delta_i)\) relating \(V_j(1)\) to, say, the 0.95 fractile of the distribution of \(X_j(2)\), and then \(\delta_i\) could be thought of as the rate of discount for the 0.95 fractile. The important point is that in the primitive models of this chapter, an expression like Equation (4.22) is just a definition; it does not imply that \(V_j(1)\) was determined in the market by applying a proportionality factor or a discount rate to the period 2 expected earnings.\(^4\)

If we restrict attention to the risk class model, a similarly limited concept of the cost of capital can be introduced and again more or less in the way described by Modigliani and Miller. To accomplish this, we must further assume that all production plans available to a firm at period 1 are associated with probability distributions on earnings at period 2 from the same risk class; that is, in the manner of Equation (4.1), the period 2 earnings provided by any given production plan would always be proportional to those of any other production plan available to the firm. Thus if \(X_j(2)\) and \(X_j(2)\)' are the earnings at period 2 for two such production plans, we have

\[
X_j(2)' = \lambda' X_j(2); \tag{4.23}
\]

that is, regardless of what the value of \(X_j(2)\) turned out to be at period 2, \(X_j(2)\)' would always be proportional to it by the factor \(\lambda'\).

Thus the different production plans available to a firm can be viewed as potentially generating different firms in the same risk class. And from our earlier analysis of the risk class model, we know that when Equation (4.23) holds and the capital market is perfect, we must have

\[
V_j(1)' = \lambda' V_j(1); \tag{4.24}
\]

that is, the current market values implied by the two production plans must be proportional by the same factor that relates their future earnings. Moreover, because Equation (4.23) holds for all possible values of future earnings for the two plans, it must also hold between their expected values; that is,

\[
E[X_j(2)'] = \lambda' E[X_j(2)]. \tag{4.25}
\]

Suppose now that we define the proportionality factor \(1/(1 + \rho_i)\) such that

\[
V_j(1) = \frac{1}{1 + \rho_i} E[X_j(2)]. \tag{4.26}
\]

\(^4\) Modigliani and Miller state this point rather clearly in sec. I.A. of Ref. 1. But their subsequent analysis makes such heavy use of terms like "capitalisation rate" and "cost of capital" that the important qualifications provided in their initial discussion are sometimes overlooked.
or equivalently

$$1 + \rho_j = \frac{E[X_j(2)]}{V_j(1)}.$$  

But from Equations (4.24) and (4.25) we know that

$$\frac{E[X_j(2)']} {V_j(1)'} = \frac{E[X_j(2)]} {V_j(1)}.$$  

Thus we must have

$$V_j(1)' = \frac{1}{1 + \rho_j} E[X_j(2)'].$$

(4.27)

so that, comparing Equations (4.26) and (4.27), the proportionality factor, $1/(1 + \rho_j)$, relating the period 1 market value of the firm to its expected period 2 earnings, is the same for the two production plans in the same risk class.

Suppose now that in its production decisions the firm abides by the rule, maximize $V_j(1) - I_j(1)$. Thus in comparing two production plans, one of which requires an incremental outlay of $dI_j(1) = I_j(1)' - I_j(1)$ more than the other, the plan with the higher outlay provides a larger excess of value over cost if $dV_j(1) - dI_j(1) > 0$, where $dV_j(1) = V_j(1)' - V_j(1)$.

But, letting $dE[X_j(2)] - E[X_j(2)'] = E[X_j(2)'] - E[X_j(2)]$, from Equations (4.26) and (4.27),

$$dV_j(1) = \frac{1}{1 + \rho_j} dE(X_j(2)),$$

so that

$$dV_j(1) - dI_j(1) > 0$$

implies

$$dE(X_j(2)) > dI_j(1)(1 + \rho_j).$$

In other words, the one-period expected return on the incremental outlay $dI_j(1)$ must be greater than $\rho_j$. Thus we can interpret $\rho_j$ as a cost of capital in the sense that it is the minimum required expected rate of return on incremental outlays for production activities of the particular risk class available to the firm. And using the same line of reasoning, it is easy to show that the same cost of capital applies to all firms in the same risk class.

It is evident, however, that we have more or less forced this concept of a cost of capital on the model by restricting attention to production activities that generate probability distributions of earnings from the same risk class. Even with this restrictive specification, however, we cannot interpret $\rho_j$ as a meaningful economic price that determines current values; that is, our primitive models do not imply that investors are concerned with expected
values or any other specific characteristics of probability distributions. Nor
do they imply, or rule out, any sort of discounting mechanism operating in
the market to determine market values. As noted throughout, such
implications could only be derived from a model that specified in detail the
characteristics of probability distributions that are relevant to investors
and the way in which optimal decisions by investors and firms combine to
determine the structure of equilibrium market prices.

It is important to emphasize, however, that we are not suggesting that
the market does not use discounting to determine market prices. Our only
goal here is to show that without a formal model that shows how the inter-
actions of individual economic units lead to the determination of discount
rates that in turn are used in determining market values, an investment
criterion based on a discounting procedure is not on a rigorous theoretical
footing. Although the importance of discount rates in the perfect certainty
model may lead us to suspect that they should also play a fundamental role
in any uncertainty model, this is nothing more than a hunch until such an
uncertainty model is actually available.

In short, the Modigliani-Miller capital structure propositions are much
more generally valid than the concepts of capitalization rates and cost of
capital. In essence, their important conclusion that production and financing
decisions are separable depends only on the assumption of a perfect capital
market, but a meaningful cost of capital requires a much more detailed
specification of the market context.

V.8. The Expected Return on Common Stock

It is possible, however, to get a little more mileage, in the form of addition-
als insights concerning the major capital structure propositions, from an
extension of the preceding discussion of the interpretation of expected

To make these comments more concrete, suppose that the market setting is as de-
scribed by the two-period states of the world model; that is, there are \( S \) possible states
of the world at period 2, and \( p(s) \) is the price at period 1 of a dollar delivered at
period 2 if state \( s \) occurs. If \( X_j(2,s) \) is the earnings of firm \( j \) at period 2 if state \( s \)
occurrs, the market value of the firm at period 1 is

\[
V_j(1) = \sum_{s=1}^{S} p(s) X_j(2,s).
\]

In this model the prices for contingent claims \( p(s) \) provide direct period 1 valuations
of the market values in each period 2 state generated by a production activity, and the
notion of a cost of capital relating current market values to the expectation of the dis-
btribution of future market values is somewhat out of place. Nevertheless, such a cost of
capital can be forced on the model in exactly the manner described above if one is willing
to assume that for any state \( s \) the relative payoff \( X_j(2,s)/X_j(2,1) \) is the same for all
production plans. And this is precisely the assumption necessary to ensure that all dis-
btributions of period 2 earnings that the firm can generate are from the same risk class.
rates of return in the context of the two-period risk class model. Rearranging expression (4.26), we can get

$$\rho_i = \frac{E[X_i(2)] - V_i(1)}{V_i(1)}; \quad (4.28)$$

that is, the cost of capital $\rho_i$ is just the expected one-period, or percentage, return that the firm earns on its period 1 market value $V_i(1)$. We can likewise define the expected one-period returns on the firm’s bonds and on its common stock as

$$r_j = \frac{E[R_j(2)] - B_j(1)}{B_j(1)}, \quad (4.29)$$

$$i_j = \frac{E[X_j(2)] - E[R_j(2)] - S_j(1)}{S_j(1)}; \quad (4.30)$$

where $E[R_j(2)]$ is the expected, or mean, value of all payments to all bondholders at period 2 and $B_j(1)$ and $S_j(1)$ are the total period 1 market values of all the firm’s debt and common stock.

The analysis in the preceding section established, in effect, that in a given risk class the period 1 market values of firms must be such that $\rho_i$, the expected one-period return on the total market value of firm $j$, is the same for all firms in the class. Moreover, the earlier analyses also established that, given its operating decisions, the firm’s financing decision at period 1 does not affect the period 1 wealths of those who hold its bonds and shares outstanding from period 0. This does not mean, however, that the expected one-period returns on the firm’s common stock and on all the bonds, new and old, in its capital structure at period 1 are independent of the period 1 financing decision. In particular, the risk of the firm’s shares as well as the risk of any new debt issued at period 1 certainly depends on the firm’s financing decisions, and this might affect the expected returns on the bonds and the shares. Let us now see if we can give these rather vague suspicions some formal content.

From Equations (4.28) and (4.29) we can obtain

$$E[X_i(2)] = (1 + \rho_i) V_i(1) \quad \text{and} \quad E[R_j(2)] = (1 + r_j) B_j(1).$$

Thus Equation (4.30) for the expected one-period return on the shares of firm $j$ can be rewritten as

$$i_j = \frac{(1 + \rho_i) V_i(1) - (1 + r_j) B_j(1) - S_j(1)}{S_j(1)};$$
or because \( S_j(1) = V_j(1) - B_j(1) \) and \( V_j(1) = S_j(1) + B_j(1) \),

\[
i_j = \rho_j \left[ \frac{S_j(1) + B_j(1)}{S_j(1)} \right] - r_j B_j(1) = \rho_j + (\rho_j - r_j) \frac{B_j(1)}{S_j(1)}. \tag{4.31}
\]

Because the total debt of the firm can never be more "risky" than the shares of an equivalent unlevered firm from the same risk class,\(^{48}\) in general we would expect that in a market dominated by risk averters \( \rho_j > r_j \). Thus Equation (4.31) says that the expected one-period, or percentage, return on a levered share in firm \( j \) is just \( \rho_j \), the percentage yield on the expected income from a share in an equivalent unlevered firm in the same risk class, plus a risk premium that depends on the debt/equity ratio of firm \( j \).

The discussion can be summarized as follows. The probability distribution and thus the risk of the period 2 earnings before interest of the firm depends on the operating decision that the firm makes at period 1. The risk implied by the operating decision is not affected by the way that it is financed; that is, the risk implied by the probability distribution on the total earnings of the firm depends on the nature of the operating decision, which alone generates the probability distribution of earnings. This probability distribution cannot be changed by different ways of financing the operating decision. The effect of the financing decision is merely to package the given total risk of the firm into different bundles; that is, for given assets the more bond financing used, the riskier the earnings that accrue to the common shareholders and to the bondholder. Specifically the more bond financing used, the higher the chance of default on at least part of the firm's debt and the higher the chance that the stockholders receive nothing. To reflect this risk, the shares and bonds of highly levered firms sell to yield higher expected returns than those of less highly levered firms.

At this point, however, it should be clear that the fact that the firm can affect the risks of its securities through its financing decisions does not imply that its security holders are concerned with these decisions. The important fact is that in a perfect capital market the firm's financing decisions do not affect the wealth of its security holders, nor do they provide or preclude any investment opportunities, that is, levels of risk, that its security holders could not obtain either from other firms or on personal account. Thus given its operating decisions, the firm's financing decisions are a matter of indifference to its shareholders, so that the operating and financing decisions are separable.

\(^{48}\) Note that when a firm's capital structure is 100 percent debt, this debt is identical with the common stock of an equivalent unlevered firm from the same risk class. Thus the total risk of the firm is the upper limit on the risk of the firm's debt.
VI. SUMMARY AND CONCLUSIONS

But the discussion of risk and risk aversion in the preceding section is a little vague, as indeed it must be at this stage of the game. A rigorous treatment of risk requires a definition of risk, which in turn requires a more detailed specification of investor tastes than has been made in this chapter; that is, in order to define the risk of an investment instrument, we have to say something about the characteristics of investor tastes, which then allow us to specify the characteristics of the investment instrument that determine its risk.

All that we have assumed about investor tastes in this chapter is the non-satiation axiom in Chapter 1; that is, other things equal, the investor always prefers more consumption to less. In combination with a perfect capital market, this simple assumption about tastes has allowed us to establish two important separation principles analogous to those obtained for the perfect certainty model in Chapter 2; that is, (1) given its operating (production-investment) decisions, the firm’s financing decisions are a matter of indifference to its security holders, so that operating and financing decisions are separable, and (2) the operating decisions of the firm can be made according to the market value rule and without regard to the details of security holder tastes.

In the absence of a theory that tells us how market values are determined, however, to say that operating decisions should be made according to the market value rule is to say very little. The goal of the next three chapters is to develop such a theory of market value determination. In line with the development of the perfect certainty model in the first part of this book, we first present a more detailed theory of consumer choice under uncertainty (Chapter 5). This is then combined with a detailed treatment of market opportunities into a theory of investor decision making (Chapter 6). And finally (Chapter 7) the theory of investor equilibrium is used to develop a model of market value determination and a theory of optimal operating decisions by firms.

REFERENCES

The first rigorous treatment of the capital structure propositions presented in this chapter is in


Their other important works in this area include


The Modigliani–Miller work is within the context of the risk class model. Their capital structure propositions were derived from the states of the world model in


General, and the original, discussions of the states of the world model are in


Analyses of the idiosyncrasies of United States tax laws and their effects on the perfect markets capital structure propositions can be found in


For a discussion of more traditional, that is, pre-Modigliani–Miller, views on the effects of financing decisions on market values see
