6

THE TWO-PERIOD CONSUMPTION-INVESTMENT MODEL

A. INTRODUCTION

In the previous chapter the expected utility model was presented, in somewhat general terms, as a model of choice under uncertainty. In this chapter the model is applied to the consumption-investment problem of an individual consumer. Throughout this chapter we are concerned with a simplified two-period case in which the individual must divide a given amount of wealth \( w_1 \) between consumption \( c_1 \) for the current period (period 1) and a portfolio investment \( h_1 = w_1 - c_1 \) that will provide a level of terminal wealth \( w_2 \) to be completely consumed during period 2, that is, \( c_2 = w_2 \).\(^1\)

As always, \( w_1 \) is just the market value at period 1 of the consumer's resources. Thus it includes the market value of portfolio assets carried forward from previous periods plus the market value of any occupational income to be

\(^1\) As in Chap. 5, tildes (~) are used throughout to denote random variables.
earned. To keep things simple, we assume that the consumer only sells labor at period 1; in period 2 he only consumes. And he receives payment at the beginning of period 1 for any labor services to be rendered during the period. Alternative approaches to the occupational decision are, however, considered briefly in a later section.

Finally, we assume throughout that the consumer is faced with a perfect capital market in the sense that he can buy as much as he wants of any investment asset without affecting its price; all investment assets are infinitely divisible; and there are no transactions costs or taxes.

At first glance the two-period model would seem to be a rather special case and of little general interest. In fact, however, we show in Chapter 8 that the results for the two-period model provide the core of the analysis for the general multiperiod problem. Thus at this point we bypass detailed justification and turn directly to the two-period model.

As in previous chapters, we proceed from simpler to more complex cases. For example, initially we treat a situation in which the individual's investment opportunities at period 1 are limited to two assets, one of which is riskless. Later in the chapter, however, this restriction is dropped, and the general N-asset problem is considered. In addition, as in previous chapters, the analysis is presented verbally, geometrically, and algebraically whenever possible. Indeed, the formal analysis may seem to become quickly rather complicated; thus we begin here with an intuitive discussion of the major results to be obtained in more rigorous manner later.

1.A. The Mean-Standard Deviation Model: An Overview

In principle the expected utility hypothesis itself is a complete theory of choice under uncertainty; that is, the individual just examines the expected utility associated with every possible consumption-investment decision and then chooses the one that maximizes expected utility. But this prescription is empty. From a normative or decision-making viewpoint, the individual faces the impossible task of examining in complete detail the probability distribution on period 2 consumption associated with each possible consumption-investment choice—of which there are also in principle an infinite number. And, from a substantive viewpoint, the expected utility model per se provides no observable or testable propositions about consumer behavior. In order to make the model practicable and to give it economic substance, we must impose more structure on the problem.

In the present chapter the consumption-investment problem is simplified by considering only situations in which the individual finds it possible to summarize his investment opportunities solely in terms of means and some measure of dispersion, usually standard deviations, of the distributions of the one-period percentage returns on different portfolios; that is, we are
concerned with situations in which, given the total amount of funds to be invested, the individual can rank a portfolio relative to other portfolios by looking only at two parameters of the distribution of the return on the portfolio, and thus ignoring other aspects of the distribution.

One special case in which such an approach is legitimate is when distributions of returns on all portfolios are normal. A normal distribution can be fully described once its mean (expected value) and standard deviation are known. Thus all the differences between any number of normal distributions can be determined from their means and standard deviations. In the consumption-investment model this implies that all portfolios can be ranked by the individual on the basis of these two parameters of their return distributions.

But the assumption that return distributions are normal is just one way to obtain a two-parameter portfolio model. We show later that there are other two-parameter distributions that can serve the same role in our analysis, and we even argue that these alternative distributions seem to fit the available return data better than the normal. The properties of normal distributions are probably more familiar to most readers than those of the alternatives, however, and for this reason most of this chapter deals with a consumption-investment model based on the assumption of normally distributed portfolio returns. But little is lost in this approach; we show later that the major results of the normal model are easily obtained from corresponding models based on other two-parameter distributions. In short, on the side of the opportunity set, the critical ingredient of the model is that return distributions can be fully described in terms of two parameters, means and some measure of dispersion like the standard deviation, with the specific distribution assumed having little effect on the analysis.

The one-period two-parameter consumption-investment model also requires some specifications of the individual’s tastes. In particular, he is assumed to behave as if he wished to make a consumption-investment decision that maximized expected utility, computed from the function \( U(c_1, c_2) \), which is assumed to be monotone-increasing and strictly concave in \((c_1, c_2)\). We emphasize that the individual’s behavior is as if he were an expected utility maximizer; as in all utility theory (see Chapter 1), in making his decisions, he need not have a utility function or expected utility consciously in mind. The assumption is that his observable behavior is indistinguishable from that of an expected utility maximizer.

If, for mathematical simplicity, we assume that the first partial derivatives of the utility function \( U(c_1, c_2) \) exist for all values of \((c_1, c_2)\), monotonicity implies

\[
\frac{\partial U(c_1, c_2)}{\partial c_1} > 0 \quad \text{and} \quad \frac{\partial U(c_1, c_2)}{\partial c_2} > 0. \tag{6.1}
\]
On the other hand, strict concavity implies that for any two nonidentical points \((c_1, c_2)\) and \((c_1', c_2')\)

\[ U(xc_1 + (1 - x)c_2', xc_2 + (1 - x)c_2') > xU(c_1, c_2) + (1 - x)U(c_1', c_2'), \quad 0 < x < 1. \quad (6.2) \]

Geometrically, (6.1) says that \(U\) is positively sloping in the direction of both \(c_1\) and \(c_2\); and (6.2) says that a straight line between any two points on the function lies everywhere below the function. Economically, monotonicity implies that the marginal utility of consumption is always positive; concavity implies that the marginal utility decreases as consumption in either period increases. As in the timeless expected utility of wealth model, under uncertainty concavity of the utility function is characteristic of a risk averter, and indeed the theory is directed entirely toward such risk-averse consumer-investors.

Let \(\bar{R}_p\) be the one-period percentage return—alternatively, the one-period return, or more simply just the return—on the portfolio \(p\). Then if \((w_1 - c_1)\) is invested in \(p\) at period 1, consumption in period 2 is

\[ c_2 = (w_1 - c_1)(1 + \bar{R}_p). \]

If the distribution of \(\bar{R}_p\) is normal, the distribution of \(\bar{c}_2\) is normal, and the mean \(E(\bar{c}_2)\) and standard deviation \(\sigma(\bar{c}_2)\) of \(\bar{c}_2\) are related to \(E(\bar{R}_p)\) and \(\sigma(\bar{R}_p)\) according to

\[ E(\bar{c}_2) = (w_1 - c_1)[1 + E(\bar{R}_p)] \quad \text{and} \quad \sigma(\bar{c}_2) = (w_1 - c_1)\sigma(\bar{R}_p). \]

\(^*\) As always, the one-period percentage return on any investment is just the market value of the investment at period 2 less its market value at period 1, all divided by the market value at period 1.
We now argue, in intuitive terms, that, given initial consumption \( c_i \), total investment \((w_1 - c_i)\), and normally distributed portfolio returns, a risk-averse consumer's expected utility is an increasing function of mean return \( E(\tilde{R}_p) \) and a decreasing function of standard deviation or dispersion of return \( \sigma(\tilde{R}_p) \).

First, with \((w_1 - c_i)\) and \( \sigma(\tilde{R}_p) \) constant, if \( E(\tilde{R}_p) \) is increased to, say, \( E(\tilde{R}_p)' \), with normally distributed portfolio returns the net effect is a shift in the distribution of \( \xi_1 \) toward higher values of \( c_2 \); that is, as illustrated in Figure 6.1, a normal distribution of given dispersion is simply moved to the right along the \( c_2 \) line: In other words, the probability that \( \xi_1 \) exceeds any given value is greater for the distribution with the higher expected return. Thus given positive marginal utility of \( c_2 \), it seems that, other things equal, in particular, \( c_1 \) and \( \sigma(\tilde{R}_p) \), the consumer must prefer more expected return to less; expected utility is an increasing function of expected portfolio return.

On the other hand, other things equal, in this case, \( c_1 \) and \( E(\tilde{R}_p) \), an increase in standard deviation of return from any \( \sigma(\tilde{R}_p) \) to \( \sigma(\tilde{R}_p)' \) results in a flattening of the distribution of \( \xi_1 \) about a given expected value, as illustrated in Figure 6.2. The chance of extremely high levels of period 2 consumption is increased with the higher \( \sigma(\tilde{R}_p)' \), but from the symmetry of the normal distribution there is an equal increase in the chance of extremely low levels of consumption. Given a risk-averse consumer—and thus decreasing marginal utility of period 2 consumption—the better chance of high levels of period 2 consumption does not increase expected utility so much as the better chance of low levels of consumption decreases it: In short, with normally distributed portfolio returns for a risk averter expected utility is a declining function of standard deviation of return \( \sigma(\tilde{R}_p) \).

\[ \text{Density of } c_2 \]

\[ \text{Distribution for } \sigma(\tilde{c}_2)' = (w_1 - c_i) \sigma(\tilde{R}_p)' \]

\[ \text{Distribution for } \sigma(\tilde{c}_2)' = (w_1 - c_i) \sigma(\tilde{R}_p)' \]

Figure 6.2
These results have an important implication. Consider the set of $E(\bar{R})$, $\sigma(\bar{R})$ efficient portfolios, where, by definition, a portfolio is $E(\bar{R})$, $\sigma(\bar{R})$ efficient if no portfolio with the same or higher expected return $E(\bar{R}_p)$ has lower standard deviation $\sigma(\bar{R}_p)$. Then for any given initial consumption $c_1$ and thus total investment $w_1 - c_1$, if expected utility is an increasing function of $E(\bar{R}_p)$ and a decreasing function of $\sigma(\bar{R}_p)$, the expected utility-maximizing, or optimal, portfolio must be a member of the efficient set. The particular efficient portfolio that is optimal depends on the level of $c_1$, but whatever the optimal level of $c_1$, with normally distributed portfolio returns:

**Efficient Set Theorem.** The optimal portfolio for a risk-averse consumer must be $E(\bar{R})$, $\sigma(\bar{R})$ efficient.

In brief, the assumptions of consumer risk aversion and normally distributed portfolio returns narrow down substantially the portfolios that the individual must consider in order to make an expected utility-maximizing consumption-investment decision. The assumption of normally distributed portfolio returns allows him to rank portfolios on the basis of means and standard deviations of returns; the assumption of risk aversion further allows him to restrict attention to $E(\bar{R})$, $\sigma(\bar{R})$ efficient portfolios.

**2B. A Familiar Picture**

The efficient set theorem is easily given a geometric interpretation. First, for any given level of initial consumption $c_1$ and thus investment $w_1 - c_1$, an indifference curve of $E(\bar{R})$ against $\sigma(\bar{R})$ is defined by the set of co-
bimations of $E(\bar{R})$ and $\sigma(\bar{R})$ that yield some fixed level of expected utility. The fact that expected utility is an increasing function of expected return $E(\bar{R})$ and a decreasing function of standard deviation $\sigma(\bar{R})$, implies that any such indifference curve must be positively sloping; to get the consumer to take on more $\sigma(\bar{R})$, he must be compensated with greater $E(\bar{R})$. In addition, as indicated by the arrow in Figure 6.3, expected utility increases from lower to higher indifference curves, that is, upward and to the left. We also show later that consumer risk aversion and normally distributed portfolio returns imply that indifference curves are convex as shown.

Intuitively, it is clear that $E(\bar{R})$, $\sigma(\bar{R})$ efficient portfolios must lie somewhere along the upper left boundary of the set of all feasible portfolios. We show later that a reasonable general representation of the left boundary of the feasible set is the curve abcd in Figure 6.4. Only portfolios along the positively sloping segment abcd are efficient, however, because, as the reader can easily check, portfolios along ab and de do not satisfy the efficiency criterion. We also show later that, like the curve abcd, the efficient boundary must be concave.

The optimal portfolio for the given period 1 consumption $c_1$ and investment $w_1 - c_1$ is that which allows the consumer to attain the highest possible indifference curve. Because the set of efficient portfolios traces a positively sloping concave curve in the $E(\bar{R})$, $\sigma(\bar{R})$ plane and the indifference curves are convex, the optimal portfolio is generally given by a tangency between an indifference curve and the efficient set curve, as illustrated, for example, by the point e in Figure 6.5. Except for the new variables $E(\bar{R})$ and $\sigma(\bar{R})$ on the axes, the picture here is similar to many we have met in previous chapters.

A graph like Figure 6.5 shows the optimal portfolio for a given split of
initial wealth \( w_1 \) between consumption \( c_1 \) and investment \( w_1 - c_1 \). Using this geometric procedure, one could in principle determine the optimal portfolio for every possible choice of \( c_1 \) and in this way determine the overall consumption-investment decision that maximizes expected utility. Although our analysis continues to concentrate on the characteristics of the optimal portfolio decision, we should always keep in mind that the choice of period 1 consumption \( c_1 \) is made simultaneously with the portfolio decision, and the two are of course interrelated.

In fact, we really have nothing of much substance to say about the characteristics of an optimal split of initial wealth between consumption and investment. To say anything specific here would require that we impose additional restrictions on the form of the individual's utility function, and we have chosen not to do so here. Rather, the formal analysis, in this chapter and the next, is primarily concerned with presenting, in more rigorous terms than above, the major implications for optimal portfolio decisions of the two-parameter consumption-investment model. In simplest terms, in this chapter we are concerned primarily with using the assumptions of consumer risk aversion and normally distributed portfolio returns to establish the characteristics of consumer tastes and market opportunities that lead to the efficient set theorem and the view of consumer equilibrium provided by Figure 6.5.

II. THE CONSUMER'S TASTES

In this section we formally derive the properties of the consumer's tastes that follow from the assumptions of risk aversion and normally
distributed portfolio returns. The first step is to show that with normally distributed portfolio returns, an optimal consumption-investment decision amounts to optimal choices of \( c_1 \) and a feasible combination of \( E(\bar{R}_p) \) and \( \sigma(\bar{R}_p) \). Next we show that with risk aversion and normally distributed portfolio returns, expected utility is an increasing function of expected return \( E(\bar{R}_p) \) and a decreasing function of standard deviation \( \sigma(\bar{R}_p) \), which leads directly to the efficient set theorem; that is, the consumer's optimal portfolio is \( E(\bar{R}) \), \( \sigma(\bar{R}) \) efficient. Finally, the properties of indifference curves of \( E(\bar{R}) \) against \( \sigma(\bar{R}) \) are established.\(^8\)

**II.A. Portfolio Decisions Based on \( E(\bar{R}) \) and \( \sigma(\bar{R}) \)**

Define the standardized variable

\[
\tau = \frac{\bar{R}_p - E(\bar{R}_p)}{\sigma(\bar{R}_p)}.
\]

(6.3)

If the distribution of \( \bar{R}_p \) is normal with mean \( E(\bar{R}_p) \) and standard deviation \( \sigma(\bar{R}_p) \), the distribution of \( \tau \) is normal with mean \( E(\tau) = 0 \) and standard deviation \( \sigma(\tau) = 1 \). In the statistical literature \( \tau \), defined as in Equation (6.3), is usually called the "unit normal variable."

Consumption in period 2 is related to the one-period return \( \bar{R}_p \) on the consumer's portfolio according to

\[
\varepsilon_2 = (w_1 - c_1)(1 + \bar{R}_p).
\]

But because the distribution of \( \bar{R}_p \) is assumed to be normal, making use of Equation (6.3), \( \varepsilon_2 \) can be written in terms of \( E(\bar{R}_p) \), \( \sigma(\bar{R}_p) \), and the unit normal variable \( \tau \) as

\[
\varepsilon_2 = (w_1 - c_1)[1 + E(\bar{R}_p) + \sigma(\bar{R}_p)\tau].
\]

(6.4)

The expected utility associated with a choice of current consumption \( c_1 \) and portfolio \( p \) is then

\[
E[U(c_1, \varepsilon_2)] = \int_{-\infty}^{\infty} U(c_1, (w_1 - c_1)[1 + E(\bar{R}_p) + \sigma(\bar{R}_p)\tau])f(\tau) \, d\tau,
\]

(6.5)

where \( f(\tau) \) is the density function of \( \tau \).

Because the distributions of \( \bar{R}_p \) for all portfolios are normal, in the

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\(^8\) The remainder of Sec. II relies heavily on concepts from elementary calculus. It is possible for the mathematically wary reader to skip directly to Sec. III. We encourage even the mathematically wary to continue on in this section, however; the verbal arguments presented will probably help in understanding the intuitive analysis presented in Sec. I.
definition of expected utility provided by Equation (6.5), the variable \( r \) and the density function \( f(r) \) are the same for all portfolios. Differences between the expected utilities for different consumption-investment decisions depend entirely on \( \pi, E(\hat{R}_p), \) and \( \sigma(\hat{R}_p) \), so that consumption-investment alternatives can be ranked on the basis of these three variables. Formally,
\[
E[U(\pi, \pi_0)] = V(\pi, E(\hat{R}_p), \sigma(\hat{R}_p)).
\]

(6.6)

*II.B. Marginal Expected Utilities and the Efficient Set Theorem*

We are interested now in the properties of expected utility as a function of expected return \( E(\hat{R}_p) \) and standard deviation or dispersion \( \sigma(\hat{R}_p) \). First, as long as the consumer invests some of his initial wealth, from Equation (6.5), the marginal expected utility of expected return \( E(\hat{R}_p) \)
\[
\frac{\partial E[U(\pi, \pi_0)]}{\partial E(\hat{R}_p)} = (w_1 - \pi_0) \int_{-\infty}^{\infty} \frac{\partial U(\pi, \pi_0)}{\partial \pi_0} f(r) \, dr > 0,
\]

(6.7)
because the marginal utility of \( \pi_0 \) is positive for all values of \( (\pi, \pi_0) \). Thus, other things equal, expected utility is an increasing function of expected return.

On the other hand, the marginal expected utility of return dispersion or standard deviation \( \sigma(\hat{R}_p) \)
\[
\frac{\partial E[U(\pi, \pi_0)]}{\partial \sigma(\hat{R}_p)} = (w_1 - \pi_0) \int_{-\infty}^{\infty} \frac{\partial U(\pi, \pi_0)}{\partial \pi_0} \left[ 1 + E(\hat{R}_p) + \sigma(\hat{R}_p) \right] f(r) \, dr < 0.
\]

(6.8)

Here the inequality is a little more difficult to see. It follows from (1) the fact that \( U(\pi, \pi_0) \) is concave, so that \( \partial U(\pi, \pi_0)/\partial \pi_0 \) is a positive but decreasing function of \( \pi_0 \) and thus of \( \pi \), and (2) \( f(r) \) is symmetric about 0. The result is perhaps best illustrated by reference to Figure 6.6. In the integral of Equation (6.8), for each value of \( r \), the variable over which we are integrating, we have a product of three terms, \( \partial U(\pi, \pi_0)/\partial \pi_0, r, \) and \( f(r) \), and this product is negative for \( r < 0 \) and positive for \( r > 0 \). But \( \partial U(\pi, \pi_0)/\partial \pi_0 \) is a positive but decreasing function of \( \pi_0 \) and thus of \( \pi \), and
\[ f(r) \text{ is symmetric about } 0, \text{ for } r > 0, \text{ so that} \]
\[ \left| \frac{\partial U(c_t, (u_1 - c_1)[1 + E(R_p) + \sigma(R_p)(-r)]}{\partial c_t} \right| (-r)f(-r) \]
\[ > \frac{\partial U(c_t, (u_1 - c_1)[1 + E(R_p) + \sigma(R_p)r])}{\partial c_t} rf(r). \]

But this implies that, as indicated in the bottom part of Figure 6.6, the product of the three terms, labeled \( g(r) \) in the figure, is more negative at any point \(-r\) than it is positive at the corresponding point \( r \). Thus the integral in Equation (6.8) is negative; expected utility is a declining function of standard deviation \( \sigma(R_p) \).

In short, given a value of initial consumption \( c_t \) and thus investment \( u_1 - c_1 \), Equation (6.7) says that for any given value of standard deviation \( \sigma(R_p) \) the consumer prefers more expected return \( E(R_p) \) to less; Equation (6.8) says that for given \( E(R_p) \) he prefers less \( \sigma(R_p) \) to more. But this implies that the optimal portfolio must be such that no portfolio with the same or higher expected return has lower standard deviation of return; that is, the optimal portfolio for any given split of \( u_1 \) between consumption \( c_t \) and investment \( u_1 - c_1 \) must be \( E(R) \), \( \sigma(R) \) efficient. Thus the best portfolio for the optimal split of \( u_1 \) between \( c_t \) and investment \( u_1 - c_1 \) must also be \( E(R) \), \( \sigma(R) \) efficient. But this of course is just the efficient set theorem.

*II.C. Properties of Indifference Curves in the Two-Parameter Model*

Finally, we now want to show that the properties of the consumer’s indifference curves of \( E(R) \) against \( \sigma(R) \) are as hypothesized in Figure 6.3. Recall that for any given level of period 1 consumption \( c_t \) and thus investment \( u_1 - c_1 \), an indifference curve of \( E(R) \) against \( \sigma(R) \) is defined by the set of \( E(R), \sigma(R) \) combinations that yields some fixed level of expected utility, say, \( V \). From Equations (6.7) and (6.8) we know already that, as indicated in Figure 6.3, any such indifference curve must be positively sloping and that the direction of increasing levels of expected utility must be upward and to the left, that is, from lower to higher curves. Thus the task of this section is completed if we can show that indifference curves must also be convex.4

4 But it is well to note that the conditions on marginal expected utilities given by Equations (6.7) and (6.8) are all we need to establish the important efficient set theorem. The additional convexity property of indifference curves just helps to make geometric analyses a lot neater.
To establish convexity it is sufficient to show that if \((E(\bar{R}), \sigma(\bar{R}))\) and \((E(\bar{R})', \sigma(\bar{R})')\) are any two points on some arbitrarily chosen indifference curve, say, the curve for expected utility \(V_1\) in Figure 6.3, the expected utility of any point
\[
(E(\bar{R})'', \sigma(\bar{R})'') = (x\bar{E}(\bar{R}) + (1 - x)\bar{E}(\bar{R})', x\sigma(\bar{R}) + (1 - x)\sigma(\bar{R})'),
\]
\[0 < x < 1,
\]
is greater than \(V_1\). With expected utility increasing upward and to the left in the \(E(\bar{R}), \sigma(\bar{R})\) plane, this implies that \((E(\bar{R})'', \sigma(\bar{R})'')\) is on a higher indifference curve than \(V_1\), which in turn implies convexity.

First let
\[
c_1 = (w_1 - c_1)[1 + E(\bar{R}) + \sigma(\bar{R})r],
\]
\[
c_2 = (w_1 - c_1)[1 + E(\bar{R})' + \sigma(\bar{R})'r],
\]
\[
c_3 = x\alpha_2 + (1 - x)c_2.
\]
But substituting from Equations (6.9) and (6.10) and simplifying, we obtain
\[
c_4'' = (w_1 - c_1)[1 + E(\bar{R})'' + \sigma(\bar{R})''r].
\]
Because we are concerned with a particular indifference map, the value of \(c_1\) is fixed. The concavity of \(U(c_1, c_2)\) then implies
\[
U(c_1, c_2') > xU(c_1, c_2) + (1 - x)U(c_1, c_2'), \quad 0 < x < 1.
\]
Substituting from Equations (6.9) to (6.11), for any given \(r\),
\[
U(c_1, (w_1 - c_1)[1 + E(\bar{R})'' + \sigma(\bar{R})''r])
\]
\[
> xU(c_1, (w_1 - c_1)[1 + E(\bar{R}) + \sigma(\bar{R})r])
\]
\[
+ (1 - x)U(c_1, (w_1 - c_1)[1 + E(\bar{R})' + \sigma(\bar{R})'r]). \quad (6.12)
\]
Because the density function \(f(r)\) is nonnegative for all values of \(r\), taking expectations over \(r\) in the manner of Equation (6.5) preserves the direction of the inequality in (6.12), so that
\[
E\{U(c_1, (w_1 - c_1)[1 + E(\bar{R})'' + \sigma(\bar{R})''r])\}
\]
\[
> xE\{U(c_1, (w_1 - c_1)[1 + E(\bar{R}) + \sigma(\bar{R})r])\}
\]
\[
+ (1 - x)E\{U(c_1, (w_1 - c_1)[1 + E(\bar{R})' + \sigma(\bar{R})'r])\}, \quad (6.13)
\]
which implies
\[
E\{U(c_1, (w_1 - c_1)[1 + E(\bar{R})'' + \sigma(\bar{R})''r])\} > V_1.
\]
Thus we have reached our goal: As indicated in Figure 6.3, \( (E(\widehat{R}'), \sigma(\widehat{R}')) \) is indeed on a higher indifference curve than \( (E(\widehat{R}), \sigma(\widehat{R})) \) and \( (E(\widehat{R}')', \sigma(\widehat{R}')') \), so that the curve on which these points fall must be convex.

Finally, an indifference map like Figure 6.3 provides the indifference curves for a given level of initial consumption \( c_0 \). There are, of course, similar indifference maps for other levels of \( c_0 \), but all indifference curves and indifference maps have the general properties discussed above. In determining the individual's optimal consumption-investment decision, we do not, of course, consider \( c_0 \) fixed, so that it is necessary to examine the indifference maps for all levels of consumption. This is getting a little ahead of ourselves, however; in order to determine the optimal consumption-investment decision, we must first consider the opportunity set facing the individual.

**III. THE INVESTMENT OPPORTUNITY SET: TWO-ASSET CASE**

For simplicity, in the initial consideration of the opportunity set, we assume that only two investment assets are available to the individual: a riskless asset \( f \) that yields the one-period return \( R_f \) with perfect certainty\(^4\) and a risky asset \( a \) whose one-period return \( R_a \) has a normal distribution with mean \( E(R_a) \), assumed to be greater than \( R_f \), and standard deviation \( \sigma(R_a) \). We do not claim, of course, that such a two-asset opportunity set is at all realistic. The goal here is simply to introduce some elementary tools and concepts that are used throughout our later discussions. We eventually consider more general cases in which there are many types of investment assets.

Although there are only two investment assets available to the individual, the fact that the assets are infinitely divisible gives him an infinite number of portfolio possibilities. Specifically, if his total portfolio investment is \( h_1 = w_1 - c_0 \) and if \( h_f \) and \( h_a \) are the funds allocated to the assets \( f \) and \( a \), he can invest any fraction \( z = h_f/h_1 \), \( 0 \leq z \leq 1 \), in the riskless asset \( f \) and put the remainder \( 1 - z = (h_1 - h_f)/h_1 = h_a/h_1 \) in the risky asset \( a \). The return on such a portfolio is

\[
\widehat{R}_p = \frac{w_2 - h_1}{h_1} = \frac{h_f(1 + R_f) + h_a(1 + R_a) - h_1}{h_1}
\]

\(^4\) Note that an asset that is riskless for one horizon period need not be riskless for another. A government bill with one year to maturity and no intermediate coupon payments is riskless for an individual with a one-year horizon; at the end of the year he just cashes it in for its face value. The same bill is not riskless for an individual with a shorter horizon, however, because in selling the bill before maturity, he faces a probability distribution on price that results from the fact that short-term interest rates are to some extent random.
or
\[
R_p = xR_f + (1 - x)R_s, \quad 0 \leq x \leq 1. \tag{6.14}
\]

Because \( R_f \) is a constant and the distribution of \( R_s \) is normal with mean \( E(R_s) \) and standard deviation \( \sigma(R_s) \), the one-period portfolio return \( R_p \) also has a normal distribution with mean
\[
E(R_p) = xR_f + (1 - x)E(R_s), \quad 0 \leq x \leq 1, \tag{6.15a}
\]
and standard deviation
\[
\sigma(R_p) = (1 - x)\sigma(R_s), \quad 0 \leq x \leq 1. \tag{6.15b}
\]

It is easy to see that by varying the value of \( x \) in Equations (6.15a) and (6.15b), we can define the trade-off between mean and standard deviation obtained in forming portfolios according to Equation (6.14). An exact description of this trade-off can be obtained as follows: First solve Equations (6.15a) and (6.15b) for \( x \) to obtain
\[
x = \frac{E(R_p) - E(R_s)}{R_f - E(R_s)}, \quad \tag{6.16a}
\]
\[
x = \frac{\sigma(R_s) - \sigma(R_p)}{\sigma(R_s)}, \quad \tag{6.16b}
\]
Equating the right-hand sides of these two expressions and then solving for \( E(R_p) \), we get
\[
E(R_p) = R_f + \left(\frac{E(R_s) - R_f}{\sigma(R_s)}\right)\sigma(R_p). \tag{6.17}
\]

Thus Equations (6.15a) and (6.15b) imply a linear relationship between \( E(R_p) \) and \( \sigma(R_p) \) for portfolios defined by Equation (6.14). A plot of Equation (6.17) for some assumed values of the parameters \( R_f, E(R_s), \) and \( \sigma(R_s) \) would look something like the line \( R_f \alpha \) in Figure 6.7. The line terminates on the \( E(R) \) axis at the point \( R_f \); from Equation (6.15b) we see that the portfolio with \( \sigma(R_p) = 0 \) corresponds to the value \( x = 1 \); that is, all portfolio funds are invested in the riskless asset \( f \). On the other hand, at the point \( \alpha, x = 0 \) and all portfolio funds are invested in the risky asset \( \alpha \); thus the expected return and standard deviation for this portfolio are \( E(R_s) \) and \( \sigma(R_s) \). By varying \( x \) between 1 and 0, which is the same as varying \( \sigma(R_p) \) between 0 and \( \sigma(R_s) \), we obtain portfolios along the line \( R_f \alpha \) in Figure 6.7. From Equation (6.17) the slope of the line is \( (E(R_s) - R_f)/\sigma(R_s) \); when portfolios are formed according to Equation
(6.14), a unit increase in $\sigma(\bar{R}_p)$ leads to an increase of $(E(\bar{R}_a) - R_f)/\sigma(\bar{R}_a)$ units of $E(\bar{R}_p)$.

By varying our assumptions slightly, we can obtain some useful additional insights into the investment opportunity set. For example, suppose that it is possible to sell-short either or both of the assets $f$ and $a$. For our purposes a short sale is defined as an exchange in which the individual borrows units of an asset at the beginning of period 1, agreeing to repay the lender the market value of these units at the beginning of period 2.

For the riskless asset $f$, a short sale is simply borrowing in which the individual agrees to repay $1 + R_f$ dollars at period 2 per dollar borrowed at period 1. To simplify the arguments, suppose that the individual can borrow as much as he likes at the rate $R_f'$ and that he uses his borrowings to increase his investment in the risky asset $a$. In this case Equations (6.14) to (6.17) remain valid, but now $x$ is allowed to take on negative values. In terms of Figure 6.7, negative values of $x$ yield portfolios along the extension of $R_f'a$ through the point $a$, that is, along the line from $a$ through $z$ in the

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* If these statements are not obvious, the reader may find it helpful to illustrate them with a numerical example of his own construction. He will then probably also find it helpful to extend the example to the analyses in subsequent paragraphs.

* A note for the more sophisticated: It is interesting to ponder why anyone would lend to our investor at the riskless rate, because there is some chance that he will not be able to repay the debt. But the primary goal here is simply to develop familiarity with the geometric representation of the opportunity set; thus for the moment we bypass this issue.
figure. Because it is assumed that any amount of borrowing is possible, the line can be extended indefinitely.

For a short sale of the risky asset \( a \), if we assume that the investor sells the borrowed units of \( a \) in the market and uses the proceeds to increase his investment in the riskless asset \( f \), the one-period portfolio return and its mean are still given by Equations (6.14) and (6.15a), except that \( x \) can now take values greater than 1. But the standard deviation of the portfolio return becomes

\[
\sigma(\bar{R}_p) = |1 - x| \sigma(\bar{R}_a) = (x - 1)\sigma(\bar{R}_a), \quad x \geq 1. \tag{6.18}
\]

The relationship between \( E(\bar{R}_p) \) and \( \sigma(\bar{R}_a) \) for portfolios involving short sales of asset \( a \) can be obtained by solving Equation (6.18) for \( x \) to get

\[
x = \frac{\sigma(\bar{R}_p) + \sigma(\bar{R}_a)}{\sigma(\bar{R}_a)}, \quad x \geq 1. \tag{6.19}
\]

Equating the right-hand sides of Equations (6.19) and (6.16a) and then solving for \( E(\bar{R}_p) \), we get the linear equation

\[
E(\bar{R}_p) = R_f + \frac{R_f - E(\bar{R}_a)}{\sigma(\bar{R}_a)} \sigma(\bar{R}_p). \tag{6.20}
\]

Note that the slope \((R_f - E(\bar{R}_a))/\sigma(\bar{R}_a)\) is just the negative of the slope in Equation (6.17). Geometrically, portfolios involving short sales of asset \( a \) fall along the line \( \bar{R}_b \) in Figure 6.7.

Finally, let us admit one more type of investment into the opportunity set. In particular, suppose that the investor has the option of holding some or all of his portfolio funds in money, which is assumed to be riskless; that is, a dollar now is a dollar at the end of the period. To simplify the analysis, assume that if the investor holds some cash, the remainder of his portfolio funds must be put either into the riskless asset \( f \) or into the risky asset \( a \), but not both. Portfolios of cash and \( f \) with the proportion \( x \) of portfolio funds held in cash and \( (1 - x) \) in \( f \) produce the sure return

\[
\bar{R}_p = (1 - x)R_f, \quad 0 \leq x \leq 1;
\]

portfolios of cash and the risky asset \( a \) have return

\[
\bar{R}_p = (1 - x)\bar{R}_a, \quad 0 \leq x \leq 1,
\]

with mean

\[
E(\bar{R}_p) = (1 - x)E(\bar{R}_a), \quad 0 \leq x \leq 1,
\]

and standard deviation given by Equation (6.15b). Geometrically, portfolios involving cash and the riskless asset \( f \) lie along the \( E(\bar{R}) \) axis between 0 and \( R_f \); portfolios involving cash and the risky asset \( a \) lie along the line \( \bar{R}_a \).
The \( E(\tilde{R}) \), \( \sigma(\tilde{R}) \) efficient set for the portfolio opportunity set represented in Figure 6.7 can be determined by inspection: All efficient portfolios lie along the line from \( R_f \) through \( a \) and \( z \); that is, the only efficient portfolios are those involving the riskless asset \( f \), in either positive or negative amounts, and nonnegative amounts of the risky asset \( a \). Portfolios along the line from \( R_f \) through \( b \), that is, those involving positive amounts of \( f \) and short sales of \( a \), are not efficient, because for any point along \( R_f b \) there is a point along \( R_f z \) with the same value of \( \sigma(\tilde{R}) \) but higher \( E(\tilde{R}) \). For the same reason points along \( OR_f \) (portfolios of cash and the riskless asset \( f \)) and points along \( Oa \) (portfolios of cash and the risky asset \( a \)) cannot be efficient.

Thus the only efficient portfolios in our simplified opportunity set are combinations of the riskless asset \( f \) and the risky asset \( a \). Less risky efficient portfolios are formed by mixing positive amounts of \( f \) and \( a \), obtaining points along the segment \( R_f a \); more risky efficient portfolios are formed by borrowing at the rate \( R_f \) and then investing both the borrowings and the initial portfolio funds in the risky asset \( a \), obtaining portfolios along the segment from \( a \) through \( z \).

Figure 6.8 shows how the optimal portfolio is determined for a given value of initial consumption \( c_1 \). The investor wants to reach the highest possible indifference curve, that is, to maximize expected utility conditional on \( c_1 \). In general, the optimal portfolio is at the tangency point between an indifference curve and the efficient set, as at the point \( d \) in Figure 6.8a. If such a tangency exists, it is unique, because the efficient set is linear and the indifference curves are strictly convex.

But it is also possible that the investor’s optimal portfolio for the given level of \( c_1 \) is a “corner” portfolio, as illustrated in Figure 6.8b. Here the highest attainable indifference curve is everywhere steeper than the efficient set line; for the given value of \( c_1 \), the individual is unwilling to accept any amount of \( \sigma(\tilde{R}) \) at the rate of exchange of \( E(\tilde{R}) \) for \( \sigma(\tilde{R}) \).
implied by the efficient set line. Thus he concentrates all his portfolio investment in the riskless asset \( f \).

If borrowing portfolios are prohibited, another type of corner solution is possible. Here the efficient set is just the line segment \( R_f a \) in Figure 6.8c. For the indifference curves shown in the figure the optimum portfolio is just the risky asset \( a \). Moreover, the optimum is not a tangency point, because the indifference curve at \( a \) is flatter than the efficient set. The individual would be willing to take on more \( \sigma(\bar{R}) \) in exchange for \( E(\bar{R}) \), but this possibility is not open to him.

**IV. THE EFFICIENT SET AND CONSUMER EQUILIBRIUM WITH \( N \) ASSETS**

The next step in the development of a model for the two-period two-parameter consumption-investment problem is to expand the opportunity
set to include an arbitrary number \( N \) of investment assets. We maintain the assumption that the distributions of one-period returns on all assets and portfolios are normal, although this assumption is discussed in detail later.

The procedure is to introduce first the definitions and elementary expressions for the \( N \)-asset model. Then a geometric representation of the \( N \)-asset opportunity set is developed and combined in the usual way with the investor's indifference curves to get a picture of consumer equilibrium. The determination of equilibrium in the \( N \)-asset problem is then analyzed algebraically. The section concludes with a discussion of some peripheral issues, for example, what is the meaning of the statement "diversification pays"?

IV.A. Definitions and Elementary Expressions for the \( N \)-Asset Model

If \( h_1 = w_i - c_i \) is the initial wealth invested in the portfolio, let \( h_{1i} \) be the initial wealth invested in asset \( i \). Then the level of wealth \( \bar{w}_2 \), and thus consumption \( \bar{c}_2 \), produced by the portfolio at the beginning of period 2 is

\[
\bar{w}_2 = \sum_{i=1}^{N} h_{1i}(1 + \bar{R}_i) = \bar{c}_2,
\]

(6.21)

where \( \bar{R}_i \) is the one-period return on asset \( i \) and \( N \) is the number of investment assets available to the individual. Thus \( \bar{R}_p \), the one-period return on the individual's portfolio, is

\[
\bar{R}_p = \frac{\bar{w}_2 - h_1}{h_1} = \frac{\sum_{i=1}^{N} h_{1i}(1 + \bar{R}_i) - h_1}{h_1}.
\]

(6.22)

We assume that all \( h_1 \) is invested, so that

\[
\sum_{i=1}^{N} h_{1i} = h_1.
\]

(6.23)

Short sales can be introduced, when appropriate, by letting \( h_{1i} \) be negative. If

\[
x_i = \frac{h_{1i}}{h_1}
\]

(6.24)

is the proportion of the portfolio investment in asset \( i \), from Equations
Two-Period Consumption-Investment Model

(6.22) and (6.23)

\[ R_p = \sum_{i=1}^{N} x_i R_i, \quad (6.25) \]

and

\[ \sum_{i=1}^{N} x_i = 1. \quad (6.26) \]

Thus the one-period return on the portfolio is just the weighted average of the one-period returns on the individual assets in the portfolio, and the weights are the proportions of investment funds in the individual assets. If the one-period returns on the individual assets have normal distributions, the returns on portfolios of these assets also have normal distributions. From Equation (6.25) the expected return on a portfolio is

\[ E(R_p) = \sum_{i=1}^{N} x_i E(R_i), \quad (6.27) \]

where \( E(R_i) \) is the expected return on asset \( i \).

Let

\[ \sigma_{ij} = \text{cov}(R_i, R_j) = E[(R_i - E(R_i))(R_j - E(R_j))] \quad (6.28) \]

be the covariance between the one-period returns on assets \( i \) and \( j \). Note that the variance of the one-period return on asset \( i \) is just

\[ \sigma^2(R_i) = E[(R_i - E(R_i))^2] = \sigma_{ii}. \quad (6.29) \]

Thus the variance can be regarded as the covariance of asset \( i \) with itself. From Equations (6.25), (6.28), and (6.29), the variance of the one-period return on the portfolio can be written

\[ \sigma^2(R_p) = \sum_{i=1}^{N} x_i^2 \sigma^2(R_i) + \sum_{i=1}^{N} \sum_{j \neq i}^{N} x_i x_j \sigma_{ij} = \sum_{i=1}^{N} x_i^2 \sigma_{ii}. \quad (6.30) \]

Thus the expected return on the portfolio is just a weighted average of the expected returns on the individual assets in the portfolio, and the variance of the one-period return on the portfolio is just a weighted average of the variances of the returns on the individual assets and the covariances between the returns on the assets.\(^8\)

\(^8\) Note that the covariance terms in Equation (6.30) are far more numerous than the variance terms. There are \( N(N - 1) \) terms involving the \( \sigma_{ij}, j \neq i \), whereas there are only \( N \) terms involving the \( \sigma^2(R_i) = \sigma_{ii} \). The implications of this fact are discussed below.
We begin the essentially empirical question of where we obtain the basic inputs, that is, the means, variances, and covariances of the distributions of one-period returns on individual assets, which allow us to define the mean and variance of the distribution of the one-period return provided by a given portfolio. For our purposes, which are completely theoretical, the inputs for the analysis are taken as given.

IV.B. Geometric Representation of Consumer Equilibrium

In the geometric treatment of the N-asset model, the first order of business is to develop a picture of the efficient set.

Consider first two arbitrarily chosen assets or portfolios\(^{10}\) a and b with expected returns \(E(R_a)\) and \(E(R_b)\) and standard deviations \(\sigma(R_a)\) and \(\sigma(R_b)\). We want to examine geometrically the combinations of expected return \(E(R_p)\) and standard deviation of return \(\sigma(R_p)\) that can be obtained from portfolios formed by investing the proportion \(x\) of \(h_a\) in asset a and \((1 - x)\) in b. The one-period return on such portfolios is

\[
R_p = xR_a + (1 - x)R_b
\]

with mean

\[
E(R_p) = xE(R_a) + (1 - x)E(R_b),
\]

and standard deviation

\[
\sigma(R_p) = [x^2\sigma^2(R_a) + (1 - x)^2\sigma^2(R_b) + 2x(1 - x)\text{cov}(R_a, R_b)]^{1/2}
\]

\[
= [x^2\sigma^2(R_a) + (1 - x)^2\sigma^2(R_b) + 2x(1 - x)k_{ab}\sigma(R_a)\sigma(R_b)]^{1/2},
\]

where we make use of the fact that \(k_{ab}\), the correlation coefficient between \(R_a\) and \(R_b\), is just

\[
k_{ab} = \frac{\text{cov}(R_a, R_b)}{\sigma(R_a)\sigma(R_b)}
\]

From the earlier discussion of the two-asset model we know that the relationship between \(E(R_p)\) and \(\sigma(R_p)\) for portfolios formed according to Equation (6.31) is linear if either a or b is riskless, that is, its standard deviation is 0. We now show that the relationship is also linear when the returns on the assets are perfectly positively correlated, that is, \(k_{ab} = 1\).

\(^{10}\) For present purposes the distinction between assets and portfolios is completely irrelevant.
In this case
\[
\sigma(R_p) = [x^2 \sigma(R_a) + (1 - x)^2 \sigma(R_b) + 2x(1 - x) \sigma(R_a) \sigma(R_b)]^{1/2}
\]
\[
= x \sigma(R_a) + (1 - x) \sigma(R_b).
\]
(6.35)

Solving both Equations (6.32) and (6.35) for \(x\) and then equating the resulting expressions, we get an expression for \(E(R_p)\) that is linear in \(\sigma(R_p)\):
\[
E(R_p) = E(R_a) + \frac{E(R_a) - E(R_b)}{\sigma(R_a) - \sigma(R_b)} (\sigma(R_p) - \sigma(R_b)).
\]
(6.36)

For \(0 \leq x \leq 1\) in Equation (6.31), the plot of Equation (6.36) for some assumed values of \(E(R_a), E(R_b), \sigma(R_a),\) and \(\sigma(R_b)\) is the line \(ab\) in Figure 6.9.

The opposite extreme from perfect positive correlation is, of course, perfect negative correlation, that is, \(\rho_{ab} = -1\). In this case the standard deviation of the portfolio return is
\[
\sigma(R_p) = [x^2 \sigma(R_a) + (1 - x)^2 \sigma(R_b) - 2x(1 - x) \sigma(R_a) \sigma(R_b)]^{1/2},
\]
which has the two roots
\[
\sigma(R_p) = x \sigma(R_a) - (1 - x) \sigma(R_b) = -\sigma(R_b) + x[\sigma(R_a) + \sigma(R_b)]
\]
(6.37)
and
\[
\sigma(R_p) = (1 - x) \sigma(R_b) - x \sigma(R_a) = \sigma(R_b) - x[\sigma(R_a) + \sigma(R_b)].\]
(6.38)

For both Equations (6.37) and (6.38), \(\sigma(R_p) = 0\) when \(x\) is equal to
\[
\frac{\sigma(R_b)}{\sigma(R_a) + \sigma(R_b)}.
\]
(6.39)

When \(x\) is greater than the value implied by (6.39), Equation (6.37) yields positive values of \(\sigma(R_p)\), and (6.38) yields negative values; the reverse is true when \(x\) is less than (6.39). Because standard deviations must always be nonnegative, Equation (6.37) is relevant when \(x\) is greater than the value given by (6.39), and Equation (6.38) is relevant when \(x\) is less. Thus solving Equations (6.32) and (6.37) for \(x\) and equating the resulting

\[\sigma(R_p) = |x \sigma(R_a) - (1 - x) \sigma(R_b)|.\]
(6.38a)
expressions gives
\[ E(R_p) = E(R_a) + \frac{E(R_a) - E(R_b)}{\sigma(R_a) + \sigma(R_b)} (\sigma(R_p) + \sigma(R_a)), \] (6.40)

which is relevant for portfolios of a and b with t greater than \( \sigma(R_a)/[\sigma(R_a) + \sigma(R_b)] \). On the other hand, solving Equations (6.32) and (6.33) for t and equating the resulting expressions gives
\[ E(R_p) = E(R_a) - \frac{E(R_a) - E(R_b)}{\sigma(R_a) + \sigma(R_b)} (\sigma(R_p) - \sigma(R_a)), \] (6.41)

which is relevant for portfolios of a and b with t less than \( \sigma(R_a)/[\sigma(R_a) + \sigma(R_b)] \). In Figure 6.9 the line ca is the plot of Equation (6.40), and cb is the plot of Equation (6.41). Thus acb represents the combinations of \( E(R_p) \) and \( \sigma(R_p) \) that can be obtained when portfolios are formed according to Equation (6.31) and \( k_{ab} = -1 \).

In the general case, however, the correlation between the returns on two assets or portfolios is not perfect, either positive or negative. From Equation (6.32) we see that the expected portfolio return is unaffected by the degree of correlation. Inspection of expression (6.33), however, tells us that for any given t (0 < t < 1), the maximum value of the standard deviation of portfolio returns occurs when \( k_{ab} = 1 \) and that the minimum value occurs when \( k_{ab} = -1 \). It follows that when 0 < t < 1 and -1 < \( k_{ab} < 1 \), portfolios of a and b lie strictly to the left of the line ab in Figure 6.9 and to the right of the lines cb and ca; that is, with -1 < \( k_{ab} < 1 \), for any given value of \( E(R_p) \), the value of \( \sigma(R_p) \) must be to the left of the relevant
point along the perfect positive correlation line \( ab \) and to the right of the relevant point along the perfect negative correlation lines \( cb \) and \( ca \).\(^{12}\)

In fact, continuing these arguments, in the case of less than perfect correlation, different combinations of any two assets or portfolios \( a \) and \( b \) always fall along a curve between the two points in the \( E(R) \), \( \sigma(R) \) plane. And any positively sloping segment of this curve must be strictly concave; any negatively sloping segment must be strictly convex. To see this, suppose, on the contrary, that the curve describing combinations of \( a \) and \( b \) is positively sloping but contains a wiggle, that is, a nonconcave segment, like the curve \( aefgb \) in Figure 6.9. Applying the arguments of the previous paragraphs, however, we know that weighted combinations of the portfolios \( e \) and \( g \) must lie on or to the left of a straight line between \( e \) and \( g \) in Figure 6.9. Thus the segment \( efg \) cannot represent combinations of portfolios \( e \) and \( g \). Because \( e \) and \( g \) are in turn just combinations of \( a \) and \( b \), the segment \( efg \) cannot represent weighted combinations of \( a \) and \( b \). This sort of argument establishes the concavity of any positively sloping segment of the curve representing combinations of assets or portfolios \( a \) and \( b \). It also implies convexity for any negatively sloping segment of the curve. Thus, for example, in Figure 6.10 the curve \( aefgb \) cannot represent combinations of \( a \) and \( b \), because with \(-1 < k_{ab} < 1\), combinations of \( e \) and \( g \), which are in turn just combinations of \( a \) and \( b \), must lie to the left of a straight line between \( e \) and \( g \).\(^{13}\)

These geometric results provide our first important insights into the effects of diversification on the distribution of the one-period return on a portfolio. When we form a portfolio by taking combinations of the assets or portfolios \( a \) and \( b \), the expected return on the portfolio is just the weighted average of the expected returns on \( a \) and \( b \), where the weights are the proportions of \( h_1 \) invested in \( a \) and \( b \). As long as the correlation between the returns on \( a \) and \( b \) is less than 1, however, the standard deviation of the return on the portfolio is less than the weighted average of the standard

\(^{12}\) Again the reader for whom the preceding analysis is not obvious should illustrate the results numerically. Thus, for example, let \( E(R_a) = 0.20 \), \( E(R_b) = 0.10 \), \( \sigma(R_a) = 0.50 \), and \( \sigma(R_b) = 0.30 \). Then examine the combinations of \( E(R_p) \) and \( \sigma(R_p) \) obtained for different values of \( x \) \((0 \leq x \leq 1)\) in Equation (6.31) when \( k_{ab} = 1 \). In doing this, it is probably simplest to work with Equations (6.32) and (6.35) rather than (6.36). Next look at the case \( k_{ab} = -1 \), working now with Equations (6.32) and (6.38a). Finally, consider some value of \( k_{ab} \) in the interval \(-1 < k_{ab} < 1\), using Equations (6.32) and (6.33) to determine the values of \( E(R_p) \) and \( \sigma(R_p) \) obtained as \( x \) is varied. In this case the simplest choice is of course \( k_{ab} = 0 \), though it may also be helpful to look at values of \( k_{ab} \) on either side of 0.

\(^{13}\) The skeptical reader is encouraged to provide himself with a formal representation of the details of these arguments.
deviations of the returns on a and b. In essence, in general the process of mixing assets or portfolios to form new, more “diversified” portfolios is a dispersion-reducing activity, a point that we develop in more detail later.

It is important to note that this geometric analysis of weighted combinations applies to either assets or portfolios; that is, the points a and b could be individual assets or portfolios of assets. In fact, we could in principle consider all possible combinations of assets and portfolios and so develop a geometric representation of the set of feasible portfolios. From the efficient set theorem, however, we know that, as long as we are concerned with risk-averse investors, we can restrict attention to the efficient subset of feasible portfolios. Recall that an efficient portfolio has the following property: No portfolio with the same or higher expected return has lower standard deviation of return. Thus, in the $E(\bar{R}), \sigma(\bar{R})$ plane, efficient portfolios must lie along a positively sloping segment of the left boundary of the set of all feasible portfolios. A reasonable general representation of the left boundary of the feasible set is the curve $abcde$ in Figure 6.4. Only portfolios along the positively sloping segment $bcd$ are efficient, however, because portfolios along $ab$ and $de$ do not satisfy the efficiency criterion. Although it can include many different portfolios, the efficient set $bcd$ is drawn concave. This concavity is an implication of the earlier analysis of weighted combinations of assets and portfolios.¹⁴

¹⁴ And again it is good practice for the reader to convince himself that this is true.
As drawn in Figure 6.4, the shape of the left boundary of the feasible set implies further assumptions about portfolio opportunities. In particular, there is no riskless asset available, and no two assets have perfectly negatively correlated returns. If there were a riskless asset or if returns on two assets had perfect negative correlation, the graph of the set of feasible portfolios would have to meet the vertical axis at some point. Moreover, the fact that the efficient set contains no linear segments implies that there are no efficient portfolios that are perfectly positively correlated. Although it is a little more difficult to see, Figure 6.4 also implicitly prohibits indefinite short selling of any asset or portfolio. If this were not the case, the top of the efficient set would be unbounded; that is, there would be an efficient portfolio for any arbitrarily high level of expected return.\footnote{For example, if indefinite short selling were allowed, portfolios with expected returns higher than \( E(R_d) \) could be obtained by selling portfolio \( b \) short and using the proceeds of the short sale, along with initial investment funds \( h_i \), to buy units of portfolio \( d \). Although such portfolios need not be efficient, they illustrate that with indefinite short selling one could obtain portfolios with any level of expected return above \( E(R_d) \).}

It is also an easy matter to introduce a riskless asset into the analysis. From the two-asset case discussed earlier we know that portfolios involving different proportions of a riskless asset and a risky asset or portfolio plot as a straight line. Thus, if there is a riskless asset \( f \) with the certain non-negative return \( R_f \), the new efficient set is determined by rotating a line with end point at \( R_f \) from right to left through the feasible set of risky portfolios, stopping when the line has been moved as far as possible, that is, when its slope is as large as possible, short of leaving the feasible set. From the concavity of the efficient set of risky portfolios, we know that following this procedure the eventual resting place of the line is along the efficient set of risky portfolios.

The situation in the presence of the riskless asset is represented in Figure 6.11. The line from \( R_f \) that just touches the efficient set at \( c \) represents the maximum leftward rotation of lines from \( R_f \) consistent with remaining in the feasible set. Portfolios between \( R_f \) and \( c \) along this line represent combinations of the riskless asset \( f \) and the portfolio \( c \), where both \( f \) and \( c \) are held at nonnegative levels. It is clear that points below \( c \) along \( bcd \) no longer represent efficient portfolios, because there are combinations of \( f \) and \( c \) that have higher expected returns at the same levels of standard deviation. Similarly, if it is possible to borrow as well as lend at the rate \( R_f \), points above \( c \) on \( bcd \) no longer represent efficient portfolios, because in this case there are feasible combinations of \( f \) and \( c \), involving negative holdings of asset \( f \), along the extension of \( R_f c \), through \( c \), that represent portfolios with higher expected returns but no higher standard deviation than corresponding points along \( cd \).
Thus, when it is possible to lend but not to borrow at the riskless rate $R_f$, the efficient set of portfolios is represented by $R_fcd$ in Figure 6.11. When it is possible to both borrow and lend at the riskless rate $R_f$, the efficient set is the line from $R_f$ through $c$ and $g$ in Figure 6.11. In this extreme and rather unrealistic situation, all efficient portfolios involve only combinations of $f$ and $c$. The only difference between efficient portfolios is in the proportion of investment funds $h_i$ that is invested in the riskless asset $f$. Less risky efficient portfolios—those along $R_fc$—involve lending some funds at the rate $R_f$ and investing remaining funds in the risky portfolio $c$. More risky efficient portfolios—those along the extension of $R_fc$ through $c$—involve borrowing at the riskless rate and investing both the borrowings and $h_i$ in the risky portfolio $c$.

16 In both Figs. 6.4 and 6.11 it is assumed that the individual who views the efficient set in this way feels that there is a trade-off between expected return and standard deviation of return in the market. In order to get more expected return, it is necessary to accept more standard deviation of return. It is not necessarily the case, however, that all individuals view the market in this way. An individual may feel that there is a portfolio that provides both highest expected return and minimum standard deviation from among the set of all feasible portfolios. In this case this portfolio would be the only member of the efficient set. (He may have this feeling because he believes that there are securities in this portfolio that are greatly "underpriced" relative to their "true worth.") In a market dominated by risk averters, however, we should not expect that this is the general view. Most investors would probably feel that there is some sort of trade-off between risk and return like that represented in Fig. 6.11.
The geometric representation of the optimal portfolio choice for a given value of $c_1$ is now very similar to the two-asset case discussed earlier. If the optimal portfolio is not at either of the end points of the efficient set, the optimum is a point of tangency between the highest attainable indifference curve and the efficient set. Such a solution is shown in Figure 6.12 for the case in which there is a riskless asset and in Figure 6.5 for the case in which there is no riskless asset.

Thus the goal of the geometric analysis has now been attained: We have provided a rigorous justification for the picture of consumer equilibrium given in Figure 6.5. We can turn now to an algebraic analysis of the same problem.

*IV.C. Algebraic Representation of Consumer Equilibrium

With the geometric approach, to determine the combined optimal consumption-investment decision, in principle we must find the optimal

\[ \text{Figure 6.12 Consumer Equilibrium When There Is a Riskless Asset} \]

\[ \text{\textsuperscript{\textith}} \text{We should note, for the mathematically more sophisticated, that the efficient set curve need not be differentiable everywhere, so that, strictly speaking, the representation of equilibrium in terms of a "tangency" could be incorrect. It can be shown, however, that the maximum number of points at which the efficient set curve is not differentiable cannot be greater than the number } N \text{ of available assets. With infinitely divisible assets, the number of efficient portfolios is infinite; that is, the efficient set curve is continuous. Thus these nondifferentiable points do not greatly detract from our conclusions; in mathematical terms, they constitute a set of measure } 0. \]
portfolio decision and its associated expected utility for each possible level of consumption $c_1$ and then choose the consumption level and its associated optimal portfolio that provides the global maximum of expected utility. With an algebraic approach, however, the problem can be solved in one step, and at least some insight into the interdependence between optimal consumption and investment decisions can be obtained.

Several equivalent mathematical statements of the consumption-investment problem are possible. For example, in the preceding analyses the decision variables were assumed to be period 1 consumption $c_1$ and the proportions $x_i, i = 1, 2, \ldots, N$, of total investment $h_t = w_1 - c_1$ that are in each of the $N$ available assets. The goal of the consumer is to choose values of $c_1$ and the $x_i$ that

$$\max E[U(c_1, \xi)]$$

$$= \int_{-\infty}^{\infty} U(c_1, (w_1 - c_1)[1 + E(\tilde{R}_p) + \sigma(\tilde{R}_p)r)]f(r) \, dr$$

$$= \int_{-\infty}^{\infty} U(c_1, (w_1 - c_1)[1 + \sum_{i=1}^{N} x_iE(\tilde{R}_i) + (\sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \sigma_{ij})^{1/2}r)]f(r) \, dr$$

subject to the constraints

$$0 \leq c_1 \leq w_1, \quad (6.43)$$

$$\sum_{i=1}^{N} x_i = 1. \quad (6.44)$$

But for current purposes it is more convenient to take a slightly different approach. In particular, we now view the decision variables as $c_1$ and the total investments $h_{1i} = (w_1 - c_1)x_i$ in each of the $N$ assets. In these terms the goal of the consumer is to choose values of $c_1$ and the $h_{1i}, i = 1, 2, \ldots, N$, that

$$\max E[U(c_1, \tilde{\xi})]$$

$$= \int_{-\infty}^{\infty} U(c_1, \sum_{i=1}^{N} h_{1i}[1 + E(\tilde{R}_i)] + [\sum_{i=1}^{N} \sum_{j=1}^{N} h_{1i} h_{1j} \sigma_{ij}]^{1/2}r)f(r) \, dr \quad (6.45a)$$

$$= \int_{-\infty}^{\infty} U(c_1, E(\tilde{\xi}) + \sigma(\tilde{\xi})r)f(r) \, dr \quad (6.45b)$$
subject to the constraint\textsuperscript{18}

\begin{equation}
    c_1 + \sum_{i=1}^{N} h_{1i} = w_1. \tag{6.46}
\end{equation}

To solve the problem, we first form the lagrangian

\[ Z = \int_{-\infty}^{\infty} U(c_t, \sum_{i=1}^{N} h_{ti}[1 + E(\bar{R}_i)]) + [\sum_{i=1}^{N} \sum_{j=1}^{N} h_{ji} h_{ij} \sigma_{ij}]^{1/2} f(r) dr \]

\[ + \lambda(w_1 - c_1 - \sum_{i=1}^{N} h_{1i}), \tag{6.47} \]

then differentiate partially with respect to $c_1$, $h_{1i}$, $i = 1, 2, \ldots, N$, and $\lambda$, and set these derivatives equal to 0, obtaining the necessary conditions for a maximum

\begin{align*}
    \frac{\partial Z}{\partial c_1} &= \frac{\partial E[U(c_1, s_2)]}{\partial c_1} = \lambda, \tag{6.48a} \\
    \frac{\partial Z}{\partial h_{1i}} &= \frac{\partial E[U(c_1, s_2)]}{\partial h_{1i}} = \lambda, \quad i = 1, 2, \ldots, N, \tag{6.48b} \\
    w_1 - c_1 - \sum_{i=1}^{N} h_{1i} &= 0. \tag{6.48c}
\end{align*}

\textsuperscript{18}In going from Equation (6.42) to (6.45a) and (6.45b), we make use of the facts that

\[ E(\bar{s}_t) = \sum_{i=1}^{N} \lambda_{ti}[1 + E(\bar{R}_i)] = (w_1 - c_1)[1 + \sum_{i=1}^{N} \lambda_i E(\bar{R}_i)] = (w_1 - c_1)[1 + E(\bar{R}_p)] \]

and

\[ \sigma(\bar{s}_t) = \left[ \sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_{ti} \lambda_{tj} \sigma_{ij} \right]^{1/2} = (w_1 - c_1)\left[ \sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_i \lambda_j \sigma_{ij} \right]^{1/2} = (w_1 - c_1)\sigma(\bar{R}_p). \]

Remember also that if the distribution of the one-period portfolio return $\bar{R}_p$ is normal, the distribution of period 2 consumption

\[ \bar{s}_t = (w_1 - c_1)(1 + \bar{R}_p) \]

must also be normal. And portfolios that are efficient in terms of $E(\bar{R}_p)$ and $\sigma(\bar{R}_p)$ must be efficient in terms of $E(\bar{s}_t)$ and $\sigma(\bar{s}_t)$.\]
In short, an optimal consumption-investment decision requires that the marginal expected utilities of period 1 consumption and period 1 investments in each of the \( N \) available assets be equal. This result, which is, after all, just a reexpression in terms of expected utility of a familiar economic principle, is really our sole insight into the interdependence of optimal consumption and investment decisions.

But the algebraic representation of the conditions of consumer equilibrium given by Equations (6.48) does allow us to rederive the properties of an optimal portfolio decision obtained geometrically earlier. In particular, the conditions (6.49) are now shown to imply the important efficient set theorem.

First, from Equation (6.45) note that

\[
\frac{\partial E[U(c_1, c_2)]}{\partial h_{1i}} = \int_{-\infty}^{\infty} \frac{\partial U(c_1, c_2)}{\partial c_1} \left[ 1 + E(\tilde{R}_i) + \frac{\partial \sigma(\tilde{R}_i)}{\partial h_{1i}} r \right] f(r) \, dr \tag{6.49a}
\]

\[
= \int_{-\infty}^{\infty} \frac{\partial U(c_1, c_2)}{\partial c_1} \left[ 1 + E(\tilde{R}_i) + \frac{\partial \sigma(\tilde{R}_p)}{\partial x_i} r \right] f(r) \, dr \tag{6.49b}
\]

\[
= \frac{\partial E[U(c_1, c_2)]}{\partial E(\tilde{R}_p)} \left[ 1 + E(\tilde{R}_i) \right] + \frac{\partial E[U(c_1, c_2)]}{\partial \sigma(\tilde{R}_p)} \frac{\partial \sigma(\tilde{R}_p)}{\partial x_i} \tag{6.49c}
\]

\[
\text{In going from Equation (6.49a) to (6.49b), we make use of the fact that}
\]

\[
\frac{\partial \sigma(\tilde{R}_i)}{\partial h_{1i}} = \frac{\sum_{j=1}^{N} h_{ij} \sigma_{ij}}{\sigma(\tilde{R}_i)} = \frac{(w_1 - c_1) \sum_{j=1}^{N} x_{ij} \sigma_{ij}}{(w_1 - c_1) \sigma(\tilde{R}_p)} = \frac{\partial \sigma(\tilde{R}_p)}{\partial x_i} \tag{6.50}
\]

And in going from Equation (6.49b) to (6.49c), we simply make use of expressions (6.7) and (6.8) for the marginal expected utilities of \( E(\tilde{R}_p) \) and \( \sigma(\tilde{R}_p) \).

Next note that because \( \lambda \) is the same for all assets, Equation (6.48b) implies that for all \( i \) and \( j \)

\[
\frac{\partial E[U(c_1, c_2)]}{\partial h_{1i}} = \frac{\partial E[U(c_1, c_2)]}{\partial h_{1j}}.
\]
which, with Equation (6.49c), reduces to

$$\frac{\partial E[U(c_1,c_2)]}{\partial \sigma(\bar{R}_p)} = \frac{E(\bar{R}_p) - E(\bar{R}_i)}{\partial \sigma(\bar{R}_p)} - \frac{\partial E[U(c_1,c_2)]}{\partial \sigma(\bar{R}_p)} \frac{\partial \sigma(\bar{R}_p)}{\partial \bar{r}_i}.$$  \hspace{1cm} (6.51)

The expression on the left in Equation (6.51) is just the marginal rate of substitution of \(E(\bar{R}_p)\) for \(\sigma(\bar{R}_p)\) along an indifference curve; that is, it is the slope of an indifference curve.\textsuperscript{10} Because Equation (6.51) is an optimality condition, this slope is in fact the slope of the indifference curve at the point corresponding to the optimal feasible combination of \(E(\bar{R}_p)\) and \(\sigma(\bar{R}_p)\). We now show that the expression on the right in Equation (6.51) is just the slope of the efficient set at the point corresponding to the consumer's optimal portfolio choice. It then follows that Equation (6.51) is the familiar equilibrium tangency condition between an indifference curve and the efficient set (as shown, for example, in Figure 6.5).

Suppose that in the optimal consumption-investment decision the expected return on the portfolio is \(E(\bar{R}_p)^*\). The efficient portfolio with expected return \(E(\bar{R}_p)^*\) must be the solution to the problem: Choose \(x_i, i = 1, 2, \ldots, N\), that\textsuperscript{*}

$$\min \sigma(\bar{R}_p) = \left(\sum_{i} \sum_{j} x_i x_j \sigma_{ij}\right)^{1/2}$$ \hspace{1cm} (6.52a)

\textsuperscript{10} For a given level of \(c_1\), an indifference curve is defined by the set of combinations of \(E(\bar{R})\) and \(\sigma(\bar{R})\) with a given level of expected utility, say, \(V\). The curve is implicitly defined by the function

\[ F = E[U(c_1,c_2)] - V = 0. \]

The slope of the curve, which is the marginal rate of substitution of \(E(\bar{R}_p)\) for \(\sigma(\bar{R}_p)\), is then obtained from the differential of \(F\) as follows:

$$dF = \frac{\partial E[U(c_1,c_2)]}{\partial \sigma(\bar{R}_p)} dE(\bar{R}_p) + \frac{\partial E[U(c_1,c_2)]}{\partial \sigma(\bar{R}_p)} d\sigma(\bar{R}_p) = 0,$$

$$dE(\bar{R}_p) = \frac{\sigma(\bar{R}_p)}{\partial \sigma(\bar{R}_p)}/\frac{\partial E[U(c_1,c_2)]}{\partial E(\bar{R}_p)}.$$  

\textsuperscript{*} In the rest of this section, we simplify the notation by writing \(\sum_i\) and \(\sum_j\) for \(\sum_{i=1}^N\) and \(\sum_{j=1}^N\).
subject to the constraints

\[ E(\bar{R}_p) = E(\bar{R}_p)^* = \sum_i x_i E(\bar{R}_i), \quad (6.52b) \]

\[ \sum_i x_i = 1. \quad (6.52c) \]

Forming the lagrangian

\[ L = \sigma(\bar{R}_p) + \lambda_1 [E(\bar{R}_p)^* - \sum_i x_i E(\bar{R}_i)] + \lambda_2 [1 - \sum_i x_i], \]

differentiating partially with respect to \( x_i \), \( i = 1, 2, \ldots, N \), \( \lambda_1 \), and \( \lambda_2 \), and setting these derivatives equal to zero, we obtain the necessary conditions for a minimum:

\[ \frac{\partial L}{\partial x_i} = \frac{\partial \sigma(\bar{R}_p)}{\partial x_i} - \lambda_1 E(\bar{R}_i) - \lambda_2 = 0, \quad i = 1, 2, \ldots, N. \quad (6.53a) \]

\[ \frac{\partial L}{\partial \lambda_1} = E(\bar{R}_p)^* - \sum_i x_i E(\bar{R}_i) = 0. \quad (6.53b) \]

\[ \frac{\partial L}{\partial \lambda_2} = 1 - \sum_i x_i = 0. \quad (6.53c) \]

Because \( \lambda_2 \) is the same for all assets, Equation (6.53a) implies that for any two assets \( i \) and \( j \)

\[ \frac{\partial \sigma(\bar{R}_p)}{\partial x_i} - \lambda_1 E(\bar{R}_i) = \frac{\partial \sigma(\bar{R}_p)}{\partial x_j} - \lambda_1 E(\bar{R}_j), \quad (6.54) \]

or

\[ \frac{1}{\lambda_1} = \frac{E(\bar{R}_i) - E(\bar{R}_j)}{\frac{\partial \sigma(\bar{R}_p)}{\partial x_i} - \frac{\partial \sigma(\bar{R}_p)}{\partial x_j}}. \quad (6.55) \]

The expression on the right of the equality in Equation (6.55) is identical with that on the right in Equation (6.51). Thus we are able to interpret Equation (6.51) once we give some meaning to \( \lambda_1 \) in Equation (6.55). But the Lagrange multiplier \( \lambda_1 \) is just the shadow price of the constraint (6.52b). It tells us how the minimum value of the standard deviation would change for "small" changes in \( E(\bar{R}_p) \) in the neighborhood of \( E(\bar{R}_p)^* \). Thus \( \lambda_1 \) is just the rate of exchange of \( \sigma(\bar{R}_p) \) for \( E(\bar{R}_p) \) at the point \( E(\bar{R}_p)^* \) in the efficient set. Its reciprocal \( 1/\lambda_1 \), then, is just the slope of the efficient set.
at $E(\bar{R}_p)$. Thus we have the desired result: Condition (6.51) implies that
the optimal portfolio is at a point of tangency between an indifference curve
and the efficient set.
Expression (6.55) is the “balance equation” for the efficient portfolio
with expected return $E(\bar{R}_p)$; that is, it tells us how the various available
assets must be utilised in order to produce the efficient portfolio with
expected return $E(\bar{R}_p)$. The elements of Equation (6.55) are the partial
derivatives

$$\frac{\partial E(\bar{R}_p)}{\partial x_i} = E(\bar{R}_i) \quad (6.56)$$

and

$$\frac{\partial \sigma(\bar{R}_p)}{\partial x_i} = \frac{x_i \sigma^2(\bar{R}_i) + \sum_{k \neq i} x_k \sigma_{ki}}{\sigma(\bar{R}_p)} \quad (6.57)$$

from Equation (6.50). Thus Equation (6.55) can be rewritten as

$$\lambda_1 = \frac{\frac{\partial \sigma(\bar{R}_p)}{\partial x_j} - \frac{\partial \sigma(\bar{R}_p)}{\partial x_i}}{\frac{\partial E(\bar{R}_p)}{\partial x_j} - \frac{\partial E(\bar{R}_p)}{\partial x_i}} \times \frac{(x_j \sigma^2(\bar{R}_j) + \sum_{k \neq i} x_k \sigma_{kj}) - (x_i \sigma^2(\bar{R}_i) + \sum_{k \neq i} x_k \sigma_{ki})}{\sigma(\bar{R}_p)} \quad (6.58)$$

In other words, for any pair of securities $j$ and $i$, in forming the efficient portfolio with expected return $E(\bar{R}_p)$, the values of the $x$'s must be
chosen so that the ratio of the difference between the marginal effects of
assets $j$ and $i$ on the standard deviation of the return on the portfolio to the
difference between the marginal effects of the two assets on the mean
portfolio return is the same for all pairs of assets and equal to $\lambda_1$, the reciprocals of the slope of the efficient set at the point $E(\bar{R}_p) = E(\bar{R}_p)$.

From Equation (6.56) the marginal effect of an asset on the mean port-
folio return is just the asset’s expected return. From Equation (6.57),
however, the marginal effect of an asset on the standard deviation of the
portfolio return depends on the asset’s variance $\sigma^2(\bar{R}_i)$, the covariances
of its return with those of other assets in the portfolio $\sigma_{ki}$, $k = 1, 2, \ldots, N$,
$k \neq i$, the proportions in which the assets are held (the $x$) and the level of
$\sigma(\bar{R}_p)$. Moreover, in Equation (6.57) the variance $\sigma^2(\bar{R}_i)$ is only one of the
$N$ terms in the numerator; in general, the $N - 1$ covariance terms are much more important in the determination of $\partial \sigma(\bar{R}_p)/\partial x_i$ than the term involving $\sigma^2(\bar{R}_i)$. And this fact is of fundamental importance in later analyses.

In the case in which short selling is prohibited, so that the $x_k$ must be nonnegative, Equation (6.58) must hold between any pair of assets that appear in the given efficient portfolio at a positive level. Assets would be excluded from the portfolio when their values of $x$ in Equation (6.58) are negative; intuitively, at least for this particular efficient portfolio, the marginal effects of such assets on $\sigma(\bar{R}_p)$ are too large relative to their marginal effects on $E(\bar{R}_p)$.

Finally, it is interesting to note that when short selling of all assets is permitted, because Equation (6.58) must hold between every pair of assets, in general all assets are included in any efficient portfolio. Some assets are held long and others are sold short, but in general no asset is excluded from an efficient portfolio. On the other hand, when short selling is prohibited, most efficient portfolios do not involve holdings of all assets. For example, in this case the efficient portfolio with the highest possible level of expected return is generally just a single asset, and other efficient portfolios with high levels of expected return also involve zero holdings of many assets.

IV.D. Some Odds and Ends of the Mean–Standard Deviation Model

The main results of the mean–standard deviation version of the two-parameter consumption-investment model have now been presented. Thus we turn to a discussion of some peripheral points. All the issues to be considered should contribute to an understanding of the model, but for the most part this is all they have in common.

IV.D.1. The investment assets to be included in portfolio models

Having completed a formal analysis of the $N$-asset model in somewhat abstract terms, we can now consider the more mundane question of just which assets should be included. The answer is direct. In the two-period model consumption $c_2$ at period 2 is just the market value at that time of the portfolio chosen at period 1. A feasible portfolio, then, must imply a complete strategy for all activities that could possibly affect the market value of the individual's consumption (= wealth) at the beginning of period 2. Thus a portfolio decision implies a complete set of simultaneous sub-decisions concerning how the investment $h_1 = w_1 - c_1$ is allocated among, for example, common stocks, bonds, real estate, insurance policies, and any other activities relevant to $c_2$. 
Moreover, it is important to emphasize that the subdecisions involved in a portfolio choice cannot usually be considered independently. For example, in an optimal decision resources cannot be allocated to common stocks or bonds without at the same time considering the insurance and real estate decisions. In the two-period model the individual chooses from among the available portfolios on the basis of their associated probability distributions of wealth \( w_2 \). It is the distribution of total wealth \( w_2 \) that matters. Thus a given subdecision, which comprises part of a portfolio choice, can be evaluated only in terms of its effects on the distribution of total terminal wealth, which in turn depends on many other subdecisions.\footnote{This point is perhaps obvious to the readers of this book. But in the economics literature it is easy to find studies concerned, for example, with optimal insurance decisions of various types, as if these could be considered separately from the rest of the consumer’s portfolio problem.}

There are two types of assets that warrant more detailed attention, however, although the problems that they create are somewhat more relevant in multiperiod than in two-period models. These assets are durable consumer goods—for example, homes, automobiles, and appliances—and human capital. With respect to durable consumer goods, there are no special problems if the services of these items can be purchased, or rented, on a period-by-period basis and if there are markets in which the goods can be sold at any time. In this case if the individual purchases a durable consumer item at the beginning of period 1, the market value of its services during period 1 is subtracted from its price and included in \( c_1 \). The remainder is then included in total investment \( h_1 \). The one-period return on the investment in the durable is then determined by its market price at the beginning of period 2. And it is worth mentioning that there are, of course, well-developed rental markets for housing and automobile services, and at least in major cities, rental markets for such durable items as refrigerators and automatic dishwashers are becoming more common. Moreover, there are primary and secondary markets in which these assets can be purchased and sold.

A thorough treatment of the optimal use of the consumer’s human capital leads to much more difficult problems, however. We have avoided these in the two-period model by assuming that the consumer is paid at the beginning of period 1 for any labor services to be rendered during the period and that he sells no labor at period 2. In this case his labor income at period 1 is just one of the components of the total wealth \( w_1 \) that he uses to make his consumption-investment decision.

But suppose either that the consumer is paid at period 2 for labor sold at period 1 or that he sells additional labor services, and is paid for them, at
period 2. In either case the amount of labor income to be received at period 2 may not be completely certain. This creates no problems if there is a perfect market for human capital; that is, at period 1 the consumer can issue equity in, and not just borrow against, his future labor income, and there is a perfectly competitive market for shares in labor income of his type. In this case the market value of the consumer's human capital is just part of \( w_t \), the total period 1 market value of all his resources, and he ends up holding shares in his own human capital only if these happen to be part of his optimal portfolio. If the consumer decides not to sell off all his human capital at period 1, the market value of whatever he holds is treated as part of his investment \( k_1 \). In short, with a perfect market for human capital, the latter turns out to be no different from any other wealth-producing asset that the consumer carries into period 1.

On the other hand if there is no market for human capital, the occupational income received at period 2 has no market value at the beginning of period 1. In this case there is no contribution to either \( w_t \) or \( k_1 \) from the occupational income of period 2. This income, however, must be included as part of the total wealth \( w_t \) provided by any feasible portfolio. In essence, the absence of a market for human capital constrains the individual to include his human capital in any feasible portfolio.

Finally, it is clear that the presence or absence of a market for human capital can affect the occupational decision itself. If there is a perfect market for human capital, the individual's attitudes toward risk do not affect his occupational decision, because he can sell the probability distribution on his income for period 2 and use the proceeds to purchase assets whose return distributions conform better to his own tastes. In this case, the best occupation is simply the one that maximizes the period 1 market value of his future income,\(^2\) and the occupation decision can be made independently of the portfolio decision. If there is no market for human capital, however, the occupational decision is affected by the individual's attitudes toward risk, because the characteristics of the distribution of occupational income in period 2 affect the distribution of terminal wealth for every feasible portfolio.

This discussion of occupational decisions barely scratches the surface of this important area of economics. But our main interest is in the characteristics of optimal portfolio decisions and in the implications of these for the structure of equilibrium market prices of investment assets. Our discussion of occupational decisions and consumer durables is meant (1) to

\(^2\) Even here we are oversimplifying, however. When we take account of the fact that the choice of occupation in part determines the individual's consumption of leisure, the occupational decision cannot be made on the basis of wealth alone.
provide a little more feeling for the framework in which the consumption-investment decision takes place and (2) to emphasize that an analysis concerned more directly with these variables would have to take into account that they also occur in the context of the overall consumption-investment decision. But because we are primarily interested here in the portfolio part of the decision, we have worked with assumptions that allow us to bypass the interesting problems that arise in the area of optimal occupational decisions.

IV.D.2. The effects of diversification: algebraic treatment

One of the conclusions in the geometric treatment of the N-asset model was that the process of mixing two assets or portfolios $a$ and $b$ in order to form a new portfolio is in general a dispersion-reducing activity; that is, the expected return on the new portfolio is just the weighted average of the expected returns on $a$ and $b$, where the weights are the proportions of portfolio funds $w_1$ invested in $a$ and $b$. When the correlation between the returns on $a$ and $b$ is less than 1, however, the standard deviation of the return on the new portfolio is less than the weighted average of $\sigma(R_a)$ and $\sigma(R_b)$. We now want to examine this conclusion in a slightly different way. Specifically, we are concerned with the behavior of the standard deviation of the return on the portfolio as the number of assets in the portfolio is increased.

Consider first the case in which an equal share of $w_1 = w_i = c_1$ is invested in each of $K < N$ assets, that is, $x_i = 1/K$, $i = 1, \ldots, K$, and in which the returns on different assets are independent, so that $\sigma_{ij} = 0$, all $i$ and $j$ and $i \neq j$. The variance of the return on the portfolio is then

$$\sigma^2(\bar{R}_p) = \sum_{i=1}^{K} x_i^2 \sigma^2(\bar{R}_i) = \frac{1}{K^2} \sum_{i=1}^{K} \sigma^2(\bar{R}_i).$$

Suppose now that, of all the assets in the market, the distribution of the return on asset $g$ has the largest variance, $\sigma^2(\bar{R}_g) = M$, and $M < \infty$. The variance on the return on a portfolio of $K$ assets must then satisfy

$$\sigma^2(\bar{R}_p) \leq \frac{KM}{K^2} = \frac{M}{K}.$$

Now this expression is smaller, the larger the value of $K$. In fact, if, simply to make a point, we assume that there are an infinite number of assets with independent returns in the market, it is possible to attain a portfolio whose variance of return is arbitrarily close to zero. In any case, it is clear from Equation (6.59) that in general the variance of the one-period return on the
portfolio falls as the number of assets in the portfolio is increased. Thus increased diversification has the effect of making the one-period return on a portfolio more certain; the probability distribution on the return is more closely concentrated about its mean or expected value.

From a practical point of view, it is important to note that substantial reduction in the dispersion of portfolio return can be achieved with a relatively small amount of diversification. Expression (6.59) essentially implies that with equal weighting of assets whose returns are independent, the standard deviation of the return on a portfolio behaves like, or is proportional to, $1/\sqrt{K}$ as $K$, the number of assets in the portfolio, is increased. The function $1/\sqrt{K}$ is plotted in Figure 6.13. The function moves toward 0 more and more slowly as $K$ is increased. For example, it is clear from the graph that increasing the size of the portfolio from 1 to 9 assets brings about a much larger decrease in dispersion than increasing $K$ from 9 to 100.

Maintaining the assumption that equal amounts of $h_i$ are invested in each asset, let us now examine the effects of diversification on the variance of the one-period return on a portfolio when the returns on individual assets are dependent. With equal weighting

$$\sigma^2(\bar{R}_p) = \sum_{i=1}^{K} \sum_{j=1}^{K} x_i x_j \sigma_{ij} = \frac{1}{K^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sigma_{ij}. $$

The double sum in this expression has $K^2$ terms; thus the variance of the one-period return on the portfolio is, in the case of equal weighting, just the average of the individual variances and covariances.

Now, however, let us rewrite $\sigma^2(\bar{R}_p)$ in a slightly different way. Specifically, let us separate out those terms which involve the variances of the distributions of the one-period returns on individual assets:

$$\sigma^2(\bar{R}_p) = \frac{1}{K^2} \sum_{i=1}^{K} \sigma^2(\bar{R}_i) + \frac{1}{K^2} \sum_{j=i}^{K} \sigma_{ij}. $$

![Figure 6.13](image)
Again, if the variances of the one-period returns on individual assets, $\sigma^2(R_i)$, have a finite upper bound, then as the number of assets in the portfolio is increased, the first sum approaches 0; that is, in a portfolio of many securities, each weighted equally,

$$\sigma^2(\bar{R}_p) \approx \frac{1}{K^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sigma_{ij}, \quad K \text{ "large"}.$$  

Now note that the average of the covariances between the returns of the assets in the portfolio is

$$\bar{\sigma}_{ij} = \frac{\sum_{i=1}^{K} \sum_{j=1}^{K} \sigma_{ij}}{K(K - 1)}.$$

Thus we can write

$$\sigma^2(\bar{R}_p) \approx \frac{1}{K^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \sigma_{ij} = \frac{K - 1}{K^2(K - 1)} \sum_{i=1}^{K} \sum_{j=1}^{K} \sigma_{ij} = \frac{K - 1}{K} \bar{\sigma}_{ij}.$$

As $K$ is increased $(K - 1)/K$ approaches 1, so that the variance of the distribution of the return on the portfolio approaches the average covariance between the one-period returns on the individual assets in the portfolio. Thus the dispersion in the distribution of the one-period return on a portfolio of many assets depends primarily on the relationships between the returns on the individual assets. The contribution of an individual asset to the dispersion of the portfolio return depends primarily on the relationships between this asset and other assets in the portfolio, rather than on the dispersion in the unconditional distribution of the return on the asset itself. When added to a large portfolio, an asset whose return has an extremely high variance may actually reduce the variance of the one-period return on the portfolio if the asset has low covariances with other assets.

Thus in a diversified portfolio the "riskiness" of an individual asset depends more on the covariability of the return on this asset with the returns on other assets than on the variance of the distribution on the return of the asset itself.\footnote{Another way to explain this result is as follows: When the number of securities in the portfolio is increased by one, the number of terms involved in the variance of the portfolio return increases by $2K - 1$, or in other words, a row and a column are added to the variance-covariance matrix. Thus the variance of the return on the new asset is only one of $2K - 1$ new terms, and its contribution to portfolio variance is smaller, the larger the value of $K$.} (And this is a result that reappears prominently in the next chapter.)
IV.D.3. Quadratic utility functions

The main result concerning optimal portfolio decisions obtained thus far in this chapter is of course the efficient set theorem; that is, if the probability distributions of one-period returns on all portfolios are normal, the optimal portfolio for a risk-averse investor is mean-standard deviation efficient. We now show that the normality assumption is not necessary to obtain this result. If, alternatively, we are willing to assume that for any given value of \( c_0 \), the individual's utility function can be well approximated by a function that is quadratic in \( c_0 \), then the optimal portfolio for a risk averter is again efficient.

To prove this statement, it is again sufficient to show that with a quadratic approximation to the utility function, (1) for any given value of \( c_0 \), the consumer can rank the probability distributions of \( \mathbf{x} \) associated with different portfolios by looking only at the means and standard deviations of distributions of portfolio returns, and (2) given \( c_0 \), expected utility is an increasing function of expected portfolio return \( E(\mathbf{R}_p) \) and a decreasing function of standard deviation \( \sigma(\mathbf{R}_p) \).

For a given value of \( c_0 \), say, \( c_0' \), a quadratic approximation to the utility function \( U(c_0',c_2) \) can be obtained from a Taylor series expansion, treating \( c_0 \) as a constant. If the expansion is taken about \( c_0 = 0 \), we get

\[
U(c_0',c_2) = U(c_0',0) + \frac{\partial U(c_0',0)}{\partial c_2} c_2 + \frac{\partial^2 U(c_0',0)}{2 \partial^2 c_2} c_2^2 + \text{higher-order terms},
\]

where \( \partial U(c_0',0)/\partial c_2 \) and \( \partial^2 U(c_0',0)/\partial c_2^2 \) are the first and second partial derivatives of \( U \) evaluated at the point \((c_0',0)\). If the higher-order terms in this expression are assumed to be negligible, we obtain the quadratic approximation

\[
U(c_0',c_2) \approx V(c_0',c_2) = a_0 + a_1 c_2 - a_2 c_2^2, \tag{6.60}
\]

where, under the assumption that the individual is a risk averter,

\[
a_0 = U(c_0',0) > 0, \quad a_1 = \frac{\partial U(c_0',0)}{\partial c_2} > 0, \quad a_2 = -\frac{1}{2} \frac{\partial^2 U(c_0',0)}{\partial c_2^2} > 0. \tag{6.61}
\]

There is one major objection to quadratic utility functions. At some level of \( c_0 \) the slope of a quadratic with \( a_2 > 0 \) becomes negative, and this

\[\text{For a risk preferer} a_2 \text{ would be negative.}

Note that we are only approximating the utility function in the \( c_0 \) direction. Essentially, we are holding \( c_0 \) fixed and then developing a quadratic approximation for the projection of the utility function in the \( c_0 \) dimension. Note also that each term in Equation (6.61) generally depends on the chosen value of \( c_0 \), so that the constants in Equation (6.60) are different for different choices of \( c_0 \).
violates the assumption that marginal utility must always be positive. Negative marginal utility occurs when

\[ \frac{\partial V(c_1', c_2)}{\partial c_2} = a_1 - 2a_{sc_2} < 0, \quad a_2 > 0, \]

that is, when

\[ c_2 > \frac{a_1}{2a_2}. \]  

(6.62)

Because a quadratic function implies negative marginal utility of consumption beyond some point, the quadratic is always at best an approximation to the true utility function of a risk averter, and this is why we always treat it as such here.

The main attraction of a quadratic utility function is the ease with which comparisons of different probability distributions of period 2 consumption can be made. With the quadratic in Equation (6.60), the approximate expected utility of the probability distribution of \( \xi_2 \) associated with a particular portfolio decision is just

\[ E[U(c_1', \xi_2)] = E[V(c_1', \xi_2)] = a_0 + a_1E(\xi_2) - a_2E(\xi_2^2), \]  

(6.63)

where \( E(\xi_2) \) and \( E(\xi_2^2) \) are the mean and second moment of the distribution of \( \xi_2 \). Because

\[ \sigma^2(\xi_2) = E(\xi_2^2) - E(\xi_2)^2, \]

the approximate expected utility given by Equation (6.63) can be restated as

\[ E[V(c_1', \xi_2)] = a_0 + a_1E(\xi_2) - a_2[\sigma^2(\xi_2) + E(\xi_2^2)]. \]  

(6.64)

But the mean and variance of \( \xi_2 \) are related to those of the underlying portfolio return \( \bar{R}_p \) according to

\[ E(\xi_2) = (w_1 - c'_1)[1 + E(\bar{R}_p)] \quad \sigma^2(\xi_2) = (w_1 - c'_1)^2\sigma^2(\bar{R}_p), \]

so that

\[ E[V(c_1', \xi_2)] = a_0 + a_1(w_1 - c'_1)[1 + E(\bar{R}_p)] \]

\[ - a_2(w_1 - c'_1)^2[\sigma^2(\bar{R}_p) + (1 + E(\bar{R}_p))^2]. \]  

(6.65)

Thus with the quadratic approximation to the consumer's utility function, given \( w_1 \) and \( c_1 = c'_1 \), differences between the approximate expected utilities associated with different portfolio choices can be determined from knowledge of only the means and variances, or standard deviations, of the
portfolio return distributions, so that portfolios can be ranked on the basis of these two parameters. Moreover,

\[
\frac{\partial E[V(c_1', \varepsilon_2)]}{\partial \sigma(\bar{R}_p)} = -2a_2(w_1 - c_1')^2\sigma(\bar{R}_p) < 0,
\]

so that expected utility is indeed a declining function of \( \sigma(\bar{R}_p) \).

On the other hand, the approximate expected utility computed from \( V \) is not a monotonically increasing function of expected return \( E(\bar{R}_p) \). For a given value of \( \sigma^2(\bar{R}_p) \), \( E[V(c_1', \varepsilon_2)] \) declines with increases in expected return when

\[
\frac{\partial E[V(c_1', \varepsilon_2)]}{\partial E(\bar{R}_p)} = a_1(w_1 - c_1') - 2a_2(w_1 - c_1')^2[1 + E(\bar{R}_p)] < 0
\]

or when

\[
(w_1 - c_1')[1 + E(\bar{R}_p)] > \frac{a_1}{2a_2}.
\]

Thus, comparing (6.62) and (6.66), expected quadratically approximated utility declines with increases in \( E(\bar{R}_p) \) when \( E(\varepsilon_2) = (w_1 - c_1')[1 + E(\bar{R}_p)] \) is in the downward sloping part of the quadratic utility function.

In short, with a quadratic approximation to the individual's utility function, expected utility rises with expected return \( E(\bar{R}_p) \) as long as \( E(\varepsilon_2) \) is within the range of values for which the quadratic is a monotone-increasing function of \( c_2 \). But this is precisely the range of values of \( c_2 \) for which the quadratic is a valid approximation to the consumer's utility function. Indeed, the implicit assumption underlying the use of a quadratic approximation to the true utility function is that in the approximate function negative marginal utility occurs at levels of consumption \( c_2 \) that are so high and so unlikely that their effect on the analysis is negligible. But this means that if the use of quadratics is valid for the consumer at hand, the optimal portfolio for \( c_1 = c_1' \) must yield expected period 2 consumption within the range of the quadratic approximation that implies positive marginal utility of \( c_2 \).

Within this range the quadratically approximated expected utility is an increasing function of expected return and a decreasing function of standard deviation of return. It follows that the optimal portfolio for \( c_1 = c_1' \) must be such that no portfolio with the same or higher expected return has lower standard deviation of return; that is, the optimal portfolio must be \( E(\bar{R}) \), \( \sigma(\bar{R}) \) efficient. And because this analysis applies to any given value of \( c_0 \), it applies to the optimal value, and thus the efficient set theorem holds:
When a risk averter's utility function is well approximated by a quadratic, his optimal portfolio is a member of the $E(R)$, $\sigma(R)$ efficient set.

But at least to us it seems that obtaining the efficient set theorem by means of quadratic approximations to utility functions is not so appealing as the approach based on the assumption that portfolio return distributions are all of the same two-parameter type, for example, normal. And our objection to the quadratic approach is not based on the fact that, strictly speaking, quadratics are not legitimate utility functions. All economic theory involves assumptions that amount to approximations of one sort or another, and this one seems no worse than most. Rather, our preference for the approach based on two-parameter portfolio return distributions comes from the fact that this distributional assumption can be tested directly on market data, whereas the assumption that utility functions can be well approximated by quadratics is at best tested only with extreme difficulty.

And we argue now that a class of two-parameter distributions does indeed seem to provide a good description of actual data on security returns. Moreover, we also contend that such results are not totally unexpected from a theoretical viewpoint; that is, there are also good theoretic arguments in support of two-parameter return distributions.

**IV.D.4. Why normality?**

The one-period return on a portfolio is just a weighted sum of the returns on the individual assets in the portfolio. Weighted sums of normal variables are themselves normally distributed; that is, the normal distribution is stable in the sense that it reproduces itself under weighted addition. Thus if distributions of asset returns are normal, distributions of portfolio returns are normal. Moreover, it is clear that stability, or invariance under addition, is a necessary property of any distribution assumed in the two-parameter portfolio model; that is, because assets are themselves portfolios, if all portfolio return distributions are to be of the same two-parameter type, the distributions must be stable or invariant under addition.

But the natural question then is: Why would one expect asset returns to have stable or, even more specifically, normal distributions? Suppose the one-period horizon in the consumption-investment problem is a year. The one-period return on an asset is just the change during the year in the market value of a dollar invested in the asset at the beginning of the year, that is, cash payments plus capital gains, all divided by the initial price. But the change in the market value of a unit of an asset over the entire one-year period is just the sum of the changes in market value from one trading point to the next during the year. For example, suppose that a particular asset $j$ is traded at each of 100 equally spaced points in time during
the year and the prices observed at these trades are \( p_{jt}, t = 1, 2, \ldots, 100 \). The one-period return on the asset for the year can then be written as either

\[
\bar{R}_j = \frac{p_{j,100} - p_{j,1}}{p_{j,1}}
\]

or

\[
\bar{R}_j = \frac{(p_{j,100} - p_{j,90}) + (p_{j,90} - p_{j,80}) + \cdots + (p_{j,2} - p_{j,1})}{p_{j,1}}
\]

If the price changes \( p_{j,t+1} - p_{j,t} \) can be regarded as drawings from a distribution for which the variance exists, the central limit theorem would lead us to suppose that the sum of price changes in this expression would have a distribution that is approximately normal; that is, the central limit theorem tells us that under fairly general conditions, in the limit of sums of identically distributed random variables approach a normal distribution if the variance of the elements of the sum exists. Thus the annual one-period return, which is just the sum of price changes between trading points multiplied by the constant \( 1/p_{jt} \), should have a distribution that is approximately normal.

But these limiting arguments that lead to the supposition of normal distributions for the one-period returns on assets or portfolios depend critically on the assumption that the variances of the distributions of price changes and one-period returns exist; that is, the expected values that define the variance are finite. There is much empirical evidence, however, that suggests rather strongly that this assumption may be inappropriate, at least for such important assets as common stocks and government bonds. See, for example, Blume [11], Fama [12], Mandelbrot [15], and Roll [17].

If the variances of price changes and one-period returns are infinite, the limiting arguments presented above lead to a much broader class of limiting distributions for asset and portfolio returns. Interestingly, all members of this class of limiting distributions have the stability property so critical for the two-parameter portfolio models, and in fact in the statistical literature these distributions are referred to as the “stable class.”

Most important, the empirical evidence cited above suggests that observed distributions of asset and portfolio returns conform well to the two-parameter members of this stable class. Thus we now conclude this chapter by extending the two-parameter portfolio model to allow for two-parameter distributions that are stable but might have infinite variances. We find that the extended model has all the important properties of the two-parameter model based on normal return distributions.
V. THE TWO-PARAMETER MODEL WITH SYMMETRIC STABLE DISTRIBUTIONS OF PORTFOLIO RETURNS

Indeed, the development of the general two-parameter stable model will be much along the lines of that of the normal model. In particular, we first consider the characteristics of consumer tastes, and we show that with two-parameter stable return distributions, the consumer is able to rank distributions of portfolio returns on the basis of expected returns \( E(\tilde{R}_p) \) and a measure of return dispersion \( \sigma(\tilde{R}_p) \), which in the general stable model is no longer interpreted as the standard deviation. Moreover, we show that for a risk-averse consumer, expected utility is an increasing function of \( E(\tilde{R}_p) \) and a decreasing function of \( \sigma(\tilde{R}_p) \), so that the efficient set theorem holds; that is, the optimal portfolio must be \( E(\tilde{R}) \), \( \sigma(\tilde{R}) \) efficient, and a portfolio is efficient if no portfolio with the same or higher expected return \( E(\tilde{R}) \) has lower return dispersion \( \sigma(\tilde{R}) \). Even more specifically, we show that risk aversion and two-parameter stable portfolio return distributions imply indifference curves of \( E(\tilde{R}) \) against \( \sigma(\tilde{R}) \) that are positively sloping and convex, just as we saw in Figure 6.3 for the normal model.

Likewise on the opportunity side, we show that the efficient set of portfolios describes a positively sloping concave curve in the \( E(\tilde{R}), \sigma(\tilde{R}) \) plane, much like the curve \( bcd \) shown for the normal model in Figure 6.4. Thus the fundamental result for the two-parameter stable portfolio model is the picture of consumer equilibrium in Figure 6.5; that is, the optimum is represented by a point of tangency between a convex indifference curve and the concave efficient set curve. The results are obviously essentially identical with those obtained for the normal model. The primary difference is that in the two-parameter stable model, \( \sigma(\tilde{R}) \) is more generally interpreted as return dispersion rather than as standard deviation.

Before proceeding with the development of the stable model of consumer equilibrium, however, we must first discuss some of the relevant statistical properties of the stable class of distributions.

V.A. Properties of Stable Distributions: A Brief Review

In the portfolio model to be presented below, it is assumed that all random variables have symmetric stable distributions. A symmetric stable distribution has three parameters that here are denoted \( \alpha, E(\tilde{g}), \) and \( \sigma(\tilde{g}) \).

The definition and original treatment of the class of stable distributions is due to Lévy [8]. A compact treatment of most of the available statistical theory is in Ref. 7. Derivations of the properties of these distributions summarized in this section are readily available in Ref. 7. The symmetric stable distributions are tabulated in Ref. 9, and procedures for estimating their parameters are discussed in Ref. 10.
where $\tilde{y}$ is the random variable. The parameter $\alpha$, which must be in the range $0 < \alpha \leq 2$, is called the "characteristic exponent," and it determines the type of a stable distribution. The stable distribution with $\alpha = 2$ is the normal; the symmetric stable distribution with $\alpha = 1$ is the Cauchy. The normal is the only stable distribution for which second- and higher-order absolute moments exist. When $\alpha < 2$, absolute moments of order less than $\alpha$ exist; those of order equal to or greater than $\alpha$ do not.

The absolute moment of order $k$ is

$$E(|\tilde{y}|^k) = \int_{-\infty}^{\infty} |y|^k f(y) \, dy,$$

where $f(y)$ is the density function of $\tilde{y}$. When we say that the absolute moment of order $k$ does not exist, we mean that the integral that defines the moment does not exist; or more simply $E(|\tilde{y}|^k) = \infty$. It is a theorem of probability theory that the ordinary moment of order $k$ defined as

$$E(\tilde{y}^k) = \int_{-\infty}^{\infty} y^k f(y) \, dy$$

exists only if the absolute moment of order $k$ exists. Thus the statements in the preceding paragraph imply that for stable distributions the variance

$$\sigma^2(\tilde{y}) = E(\tilde{y}^2) - E(\tilde{y})^2$$

exists only when $\alpha = 2$; the mean $E(\tilde{y})$ exists only when $\alpha > 1$. In our portfolio models we always assume that $\alpha > 1$, so that the expected value $E(\tilde{y})$ can be used as a measure of the location of the distribution of $\tilde{y}$.

The nonnegative parameter $\sigma(\tilde{y})$ defines the scale of a stable distribution. When $\alpha = 2$, $\sigma(\tilde{y})$ is the standard deviation divided by $\sqrt{2}$. When $\alpha < 2$, the standard deviation of the stable distribution does not exist, and $\sigma(\tilde{y})$ must be given some other interpretation. In Ref. 9 it is shown that when $1 < \alpha \leq 2$, which is the important range of values of $\alpha$ for portfolio models, $\sigma(\tilde{y})$ corresponds approximately to the semi-interquartile range (half the difference between the 0.75 and 0.25 fractiles) of the distribution of $\tilde{y}$.

The standardized variable

$$t = \frac{\tilde{y} - E(\tilde{y})}{\sigma(\tilde{y})}$$

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The standardized variable

$$t = \frac{\tilde{y} - E(\tilde{y})}{\sigma(\tilde{y})}$$

(6.67)

The stable class also includes asymmetric distributions. A fourth parameter, for skewness, is zero in the symmetric case. From a theoretical viewpoint the symmetry assumption reduces the generality of the model. But in fact the assumption seems to be justified by the data. See, for example, Blume [11], Fama [12], Kendall [13], Moore [16], and Roll [17].
Figure 6.14 Cauchy and Normal Density Functions for the Standardised Variable $r$

provides an excellent means of studying the role of the characteristic exponent $\alpha$ in determining the type of a symmetric stable distribution. If $\tilde{\gamma}$ is symmetric-stable with parameters $\alpha$, $E(\tilde{\gamma})$, and $\sigma(\tilde{\gamma})$, then $\tilde{r}$ is symmetric stable with parameters $\alpha$ (unaffected by the transformation), $E(\tilde{r}) = 0$, and $\sigma(\tilde{r}) = 1$. Distributions with $\alpha < 2$ depart from the normal distribution ($\alpha = 2$) in the following ways: (1) For some range of $\tilde{r}$ close to $E(\tilde{r})$ a stable distribution with $\alpha < 2$ is more peaked (has higher densities) than the normal. (2) For $|\tilde{r}|$ "large," stable distributions with $\alpha < 2$ have higher tails than the normal distribution. (3) For some intermediate range of $|\tilde{r}|$, stable distributions with $\alpha < 2$ have lower densities than the normal distribution. These properties are illustrated in Figure 6.14, which presents the density functions of $\tilde{r}$ for the normal ($\alpha = 2$) and Cauchy ($\alpha = 1$) distributions.

By definition, stable random variables are stable or invariant under weighted addition; that is, let $\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n$ be independent symmetric stable variables whose distributions have the same characteristic exponent $\alpha$, but possibly different location and scale parameters $E(\tilde{\gamma}_j), \sigma(\tilde{\gamma}_j)$, $j = 1, 2, \ldots, n$. Then if $d_j$, $j = 1, 2, \ldots, n$, are fixed weights, the sum

$$\tilde{\gamma} = \sum_{j=1}^{n} d_j \tilde{\gamma}_j$$

is symmetric stable with characteristic exponent $\alpha$ (unaffected by the
weighted addition) and with location and scale parameters

\[ E(\bar{y}) = E(\sum_{j=1}^{n} d_j \bar{y}_j) = \sum_{j=1}^{n} d_j E(\bar{y}_j), \]  
\[ \sigma(\bar{y}) = \sigma(\sum_{j=1}^{n} d_j \bar{y}_j) = \left[ \sum_{j=1}^{n} \sigma^2(\bar{y}_j) \right]^{1/\alpha}. \]

(6.68)  
(6.69)

In short, the property of stability means that a weighted sum of independent stable variables, each distributed with the same value of \( \alpha \), is of the same type; that is, it is stable with the same value of \( \alpha \) as the distributions of the individual summands.\(^{27}\)

It is also well to note that the class of stable distributions provides a generalization of the normal "central limit theorem"; that is, if a weighted sum of random variables has a limiting distribution, the limiting distribution is a member of the stable class. In the case in which variances are assumed to exist, under fairly general conditions, distributions of weighted sums of random variables approach a normal distribution as the number of variables in the sum is increased; this is the central limit theorem as it applies to normal variables. The stable nonnormal distributions generalize the central limit theorem to the case in which the variances of the underlying variables do not exist.

Thus the entire stable class of distributions, which includes the normal as a special case, has the two properties that make the normal distribution so desirable in portfolio models. First, we soon see again that the stability property of these distributions is critical in ensuring that distributions of portfolio returns are all of the same two-parameter type. Second, the generalized central limit theorem provides some theoretical justification for the fact that observed return distributions seem to conform well to members of the stable class.

Finally, in our portfolio models it is assumed that distributions of returns on all assets and portfolios are symmetric stable with characteristic exponent \( \alpha > 1 \). The reason that \( \alpha \) must be greater than 1 becomes clear later.\(^{28}\) But we also assume that whatever the value of \( \alpha \), it is the same for

\(^{27}\) The standardized, that is, \( E(\bar{r}) = 0, \sigma(\bar{r}) = 1 \), variable \( \bar{r} \) defined by (6.67) and expressions (6.68) and (6.69) for the location and scale parameters of weighted sums of stable variables are used extensively in later analyses. Setting \( \alpha = 2 \) in (6.68) and (6.69) and interpreting the \( \sigma's \) as standard deviations, one obtains the familiar expressions for the expected value and standard deviation of a weighted sum of independent normal, or other finite variance, variables.

\(^{28}\) In fact the estimated values of \( \alpha \) for returns on common stocks and United States Treasury bills have generally been well in excess of 1. From the evidence of Blume [11], Fama [12], and Roll [17], values of \( \alpha \) in the neighborhood of 1.6 seem reasonable.
all assets and portfolios. This assumption is necessary to obtain two-parameter return distributions—only the values of \( E(\bar{R}) \) and \( \sigma(\bar{R}) \) vary from one portfolio to another—but it clearly reduces the generality of the model. Note, however, that the mean–standard deviation portfolio model in Sections II to IV assumed normality, that is, \( \alpha = 2 \), for all return distributions. In essence, we now generalize this model to allow the characteristic exponent \( \alpha \) of return distributions to take any (given) value in the interval \( 1 < \alpha \leq 2 \).

V.B. Representation of the Consumer's Tastes: The Efficient Set Theorem

The first step is to discuss the way in which consumer tastes can be characterized in a portfolio model with two-parameter stable return distributions. The arguments are almost identical with those for the normal model in Section II, so that the present discussion can be quite brief. In fact just the first few steps of the analysis are presented, and the remainder is summarized with reference to the results in Section II.

Again the individual is assumed to have a given quantity of initial wealth \( w_1 \) that he must allocate between consumption \( c_1 \) and a portfolio investment \( w_1 - c_1 = h_1 \), the return on which is the basis of his consumption in period 2. The individual is assumed to behave as if he wished to make a consumption–investment decision that maximized expected utility, computed from the utility function \( U(c_1, c_2) \). He is assumed to be a risk averter, so that \( U(c_1, c_2) \) is monotone-increasing and strictly concave in \((c_1, c_2)\).

Consumption in period 2 is related to the one-period return \( \bar{R}_p \) on the individual's portfolio according to

\[
\bar{c}_2 = (w_1 - c_1)(1 + \bar{R}_p).
\]

But using the standardized variable

\[
\bar{r} = \frac{\bar{R}_p - E(\bar{R}_p)}{\sigma(\bar{R}_p)},
\]

\( \bar{c}_2 \) can be written in terms of \( E(\bar{R}_p), \sigma(\bar{R}_p) \), and \( \bar{r} \) as

\[
\bar{c}_2 = (w_1 - c_1)[1 + E(\bar{R}_p) + \sigma(\bar{R}_p)\bar{r}].
\]

The expected utility associated with a choice of current consumption \( c_1 \) and portfolio \( p \) is then

\[
E[U(c_1, \bar{c}_2)] = \int_{-\infty}^{\infty} U(c_1, (w_1 - c_1)[1 + E(\bar{R}_p) + \sigma(\bar{R}_p)\bar{r}]) f(\bar{r}) d\bar{r},
\]

where \( f(\bar{r}) \) is the density function of the variable \( \bar{r} \).
Because the distributions of \( R_p \) for all portfolios are assumed to be symmetric stable with the same value of the characteristic exponent \( \alpha \), the standardized variable \( \tau \), as defined by Equation (6.3), has a symmetric stable distribution with the same value of \( \alpha \). Thus with the definition of expected utility provided by Equation (6.5), the variable \( \tau \) and the density function \( f(\tau) \) are the same for all portfolios. Differences between the expected utilities associated with different consumption-investment decisions depend entirely on \( c_1, E(R_p) \), and \( \sigma(R_p) \), so that consumption-investment alternatives can be ranked solely on the basis of these three variables. Formally, we can write

\[
E[U(c_1, c_2)] = V(c_1, E(R_p), \sigma(R_p)). \tag{6.6}
\]

But except for the fact that the scale parameter \( \sigma(R_p) \) is no longer the standard deviation, this analysis is identical with that in Section II, and in fact Equations (6.3) to (6.6) are reproduced directly from Section II. And the remainder of the analysis of the present model would go precisely as in the case of the model based on normally distributed portfolio returns. Again the analysis of expressions (6.7) and (6.8) in Section II would lead us to conclude here that

\[
\frac{\partial EU(c_1, c_2)}{\partial E(R_p)} > 0 \quad \text{and} \quad \frac{\partial EU(c_1, c_2)}{\partial \sigma(R_p)} < 0;
\]

that is, for a risk averter expected utility is an increasing function of expected portfolio return and a decreasing function of return dispersion. Again it follows that the optimal portfolio corresponding to any, and thus the optimal choice of \( c_1 \) must be \( E(R), \sigma(R) \) efficient; that is, the optimal portfolio must be such that no portfolio with the same or higher expected return \( E(R) \) has lower return dispersion \( \sigma(R) \). But this is just the efficient set theorem. Finally, the analysis in Section II in the present model again implies that for any given \( c_1 \) indifference curves of \( E(R) \) against \( \sigma(R) \) are positively sloping and convex, just as shown in Figure 6.3 for the normal model.

In checking that these properties of consumer tastes when portfolio returns are symmetric stable are direct implications of the results for the normal model in Section II, note that the critical assumptions in the analysis in Section II are that (1) \( U(c_1, c_2) \) is concave, which is equivalent to assuming risk aversion; (2) distributions of portfolio returns are symmetric; and (3) differences between distributions of portfolio returns can be completely summarized by two parameters, a measure of location \( E(R) \) and a measure of scale or dispersion \( \sigma(R) \). The fact that in the model based on normally distributed portfolio returns \( \sigma(R) \) was interpreted as the
standard deviation was irrelevant. The important assumption was that
differences between distributions of returns on portfolios can be completely
summarized in terms of two parameters.

Or equivalently, the analysis of the properties of consumer tastes in
Section II requires that one period returns on all assets and portfolios must
be generated by two-parameter distributions of the same type. The assump-
tion made here, that distributions of returns on all assets and portfolios are
symmetric stable with the same value of the characteristic exponent \( \alpha \),
reduces these to two-parameter distributions of the same type, and it is
this which makes the analysis in Section II, with a change in the inter-
pretation of the scale parameter \( \sigma(R) \), directly applicable to the more
general stable model.

V.C. The Opportunity Set with Stable Return
Distributions

Without an exact description of his utility function we cannot, of course,
determine which efficient portfolio the consumer chooses, but we do know
that the optimal choice is a member of the efficient set. The next order of
business, then, is to develop a representation of the efficient set for a market
in which returns on all assets and portfolios conform to symmetric stable
distributions. In order to do this, however, we must first discuss a model
for describing how these returns are generated.

V.C.1. The market model

The one-period return on a portfolio is just a weighted average of the
returns on the assets in the portfolio; that is,

\[
\tilde{R}_p = \sum_{j=1}^{N} x_j \tilde{R}_j,
\]

(6.70)

where \( x_j \) is the proportion of investment funds \( w_i - c_i \) invested in asset \( j \),
\( N \) is the number of assets available, and

\[
\sum_{j=1}^{N} x_j = 1.
\]

The probability distribution of \( \tilde{R}_p \) depends on the distributions of the
\( \tilde{R}_j \), the interrelationships among the \( \tilde{R}_j \), and the set of \( x \)'s chosen. In the
models in this section it is assumed that the \( \tilde{R}_j \) are symmetric stable random
variables, so that \( \tilde{R}_p \) is just a weighted sum of stable variables. There is
ample empirical evidence that one-period returns on most assets are inter-
dependent. See, for example, Kendall [13], King [14], and Blume [11].
But unfortunately a general statistical theory covering distributions of
weighted sums of dependent stable variables is unavailable. The problem, then, is to develop a stochastic model in which the \( \bar{R}_j \) are dependent, but \( \bar{R}_p \) can nevertheless be expressed as a weighted sum of independent stable variables.

Assume that all interrelationships among the returns on individual assets arise from the fact that there is a common "market factor" \( \bar{M} \) that affects the returns on all assets; that is, the returns on individual assets are generated by the market model

\[
\bar{R}_j = a_j + b_j \bar{M} + \bar{\varepsilon}_j, \quad j = 1, 2, \ldots, N. \tag{6.71}
\]

Here \( a_j \) and \( b_j \) are constants, and \( \bar{\varepsilon}_j \) is a random disturbance whose distribution is assumed to have expected value equal to 0. It is assumed that \( \bar{\varepsilon}_j, j = 1, 2, \ldots, N, \) and \( \bar{M} \) are mutually independent, symmetric stable variables, all with the same characteristic exponent \( \alpha > 1. \)

Because \( \bar{M} \) affects the returns of all assets, Equation (6.71) does indeed allow for interdependence among the returns of different assets. Nevertheless, combining Equations (6.70) and (6.71),

\[
\bar{R}_p = \sum_{j=1}^{N} x_j a_j + \sum_{j=1}^{N} x_j b_j \bar{M} + \sum_{j=1}^{N} x_j \bar{\varepsilon}_j, \tag{6.72}
\]

so that the return on a portfolio is a weighted sum of the independent random variables \( \bar{M} \) and \( \bar{\varepsilon}_j, j = 1, 2, \ldots, N. \) Because these are assumed to be symmetric stable with the same characteristic exponent \( \alpha, \bar{R}_p \) is also symmetric stable with the same value of \( \alpha, \) and from Equations (6.68) and (6.69), \( E(\bar{R}_p) \) and \( \sigma(\bar{R}_p), \) the mean and dispersion of the distribution of \( \bar{R}_p, \) are related to those of \( \bar{M} \) and the \( \bar{\varepsilon}_j \) according to

\[
E(\bar{R}_p) = \sum_{j=1}^{N} x_j a_j + \sum_{j=1}^{N} x_j b_j E(\bar{M}) = \sum_{j=1}^{N} x_j E(\bar{R}_j), \tag{6.73}
\]

\[
\sigma(\bar{R}_p) = \left[ \sigma^2(\bar{M}) \left| \sum_{j=1}^{N} x_j b_j \sigma^2(\bar{\varepsilon}_j) \right| \right]^{1/4}. \tag{6.74}
\]

Fortunately, the empirical evidence, and especially Blume [11], indicates that our specification of the return-generating process, that is, the market model with \( \alpha > 1, \) is a good description of actual return data, at least for common stocks.

The market model was originally suggested by Markowitz [1] as a way of reducing the number of parameter inputs required in the normal or mean-standard deviation portfolio model. In the general mean-standard deviation model, from Equation (6.30), the variance of the return on a portfolio of \( N \) assets requires estimates of the \( N \) variances of the individual asset returns \( \sigma^2(\bar{R}_i) \) and \( N(N - 1)/2 \) covariances; that is, there are \( N(N - 1) \) covariances, but \( \sigma_{ji} = \sigma_{ij}. \) But from Equation (6.74), with the market model, to measure portfolio dispersion we need only have estimates of \( \sum b_j \) and \( \sigma(\bar{\varepsilon}_j), \)
Finally, there would be no problem in our models in allowing returns to be linear functions of any number of independent, symmetric stable variables. For example, in line with the work of King [14], we could postulate the "market-industry" model

\[ R_j = \alpha_j + b_j Y + \sum_{k=1}^{K} c_k I_k + \bar{\varepsilon}_j, \quad j = 1, 2, \ldots, N. \]

Here the \( I_k \) are industry factors, assumed to be mutually independent, independent of \( Y \) and the \( \bar{\varepsilon}_j \), and distributed with the same value of \( \alpha \) as \( Y \) and the \( \bar{\varepsilon}_j \). But such a generalization of the return-generating process would complicate the algebra of the models to be presented without contributing additional insights. Thus we concentrate on the market model.

\[ \text{V.C.2. Diversification and the distribution of portfolio return} \]

By way of introduction to the geometric discussion of the portfolio opportunity set to be presented in the next section, we now show that as long as the characteristic exponent \( \alpha \) of the process generating asset returns is greater than 1, diversification is an effective tool for reducing the dispersion in the probability distribution of portfolio return. Consider a portfolio that includes only \( n < N \) assets at a nonzero level; that is, some of the \( x_j \) in Equation (6.70) are zero. Without loss of generality, we label the \( n \) assets in the portfolio 1, 2, \ldots, \( n \), so that

\[ \sigma(R_\pi) = \left[ \sigma^2(Y) \left| \sum_{j=1}^{n} x_j b_j \right|^\alpha + \sum_{j=1}^{n} \sigma^2(\bar{\varepsilon}_j) \left| x_j \right|^\alpha \right]^{1/\alpha}. \quad (6.75) \]

In general, how is return dispersion \( \sigma(R_\pi) \) affected as \( n \) is increased and the proportions of initial wealth invested in individual assets are reduced? Any rearrangement of the \( x_j \) affects the weighted average \( \sum x_j b_j \) and so also affects the first or "market" term in Equation (6.75). But the change in the average can go in either direction, and in any case it is not due to diversification per se. In essence, because the market factor \( Y \) affects the returns on all assets, increased diversification does not in general reduce its effects on the dispersion of the portfolio return.

On the other hand, when \( \alpha > 1 \), diversification, in general, systematically reduces the effects of the \( \bar{\varepsilon}_j \) on the dispersion of the portfolio return. Con-
Consider the simple case \( x_j = 1/n, j = 1, 2, \ldots, n \). Then the last term in Equation (6.75) becomes

\[
\sigma^\alpha(\tilde{\epsilon}_p) = \left( \frac{1}{n} \right)^\alpha \sum_{j=1}^{n} \sigma^\alpha(\tilde{\epsilon}_j).
\]

Because the \( \sigma^\alpha(\tilde{\epsilon}_j) \) are assumed to be bounded, as long as \( \alpha > 1 \) the value of \( \sigma^\alpha(\tilde{\epsilon}_p) \) decreases as \( n \) is increased. In essence, the \( \tilde{\epsilon}_j \) represent the effects of random factors, independent from asset to asset, and these effects become more and more offsetting as the portfolio is diversified. It must be emphasized, however, that this effect is only realized when \( \alpha > 1 \). When \( \alpha = 1 \), in general, diversification has no effect on \( \sigma^\alpha(\tilde{\epsilon}_p) \); when \( \alpha < 1 \), increased diversification usually causes \( \sigma^\alpha(\tilde{\epsilon}_p) \) to increase.

A numerical example makes these statements more concrete. For simplicity suppose that \( \sigma^\alpha(\tilde{\epsilon}_j) = 1 \) and \( x_j = 1/n \) for all \( j \). Then Table 1 shows the behavior of

\[
\sigma^\alpha(\tilde{\epsilon}_p) = \left( \frac{1}{n} \right)^\alpha \sum_{j=1}^{n} \sigma^\alpha(\tilde{\epsilon}_j) = n^{1-\alpha}
\]

for different values of \( \alpha \) as \( n \) is increased. It is clear from the table that when \( \alpha = 1 \), diversification is ineffective in reducing the dispersion of the distribution of the return on the portfolio. When \( \alpha = 1 \), under the simple conditions assumed in this example \( \sigma^\alpha(\tilde{\epsilon}_p) = 1 \), regardless of the number of assets in the portfolio. When \( \alpha < 1 \), the table demonstrates that the return on a more diversified portfolio may actually have a higher degree of dispersion than the return on a less diversified portfolio. When \( \alpha = 0.5 \), \( \sigma^\alpha(\tilde{\epsilon}_p) \) goes from 1.0 to 3.162 to 10.0 as \( n \) is increased from 1 to 10 and then to 100.

**Table 1**

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>10</th>
<th>100</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.00</td>
<td>1</td>
<td>0.100</td>
<td>0.010</td>
<td>0.001</td>
</tr>
<tr>
<td>1.75</td>
<td>1</td>
<td>0.178</td>
<td>0.032</td>
<td>0.006</td>
</tr>
<tr>
<td>1.50</td>
<td>1</td>
<td>0.316</td>
<td>0.100</td>
<td>0.032</td>
</tr>
<tr>
<td>1.25</td>
<td>1</td>
<td>0.562</td>
<td>0.316</td>
<td>0.176</td>
</tr>
<tr>
<td>1.00</td>
<td>1</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>0.50</td>
<td>1</td>
<td>3.162</td>
<td>10.000</td>
<td>31.623</td>
</tr>
</tbody>
</table>
When $\alpha > 1$, diversification reduces the dispersion of the distribution of the return on the portfolio. Moreover, Table 1 demonstrates that diversification is more effective, the higher the value of $\alpha$. As $n$ is increased, $\sigma^*(\bar{x}_p)$ approaches the limiting value 0 at different rates, depending on the value of $\alpha$. For example, when $\alpha = 2$, $\sigma^*(\bar{x}_p) = 0.100$ for a portfolio of 10 assets, whereas when $\alpha = 1.5$, a portfolio of 100 assets is required before $\sigma^*(\bar{x}_p)$ reaches this level.

In statistical terms all this means that when $\alpha > 1$ there is a law of large numbers at work that makes the return on a portfolio more certain as the number of assets in the portfolio is increased. The law becomes weaker as $\alpha$ moves away from 2 in the direction of 1, and when $\alpha = 1$ there is no law of large numbers at work. Finally, when $\alpha < 1$ the law of large numbers actually works in reverse, so that the return on the portfolio becomes less certain as the number of assets is increased.

An encouraging feature of Table 1 is the fact that for $\alpha > 1.25$, $\sigma^*(\bar{x}_p)$ moves relatively quickly toward its asymptotic value as the number of assets in the portfolio is increased. Because the available empirical work seems to indicate that, at least for the common stocks of large American companies and for United States government bonds, $1.25 < \alpha < 2.0$, an investor should be considerably heartened by the discussion in the previous paragraphs. When $\alpha$ is in this range, diversification is still an effective way to reduce the dispersion of the distribution of the return on a portfolio, although it is not so effective as in the case $\alpha = 2$ (the normal distribution).

**V.C.3. The efficient set**

We showed earlier that if distributions of portfolio returns are symmetric stable with the same value of $\alpha$, the efficient set theorem holds; that is, the optimal portfolio for a risk averter must be such that no portfolio with the same or higher expected return $E(\bar{R})$ has lower return dispersion $\sigma(\bar{R})$.

We now want to show that the preceding analysis of the effects of diversification on dispersion of portfolio returns implies that, as in the mean–standard deviation model, when $\alpha > 1$ the efficient set curve is positively sloping and concave in the $E(\bar{R})$, $\sigma(\bar{R})$ plane, somewhat like the curve bod in Figure 6.4.

From the discussion in Section IV.B, recall that the concavity of the efficient set curve in the mean–standard deviation portfolio model follows

---

We should note that the efficient set theorem for symmetric stable return distributions was established before the introduction of the market model specification of the return-generating process. The efficient set theorem is a consequence of consumer risk aversion and two-parameter distributions of portfolio returns and does not require the more restrictive specification of the return-generating process provided by the market model.
from the fact that any portfolio $p$ formed from two assets or portfolios $i$ and $j$ according to
\[ R_p = xR_i + (1 - x)R_j, \quad 0 < x < 1, \] (6.76)
has
\[ E(R_p) =xE(R_i) + (1 - x)E(R_j), \quad (6.77) \]
\[ \sigma(R_p) \leq x\sigma(R_i) + (1 - x)\sigma(R_j); \quad (6.78) \]
that is, the expected portfolio return is just the average of the expected returns on $i$ and $j$, whereas the standard deviation of the portfolio return is equal to or less than the corresponding average of the standard deviations of $R_i$ and $R_j$, with Equation (6.78) holding as an equality only when either (1) one of $i$ or $j$ is riskless or (2) there is an exact linear relationship between $R_i$ and $R_j$. We now argue that these results also hold when returns are generated by the market model and the characteristic exponent $1 < \alpha \leq 2$, except that now we interpret $\sigma(R_p)$ more generally as return dispersion rather than as standard deviation.

First, from Equation (6.73) we see immediately that Equation (6.77) holds in the symmetric stable market model, so that we can concentrate our attention on Equation (6.78). But remember that in the mean–standard deviation model Equation (6.78) is just a representation of the fact that diversification is a dispersion-reducing activity. The analysis of the effects of diversification in the preceding section showed that this is also true in the symmetric stable market model as long as the characteristic exponent $\alpha$ of the return-generating process is greater than 1. It may be helpful, however, to illustrate this result again and from a viewpoint a little closer to the current context.

Thus suppose that $R_i$ and $R_j$ in Equation (6.76) are the returns on two individual assets. From Equation (6.72) note that when the returns on the components of a portfolio are generated by the market model, portfolio returns also follow the market model; that is,
\[ R_p = a_p + b_pM + \tilde{\epsilon}_p, \]
where
\[ a_p = \sum_{j=1}^{N} x_j a_j, \quad b_p = \sum_{j=1}^{N} x_j b_j, \quad \tilde{\epsilon}_p = \sum_{j=1}^{N} x_j \tilde{\epsilon}_j. \]

Thus with a two-asset portfolio
\[ \sigma(R_p) = [\sigma^2(M) | b_p |^\alpha + \sigma^2(\tilde{\epsilon}_p)]^{1/\alpha} \]
(6.79a)
\[ = [\sigma^2(M) | b_i (1 - x)b_j |^\alpha + (x^2\sigma^2(\tilde{\epsilon}_i) + (1 - x^2\sigma^2(\tilde{\epsilon}_j))]^{1/\alpha}. \]
(6.79b)
Similarly, the dispersion parameters of the distributions of $\tilde{R}_i$ and $\tilde{R}_j$ are

$$\sigma(\tilde{R}_i) = \left[\sigma^2(\tilde{M}) + b_i |\alpha + \sigma^2(\tilde{\epsilon}_i)\right]^{1/\alpha}$$

and

$$\sigma(\tilde{R}_j) = \left[\sigma^2(\tilde{M}) + b_j |\alpha + \sigma^2(\tilde{\epsilon}_j)\right]^{1/\alpha}.$$

Hence when asset and portfolio returns are generated by the market model, return dispersion arises from two sources, the market factor $\tilde{M}$ and the residual term $\tilde{\epsilon}$. The contribution of the market factor term to total return dispersion depends on the magnitude of the "sensitivity coefficient" $b$, and for the portfolio of assets $i$ and $j$, $b_p = x b_i + (1 - x) b_j$ is just an average of the coefficients for the individual assets. In short, the market factor $\tilde{M}$ is common to the returns on both assets, and forming a portfolio does not in itself reduce the effects of $\tilde{M}$ on return dispersion. Rather, the effects of diversification come from the residual term $\tilde{\epsilon}_p$, where we have

$$\sigma(\tilde{\epsilon}_p) = \left[\alpha \sigma^2(\tilde{\epsilon}_i) + (1 - x) \sigma^2(\tilde{\epsilon}_j)\right]^{1/\alpha} < x \sigma(\tilde{\epsilon}_i) + (1 - x) \sigma(\tilde{\epsilon}_j) \quad (6.80)$$

as long as $\alpha > 1$ and neither $\sigma(\tilde{\epsilon}_i)$ nor $\sigma(\tilde{\epsilon}_j)$ is identically 0.

The reader can convince himself that, as in the mean–standard deviation model, Equation (6.78) holds as an equality in the symmetric stable market model with $\alpha > 1$ when (1) either $i$ or $j$ is riskless, that is, $\sigma(\tilde{R}_i) = 0$ or $\sigma(\tilde{R}_j) = 0$, or (2) there is an exact linear relationship between $\tilde{R}_i$ and $\tilde{R}_j$. And the latter can happen only when $\sigma(\tilde{\epsilon}_i) = \sigma(\tilde{\epsilon}_j) = 0$, that is, the returns on both securities can be perfectly predicted from $\tilde{M}$, a case which, according to the empirical evidence of Blume [11], can be ignored. If either (1) or (2) holds, the combinations of $E(\tilde{R}_p)$ and $\sigma(\tilde{R}_p)$ obtained by varying $x$ in Equation (6.76) lie along a straight line in the $E(\tilde{R}_p), \sigma(\tilde{R}_p)$ plane between the points representing the assets $i$ and $j$. Otherwise Equation (6.78) holds, and the $E(\tilde{R}_p), \sigma(\tilde{R}_p)$ combinations obtained by varying $x$ in Equation (6.76) lie to the left of the line between $i$ and $j$. And this is all again precisely as in the mean–standard deviation model.\footnote{The analysis is somewhat more tedious, although not more difficult, when $i$ and $j$ themselves can be portfolios. In particular, the expression for $\sigma(\tilde{\epsilon}_p)$ is more complicated than Equation (6.80). But the reader can verify that the results are the same.}

\[V.C.4. \textbf{Consumer equilibrium}\]

In short, the fact that diversification is a dispersion-reducing activity implies that Equation (6.78) holds when returns are generated by the symmetric stable market model with $\alpha > 1$. Then using arguments identical with those used earlier in the mean–standard deviation model, we can show that Equations (6.77) and (6.78) together imply that the efficient set curve is positively sloping and concave in the $E(\tilde{R}), \sigma(\tilde{R})$ plane, somewhat like the curve $bc\bar{d}$ in Figure 6.4. We have already argued that the consumer's
indifference curves of $E(R)$ against $\sigma(R)$ in the stable model have the same general properties as those of the mean-standard deviation model. Thus Figure 6.5 is again a relevant picture of consumer equilibrium; that is, again consumer equilibrium is represented by a point of tangency between a convex indifference curve and the concave efficient boundary.\[2\]

VI. CONCLUSIONS

Our discussion of the one-period two-parameter model of optimal consumption-investment decisions is now complete. We turn in the next chapter to a model of capital market equilibrium for which this consumption-investment model is the basic building block.

REFERENCES

The pioneering works on the mean-standard deviation portfolio model are


The so-called market model in the present chapter was first suggested by Markowitz [1, pp. 96–100]. The mean-standard deviation version of the model was then studied in detail by


The model was then applied to the class of symmetric stable distributions by


Statistical background on the stable class of distributions, discovered by Lévy [8], is in


\[2\] The interested reader might also check that the algebraic results obtained for the mean-standard deviation model in Sec. IV.C also carry over easily to the symmetric stable model in this section.


References 9 and 10 are particularly concerned with problems of estimation.

Empirical work cited in this chapter on the distributions of security returns is in


