I. INTRODUCTION

In Chapter 6 a model of consumer equilibrium for the two-period consumption-investment problem was presented. In the present chapter the implications of this model for a theory of capital market equilibrium under uncertainty are considered. Specifically, we are concerned with the following questions: Given a market of risk-averse, expected utility-maximizing consumers, (1) what is the appropriate measure of the risk of an asset, and (2) what is the relationship in equilibrium between this measure of the asset's risk and its one-period expected return?

The asset-pricing model to be presented in the present chapter is the natural extension of the consumption-investment model in Chapter 6. The consumption-investment model is concerned with how the risk-averse, expected utility-maximizing consumer should allocate his initial wealth $w_1$ between consumption $c_t$ and a portfolio invest-
ment $h_t = w_t - c_t$ in the various assets available in the market. The asset-pricing model then draws the aggregate implications of the wealth allocation decisions of individual consumers for the equilibrium relationship between risk and expected return for assets and portfolios.

II. THE MARKET SETTING

The market setting in which equilibrium must be established is assumed to be as follows.

Perfect Markets. First, markets for consumption goods and investment assets are assumed to be perfect in the sense that all goods and assets are infinitely divisible; any information is costless and available to everybody; there are no transactions costs or taxes; all individuals pay the same price for any given commodity or asset; no individual is wealthy enough to affect the market price of any asset; and no firm is large enough to affect the opportunity set facing consumers. In short, as in most of the rest of this book, individual consumers and firms are assumed to be price takers in frictionless markets.

Firms. It is assumed that all production is organized by "firms." At the beginning of period 1 firms purchase, and pay for, the services of inputs—labor, machinery, and so on—and use these to produce consumption goods and services to be sold at the beginning of period 2. Firms finance their period 1 outlays for production by issuing shares in their period 2 market values (=sales of output), and these shares are the investment assets held by consumers. It is the process by which the period 1 market prices of such assets are determined that is the main concern here.

Consumers. The model of consumer equilibrium is that in Chapter 6. Briefly, at the beginning of period 1, consumers are assumed to have given quantities of resources—labor, which will be sold to some firm, and portfolio assets, that is, shares of firms, carried forward from previous periods—that must be allocated to current consumption $c_t$, measured in terms of some numéraire—for example, "dollars"—and a portfolio investment whose market value at the beginning of period 2 determines the level of consumption $c_t$ for period 2. Consumers are all risk-averse expected utility maximizers; that is, in his period 1 consumption-investment decision, each consumer behaves as if he were trying to maximize expected utility with respect to a utility of consumption function $U(c_1,c_2)$ that is monotone-increasing and strictly concave in $(c_1,c_2)$. 
Moreover, every consumer believes that, or better, behaves as if distributions of one-period returns on all portfolios can be fully described in terms of two parameters, expected return \( E(\bar{R}_p) \) and some measure of return dispersion \( \sigma(\bar{R}_p) \). From Chapter 6 we know that this implies that the consumer behaves as if distributions of returns on all portfolios are symmetric stable with the same value of the characteristic exponent \( \alpha \), for example, normal distributions when \( \alpha = 2 \). From the previous chapter we also know that the combination of consumer risk aversion and two-parameter return distributions implies the "efficient set theorem": that is, the optimal portfolio for a given consumer is \( E(\bar{R}_p), \sigma(\bar{R}_p) \) efficient, where portfolio efficiency requires that no portfolio with the same or higher expected return \( E(\bar{R}_p) \) has lower return dispersion \( \sigma(\bar{R}_p) \).

*Market Equilibrium.* Equilibrium at the beginning of period 1 is assumed to be reached through a process of tâtonnement with recontracting; that is, investors come to market with their resources and tastes, and firms bring their production opportunity sets. A tentative set of prices for consumption goods, labor, and shares is announced, firms make tentative production decisions, and investors offer their labor to firms and begin bidding for consumption goods and investment assets. Prices and decisions are tentative; it is agreed that no decisions are executed until an equilibrium set of prices, that is, a set of prices at which all markets can clear, has been determined.

Our treatment of this model concentrates on the nature of equilibrium in the capital market; that is, we take equilibrium in the markets for labor and consumption goods as given. And the order of presentation is a bit illogical. In particular, to emphasize that all prices and decisions are simultaneously determined and executed, it would be best, if possible, to consider first the characteristics of capital market equilibrium, that is, the relationship between risk and return, and then move on to partial equilibrium studies of the nature of optimal consumption-investment decisions by individuals and optimal production decisions by firms.

But the analysis cannot proceed in this way. The "risk structure" of equilibrium expected returns on shares can only be determined from properties of optimal consumption-investment decisions by individuals. Then the implications of the risk structure of equilibrium expected returns for optimal production decisions by firms can be considered. In short, although in our world everything happens at once, the analysis proceeds from partial equilibrium (consumer-investors) to market equilibrium to partial equilibrium (firms). Thus the first step is to discuss the appropriate measure of the risk of an individual asset and the relationship between risk
and expected return from the viewpoint of an individual consumer. We then
generalize these concepts to the level of the market.\footnote{A note for the more sophisticated: We use the term “market equilibrium” in recognition of the fact that we always work conditional on an assumed equilibrium in the markets for labor and consumption goods. A full general equilibrium model would allow for simultaneous determination of prices in these markets as well as in the capital market.}

We concentrate initially on the case in which all portfolio return distributions are assumed to be normal. Or more accurately, consumers are assumed to behave as if all portfolio return distributions are normal. But we show later that the major results also hold when the two-parameter portfolio return distributions are symmetric stable with characteristic exponent \( \alpha < 2 \).

III. RISK AND EXPECTED RETURN FROM THE VIEWPOINT OF A CONSUMER

A consumer comes to market at period 1 with resources, that is, his labor and shares of firms purchased in earlier periods, whose value will not be known until a set of equilibrium prices has been determined. When a market equilibrium is reached, the market value of the consumer’s resources or his wealth \( w_1 \) will be determined, and there will be an optimal allocation of \( w_1 \) between initial consumption \( c_1 \) and investment \( h_1 = w_1 - c_1 \). We show now how the risk of an asset to the consumer and the relationship between its risk and expected return can be obtained from what we already know about the properties of an optimal portfolio decision.

III.A. The Risks of Assets and Portfolios

The consumer invests at period 1 only to obtain consumption for period 2. His consumption for period 2 is the period 2 market value of his portfolio. Thus the consumer’s sole concern in his period 1 investment decision is the probability distribution on period 2 portfolio market value that he obtains. It is logical, then, that the risk of an individual asset to the consumer must be measured by its contribution to the risk of his portfolio.

With normally distributed portfolio returns, we know that the consumer finds it possible to summarize the distribution of the period 2 market value of any portfolio in terms of two parameters—the expected one-period return on the portfolio, \( E(R_p) \), and the standard deviation of the one-period return,
σ(\bar{R}_p). In this context, it is relevant to think of σ(\bar{R}_p) as a measure of the risk of the portfolio p. Thus the risk of an individual asset in the portfolio p is measured by its contribution to σ(\bar{R}_p). Likewise in looking at the expected return on an asset, the consumer is only concerned with how it contributes to the expected return on the portfolio.

The mean and standard deviation of the return on the portfolio p are

\[ E(\bar{R}_p) = \sum_{j=1}^{N} x_{jp} E(R_j), \quad (7.1a) \]
\[ \sigma(\bar{R}_p) = \left[ \sum_{i=1}^{N} \sum_{j=1}^{N} x_{ip} x_{jp} \sigma_{ij} \right]^{1/2}. \quad (7.1b) \]

Here \( \bar{R}_j \) is the return on asset \( j \), and the tilde again indicates that the return is a random variable; \( \sigma_{ij} = \text{cov}(\bar{R}_i, \bar{R}_j) \); and \( x_{jp} \) is the proportion of asset \( j \) in the portfolio p, so that

\[ \sum_{j=1}^{N} x_{jp} = 1. \]

The expected return on the portfolio p is just the weighted average of the expected returns on the individual assets in the portfolio. Thus the contribution of any asset to the expected portfolio return depends directly on the expected return on the asset.

Likewise, the standard deviation of the return on p is made up of a weighted average of the pairwise covariances between the returns on individual assets. To determine the contribution of an individual asset, say, asset \( i \), to σ(\bar{R}_p), it is helpful to break Equation (7.1b) down as follows:

\[ \sigma(\bar{R}_p) = \frac{\sigma^2(\bar{R}_p)}{\sigma(\bar{R}_p)} = \frac{\sum_{i=1}^{N} \sum_{j=1}^{N} x_{ip} x_{jp} \sigma_{ij}}{\sigma(\bar{R}_p)} = \sum_{i=1}^{N} x_{ip} \left( \frac{\sum_{j=1}^{N} x_{jp} \sigma_{ij}}{\sigma(\bar{R}_p)} \right). \quad (7.2) \]

\* This is a change in notation from Chap. 6, where \( x_j \) was used to represent the proportion of investment funds put into asset \( j \). The usefulness of the additional portfolio subscript in the more complicated models in the present chapter should soon become apparent.
Thus the contribution of asset \( i \) to the risk or standard deviation of the return on portfolio \( p \) depends directly on \( \sum_{j=1}^{N} x_{ij} \sigma_{ij} \), the weighted average of the pairwise covariances between the return on asset \( i \) and the return on each of the assets in the portfolio. If we measure the risk of asset \( i \) in the portfolio \( p \) as

\[
\frac{\sum_{j=1}^{N} x_{ij} \sigma_{ij}}{\sigma(\bar{R}_p)} \quad \text{(risk of asset } i \text{ in portfolio } p) \quad (7.3)
\]

then we see from expressions (7.2) and (7.3) that the portfolio's risk is just the weighted average of the risks of the individual assets.

Two points should be made here. First, note that when we take a portfolio viewpoint, the risk of an individual asset depends on the weights \( x_{ip} \) and on \( \sigma(\bar{R}_p) \), both of which vary from portfolio to portfolio. Thus the risk of an asset is not unique; it must always be measured with reference to some specific portfolio. Second, note that the risk of asset \( i \) depends on the variance of its return, \( \sigma^2(\bar{R}_i) = \sigma_{ii} \), and on the \( N-1 \) pairwise covariances of its return with the returns on other assets. Thus in a well-diversified portfolio, that is, a portfolio of many assets with no individual asset accounting for a large part of the total investment, the risk of an asset is likely to depend much more on these covariances than on the variance of the asset's return. But this result was already suggested in Chapter 6 (in the discussion of the "effects of diversification"), and we meet up with it again later.

The development of expressions (7.1) to (7.3) has given us a way to measure the risk of an individual asset. We now want to determine the relationship between risk and expected return that is relevant for a consumer. Again we see that the appropriate expected return–risk relationship can be obtained from what we already know about the properties of optimal portfolio decisions.

III.B. The Relationship between Risk and Expected Return

Because the consumer is assumed to be risk-averse and capable of summarizing distributions of one-period portfolio returns in terms of mean \( E(\bar{R}_p) \) and standard deviation \( \sigma(\bar{R}_p) \), we know that his optimal portfolio must be efficient. Suppose that in fact the situation is as shown in Figure 7.1; that is, the optimal portfolio is the efficient portfolio with expected return \( E(\bar{R}_e) \) and dispersion \( \sigma(\bar{R}_e) \). The fundamental idea in the analysis of risk from the viewpoint of the consumer is that because the consumer
holds the efficient portfolio $e$, the risk of any asset to him should be measured by its contribution to the total risk of $e$. We now show that for the consumer who chooses $e$ the appropriate relationship between risk and expected return is that implicit in the condition that $e$ is efficient. 

Reviewing briefly the results in Section IV.C in Chapter 6, the efficient portfolio $e$ is given by the solution to the problem: Choose the proportions $x_{jp}, j = 1, 2, \ldots, N,$ invested in the shares of each of the $N$ available firms that

$$\min \sigma(\tilde{R}_p) = \left[ \sum_{j=1}^{N} \sum_{i=1}^{N} x_{ijp} \sigma_{ij} \right]^{1/2} \quad (7.4a)$$

subject to the constraints

$$E(\tilde{R}_p) = \sum_{j=1}^{N} x_{jp} E(\tilde{R}_j) = E(\tilde{R}_e), \quad (7.4b)$$

$$\sum_{j=1}^{N} x_{jp} = 1. \quad (7.4c)$$

\footnote{The next three paragraphs unavoidably involve the use of partial derivatives. The mathematically wary reader should nevertheless be able to follow the logic of the analysis from the verbal explanations.}
The values of \( x_{ip} \) that provide the solution to this problem are \( x_{ip}, \ j = 1, 2, \ldots, N \), and the minimum value of \( \sigma(\bar{R}_p) \) is of course \( \sigma(\bar{R}) \). The \( x_{ip} \) must satisfy Equations (7.4b), (7.4c), and the “balance equations”

\[
E(\bar{R}_i) - E(\bar{R}_p) = S_e \left( \frac{\partial \sigma(\bar{R}_p)}{\partial x_{ip}} - \frac{\partial \sigma(\bar{R}_p)}{\partial x_{ip}} \right), \quad i, j = 1, 2, \ldots, N, \tag{7.5}
\]

where \( S_e \) is the slope of the efficient set at the point \( e \) in Figure 7.1 and \( \partial \sigma(\bar{R}_p)/\partial x_{ip} \) is \( \sigma(\bar{R}_p)/\partial x_{ip} \) evaluated at the optimal values \( x_{ip} = x_{ip}, \ j = 1, 2, \ldots, N \).\(^4\)

In words, the partial derivative \( \partial \sigma(\bar{R}_p)/\partial x_{ip} \) is the rate of change of the standard deviation of the portfolio return with respect to changes in the proportion of portfolio funds invested in asset \( j \). Equivalently, it is the marginal effect of asset \( j \) on the dispersion or risk of the portfolio return. Thus (7.5) is a balance equation in the sense that it tells us how the values of the \( x \)'s, the proportions invested in individual assets, must be chosen in order to balance properly the differences between the expected returns on assets and the differences between their marginal effects on the dispersion of the portfolio return.

It is, however, useful to transform Equation (7.5) into an expression involving the expected portfolio return \( E(\bar{R}_p) \) and standard deviation \( \sigma(\bar{R}_p) \). Thus first multiply both sides of Equation (7.5) by \( x_{ip} \) and then sum over \( i \) to obtain

\[
\sum_{i=1}^{N} x_{ip} [E(\bar{R}_i) - E(\bar{R}_p)] = S_e \left( \sum_{i=1}^{N} x_{ip} \frac{\partial \sigma(\bar{R}_p)}{\partial x_{ip}} - \sum_{i=1}^{N} x_{ip} \frac{\partial \sigma(\bar{R}_p)}{\partial x_{ip}} \right).
\]

Or because \( \sum_{i=1}^{N} x_{ip} = 1 \),

\[
E(\bar{R}_i) - E(\bar{R}_p) = S_e \left( \frac{\partial \sigma(\bar{R}_p)}{\partial x_{ip}} - \sum_{i=1}^{N} x_{ip} \frac{\partial \sigma(\bar{R}_p)}{\partial x_{ip}} \right). \tag{7.6}
\]

\(^4\) That is, to solve the problem in Equation (7.4) we form the lagrangian

\[
L = \sigma(\bar{R}_p) + \lambda_1 \left[ E(\bar{R}_i) - \sum_{i=1}^{N} x_{ip} E(\bar{R}_p) \right] + \lambda_2 \left[ 1 - \sum_{i=1}^{N} x_{ip} \right],
\]

differentiate partially with respect to \( \lambda_1, \lambda_2, \) and \( x_{ip}, \ j = 1, 2, \ldots, N \), and set these derivatives equal to 0, thus obtaining Equations (7.4b), (7.4c), and

\[
\frac{\partial L}{\partial x_{ip}} = \frac{\partial \sigma(\bar{R}_p)}{\partial x_{ip}} - \lambda_1 E(\bar{R}_p) - \lambda_2 = 0, \quad j = 1, 2, \ldots, N.
\]

Because the shadow price \( \lambda_1 \) is the same for all \( j \) and is in fact equal to \( 1/S_e \), this expression leads directly to Equation (7.5).
But note that, using Equation (7.1b) and the chain rule for differentiation,

\[
\frac{\partial \sigma (\overline{R}_p)}{\partial z_{i*}} = \frac{\partial \sigma (\overline{R}_p)}{\partial \sigma^2 (\overline{R}_p)} \frac{\partial \sigma^2 (\overline{R}_p)}{\partial z_{i*}} = \frac{\sum_{j=1}^{N} x_{ij} \sigma_{ij}}{\sigma (\overline{R}_p)}.
\]  

(7.7)

Thus the marginal effect of asset \( i \) on the standard deviation of the portfolio return is what we have already found to be the risk of asset \( i \), that is, its contribution to the risk (standard deviation) of the portfolio return. And, from Equation (7.2),

\[
\sum_{i=1}^{N} x_{i*} \frac{\partial \sigma (\overline{R}_p)}{\partial z_{i*}} = \sum_{i=1}^{N} \frac{x_{ij} \sigma_{ij}}{\sigma (\overline{R}_p)} = \sigma (\overline{R}_p).
\]  

(7.8)

Substituting Equation (7.8) into Equation (7.6), we obtain the new balance equation

\[
E(\overline{R}_j) - E(\overline{R}_p) = S_*(\frac{\partial \sigma (\overline{R}_p)}{\partial z_{j*}} - \sigma (\overline{R}_p)), \quad j = 1, 2, \ldots, N.
\]  

(7.9)

Or equivalently, using Equation (7.7),

\[
E(\overline{R}_j) - E(\overline{R}_p) = S_* \left( \frac{\sum_{i=1}^{N} x_{ij} \sigma_{ij}}{\sigma (\overline{R}_p)} - \sigma (\overline{R}_p) \right).
\]  

(7.10)

In words, to form the efficient portfolio with expected return \( E(\overline{R}_p) \), the proportions \( z_{i*} \) invested in individual assets must be such that the difference between the expected return on the asset and the expected return on the portfolio is proportional to the difference between the marginal effect of the asset on \( \sigma (\overline{R}_p) \) and the weighted average of these marginal effects, which weighted average is just \( \sigma (\overline{R}_p) \), and where the proportionality factor \( S_* \) is the slope of the efficient set at the point \( e \).

But the balance equation (7.10) can also be interpreted as the relevant relation between the risk of an asset and its expected return for a consumer who chooses the portfolio \( e \). In a market in which one-period returns on all portfolios are normal, a risk-averse consumer finds it possible to summarize the distribution of return for any portfolio in terms of two parameters, expected return \( E(\overline{R}_p) \) and standard deviation \( \sigma (\overline{R}_p) \). In this context it is valid to think of \( \sigma (\overline{R}_p) \) as measuring the risk of the distribution of \( \overline{R}_p \): For a given value of \( E(\overline{R}_p) \), the risk-averse consumer always prefers a lower
value of $\sigma(R_p)$ to a higher value. If his optimal portfolio is the efficient portfolio with expected return $E(R_e)$, then $\sigma(R_e)$ measures this portfolio's risk. And the efficiency condition on the portfolio $e$ given by Equation (7.10) can then be interpreted as the relationship between expected return and risk for an individual asset, measured relative to the efficient portfolio $e$; that is, the difference between the expected returns on an asset and on the portfolio is proportional to the difference between the risk of the asset and the risk of the portfolio.  

Before concluding this discussion of risk and the relationship between risk and expected return from the viewpoint of the consumer, we wish to make two points.

First, note that

$$\text{cov} (R_i, R_e) = \text{cov} (R_i, \sum_{j=1}^{N} x_j R_j) = \sum_{j=1}^{N} x_j \sigma_{ij}.$$  

Thus, using Equation (7.7),

$$\sum_{i=1}^{N} x_j \sigma_{ij} \frac{\partial \sigma(R_e)}{\partial x_i} = \text{cov} (R_i, R_e) \frac{\sigma(R_e)}{\sigma(R_e)} = \frac{\text{cov} (R_i, R_e)}{\sigma(R_e)}.$$  

(7.11)

In short, we already knew that the risk of an asset $i$ in the portfolio $e$ can be interpreted either as the contribution of the asset to the risk of the portfolio—the first term in Equation (7.7)—or as the marginal effect of the asset on the risk of the portfolio—the second term in Equation (7.7). From (7.11) we now know that $\text{cov} (R_i, R_e) / \sigma(R_e)$ is an equivalent measure of the asset's risk, which can now be interpreted as being directly determined by the relationship between the return on the asset and the return on the portfolio.

Finally, the risk of a portfolio is measured by the standard deviation of its return. Because the efficient set curve $bed$ shown in Figure 7.1 is non-

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1 A note for the more sophisticated: If there are nonnegativity restrictions on the $x$'s, Equation (7.10) applies only to assets that appear in the portfolio $e$ at a nonzero level. But we should not expect to be able to use the portfolio $e$ to measure the risks of assets that do not appear in $e$. Moreover, the risks of such assets are not really relevant to the consumer who chooses this portfolio.

2 Recall that $\text{cov} (R_i, R_e)$ is a measure of the relationship between the return on asset $i$ and the return on the portfolio $e$. For example, the correlation coefficient $\text{corr} (R_i, R_e)$ between $R_i$ and $R_e$ is

$$\text{corr} (R_i, R_e) = \frac{\text{cov} (R_i, R_e)}{\sigma(R_i) \sigma(R_e)}.$$
IV. RISK AND EXPECTED RETURN FOR THE MARKET

From the viewpoint of the individual risk-averse consumer—and he is, after all, the one who bears the risk of his chosen portfolio—the preceding analysis provides a complete description of appropriate procedures for measuring the risks of assets and the relationship between risk and expected return. The results of such an analysis, however, are likely to depend on the
particular consumer under consideration, because (1) the risk of an asset to the consumer is measured relative to the particular portfolio that is optimal for him and (2) his picture of the efficient set itself depends on subjective estimates of the parameters of the portfolio model that may be specific to him. But although risk is ultimately an individual matter, the goal of the asset-pricing model is to develop testable aggregate or market implications of the expected return–risk relations for individual consumers that arise naturally within the context of the two-parameter models of consumer equilibrium. We have the intuitive feeling that in a market of risk-averse investors, there must be some way of deriving expected return–risk relations that also have validity for the market as a whole.

IV.A. Homogeneous Expectations and Portfolio Opportunities

One approach to the problem of deriving meaningful aggregate relationships, and one that is common to most of the literature in this area, is to restrict the model further by assuming that expectations and portfolio opportunities are “homogeneous” throughout the market; that is, all consumers have the same set of portfolio opportunities, in terms of the assets that are available, and all view the probability distributions of returns associated with the various available portfolios in the same way. In this case when market equilibrium is attained, all consumers face the same picture of the efficient set, and the expected return–risk relations derived for any given efficient portfolio are relevant for all investors who choose this portfolio. Moreover, consumers agree on the expected return–risk relations that apply to any particular efficient portfolio. Thus it seems valid to say that in this situation there are expected return–risk relations that make sense at the level of the market.

But it is important to emphasize that the fact that consumers may face different pictures of the efficient set and may perceive risk-return relations differently in itself poses no problem for the determination of a market equilibrium. Market equilibrium simply requires a set of market-clearing prices; that is, equilibrium prices of shares must be such that when consumers make their consumption-investment decisions and firms make their production decisions, all desired exchanges of investment assets take place and the market is cleared of all outstanding units of all assets; that is, demand equals supply for each asset.

See Sharpe [7],Lintner [8,9], Mosin [10], and Fama [11]. It may be helpful to think of the homogeneous expectations assumption as a way of concentrating on the pure effects of “objective” uncertainty on the pricing of investment assets.
IV.B. The Role of a Riskless Asset

But unfortunately the model still lacks empirical content. To test an equation like (7.10), it is necessary to know the exact composition, that is, the values of the \( x_{i,n} \) of some efficient portfolio and the slope of the efficient set at the point corresponding to this portfolio. The assumption of homogeneous expectations and portfolio opportunities has not provided us, as uninvolved observers, with this information. One way out of this dilemma is to restrict the model even further by assuming that there is a riskless asset \( f \), that is, \( \sigma(R_f) = 0 \), and all consumers can borrow or lend at the riskless rate \( R_f \).

As we saw in Chapter 6, the presence of riskless borrowing-lending opportunities greatly simplifies the determination of the efficient set of portfolios. Consider portfolios of \( f \) and any risky asset or portfolio \( a \) that are formed according to

\[
R_p = xR_f + (1 - x)R_a, \quad x \leq 1. \tag{7.12}
\]

The expected value and standard deviation of \( R_p \) are

\[
E(R_p) = xR_f + (1 - x)E(R_a),
\]

\[
\sigma(R_p) = (1 - x)\sigma(R_a).
\]

Thus, as shown in Figure 7.3, the graph of \( E(R_p) \) against \( \sigma(R_p) \) for port-

![Figure 7.3 Portfolio Opportunities with Riskless Borrowing and Lending](image-url)
folios formed according to Equation (7.12) is a straight line from \( R_f \) through \( a \). Points between \( R_f \) and \( a \) on the line correspond to lending portfolios (that is, \( 0 \leq x \leq 1 \)), and points above \( a \) represent borrowing portfolios (\( x < 0 \)).

The efficient set in the presence of riskless borrowing-lending opportunities is now easily determined. Take a straight line from \( R_f \) and rotate it upward and to the left in Figure 7.3 until it can be moved no further without leaving the feasible set. This leads to the line from \( R_f \) through \( m \). Except for \( m \), points along \( bmd \) no longer represent efficient portfolios; at given levels of \( E(\bar{R}) \) there are portfolios along \( R_f/mz \) that provide lower levels of \( \sigma(\bar{R}) \). Portfolios along \( R_f/mz \) are formed according to

\[
\bar{R}_p = xR_f + (1-x)\bar{R}_m, \quad x \leq 1. \tag{7.13}
\]

Thus when it is possible both to borrow and lend at the rate \( R_f \), the only difference between any two efficient portfolios is in the proportion \( x \) invested in the riskless asset \( f \). More risky efficient portfolios—those above \( m \) on the efficient set line in Figure 7.3—involves borrowing (\( x < 0 \)) and investing all available funds, including borrowings, in the risky combination \( m \). Less risky portfolios—those along the line segment \( R_f/m \)—involve lending (\( 1 \geq x \geq 0 \)) some funds at \( R_f \) and investing remaining funds in \( m \). The particular portfolio that a consumer chooses depends on his attitudes toward risk and expected return, but optimum portfolios for all investors are just combinations of \( f \) and \( m \). There is no incentive for anyone to hold risky assets not included in \( m \). Thus if Figure 7.3 is to represent market equilibrium, \( m \) must be the market portfolio; that is, \( m \) consists of all assets in the market, each entering the portfolio with weight equal to the ratio of its total market value to the total market value of all assets.\(^*\) In addition, the riskless rate \( R_f \) must be such that net borrowings are 0; that is, at the rate \( R_f \), the total quantity of funds that people want to borrow is equal to the quantity that others want to lend.

The market portfolio \( m \) is efficient; thus the balance equation (7.9) must hold for \( m \), so that

\[
E(\bar{R}_j) - E(\bar{R}_m) = S_m \left( \frac{\partial \sigma(\bar{R}_m)}{\partial x_{jm}} - \sigma(\bar{R}_m) \right), \quad j = 1, 2, \ldots, N. \tag{7.14}
\]

Noting from Figure 7.3 that \( S_m = [E(\bar{R}_m) - R_f]/\sigma(\bar{R}_m) \), with some

\(^*\) That is, for \( j = 1, 2, \ldots, N \), in the market portfolio \( m \),

\[
x_{jm} = \frac{\text{Total market value of all outstanding shares in firm } j}{\text{Total market value of all outstanding shares of all firms}}.
\]
rearrangement of terms, Equation (7.14) can be rewritten

\[ E(R_j) = R_f + \left[ \frac{E(R_m) - R_f}{\sigma(R_m)} \right] \frac{\partial \sigma(R_m)}{\partial x_{jm}}, \quad j = 1, 2, \ldots, N. \quad (7.15) \]

But this is more than a balance equation that helps determine the proportions \( x_{jm}, j = 1, 2, \ldots, N \), necessary to form the efficient portfolio with expected return \( E(R_m) \). There is no question of choosing the optimal proportions invested in each asset in forming this particular efficient portfolio; with homogeneous expectations and riskless borrowing-lending opportunities, in equilibrium market prices and expected returns on individual shares must be such that the optimal proportions are those associated with the market portfolio \( m \). Thus in this case Equation (7.15) represents a condition on market prices and expected returns that must be met in equilibrium.

Moreover, the only risky assets that any consumer holds are shares in the market portfolio \( m \). Thus it seems that for everyone who is actively in the market, the risk of an individual asset \( j \) is appropriately measured by \( \frac{\partial \sigma(R_m)}{\partial x_{jm}} \), the marginal effect of the asset on the dispersion in the distribution of \( R_m \), and Equation (7.15) is the appropriate relationship between risk and expected return. In this sense, then, Equation (7.15) is an expected return-risk relation for the market as a whole. Thus the assumptions of homogeneous expectations and riskless borrowing-lending opportunities have carried us to our goal: We have a picture of market equilibrium that implies a measure of risk for individual assets and a relationship between risk and equilibrium expected return that are relevant at both the level of the individual consumer and that of the market.

IV.C. Interpretation

These are the fundamental results of the present chapter, and it is useful to discuss them a bit more. We first reconsider briefly the measurement of risk and the relationship between risk and expected return. Then we examine the two key assumptions (1) that consumers have homogeneous expectations and (2) that there are riskless borrowing-lending opportunities.

IV.C.1. Risk and expected return in the market portfolio

From Equation (7.11) we know that

\[ \frac{\partial \sigma(R_m)}{\partial x_{jm}} = \frac{\sum_{i=1}^{N} x_{im} \sigma_{ij}}{\sigma(R_m)}; \]
that is, for any portfolio, \( \frac{\partial \sigma(R_m)}{\partial x_{jm}} \), the marginal effect of asset \( j \) on the standard deviation of the return on the market portfolio \( m \), is also the contribution of asset \( j \) to the standard deviation or total risk of the return on \( m \). And that is why we say that it measures the risk of \( j \) in \( m \). Moreover, in the view of market equilibrium presented here, the only risky assets that consumers hold are shares in \( m \); thus \( \frac{\partial \sigma(R_m)}{\partial x_{jm}} \) measures the risk of \( j \) for any consumer and the market as a whole.

It is interesting to examine the role that the variance of the one-period return on asset \( j \) plays in the determination of its risk and thus of the risk premium in its expected one-period return. The risk of asset \( j \) in the market portfolio \( m \) can also be written

\[
\frac{\partial \sigma(R_m)}{\partial x_{jm}} = \frac{\sum_{k=1}^{N} x_{km} \sigma_{kj}}{\sigma(R_m)} = \frac{x_{jm} \sigma^2(R_j) + \sum_{k \neq j} x_{km} \sigma_{kj}}{\sigma(R_m)}.
\]

Thus \( \sigma^2(R_j) = \sigma_{jj} \) is just one of the \( N \) terms that determine the risk of asset \( j \), and \( N \), the total number of assets in the market, is of course quite large. Moreover, recall that \( x_{jm} \), the weight applied to the variance term, is the total market value of all outstanding units of asset \( j \), divided by the total market value of all outstanding units of all assets, so that \( x_{jm} \) is likely to be very close to 0. Thus for most assets the variance term in the asset’s risk is likely to be trivial relative to the weighted sum of covariances.

But such a result is hardly surprising. In Chapter 6 we showed that in a diversified portfolio the contribution of an individual asset to the standard deviation of the portfolio return depends almost entirely on the average covariance between the return on the asset and the returns on other assets in the portfolio and depends very little on the variance of the asset’s return. In essence, the results of the present chapter concerning expected return–risk relationships simply say that in equilibrium an asset’s risk is measured from a portfolio point of view.

Finally, from Equation (7.11) we know that there are in fact three equivalent ways of representing the risk of asset \( j \) in the market portfolio \( m \):

\[
\frac{\partial \sigma(R_m)}{\partial x_{jm}} = \frac{\sum_{k=1}^{N} x_{km} \sigma_{kj}}{\sigma(R_m)} = \frac{\text{cov}(R_j, R_m)}{\sigma(R_m)}; \tag{7.16}
\]

that is, the risk of asset \( j \) is directly determined by the covariance of its return with the return on \( m \), and as usual, this covariance measures the relationship between the returns on \( j \) and \( m \). With Equation (7.16),
Figure 7.4  Relationship between Expected Return on an Asset and Its Risk When There Is Riskless Borrowing and Lending

Equation (7.15) can be rewritten

\[ E(R_j) = R_f + \left[ \frac{E(R_m) - R_f}{\sigma(R_m)} \right] \frac{\text{cov}(R_j, R_m)}{\sigma(R_m)} , \quad j = 1, 2, \ldots, N; \quad (7.17) \]

that is, the equilibrium expected return on any asset \( j \) is the riskless rate of interest plus a risk premium that is proportional to the asset's risk. The proportionality factor \( S_m = [E(R_m) - R_f]/\sigma(R_m) \) is the same for all assets, and it seems appropriate to interpret this factor as the market price per unit of risk.

Equations (7.17) and (7.15) give us a linear relationship between the expected return on an asset and its risk that might, for example, be as shown in Figure 7.4. It is important to recognize, however, that in two-parameter models the linearity of the relationship between the risk of an asset and its expected return does not depend on either homogeneous expectations or riskless borrowing-lending. Thus, as illustrated in Figure 7.2, Equation (7.10) is a linear expected return–risk relation, derived without either of these assumptions, that is relevant for an investor who chooses the efficient portfolio \( e \). Rather, the role of homogeneous expectations and riskless borrowing-lending is to give a relationship between risk and expected return that is relevant for every investor and so is unambiguously interpreted as a market condition. We now examine each of these assumptions in a little more detail.

IV.C.2. Homogeneous expectations and the riskless asset: a closer look

We first reemphasize that the assumption of homogeneous expectations and portfolio opportunities plays no essential role in either the determination
of a market equilibrium or the analysis of the details of consumer equilibrium. A market equilibrium requires a set of asset prices such that all desired exchanges of assets can take place and the market is cleared of all outstanding units of all assets. In short, market equilibrium simply requires a set of asset prices such that supply equals demand for every asset.

Given a set of market-clearing prices, it does not matter to the consumer whether the picture of the efficient set facing him is based on expectations, that is, parameter estimates, that he has in common with everybody in the market or whether the picture results from his personal estimates of expected returns, variances, and covariances on the assets and portfolios available to him. In either case he goes about making an optimal consumption-investment decision in the same way, and as long as he is risk-averse and assumes that portfolio return distributions are normal, his optimal portfolio is always some mean-standard deviation efficient portfolio.

Moreover, once he chooses some efficient portfolio \( e \), from his point of view Equation (7.10) is the relevant relationship between expected return and risk; that is, the fact that the means, variances, and covariances in this expression are personal estimates rather than market expectations does not change the fact that the risk of an individual asset \( j \) is measured by his estimate of its effect on the dispersion of the distribution of the return on his portfolio.

In essence, the risk of an individual asset to an investor is always measured relative to the portfolio that he holds, regardless of whether his expectations or his portfolio opportunities are the same as everyone else's. The assumption that expectations and portfolio opportunities are homogeneous throughout the market just simplifies the analysis in that one picture of the efficient set represents market equilibrium for all investors.

With homogeneous expectations and portfolio opportunities, all consumers can agree on the relationship between the expected return on an asset and its risk, measured vis-à-vis any particular efficient portfolio \( e \). But the efficient set is continuous, so that in the absence of a riskless asset, for any given asset \( j \) there are an infinite number of risk measures and expected return–risk relationships, one for each different efficient portfolio; that is, for a given asset \( j \), Equation (7.10) must hold for every efficient portfolio. The expected return \( E(\bar{R}_e) \) on asset \( j \) and the set of covariances \( \sigma_{ij}, i = 1, 2, \ldots, N \), are the same from portfolio to portfolio, but the set of weights \( w_i, i = 1, 2, \ldots, N \), is different, as are the expected return \( E(\bar{R}_e) \), the standard deviation \( \sigma(\bar{R}_e) \), and \( S_e \), the slope of the efficient set at the point \( e \).

Thus in the absence of opportunities to borrow and lend at a riskless rate, the concept of an expected return–risk relationship for the market as a
whole is not unambiguous; in this case there are many different efficient portfolios of risky assets and thus many different equilibrium expected return–risk relationships for any given asset. From the viewpoint of an individual investor it is sensible to talk about the risk of an asset and the equilibrium relationship between risk and expected return; with convex indifference curves, his choice of an optimal efficient portfolio is unique. From the viewpoint of the market, however, all we can say is that in equilibrium the trade-off of $E(R)$ for $\sigma(R)$ along the efficient set must be such that when all consumers make optimal portfolio decisions, the market is cleared of all outstanding units of all assets.

With both homogeneous expectations and riskless borrowing-lending, however, in equilibrium the only risky assets held by any investor are shares in the market portfolio $m$, so that Equation (7.17) is a relationship between expected return and risk that has unambiguous meaning at the level of the market. Thus it seems clear that the characteristics of consumer and market equilibrium when there is riskless borrowing-lending have much appeal, if only for their simplicity. Nevertheless, admittedly at least the assumption of indefinite riskless borrowing opportunities is somewhat artificial; the borrowings must be repaid from risky portfolio investments, so that, strictly speaking, there is always some chance of at least partial default.

But one must keep the proper perspective. The goal of the asset-pricing model is to develop testable aggregate or market implications of the expected return–risk relations for individual consumers that arise naturally within the context of the two-parameter models of consumer equilibrium. The general idea is that if consumers are trying to choose efficient portfolios, the pricing of investment assets must somehow reflect expected return–risk relations that are appropriate when assets are viewed as components of efficient portfolios. The assumptions of homogeneous expectations and riskless borrowing-lending opportunities are just one specific and straightforward way to give precise form to this general idea. As in any economic model, the value of such assumptions cannot be judged in the abstract but rather depends only on how well the testable implications of the assumptions stand up in the future to the market data that they are meant to explain.

Moreover, if all the implications of this specific model of market equilibrium are not upheld in empirical tests, it does not follow that there is no other model of market equilibrium based on two-parameter portfolio models that can provide a more adequate explanation of the data. For example, intuition tells us that if the pricing of investment assets is in line with portfolio considerations, the market portfolio must be a member of the efficient set. Thus even if we do not assume that there is riskless borrow-
ing and lending, Equation (7.14) is nevertheless a relevant representation of the relationship between risk and expected return. In the absence of riskless borrowing and lending, however, the expected return–risk line given by Equation (7.14) is not as shown in Figure 7.4; that is, because there is no riskless rate, we cannot interpret the intercept and slope of the line in terms of such a rate.

In most general terms, all that we ultimately expect from two-parameter models of market equilibrium is vindication for the two major implications of the two-parameter portfolio models. First, if people are generally risk-averse and make portfolio decisions according to the two-parameter model, then along the set of efficient portfolios, and presuming that we had some way to identify efficient portfolios, we should on average observe a positive trade-off of risk for return. Second, prices of individual investment assets should reflect the presumed attempts of investors to hold efficient portfolios; that is, in any given efficient portfolio we should on average observe a linear relationship between the risk of an asset and its return, where the asset's risk is measured by its contribution to the total risk in the return on the portfolio.

V. EQUILIBRIUM EXPECTED RETURN AND THE MARKET VALUE OF A FIRM

The geometric exposition of the model of market equilibrium is most conveniently carried out in terms of one-period returns on assets and portfolios. In the return form of the model, with homogeneous expectations and riskless borrowing-lending, everyone faces the same picture of the portfolio opportunity set, and this is especially useful in the derivation of an equilibrium relationship between expected return and risk that has clear-cut meaning for the market as a whole.

But of course what are determined in the capital market at period 1 are the prices of investment assets, that is, the prices of the shares of firms. It will now be shown that the expected return–risk relation (7.17) is easily transformed into a pricing equation that relates $P_j$, the equilibrium market value of firm $j$ at period 1, to its expected period 2 value $E(\bar{V}_j)$ and to the "risk" of $ar{V}_j$.

Because firms are assumed to issue only shares, the return $\bar{R}_j$ and expected return $E(\bar{R}_j)$ on the shares of firm $j$ are

$$\bar{R}_j = \frac{\bar{V}_j - P_j}{P_j} \quad \text{and} \quad E(\bar{R}_j) = \frac{E(\bar{V}_j) - P_j}{P_j}.$$  \hspace{1cm} (7.18)

Thus

$$\bar{V}_j = P_j(1 + \bar{R}_j).$$
Likewise, $P_m$ and $\bar{V}_m$, the market values at periods 1 and 2 of the market portfolio or "market wealth" are

$$P_m = \sum_{j=1}^{N} P_j \quad \text{and} \quad \bar{V}_m = \sum_{j=1}^{N} \bar{V}_j.$$  

And the relationship between $P_m$ and $\bar{V}_m$ is

$$\bar{V}_m = P_m(1 + \bar{R}_m). \quad (7.19)$$

With these expressions, the covariance between the period 2 values $\bar{V}_j$ and $\bar{V}_m$ can be obtained from the covariance between the one-period returns $\bar{R}_j$ and $\bar{R}_m$ as follows:

$$\text{cov}(\bar{V}_j, \bar{V}_m) = \text{cov}[P_j(1 + \bar{R}_j), P_m(1 + \bar{R}_m)]$$

$$= P_j P_m \text{cov}(\bar{R}_j, \bar{R}_m). \quad (7.20)$$

And, from Equation (7.19),

$$\sigma(\bar{V}_m) = P_m \sigma(\bar{R}_m). \quad (7.21)$$

Finally, from Equations (7.20) and (7.21),

$$\frac{\text{cov}(\bar{R}_j, \bar{R}_m)}{\sigma(\bar{R}_m)} = \frac{\text{cov}(\bar{V}_j, \bar{V}_m)}{P_j \sigma(\bar{V}_m)}. \quad (7.22)$$

The equilibrium market value $P_j$ implied by the equilibrium expected return–risk relation (7.17) is now easily obtained. Substituting Equations (7.18) and (7.22) into (7.17), we get

$$\frac{E(\bar{V}_j) - P_j}{P_j} = R_f + \left[\frac{E(\bar{R}_m) - R_f}{\sigma(\bar{R}_m)}\right] \frac{\text{cov}(\bar{V}_j, \bar{V}_m)}{P_j \sigma(\bar{V}_m)}.$$  

This in turn implies the pricing equation

$$P_j = \frac{E(\bar{V}_j) - S_m[\text{cov}(\bar{V}_j, \bar{V}_m)/\sigma(\bar{V}_m)]}{1 + R_f}, \quad (7.23)$$

where $S_m = [E(\bar{R}_m) - R_f]/\sigma(\bar{R}_m)$, or equivalently, $S_m = [E(\bar{V}_m) - P_m(1 + R_f)]/\sigma(\bar{V}_m)$.

To interpret Equation (7.23), first note that

$$\frac{\text{cov}(\bar{V}_j, \bar{V}_m)}{\sigma(\bar{V}_m)} = \frac{\text{cov}(\bar{V}_j, \sum_{k=1}^{N} \bar{V}_k)}{\sigma(\bar{V}_m)} = \frac{\sum_{k=1}^{N} \text{cov}(\bar{V}_j, \bar{V}_k)}{\sigma(\bar{V}_m)},$$
so that
\[ \sigma(\bar{V}_m) = \frac{\sum_{j=1}^{N} \sum_{k=1}^{N} \text{cov}(\bar{V}_j, \bar{V}_k)}{\sigma(\bar{V}_m)} = \sum_{j=1}^{N} \frac{\text{cov}(\bar{V}_j, \bar{V}_m)}{\sigma(\bar{V}_m)}. \tag{7.24}\]

Thus \( \text{cov}(\bar{V}_j, \bar{V}_m)/\sigma(\bar{V}_m) \) is the contribution of the period 1 production decision of firm \( j \) to \( \sigma(\bar{V}_m) \), the standard deviation of aggregate period 2 wealth. If we interpret \( \sigma(\bar{V}_m) \) as aggregate risk, then \( \text{cov}(\bar{V}_j, \bar{V}_m)/\sigma(\bar{V}_m) \) measures the risk of firm \( j \).

Next note that, using Equation (7.24) and the fact that \( E(\bar{V}_m) = \sum_{j=1}^{N} E(\bar{V}_j) \), if we sum over \( j \) in Equation (7.23), we get
\[
P_m = \frac{E(\bar{V}_m) - S_m \sigma(\bar{V}_m)}{1 + R_f}. \tag{7.25}\]

Thus in the two-parameter model, two dimensions, the mean and standard deviation of total period 2 wealth, are priced in the capital market at period 1. The period 1 price of a unit of mean is \( 1/(1 + R_f) \); the price of a unit of standard deviation is \( -S_m/(1 + R_f) \). And from Equation (7.23), the period 1 market value of firm \( j \) is just the sum of the market values of (1) the contribution, \( E(\bar{V}_j) \), of the firm's period 1 production decision to expected period 2 market wealth and (2) \( \text{cov}(\bar{V}_j, \bar{V}_m)/\sigma(\bar{V}_m) \), which is the contribution of the firm to the standard deviation or risk of aggregate period 2 wealth.

To look at these results in a slightly different way, note that the means, covariances, and standard deviations in the pricing equations (7.23) and (7.25) result entirely from the period 1 production decisions of firms. Given the production decisions of firms, all the capital market determines at period 1 are the two prices \( 1/(1 + R_f) \) and \( -S_m/(1 + R_f) \), from which, by way of Equation (7.23), the market values of all firms are obtained.

In aggregate, then, the objects being cleared from the capital market at period 1 are total expected market wealth and risk, \( E(\bar{V}_m) \) and \( \sigma(\bar{V}_m) \). And the units that firm \( j \) contributes to both are indistinguishable from those contributed by other firms. This shows up clearly in the pricing equation (7.23), where the market price per unit of expected value, \( 1/(1 + R_f) \), and the market price per unit of risk, \( -S_m/(1 + R_f) \), are the same for all firms. The important implication of all this is that in the two-parameter model no firm is unique in terms of the objects that it brings to the capital market for sale.

This does not mean, of course, that the consumer has no incentive to diversify. Rather, the essence of the model is that risk itself is always measured relative to an efficient portfolio. It is important always to keep in
mind that all the expected return–risk results of the present chapter follow from the fact that in the two-parameter portfolio model the risk-averse consumer’s optimal portfolio is efficient.

Finally, it is a simple matter to convert Equation (7.23) for the equilibrium period 1 market value of firm \( j \) into a pricing equation for individual shares. Let \( n_j \) be the number of shares outstanding in firm \( j \), let \( p_j \) be the period 1 price of a share, and let \( \theta_j \) be its period 2 value. Then

\[
P_j = n_j \theta_j,
\]

\[
E(\bar{V}_j) = n_j E(\theta_j),
\]

\[
\text{cov} (\bar{V}_j, \bar{V}_m) = n_j \text{cov} (\theta_j, \bar{V}_m),
\]

so that, from Equation (7.23), we easily obtain

\[
p_j = \frac{E(\theta_j) - S_m [\text{cov} (\theta_j, \bar{V}_m) / \sigma(\bar{V}_m)]}{1 + R_f}.
\]  

(7.26)

We could interpret Equation (7.26) in the same terms as Equations (7.23) and (7.25). Rather than going over old ground, however, let us take a slightly different approach, which likewise applies equally well to Equations (7.23) and (7.25). In particular, note that \( p_j (1 + R_f) \) is the market value for certain at period 2 of \( p_j \) dollars invested in the riskless asset at period 1. Thus the numerator in Equation (7.26), which is equal to \( p_j (1 + R_f) \), is the certainty equivalent of \( \theta_j \); it is the expected market value \( E(\theta_j) \) less an adjustment for the risk of \( \theta_j \). Cov \((\theta_j, \bar{V}_m)\) is indeed the appropriate measure of the risk of a share of firm \( j \), because it measures the contribution of a share to \( \sigma(\bar{V}_m) \); that is, \( \sigma(\bar{V}_m) = \sum_{j=1}^{N} n_j \text{cov} (\theta_j, \bar{V}_m) / \sigma(\bar{V}_m) \).

The period 1 price of a share is thus the present value, computed at the riskless rate, of the certainty equivalent of the distribution of the share’s period 2 value.

It is well to note that we now have at least three comparable measures of the risk of a firm: \( \text{cov} (\bar{R}_j, \bar{R}_m) / \sigma(\bar{R}_m) \), \( \text{cov} (\theta_j, \bar{V}_m) / \sigma(\bar{V}_m) \), and \( \text{cov} (\bar{V}_j, \bar{V}_m) / \sigma(\bar{V}_m) \). The first measures the contribution of firm \( j \) to the risk or standard deviation of the rate of return on the market portfolio and so can be interpreted as the risk of the rate of return on firm \( j \) and on its shares. The second measures the contribution of a share in firm \( j \) to \( \sigma(\bar{V}_m) \), the risk of period 2 market wealth, and so can be interpreted as the risk of the period 2 value of a share. Likewise \( \text{cov} (\bar{V}_j, \bar{V}_m) / \sigma(\bar{V}_m) \) can be interpreted as the risk of the period 2 market value of firm \( j \) or more simply as the total risk of the firm.
VI. OPTIMAL PRODUCTION DECISIONS BY FIRMS

Two of the three steps in the analysis have now been completed. We examined the nature of optimal consumption-investment decisions by consumers, given a market equilibrium. Then we considered the implications of consumer decisions for the risk structure of equilibrium expected returns in the market. The final step is to investigate the implications of optimal consumer decisions and the risk structure of equilibrium expected returns for the characteristics of optimal production or, if you like, investment decisions by firms. In essence, we are now concerned with the fact that outstanding quantities of investment assets (shares in firms) as well as their prices must be determined in the market at the beginning of period 1.

But before proceeding with our analysis of the firm, it is well to note again that, although of necessity we examine separately the characteristics of equilibrium from the viewpoints of the consumer and of the firm, the reader should keep in mind that the production decisions of firms, the consumption-investment decisions of consumers, and the structure of market prices for investment assets are all determined simultaneously in the market at period 1. All these are implied as soon as we say that a market-clearing set of prices has been attained. In the partial equilibrium analyses of consumers and firms, we are looking at those parts of the market or general equilibrium which represent the positions of these microentities.

The first step in the analysis of the firm is to discuss the firm's objective function (the goal of its production decisions). Then this objective function is combined with the results of the asset-pricing model to produce simple production decision rules. The assumptions of homogeneous expectations and riskless borrowing-lending opportunities are maintained throughout.

VIA. The Firm's Objective Function

At the beginning of period 1 the firm is faced with a production opportunity set, to be defined more precisely below, and it must decide on a production plan that is in the best interests of the shareholders. But this implies two further problems: (1) Which shareholders should the firm be concerned with, current or prospective, and (2) what are the best interests of the shareholders?

There is really nothing that the firm can do for prospective shareholders. When the portfolio decisions of consumers are executed at the beginning of period 1, anyone who decides to hold the shares of the firm during the period pays the equilibrium price and receives the equilibrium trade-off between risk and expected return. Thus when we speak of shareholder interests in
the firm's production decision, we refer to those who hold the firm's shares before the market is cleared at the beginning of period 1. But what are the best interests of these shareholders as far as the activities of the firm are concerned?

The shares of firms that the consumer holds before his current portfolio decisions are executed are just part of the resources that he brings to the market. In addition, the assumption of perfect commodity and capital markets presumes that no firm is large enough to have an effect on the opportunity set, that is, the types of goods and assets available and their prices, facing consumers.\textsuperscript{10} Thus given that there are no transactional costs and the only consideration in his portfolio decision is maximum expected utility, the choice of optimum portfolio is independent of the assets that happen to comprise the investor's initial resources. It follows that of the factors—initial wealth, utility function, and opportunity set—involved in the consumption-investment decision, the only one that the firm can affect with its production decision is \( w_0 \), the market value of initial wealth. Here the best interests of the consumer are clear—more initial wealth is better than less.\textsuperscript{11} Or in other words, the firm should choose the production plan that maximizes the wealth of those who come to market with its old shares at the beginning of period 1. And this is equivalent to maximizing \( P_j - I_j \), where \( P_j \) is the market value of firm \( j \) at period 1 and \( I_j \) is the "investment" that the firm must make to generate the market value \( P_j \).\textsuperscript{12}

We emphasize, however, that the maximize shareholder wealth rule is critically dependent on the perfect market assumption that the firm's production decisions do not affect the equilibrium picture of the efficient

\textsuperscript{10} For the more sophisticated, we should note that among financial economists the conditions under which the capital market of the two-parameter model can be perfect in this sense have been a subject of substantial debate. Because the issues are rather esoteric, we do not go into them here. The interested reader is referred to Ref. 13.

\textsuperscript{11} A more formal proof is as follows. If the individual's initial wealth is increased, one possible, although probably suboptimal, strategy is to use the additional wealth to increase consumption in period 1, leaving the portfolio decision unchanged. Because the marginal expected utility of \( c_0 \) is positive, the expected utility for such a strategy must be higher than that associated with the consumption-investment decision that is optimal for the lower level of wealth. Thus an optimal reallocation at the higher wealth level must do at least as well.

\textsuperscript{12} In the simplest case the firm just issues new shares at period 1 in the amount \( I_j \), so that \( P_j - I_j \) is the value of its old shares. But the reader who understood the discussion of the effects of financing decisions in Chap. 4 will realize that the simplest case is also the general case; that is, \( P_j - I_j \) is the contribution of the production decision to the wealth of the old shareholders regardless of how the production decision is financed.

The reader should also recognize that this discussion of the firm's objective function is just an application of the general discussion in Chap. 4 to the specific market context of the two-parameter model.
set. If the firm's activities affect the combinations of \( E(\bar{R}) \) and \( \sigma(\bar{R}) \) that are available in equilibrium, it affects its shareholders' expected utilities by its effects on their period 1 wealth levels and on the portfolio opportunity set. In principle, both would have to be considered in an optimal production decision, although in practice it is not at all clear how this would be done.

For example, if we dropped the assumption of homogeneous expectations, it would be easy to imagine situations in which some shareholders, who are more optimistic about the investment prospects of the firm than the market as a whole, may want the firm to invest more, although this causes the period 1 market value of their shares to be lower. For these shareholders, the expansion of their perceived opportunity sets, that is, the combinations of \( E(\bar{R}) \) and \( \sigma(\bar{R}) \) that they perceive to be available, can be more than sufficient to compensate for the loss in wealth that they suffer if the firm invests more than is implied by value maximization.\(^{13}\)

With perfect markets and homogeneous expectations, however, the maximize shareholder wealth rule has unambiguous meaning, and we now examine its implications in a little more detail.

VI.B. Optimal Production Decisions:  
Single-Activity Firms

Given equilibrium values of the capital market prices \( R_f \) and \( S_m \), the most direct way for firm \( f \) to proceed in maximizing the excess of value over cost, \( P_f - I_f \), is to make use of the pricing equation (7.23); that is, the firm looks at the outlays \( I_f \) at period 1 that are required in order to generate different combinations of expected period 2 value \( E(\bar{V}_f) \) and risk \( \text{cov}(\bar{V}_f, \bar{V}_m) / \sigma(\bar{V}_m) \) and then uses Equation (7.23) to determine the period 1 production decision that maximizes \( P_f - I_f \).

But although this is the most generally valid description of the path to optimal production decisions, we promised in Chapter 4 eventually to give a less restrictive discussion of the concepts of "risk class" and "cost of capital" than provided in that chapter. Thus we now see what meaning we can give these concepts within the context of the two-parameter model.

For simplicity we initially assume that the firm is only engaged in one "type" of activity, so that its production decision involves choosing the optimal scale of operations. What we mean when we say that the firm is

\(^{13}\) A note for the more sophisticated: The economics literature in general has little to say about the problem of optimal decision making when members of the group affected by the decision disagree about the characteristics of the opportunity set. It is surprising, then, that nonfinance economists tend to slight the finance literature for the apparent dependence of its uncertainty models on the assumption of homogeneous expectations. In fact, almost all uncertainty models in economics take this condition as given. It is just that financial economists are more careful to make the assumption explicit.
engaged in one type of activity is that the value of $\text{cov} (R_j, R_m)/\sigma(R_m)$, the risk of the return on its shares, is the same for any production decision that it might make.\(^4\) From Equation (7.17) this assumption implies that the expected return on the market value of the firm is the same for all production decisions; that is, the equilibrium expected return on the market value of the firm is

$$E(\bar{R}_j) = R_f + \left[ \frac{E(R_m) - R_f}{\sigma(R_m)} \right] \frac{\text{cov} (R_j, R_m)}{\sigma(R_m)}. \quad (7.17)$$

Thus if $\text{cov} (R_j, R_m)/\sigma(R_m)$ is the same for all production decisions, $E(\bar{R}_j)$ is the same for all production decisions.

If $E(\bar{V}_j)$ is the expected market value of firm $j$ at period 2, in equilibrium the market value of the firm at period 1, $P_j$, must be such that

$$\frac{E(\bar{V}_j) - P_j}{P_j} = E(\bar{R}_j).$$

Or equivalently

$$P_j = \frac{E(\bar{V}_j)}{1 + E(\bar{R}_j)}, \quad (7.27)$$

where the value of $E(\bar{R}_j)$ is given by Equation (7.17).

If $I_j$ is the investment at period 1 that is required in order to generate the expected market value $E(\bar{V}_j)$, the one-period expected return on $I_j$ is the value of $\rho^*_f$ that satisfies

$$I_j = \frac{E(\bar{V}_j)}{1 + \rho^*_f}. \quad (7.28)$$

It follows easily from Equations (7.27) and (7.28) that additional investment will contribute positively to $P_j - I_j$ as long as

$$\rho^*_f > E(\bar{R}_j). \quad (7.29)$$

\(^4\) The more sophisticated reader should be able to show that the assumption that $\text{cov} (R_j, R_m)/\sigma(R_m)$ is the same for all production decisions implies that the firm produces mean and risk in fixed proportions; that is, the ratio of the firm's total risk to its expected period 2 value,

$$\frac{\text{cov} (\bar{V}_j, \bar{V}_m)/\sigma(\bar{V}_m)}{E(\bar{V}_j)},$$

is the same for all possible values of $E(\bar{V}_j)$. Hint: The result can be obtained from Equations (7.22) and (7.23).
Thus the equilibrium expected return $E(\bar{R}_j)$, as defined by Equation (7.17), can be interpreted as the minimum required expected one-period return or cost of capital for the firm's production activities, which have the risk level implied by $\text{cov}(\bar{R}_j, \bar{R}_m)/\sigma(\bar{R}_m)$. And it is clear that the value of $\text{cov}(\bar{R}_j, \bar{R}_m)/\sigma(\bar{R}_m)$ determines the firm's risk class in the sense that two firms with the same value of this risk parameter have the same equilibrium expected return and thus the same cost of capital.

We leave it to the reader to show that this definition of a risk class is much less restrictive than that in Chapter 4, where in order to be in the same risk class, the period 2 market values of two firms were assumed to be perfectly correlated and, per dollar of expected value, the distributions of the period 2 values of the firms were assumed to be identical. At this point, the reader should also be able to convince himself that the less restrictive definition of a risk class obtained in the present chapter is the consequence of a more detailed specification of the market context than provided in the general models in Chapter 4.

The production condition in expression (7.29) is easily given a familiar geometric interpretation. Let the boundary of the set of production activities available to firm $j$ be expressed by the transformation function

$$T(E(\bar{V}_j), I_j) = 0,$$

which tells the maximum amounts of expected market value at the beginning of period 2, $E(\bar{V}_j)$, that can be obtained with different amounts of investment $I_j$ at the beginning of period 1. For simplicity assume that $E(\bar{V}_j)$ is a strictly concave function of $I_j$. Thus if the investment $I_j$ is considered an outlay $-I_j$, the transformation function looks something like the curve $OT$ in Figure 7.5. $P_j - I_j$ is maximized by pushing investment $I_j$ to the point at which the slope of the transformation curve is $-1 - E(\bar{R}_j)/I_j$, that is, where the marginal expected one-period return on investment is equal to

![Figure 7.5 Optimal Production Decisions for a Single-Activity Firm](image-url)
$E(R_j)$, the cost of capital for the firm's productive activities. The optimal investment is $I^*_j$, which leads to expected market value $E(\bar{V}_j)^*$ at period 2, which in turn has current market value $P_j^* = E(\bar{V}_j)^*/[1 + E(R_j)]$. The picture is obviously quite similar to that obtained in Chapter 2 in the analysis of optimal production-investment decisions in a world of perfect certainty and perfect capital markets (see Figure 2.7).

VII.C. Optimal Production Decisions: Multiple-Activity Firms

In many cases the firm itself is a portfolio of productive activities, each of which produces some probability distribution of market value at the beginning of period 2. The last step in our analysis of optimal production decisions is to develop production criteria that can be applied activity by activity.

Let $\bar{V}_{kj}$ be the period 2 market value of activity $k$ in firm $j$, so that $\bar{V}_j$, the total period 2 value of the firm, is

$$\bar{V}_j = \sum_{k=1}^{m(j)} \bar{V}_{kj},$$

(7.30)

where $m(j)$ is the number of activities. It is always possible to define "implicit" or "indirect" period 1 prices $P_{kj}$ for activities such that

$$P_j = \sum_{k=1}^{m(j)} P_{kj}.$$

(7.31)

The prices are implicit or indirect, because consumers must buy shares of firm $j$; they cannot deal directly in the firm’s individual production activities. Nevertheless it will now be shown that, like the explicit prices $P_j$, the implicit prices $P_{kj}$ satisfy Equation (7.23), where the measure of risk is $\text{cov}(\bar{V}_{kj}, \bar{V}_m)/\sigma(\bar{V}_m)$, and that optimal production decisions require evaluating each activity in accordance with its own risk and its own equilibrium expected return.

First note that, from Equation (7.30),

$$E(\bar{V}_j) = \sum_{k=1}^{m(j)} E(\bar{V}_{kj})$$

(7.32)

and

$$\text{cov}(\bar{V}_j, \bar{V}_m) = \sum_{k=1}^{m(j)} \text{cov}(\bar{V}_{kj}, \bar{V}_m) = \sum_{k=1}^{m(j)} \sum_{l=1}^{m(j)} \text{cov}(\bar{V}_{kj}, \bar{V}_{lm}).$$

(7.33)

With Equations (7.31) to (7.33), the equilibrium period 1 market value of
firm \( j \), as given by Equation (7.23), can be rewritten

\[
P_f = \sum_{k=1}^{m(j)} \frac{\sum_{k=1}^{m(j)} E(\tilde{V}_{kj}) - S_m \sum_{k=1}^{m(j)} \text{cov}(\tilde{V}_{kj}, \tilde{V}_m)/\sigma(\tilde{V}_m)}{1 + R_f}
\]

Or equivalently

\[
P_f = \sum_{k=1}^{m(j)} P_{kj} = \sum_{k=1}^{m(j)} \left\{ \frac{E(\tilde{V}_k) - S_m \text{cov}(\tilde{V}_k, \tilde{V}_m)/\sigma(\tilde{V}_m)}{1 + R_f} \right\}.
\]

(7.34)

Thus the period 1 market value of the firm is just the sum of the implicit market values of its production activities, with the implicit market value of an activity obtained by applying the pricing equation (7.23) to the expected period 2 value of the activity, \( E(\tilde{V}_k) \), and to its risk \( \text{cov}(\tilde{V}_k, \tilde{V}_m)/\sigma(\tilde{V}_m) \). Or in other words, the period 1 value of the firm is just as if the firm sold each of its production activities separately.

But there is nothing at all strange in this result. Remember that in the two-parameter model the objects being cleared from the capital market at period 1 are total expected market wealth at period 2,

\[
E(\tilde{V}_m) = \sum_{j=1}^{N} E(\tilde{V}_m) = \sum_{j=1}^{N} \sum_{k=1}^{m(j)} E(\tilde{V}_{kj}),
\]

and the total dispersion or risk of market wealth,

\[
\sigma(\tilde{V}_m) = \frac{\sum_{j=1}^{N} \text{cov}(\tilde{V}_j, \tilde{V}_m)}{\sigma(\tilde{V}_m)} = \frac{\sum_{j=1}^{N} \sum_{k=1}^{m(j)} \text{cov}(\tilde{V}_{kj}, \tilde{V}_m)}{\sigma(\tilde{V}_m)}
\]

For all firms the price of a unit of period 2 expected market value is \( 1/(1 + R_f) \); the price of a unit of risk is \( -S_m/(1 + R_f) \). Because the contributions of a firm to total expected market wealth and dispersion of market wealth are just the sums of the expected values and risks of each of its production activities, the market value of the firm is just the sum of the implicit market values of its separate activities, where these implicit prices are just as if the firm sold shares directly in each activity.

The pricing equation (7.34) provides a way for the firm to evaluate directly the contribution of individual production activities to its period 1 market value. And the optimal scale for each activity is, of course, determined by balancing period 1 costs against contributions to market value. But as in the case of single-activity firms, if the risk of the return on the market value of an activity is independent of the scale of the activity,
there is a cost of capital that can be used to determine the activity's optimal scale.

First, note that, from Equations (7.30) and (7.31), the one-period return on the shares of firm $j$ can be written in terms of the returns on its separate production activities:

$$ R_j = \frac{V_j - P_j}{P_j} = \sum_{k=1}^{m(j)} \frac{P_{kj} - \sum_{k=1}^{m(j)} P_{kj}}{\sum_{k=1}^{m(j)} P_{kj}} = \sum_{k=1}^{m(j)} \frac{P_{kj}}{\sum_{k=1}^{m(j)} P_{kj}} \left( \frac{V_{kj} - P_{kj}}{P_{kj}} \right) $$

or

$$ R_j = \sum_{i=1}^{m(j)} x_{kj} \tilde{R}_{kj}, \quad (7.35) $$

where $x_{kj} = P_{kj}/P_j$ is the proportion of the total market value of firm $j$ at period 1 that is accounted for by the production activity $k$, and $\tilde{R}_{kj} = (V_{kj} - P_{kj})/P_{kj}$ is the one-period return on the period 1 value of activity $k$. It follows that

$$ E(R_j) = \sum_{i=1}^{m(j)} x_{kj} E(\tilde{R}_{kj}). \quad (7.36) $$

With Equations (7.35) and (7.36), the covariance between the one-period return on the firm's shares and the one-period return on the market portfolio $m$ is

$$ \text{cov} (R_j, R_m) = \text{cov} \left( \sum_{i=1}^{m(j)} x_{kj} \tilde{R}_{kj}, R_m \right) $$

$$ = \sum_{i=1}^{m(j)} x_{kj} \text{cov} (\tilde{R}_{kj}, R_m), $$

so that

$$ \frac{\text{cov} (R_j, R_m)}{\sigma(R_m)} = \sum_{i=1}^{m(j)} x_{kj} \frac{\text{cov} (\tilde{R}_{kj}, R_m)}{\sigma(R_m)}. \quad (7.37) $$

We know that in equilibrium the expected return on the shares of firm $j$ must satisfy the expected return–risk relation (7.17); that is,

$$ E(R_j) = R_f + S_m \frac{\text{cov} (R_j, R_m)}{\sigma(R_m)}. $$
But with Equations (7.36) and (7.37), this can be rewritten

\[ E(\bar{E}_2) = \sum_{k=1}^{m(j)} x_{kj} E(\bar{R}_{kj}) = R_f + S_m \sum_{k=1}^{m(j)} x_{kj} \frac{\text{cov}(\bar{R}_{kj}, \bar{R}_m)}{\sigma(\bar{R}_m)}, \]

or

\[ E(\bar{E}_i) = \sum_{k=1}^{m(j)} x_{kj} E(\bar{R}_{kj}) = \sum_{k=1}^{m(j)} x_{kj} \left[ R_f + S_m \frac{\text{cov}(\bar{R}_{kj}, \bar{R}_m)}{\sigma(\bar{R}_m)} \right]. \]

In words, the expected return on the firm’s shares is just the weighted average of the \( E(\bar{R}_{kj}) \), the expected returns on its individual activities, and the values of each of the \( E(\bar{R}_{kj}) \) must satisfy the equilibrium relationship between risk and expected return.

In essence, the firm is just a portfolio of production activities, and market relationships between risk and expected return hold for each activity. And with a perfect capital market, the equilibrium expected returns \( E(\bar{R}_{kj}) \), \( k = 1, 2, \ldots, m(j) \), are the required expected one-period returns or costs of capital for the firm’s production activities. The steps required to show this are identical with those which led to expression (7.29) for the case of a single-activity firm; thus we leave formal derivation to the interested reader.

Thus although in the capital market consumers must deal directly in the shares of firm \( j \), nevertheless the implicit market values of the firm’s separate production activities and the optimal production criteria for these activities are as if consumers could deal directly in the activities themselves.

V.I.D. Optimal Production Decisions:

Some Comments

Three important points in the analysis of optimal production decisions should be mentioned.

First, although the market value at period 1 of each of the firm’s production activities depends only on the expectation and risk of the distribution of the period 2 market value of the activity, this does not imply independence among production activities in either revenues or costs. In fact, the constraints on the firm’s production opportunities are most simply represented by its transformation function

\[ T'(E(\bar{V}_{1j}), E(\bar{V}_{2j}), \ldots, E(\bar{V}_{m(j), i}), I_{1j}, I_{2j}, \ldots, I_{m(j), i}) = 0, \]

which tells the maximum expected market values at period 2 that can be generated by different outlays at period 1. The general functional form of \( T' \) allows for interdependence among the levels of any of the variables. (A realistic example of such dependencies would be the case in which the outlays \( I_{kj} \) necessary to generate a given level of \( E(\bar{V}_{kj}) \) depend on the
other production activities that a firm undertakes.) Nevertheless when the optimal levels of current outlays for all activities have been chosen, the resulting period 2 expected market value of each activity is priced in accordance with its risk.

Second, it is important to note that in the analysis of optimal production decisions the risk of a particular production activity is measured by the contribution of the activity to the dispersion in the market value of the market portfolio. The risk of the activity to the firm itself, however this might be measured, is of no consequence; the market evaluates the activity and its risk in the same way, regardless of the particular portfolio of activities that the firm happens to hold. Moreover, optimal investment decisions imply no incentive for the firm to diversify. Although consumers all hold efficient portfolios, in a perfect market consumers can combine the shares of different firms and obtain efficient portfolios with no transactions costs, so that in itself diversification on the part of an individual firm is of no particular value to them.

Third, it is interesting to examine the role of utility functions in the analysis of optimal production decisions. The utility functions of individual consumers combine in the market to determine the trade-off between \( E(R) \) and \( \sigma(R) \) along the efficient set. The properties of the efficient set lead directly to the equilibrium expected return–risk relations that in turn are the basis of optimal investment criteria for the firm. But although the firm’s optimal investment decisions must take direct account of market expected return–risk relations, its investment decisions do not directly involve utility functions. In fact, just as in a world of perfect certainty, with a perfect market the notion of a utility function for a firm is completely inappropriate in determining an optimal production plan. From the viewpoint of the shareholders, the goal of the firm is simply to maximize the period 1 market value of their shares.

**VII. ALGEBRAIC TREATMENT OF CAPITAL MARKET EQUILIBRIUM**

So far the development of the theory of capital market equilibrium has been, to whatever extent possible, geometric. In the present section the model is reconsidered from a completely algebraic viewpoint. The main purpose is to verify the completeness of the geometric analysis; that is, we must verify that in the geometric development we have imposed a sufficient number of conditions to determine the variables of interest.

The procedure is to present first the necessary conditions of consumer equilibrium. Then the equilibrium conditions for firms are discussed. Finally, the equilibrium conditions for consumers and firms are combined
with market-clearing constraints to complete the picture of capital market equilibrium.

VII.A. Consumer Equilibrium

The goal of the consumer is to make a consumption-investment decision at the beginning of period 1 that maximizes the expected utility of consumption in periods 1 and 2, subject to the constraint that period 1 consumption plus the market value of portfolio investments is equal to the market value of his resources at the beginning of period 1. There are, however, many equivalent ways in which this problem can be stated. For example, as has been the practice in the present chapter and the last, we can think of consumer $i$ as choosing optimal values of period 1 consumption $c_{1i}$ and proportions $x_{jp}, j = 1, 2, \ldots, N$, of investment funds $(w_{1i} - c_{1i})$ allocated to the shares of each of the $N$ firms.\footnote{In the present section we often have to deal with more than one consumer at a time. Thus the superscript $i$ is used to refer to a particular consumer $i$, and there are assumed to be $I$ consumers in the market.} Or, as in Section IV.C in Chapter 6, we can think of the consumer as choosing $c_{1i}$ and total dollars $h_{1j}, j = 1, 2, \ldots, N$, invested in the shares of individual firms, so that

$$\sum_{j=1}^{N} h_{1j} = w_{1i} - c_{1i}.$$ 

Finally, the consumer's decision problem can also be stated in terms of the variables $c_{1i}$ and $n_{ji}, j = 1, 2, \ldots, N$, where $n_{ji}$ is the number of shares of firm $j$ that appear in the portfolio that is optimal for consumer $i$ at the beginning of period 1. This last approach is the most convenient for present purposes.

The individual's consumption in period 2, $c_{2i}$, is related to the one-period return on his portfolio, $R_p$, as follows:

$$c_{2i} = (w_{1i} - c_{1i})(1 + R_p).$$

Because we assume that the one-period returns on all portfolios have normal distributions, $R_p$ can be written

$$R_p = E(R_p) + \sigma(R_p) \bar{\nu},$$

where $\bar{\nu}$ is the unit normal variable; that is, the distribution of $\bar{\nu}$ is normal with mean equal to 0 and standard deviation equal to 1. Thus $c_{2i}$ can be rewritten

$$c_{2i} = (w_{1i} - c_{1i})(1 + E(R_p) + \sigma(R_p) \bar{\nu})$$

or

$$c_{2i} = (w_{1i} - c_{1i})(1 + \sum_{j=1}^{N} x_{jp}E(R_j) + \sigma(\sum_{j=1}^{N} x_{jp}R_j) \bar{\nu}), \quad (7.38)$$
where \[ x_{jp} = \frac{n_j p_j}{\sum_{j=1}^{N} n_j p_j} = \frac{n_j p_j}{w_1 - c_1}, \quad j = 1, 2, \ldots, N, \] (7.39)

are the proportions of investment funds, \( w_1 - c_1 \), invested in each of the \( N \) available assets. Thus, substituting Equation (7.39) into Equation (7.38), we can express \( \epsilon_i \) in terms of the numbers of shares of each firm purchased, \( n_j \), as

\[ \epsilon_i = \sum_{j=1}^{N} n_j p_j + \sum_{j=1}^{N} n_j p_j E(R_j) + \sigma(\sum_{j=1}^{N} n_j p_j R_j) \right]. \]

The decision facing consumer \( i \) can then be stated as follows: Choose the values of \( c_i \) and \( n_j \), \( j = 1, 2, \ldots, N \), that

\[ \text{max } E[U(c_i, \epsilon_i)] \]

\[ = \int_{-\infty}^{\infty} U(c_i, \sum_{j=1}^{N} n_j p_j + \sum_{j=1}^{N} n_j p_j E(R_j) + \sigma(\sum_{j=1}^{N} n_j p_j R_j) f(r) \right] dr \]

subject to the constraint

\[ w_i = c_i + \sum_{j=1}^{N} n_j p_j. \]

The necessary conditions for a maximum are determined by first forming the lagrangian expression

\[ L = E[U(c_i, \epsilon_i)] + \lambda_i (w_i - c_i - \sum_{j=1}^{N} n_j p_j), \]

then differentiating \( L \) partially with respect to \( c_i \), \( n_j \), \( j = 1, 2, \ldots, N \), and \( \lambda_i \), and setting the partial derivatives equal to 0, obtaining

\[ \frac{\partial E[U(c_i, \epsilon_i)]}{\partial c_i} = \lambda_i, \] (7.40a)

\[ \frac{\partial E[U(c_i, \epsilon_i)]}{\partial n_j} \left/ p_j \right. = \lambda_i, \quad j = 1, 2, \ldots, N, \] (7.40b)

\[ w_i = c_i + \sum_{j=1}^{N} n_j p_j = 0. \] (7.40c)

Equations (7.40a) and (7.40b) are, of course, just the usual conditions that the marginal expected utilities per dollar allocated to period 1 con-
sumption and investments in each of the $N$ available assets must be equal. We can perhaps see this more easily by combining (7.40a) and (7.40b) to obtain the new system of equations

$$\frac{\partial E[U(c_0', s_0')]}{\partial n_i^\prime} = \frac{\partial E[U(c_0', s_0')]}{\partial c_i^\prime}, \quad j = 1, 2, \ldots, N, \quad (7.41a)$$

$$\nu_1^\prime - c_1^\prime - \sum_{j=1}^{N} n_j^\prime p_{j1} = 0. \quad (7.41b)$$

This also shows that one of the $N + 1$ equations (7.40a) and (7.40b) is redundant; given values of $N$ of the $N + 1$ variables $c_i^\prime$ and $n_i^\prime, j = 1, 2, \ldots, N$, the budget constraint (7.40c) allows us to determine the value of the $N$th variable.

VII.B. Equilibrium for the Firm

Recall that with homogeneous expectations and perfect capital markets, the objective of the firm, assumed to operate in the best interests of its shareholders, is to maximize the difference between the market value of the firm at period 1, $P_f$, and the outlays required by the production decision made at period 1. For simplicity, we assume now that a share in firm $j$ represents a claim to $S_1$ of $E(V_j)$, the expected market value of the firm at period 2. Thus $E(V_j) = n_j$, where $n_j$ is the number of shares in firm $j$ implied by the production decision of period 1. As in Section VI.B, we also assume that the firm is engaged in only one type of production activity, and the production decision involves choosing the optimal scale at which to operate this activity. Equivalently, the firm must choose the optimal number of units of period 2 expected market value to generate, but whatever the level of $E(V_j)$ chosen, each unit or share has the same set of covariances $\text{cov}(\theta_i, \theta_j), i = 1, 2, \ldots, N$, with the shares of other firms. Finally, the firm is a perfect competitor; that is, it takes its share price $p_j$ as given.14

In this context the decision problem facing the firm can be stated as follows: Choose a value of $n_j$ that

$$\max \left( P_f - I(n_j) \right),$$

or equivalently

$$\max \left( n_j p_j - I(n_j) \right),$$

14 At this point, the reader can easily convince himself that these conditions are indeed equivalent to those imposed in Sec. VI.B. In particular, they imply that $\text{cov}(\bar{R}_j, \bar{R}_m)/\sigma(\bar{R}_m)$ is independent of the scale of $E(V_j)$. In short, the firm is in a given risk class, and its only decision is to choose the optimal scale of activity.
where \( I(n_j) \) are the outlays required at period 1 to generate \( E(\bar{V}_j) = n_j \) units, or shares, of period 2 expected market value. Thus the necessary condition for a maximum is

\[
\frac{dI(n_j)}{dn_j} = p_j, \quad j = 1, 2, \ldots, N. \tag{7.42}
\]

This is hardly a new result. First note that

\[
p_j = \frac{E(\bar{v}_j)}{1 + E(\bar{R}_j)} = \frac{\$1}{1 + E(\bar{R}_j)}, \tag{7.43}
\]

where \( E(\bar{R}_j) \) is the equilibrium expected return on a share of firm \( j \). Note also that

\[
E(\bar{v}_j) = \$1 = \frac{dE(\bar{V}_j)}{dn_j}. \tag{7.44}
\]

In words, increasing the number of shares by one increases the period 2 expected market value of the firm by \$1. Thus using Equations (7.43) and (7.44), Equation (7.42) can be rewritten

\[
\frac{dI(n_j)}{dn_j} = \frac{dE(\bar{V}_j)/dn_j}{1 + E(\bar{R}_j)};
\]

that is, production should be pushed to a point at which the incremental outlay yields an one-period expected return equal to the equilibrium expected return on the firm's shares, which can thus be regarded as the firm's cost of capital. But this is, of course, precisely the condition on optimal production decisions represented in Figure 7.5.

VII.C. Market Equilibrium

Given the share prices \( p_j, j = 1, 2, \ldots, N \), Equations (7.42) provide \( N \) equations in the \( N \) unknowns \( n_j, j = 1, 2, \ldots, N \). Thus the numbers of shares supplied to the market by firms are determined. Similarly, given the share prices \( p_j \), Equations (7.41) provide \( N + 1 \) equations for the \( N + 1 \) unknowns \( c^i \) and \( n_j^i, j = 1, 2, \ldots, N \). Because there are \( I \) sets of such equations, one for each of the \( i = 1, 2, \ldots, I \) consumers in the market, the numbers of shares in firms demanded by consumers are also determined. To determine the share prices themselves, we simply add the market-clearing constraints,

\[
\sum_{i=1}^{I} n_j^i p_j = n_j^i p_j, \quad j = 1, 2, \ldots, N. \tag{7.45}
\]
In words, the demand for the shares of each firm by consumers must be equal to the quantity supplied by the firm. And here we have \( N \) equations in the \( N \) unknown prices \( p_j, j = 1, 2, \ldots, N \).

But the more sophisticated reader may be troubled by the fact that any market equilibrium model can only determine relative prices. (And relative prices, the rates of exchange of goods for each other, are, after all, the only prices that have economic meaning.) In fact, only relative prices are implied in our model. In the total system represented by Equations (7.41), (7.42), and (7.45) we have precisely enough equations to determine the quantities of shares issued by firms and the prices of these shares, the number of shares of each firm demanded by each consumer, and the value of optimal period 1 consumption \( \alpha \) for each consumer. But in concentrating on the determination of equilibrium in the capital market, we have implicitly taken equilibrium in the goods market, that is, prices of period 1 consumption goods, as given. Thus the equilibrium prices of shares determined in the capital market are conditional on or relative to the assumed prices for consumption goods.

VIII. MARKET EQUILIBRIUM WITH SYMMETRIC STABLE RETURN DISTRIBUTIONS

The last step in the discussion of market equilibrium is to generalize the two-parameter mean–standard deviation model in the preceding sections to the case in which portfolio return distributions can be symmetric stable with characteristic exponent \( 1 < \alpha \leq 2 \). In short, we extend the model of capital market equilibrium in the present chapter to the two-parameter symmetric stable portfolio model presented at the end of the last chapter. Because the analysis and conclusions are so similar to those for the two-parameter normal model, the discussion here can be brief, and in fact we just present the first few steps.

The first step is to review briefly some of the building blocks of the symmetric stable model.

VIII.A. The Market Model Return–Generating Process

In the symmetric stable portfolio model in Chapter 6, the returns on investment assets—the shares of the \( N \) firms—are assumed to be generated by the "market model"

\[
R_j = a_j + b_j \bar{M} + \epsilon_j, \quad j = 1, 2, \ldots, N. \tag{7.46}
\]

Here \( a_j \) and \( b_j \) are constants, and \( \epsilon_j \) is a random disturbance with expected value equal to 0. It is assumed that \( \bar{M} \) and the \( \epsilon_j, j = 1, 2, \ldots, N, \) are mutually independent, symmetric stable variables, all with the same characteristic exponent \( \alpha \), which is assumed to be greater than 1.
With asset returns generated according to Equation (7.46), the return on any portfolio $p$ is

$$ R_p = \sum_{j=1}^{N} x_{jp} \bar{r}_j = \sum_{j=1}^{N} x_{jp} \alpha_j + \sum_{j=1}^{N} x_{jp} b_j \bar{Y} + \sum_{j=1}^{N} x_{jp} \hat{r}_j, \quad (7.47) $$

so that the return on a portfolio is a weighted sum of the independent random variables $\bar{Y}$ and the $\hat{r}_j$. Because these are assumed to be symmetric stable with the same characteristic exponent $\alpha$, $R_p$ is also symmetric stable with the same value of $\alpha$; and $E(R_p)$ and $\sigma(R_p)$, the expected value and dispersion of the distribution of $R_p$, are related to those of $\bar{Y}$ and the $\hat{r}_j$ according to

$$ E(R_p) = \sum_{j=1}^{N} x_{jp} E(\bar{r}_j) = \sum_{j=1}^{N} x_{jp} [\alpha_j + b_j E(\bar{Y})], \quad (7.48) $$

$$ \sigma(R_p) = [\sigma^2(\bar{Y}) + \sum_{j=1}^{N} x_{jp}^2 \hat{b}_j^{\alpha} + \sum_{j=1}^{N} \sigma^2(\hat{r}_j) x_{jp} |^{\alpha}]^{1/\alpha}. \quad (7.49) $$

Now, however, we interpret $\sigma(R_p)$ as return dispersion rather than as standard deviation. For the purpose of concreteness, remember that with symmetric stable distributions $\sigma(R_p)$ corresponds approximately to the semi-interquartile range of the distribution of $R_p$.

In the two-parameter model based on normally distributed portfolio returns all the major results on the measurement of risk and the relationships between expected return and risk derive from the efficient set theorem, that is, from the fact that the optimal portfolio for a risk averter must be $E(\bar{R})$, $\sigma(\bar{R})$ efficient. But from Chapter 6 we already know that, interpreting $\sigma(\bar{R})$ as return dispersion, the efficient set theorem also holds in the two-parameter stable model. And we now use this fact, precisely as in the normal model, to develop expected return–risk relations both for the individual consumer and for the market as a whole.

VIII.B. Consumer Equilibrium and the Measurement of Risk

Suppose that Figure 7.1 now represents the picture of equilibrium for a given consumer in the two-parameter symmetric stable portfolio model.\(^{17}\)

\(^{17}\) From Chap. 6 remember that in the symmetric stable portfolio model, the consumer's indifference curves for $E(\bar{R})$ against $\sigma(\bar{R})$ are positively sloping and convex with expected utility increasing upward and to the left in the $E(\bar{R})$, $\sigma(\bar{R})$ plane, and the efficient set curve is positively sloping and concave. Thus Fig. 7.1 is indeed an appropriate representation of consumer equilibrium.
As in the normal model, the basic presumption again is that, because the consumer chooses the efficient portfolio \( e \), for him the risks of individual assets and the relationship between expected return and risk must be measured relative to \( e \). And we now see that, precisely as in the normal model, these concepts are embedded in the implications of efficiency for properties of \( e \).

The efficient portfolio with expected return \( E(R_e) \) is the solution to the problem: Choose values of \( x_{jp}, j = 1, 2, \ldots, N \), that

\[
\min \sigma(R_p) = \left[ \sigma^2(M) + \sum_{j=1}^{N} x_{jp} \sigma_j \right]^{1/2} + \sum_{j=1}^{N} \sigma^2(\varepsilon_j) x_{jp} \tag{7.50a}
\]

subject to

\[
E(R_p) = \sum_{j=1}^{N} x_{jp} E(R_j) = E(R_e) \tag{7.50b}
\]

and

\[
\sum_{j=1}^{N} x_{jp} = 1. \tag{7.50c}
\]

With the usual lagrangian methods, we could easily determine that the solution to this problem must satisfy Equations (7.50b), (7.50c), and the old familiar balance equation

\[
E(R_j) - E(R_i) = S_e \left( \frac{\partial \sigma(R_e)}{\partial x_{je}} - \frac{\partial \sigma(R_e)}{\partial x_{ie}} \right), \quad i, j = 1, 2, \ldots, N, \tag{7.5}
\]

where \( S_e \) is the slope of the efficient set at the point \( e \), and \( \partial \sigma(R_e)/\partial x_{je} \) is \( \partial \sigma(R_p)/\partial x_{jp} \) evaluated at the optimizing values \( x_{jp} = x_{je}, j = 1, 2, \ldots, N \). Multiplying both sides of Equation (7.5) by \( x_{ie} \) and summing over \( i \), we obtain the new balance equation

\[
E(R_j) - E(R_e) = S_e \left( \frac{\partial \sigma(R_e)}{\partial x_{je}} - \sum_{i=1}^{N} x_{ie} \frac{\partial \sigma(R_e)}{\partial x_{ie}} \right), \quad j = 1, 2, \ldots, N. \tag{7.6}
\]

As the equation numbers indicate, so far things are precisely as in the two-parameter normal model. The next and critical step in the development of the normal model was to show that

\[
\sum_{i=1}^{N} x_{ie} \frac{\partial \sigma(R_e)}{\partial x_{ie}} = \sigma(R_e), \tag{7.8}
\]
so that Equation (7.6) could be rewritten

$$E(R_j) - E(R_i) = S_i \left( \frac{\partial \sigma(R_i)}{\partial x_i} - \sigma(R_i) \right), \quad j = 1, 2, \ldots, N. \quad (7.9)$$

This balance equation can then be interpreted as the relevant relationship between expected return and risk for the consumer who chooses the efficient portfolio \( \sigma \); that is, in the two-parameter world, the risk of the portfolio \( \sigma \) is measured by its return dispersion \( \sigma(R) \). But \( \partial \sigma(R_i)/\partial x_i \), is just the contribution of asset \( j \) to \( \sigma(R) \); thus, it measures the risk of asset \( j \) in the efficient portfolio \( \sigma \). Then Equation (7.9) can be interpreted as the relationship between the expected returns and risks of individual assets vis-à-vis the portfolio \( \sigma \): It says that the portfolio \( \sigma \) is formed in such a way that the difference between the expected return on any asset and the expected return on the portfolio is proportional to the difference between the risk of the asset and the risk of the portfolio, where the proportionality factor is \( S_i \), the slope of the efficient set at the point \( \sigma \).

But the balance equation (7.9) and this interpretation of it as an expected return–risk relation apply equally well to the two-parameter stable model. To establish this, we need only show that Equation (7.8) holds in this model, so that (7.9) follows from (7.5), which has already been shown to hold in the stable case. But Equation (7.8) itself follows directly from

$$\frac{\partial \sigma(R_i)}{\partial x_i} = \sigma(R_i)^{1-\alpha} \left[ \sigma^*(M) \frac{b_i}{\sum_{j=1}^N x_j b_j} \sigma^*(\tilde{x}_i) \frac{x_{i \sigma}}{x_i} \right]. \quad (7.51)$$

Thus the major lines of the analysis of risk and the relationship between expected return and risk for the individual consumer are the same in the two-parameter stable model and in the model based on normally distributed portfolio returns. The only differences between the models are that in the general stable case the return dispersion \( \sigma(R) \) can no longer be interpreted as standard deviation and the risk measure \( \partial \sigma(R_i)/\partial x_i \) can no longer be interpreted as a weighted sum of covariances.

And it is important not to overstate the differences between the models. Remember that the normal distribution itself is the symmetric stable distribution with characteristic exponent \( \alpha = 2 \). If asset returns are normally distributed and generated by the market model in Equation (7.46), the analysis of this section applies to the normal model, and in particular Equation (7.51) measures the risk of an asset relative to the efficient portfolio \( \sigma \). The special nature of Equation (7.51) arises from the special nature of the market model and not from any assumption of nonnormality.
Thus in the normal model, from Equation (7.7),

\[
\frac{\partial \sigma(\overline{R}_e)}{\partial x_{ie}} = \frac{\sum_{i=1}^{N} x_{ij} \sigma_{ij}}{\sigma(\overline{R}_e)} = \frac{\sum_{i=1}^{N} x_{ie} \text{cov}(\overline{R}_i, \overline{R}_j)}{\sigma(\overline{R}_e)}.
\]

But when share returns are generated by the market model,

\[
\text{cov}(\overline{R}_i, \overline{R}_j) = \text{cov}(a_i + b_i \overline{M} + \overline{\varepsilon}_i; a_j + b_j \overline{M} + \overline{\varepsilon}_j)
\]

\[
= b_i b_j \sigma^2(\overline{M}), \quad i \neq j,
\]

\[
= b_i \sigma^2(\overline{M}) + \sigma^2(\overline{\varepsilon}_i), \quad i = j,
\]

so that

\[
\frac{\partial \sigma(\overline{R}_e)}{\partial x_{ie}} = \frac{\sigma^2(\overline{M}) b_i \sum_{j=1}^{N} x_{ij} + x_{ie} \sigma^2(\overline{\varepsilon}_i)}{\sigma(\overline{R}_e)}.
\]

But this is just Equation (7.51) for the case \(a = 2\).

Perhaps the major difference between the normal and nonnormal two-parameter portfolio models is best described as follows. With normally distributed asset returns, arbitrary relationships among these returns can be expressed quite generally in terms of covariances. On the other hand, with nonnormal symmetric stable asset returns, the absence of a general statistical theory for deriving portfolio return distributions from the distributions of dependent asset returns requires us to posit a model like the market model, wherein the returns on securities are dependent, yet the return on a portfolio can be expressed as a weighted sum of independent symmetric stable variables.

VIII.C. Risk and Return for the Market

The development of expression (7.9) gives us a theory for the measurement of risk and the relationship between risk and expected return that is relevant at the level of the individual consumer. As in the normal model, to generalize these concepts to the level of the market, we must further restrict the market context. And the restrictions that we impose are precisely those of the normal model. In particular, we assume the existence of (1) riskless borrowing-lending opportunities and (2) homogeneous expectations. And the results are again precisely as in the normal model.

Thus, with riskless borrowing-lending, portfolios of the riskless asset \(f\) and any risky asset or portfolio \(a\) that are formed according to

\[
\overline{R}_p = x \overline{R}_f + (1 - x) \overline{R}_e, \quad x \leq 1,
\] (7.12)
have expected return and dispersion

\[ E(\bar{R}_p) = xR_f + (1 - x)E(\bar{R}_m), \]
\[ \sigma(\bar{R}_p) = (1 - x)\sigma(\bar{R}_m). \]

Thus, as shown in Figure 7.3, the graph of \( E(\bar{R}_p) \) against \( \sigma(\bar{R}_p) \) for portfolios formed according to Equation (7.12) is a straight line from \( R_f \) through \( a \). Points between \( R_f \) and \( a \) on the line correspond to lending portfolios—that is, \( 0 \leq x \leq 1 \)—and points above \( a \) represent borrowing portfolios—that is, \( x < 0 \).

As in the normal model, the efficient set of portfolios will be given by the line from \( R_f \) through \( m \) and \( z \) in Figure 7.3. Thus all efficient portfolios will be combinations of the riskless asset \( f \) and the single portfolio of risky assets \( m \). And with homogeneous expectations, Figure 7.3 will be the picture of portfolio opportunities facing every investor, so that the risky component of every investor's optimal portfolio is the portfolio \( m \). It follows that market equilibrium requires that \( m \) be the market portfolio.

Applying Equation (7.9) to \( m \) and noting from Figure 7.3 that \( S_m = [E(\bar{R}_m) - R_f]/\sigma(\bar{R}_m) \), we get the familiar result,

\[ E(\bar{R}_j) = R_f + \left[ \frac{E(\bar{R}_m) - R_f}{\sigma(\bar{R}_m)} \right] \frac{\partial \sigma(\bar{R}_m)}{\partial x_{jm}}, \quad j = 1, 2, \ldots, N. \quad (7.15) \]

Since the only risky assets held by consumers are shares in the market portfolio \( m \), for all consumers, and thus for the market as a whole, the risk of an asset and the relationship between its risk and its expected return are appropriately measured relative to the market portfolio. This is done in Equation (7.15), in which the equilibrium expected return on asset \( j \) is the riskless rate of interest \( R_f \) plus a risk premium that is proportional to \( \partial \sigma(\bar{R}_m)/\partial x_{jm} \), the asset's contribution to the total risk, that is, dispersion of return, of the market portfolio.

The analysis of the nature of optimal production decisions for the two-parameter stable model of this section is identical to the analysis of optimal production decisions in the normal model, and so does not even bear any repeating.

IX. CONCLUSIONS

One-period, two-parameter models of consumer, firm, and market equilibrium have now been presented. It should be clear, however, that the
properties of consumer equilibrium are the underpinnings of all three models. And, in particular, all of the major results derive from the fact that with two-parameter portfolio return distributions, the optimal portfolios for risk-averse consumers are $E(R)$, $\sigma(R)$ efficient.

In the next chapter we conclude the book with a discussion of some multi-period models.

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