I. INTRODUCTION

This concluding chapter of the book is organized around the general theme of multiperiod models of consumer (investor) and capital market equilibrium. The analyses very soon take us to the boundaries of current knowledge in finance and for this reason are not likely to provide the reader with a very good sense of closure. But the primary goal of the book, after all, is to give an accurate picture of the current status of the theory of finance. And the vitality of the area encourages an impression of semi-maturity, with much of the best yet to come, rather than an image of full growth.

The chapter covers two topics. First we consider multiperiod models of consumer and capital market equilibrium, attempting to relate these as much as possible to the two-period models in the preceding chapters. Then the implications of a perfect capital market for the nature of the process generating prices of investment assets through time are discussed.
Although multiperiod models is the organizing theme of the chapter, the two major topics are only semirelated. The orientation of the first— multiperiod models of consumer and market equilibrium—is theoretical, and unfortunately the analysis requires a slightly higher level of mathematics than has been the case up to now. The primary goal is to provide multiperiod extensions of the two-period models in the preceding chapters. As elsewhere in the book, the analysis is presented so that the reader who cannot follow all the mathematics can nevertheless pick up the major ideas from the verbal discussions.

On the other hand, the orientation of the second major topic—the so-called efficient markets model—is primarily empirical. Most of this book is based on the assumed existence of a perfect capital market. In presenting the efficient markets model, although the empirical work is not discussed, we contend that the available empirical evidence supports many of the major implications of a perfect market, and that among the various areas of economics, the theory of finance ranks high in terms of degree of correspondence between theory and empirical evidence.

II. MULTIPERIOD CONSUMPTION–INVESTMENT DECISIONS

II.A. The Problem

The simplest version of the multiperiod consumption–investment problem, and the only version that is treated here, considers a consumer with wealth \( w_1 \), defined as the market value of his assets at the beginning of period 1, which must be allocated to consumption \( c_1 \) and a portfolio investment \( w_1 - c_1 \). The portfolio yields an uncertain wealth level \( w_2 \) at the beginning of period 2 that must be divided between consumption \( c_2 \) and investment \( w_2 - c_2 \). Consumption-investment decisions must be made at the beginning of each period, until the consumer dies and his wealth is distributed among his heirs. The consumer's objective is to maximize the expected utility of lifetime consumption.\(^1\)

\(^1\) Indeed, following the practice in earlier chapters, perhaps all Sec. II should be asterisked.

\(^2\) The simple multiperiod model to be presented here could easily be extended in several directions. For example it would not be difficult—indeed the major complications are notational—to extend the model to take account of the fact that the consumer has an asset, his "human capital," that generates income in many periods but which cannot be sold outright in the market at the beginning of any period. Such an extended model could allow the ways that the consumer employs his human capital during any period—his choice of occupation(s) and the division of his time between labor and leisure—to be at his discretion. But such complications would only confuse our attempts to bridge the gap between the two-period models in the preceding chapters and the multiperiod models to be considered here.
The goal is to present a general multiperiod consumption-investment model but one that nevertheless leads to interesting hypotheses about observable aspects of consumer behavior. The main result is the proposition that if the consumer is risk-averse, that is, his utility function for lifetime consumption is strictly concave, and markets for consumption goods and portfolio assets are perfect, the consumer's observable behavior in the market in any period is indistinguishable from that of a risk-averse expected utility maximizer who has a two-period horizon.

With this result it is then possible to provide a multiperiod setting for hypotheses about consumer behavior derived from two-period wealth allocation models, and these have been studied extensively, both here and elsewhere. Two-period models assume, of course, that consumers have two-period horizons, but in most cases their behavioral propositions require only that consumers behave as if they were risk-averse two-period expected utility maximizers, and this is the case in the multiperiod model to be presented here. Thus perhaps the major contribution of the present analysis is in providing a means for bridging the gap between two-period and multiperiod models.

The multiperiod model that we present covers much more general types of two-period models than just the two-parameter models in the preceding chapters. But we later consider in detail the adjustments to the multiperiod model that are necessary to provide a multiperiod setting for the major propositions about consumer behavior and the nature of market equilibrium associated with the two-period two-parameter models. Indeed we show that a multiperiod model in which the optimal portfolio for any period is "efficient" in terms of distributions of one-period returns requires few assumptions beyond those already made in the two-period models.

II.B. The Wealth Allocation Model

But first the multiperiod model must itself be developed. Let $\Phi_t$, the "state of the world," signify the set of information—current and past prices, and so on—available at the beginning of period $t$. Thus $\Phi_{t+1}$ is a subset of $\Phi_t$. For simplicity we assume that in any period or state the number of investment assets available is $N$—although there would be no problems in letting $N$ depend on $\Phi_t$—and the one-period returns from $t$ to $t + 1$ on these assets are represented as $R_j(\Phi_{t+1}), j = 1, 2, \ldots, N$, so that a value of $\Phi_{t+1}$ implies the values of the returns. Thus if $h_j, j = 1, 2, \ldots, N$,

---

3 That is, as always, (1) consumption goods and portfolio assets are infinitely divisible, (2) reallocations of consumption and investment expenditures are costless, and (3) the consumer is a price taker in all markets.

4 Note that the random variable is now considered to be the state of the world at $t + 1$, $\Phi_{t+1}$. And the returns are random variables because they depend on $\Phi_{t+1}$.
are the dollars invested in each asset at \( t \), the consumer's wealth at \( t + 1 \) is

\[
w_{t+1} = \sum_{j=1}^{N} h_j [1 + R_j(\Phi_{t+1})].
\] (8.1)

If for simplicity we assume that the consumer will die for certain\(^4\) at the beginning of period \( \tau + 1 \) and if the state of the world at \( \tau + 1 \) is \( \Phi_{t+1} \), the consumer's utility for lifetime consumption is assumed to be given by the function

\[
U_{\tau+1}(C_{\tau+1} | \Phi_{\tau+1}) = U_{\tau+1}(c_{\tau+1}, \ldots, c_{t+1} | \Phi_{\tau+1}),
\]

where in general

\[
C_t = (c_{t+1}, \ldots, c_{t+1}, \ldots, c_t)
\] (8.2)

is consumption from the beginning of his life, period \( 1 - k \), to period \( t \) and the consumption \( c_{\tau+1} \) is in the form of a bequest. The goal of the consumer in his consumption-investment decisions is to maximize the expected utility of lifetime consumption.

The consumer must make an optimal consumption-investment decision for period 1, taking into account that decisions must also be made at the beginning of each future period prior to \( \tau + 1 \) and that these future decisions will depend on future events. Dynamic programming, with its "backward optimization," provides a natural approach; that is, to solve the decision problem for period 1, the consumer first determines optimal decisions for all contingencies for the decision problem to be faced at period \( \tau \). Then he determines optimal decisions for \( \tau - 1 \), under the assumption that he always makes optimal decisions at \( \tau \). And so on, until he works his way back to the decision at period 1, which is then based on the assumption that optimal decisions are made at each future period for any possible contingency.

Formally, optimal decisions at period \( \tau \) for all \( w \) and \( \Phi \), can be summarized by the function

\[
U_{\tau}(C_{\tau+1}, w_{\tau} | \Phi_{\tau})
\]

subject to

\[
0 \leq c_{\tau} \leq w_{\tau}, \quad \text{and} \quad \sum_{j=1}^{N} h_j = w_{\tau} - c_{\tau},
\]

\(^4\) The model is easily extended to allow for an uncertain period of death. Indeed the major complications are notational. (See, for example, Ref. 1.)
where \( F_{\tau}(\Phi_{\tau+1}) \) is the distribution function of \( \Phi_{\tau+1} \), given state \( \Phi_\tau \) at \( \tau \), and the notation
\[
\max_{c_{\tau-1}, h}
\]
is read "maximize with respect to a feasible choice of \( c_{\tau} \) and \( h_j \), \( j = 1, 2, \ldots, N \)".

\( U_\tau(\text{C}_{\tau-1}, w_\tau \mid \Phi_\tau) \) is the maximum expected utility at \( \tau \) of lifetime consumption as a function of realized past consumption \( \text{C}_{\tau-1} \) and wealth \( w_\tau \), given that the state of the world is \( \Phi_\tau \). When period \( \tau \) actually comes along, \( \text{C}_{\tau-1} \), \( w_\tau \), and \( \Phi_\tau \) will be known, but in earlier periods this will not be the case. And just as \( U_{\tau+1}(\text{C}_{\tau+1} \mid \Phi_{\tau+1}) \) serves as the input or objective function for the decision at period \( \tau \), \( U_\tau(\text{C}_{\tau-1}, w_\tau \mid \Phi_\tau) \) now serves as the objective function for the decision problem at period \( \tau - 1 \); that is, optimal decisions for all \( \text{w}_{\tau-1} \) and \( \Phi_{\tau-1} \) can be summarized by the function
\[
U_{\tau-1}(\text{C}_{\tau-2}, w_{\tau-1} \mid \Phi_{\tau-1})
= \max_{c_{\tau-1}, h} \int_{\Phi_\tau} U_\tau(\text{C}_{\tau-1}, \sum_{j=1}^{N} h_j[1 + R_j(\Phi_\tau)] \mid \Phi_\tau) \, dF_{\tau-1}(\Phi_\tau)
\]
subject to
\[
0 \leq c_{\tau-1} \leq w_{\tau-1} \quad \text{and} \quad \sum_{j=1}^{N} h_j = w_{\tau-1} - c_{\tau-1}.
\]

Again \( U_{\tau-1}(\text{C}_{\tau-2}, w_{\tau-1} \mid \Phi_{\tau-1}) \) is the maximum expected utility of lifetime consumption if the consumer is in state \( \Phi_{\tau-1} \) at \( \tau - 1 \), his wealth is \( w_{\tau-1} \), his past consumption was \( \text{C}_{\tau-2} \), and optimal consumption-investment decisions are made first at \( \tau - 1 \) and then at \( \tau \) in whatever state of the world occurs at \( \tau \). And \( U_{\tau-1}(\text{C}_{\tau-2}, w_{\tau-1} \mid \Phi_{\tau-1}) \) in turn becomes the objective function for the decision problem at \( \tau - 2 \).

In fact for \( t = 1, 2, \ldots, \tau \), the entire process of backward optimization can be summarized by the recursive relation
\[
U_t(\text{C}_{t-1}, w_t \mid \Phi_t)
= \max_{c_{t-1}, h} \int_{\Phi_{t+1}} U_{t+1}(\text{C}_{t}, \sum_{j=1}^{N} h_j[1 + R_j(\Phi_{t+1})] \mid \Phi_{t+1}) \, dF_{t}(\Phi_{t+1}) \quad (8.4)
\]

Stieltjes integrals are used here, so that \( \Phi_{\tau+1} \) can be either a discrete or continuous random variable. For practical purposes, and speaking somewhat less than rigorously, this means that when \( \Phi_{\tau+1} \) is a discrete variable, the integral in Equation (8.3) is interpreted as a sum over all possible values of \( \Phi_{\tau+1} \), and \( dF_{\Phi_{\tau+1}}(\Phi_{\tau+1}) \) is the probability of the value \( \Phi_{\tau+1} \). On the other hand, when \( \Phi_{\tau+1} \) is a continuous random variable, it is most convenient to interpret Equation (8.3) as an ordinary Riemann integral, with \( dF_{\Phi_{\tau+1}}(\Phi_{\tau+1}) \) just a shorthand way of writing \( f_{\Phi_{\tau+1}}(\Phi_{\tau+1}) \, d\Phi_{\tau+1} \), where \( f_{\Phi_{\tau+1}}(\Phi_{\tau+1}) \) is the density function for \( \Phi_{\tau+1} \) given state \( \Phi_t \) at \( \tau \).
subject to
\[ 0 \leq c_t \leq w_t \quad \text{and} \quad \sum_{j=1}^{N} h_j = w_t - c_t. \]

The function \( U_t(C_{t-1}, w_t \mid \Phi_t) \) provides the maximum expected utility of lifetime consumption if the consumer is in state \( \Phi_t \) at period \( t \), his wealth is \( w_t \), his past consumption was \( C_{t-1} \), and optimal consumption-investment decisions are made at the beginning of period \( t \) and all future periods.

As stated in Equation (8.4), the multiperiod consumption-investment problem exemplifies a common feature of dynamic programming models. In general it is possible to represent the decision problem of any period \( t \) in terms of a derived objective function, in this case \( U_{t+1} \), which is explicitly a function only of variables for \( t + 1 \) and earlier periods but which in fact summarizes the results of optimal decisions at \( t + 1 \) and subsequent periods for all possible future events. Thus the recursive relation (8.4) represents the multiperiod problem as a sequence of two-period problems, although at any stage in the process the objective function used to solve the two-period problem summarizes optimal decisions for all future periods.

Representing the multiperiod consumption-investment problem as a sequence of two-period problems in itself says nothing about the characteristics of an optimal decision for any period. The main result here is the following.

**Theorem 1.** If the utility function for lifetime consumption \( U_{t+1}(C_{t+1} \mid \Phi_{t+1}) \) has properties characteristic of risk aversion, specifically, if for all \( \Phi_{t+1} \), \( U_{t+1}(C_{t+1} \mid \Phi_{t+1}) \) is monotone-increasing and strictly concave in \( C_{t+1} \), then for all \( t \) the derived functions \( U_t(C_{t-1}, w_t \mid \Phi_t) \) also have these properties.\(^1\)

The proof of the theorem is presented in a later section. We now turn to a study of its implications.

**II.C. Implications: Bridging the Gap between Two-Period and Multiperiod Models**

Although at this point its importance is far from obvious, it is the concavity of the functions \( U_t(C_{t-1}, w_t \mid \Phi_t) \) for all \( t \) and \( \Phi_t \), as stated in Theorem

\[ U_{t+1}(aC_{t+1} + (1 - a)\tilde{C}_{t+1} \mid \Phi_{t+1}) > aU_{t+1}(C_{t+1} \mid \Phi_{t+1}) + (1 - a)U_{t+1}(\tilde{C}_{t+1} \mid \Phi_{t+1}), \]

where \( C_{t+1} \) and \( \tilde{C}_{t+1} \) are any two consumption vectors that differ in at least one element. Geometrically, as always, concavity says that a straight line between any two points on the function \( U_{t+1} \) lies below the function. As in the case of the more familiar utility of money function, the concavity of \( U_{t+1} \) implies risk aversion.

\(^1\) The monotonicity of \( U_{t+1} \) says that the marginal utility of consumption in any period is positive. Strict concavity implies that for \( 0 < \alpha < 1 \),
1, that now allows us to bridge the gap between two-period and multiperiod wealth allocation models.

II.C.1. The utility of money function

A foretaste of the discussion can be obtained by using the multiperiod model to derive the familiar utility of wealth function, most often discussed in the literature in connection with the expected utility model. If the state of the world at period 1 is \( \Phi_1 \) and the consumer's past consumption has been \( \tilde{C}_0 \), then for \( t = 1 \) expression (8.4) yields

\[
U_1(\tilde{C}_0, w_1 \mid \Phi_1) = \max_{c_1, w_1} \int u_2(\tilde{C}_1, \sum_{j=1}^N h_j[1 + R_j(\Phi_1)]) \mid \Phi_2 \) \ dF_{\Phi_1}(\Phi_2)
\]

subject to

\[
0 \leq c_1 \leq w_1 \quad \text{and} \quad \sum_{j=1}^N h_j = w_1 - c_1.
\]

But \( \tilde{C}_0 \) is known at period 1; thus we might just as well write

\[
v_1(w_1 \mid \Phi_1) = U_1(\tilde{C}_0, w_1 \mid \Phi_1).
\]

\( v_1 \) is the relevant utility function for timeless gambles taking place at period 1, that is, gambles in which the outcome is known before the consumption-investment decision of period 1 is made (see Chapter 5). From Theorem 1, \( v_1 \) has the characteristics of a risk averter's utility of wealth function; that is, it is monotone-increasing and strictly concave in \( w_1 \). Thus, although he obtains his utility of wealth function by a complicated process of backward optimization and although his utility of wealth function in fact shows the expected utility of lifetime consumption associated with a given level of wealth at period 1, the consumer's behavior in choosing among timeless gambles is indistinguishable from that of a risk averter making a once and for all decision. Or in other words, our analysis provides a multiperiod setting for the more traditional discussions of utility of wealth functions for risk averters, most of which abstract from the effects of future decisions.

II.C.2. Two-period and multiperiod models: general treatment

More generally, when it comes time to make a decision at the beginning of any period \( t \), \( t = 1, 2, \ldots, r \), past consumption, equal, say, to \( \tilde{C}_{t-1} \), is known, so that the decision at \( t \) can be based on the function

\[
v_{t+1}(c_t, w_{t+1} \mid \Phi_{t+1}) = U_{t+1}(\tilde{C}_{t-1}, c_t, w_{t+1} \mid \Phi_{t+1}).
\]

Thus, for given wealth \( w_t \) and state of the world \( \Phi_t \), the consumer's problem
at $t$ can be expressed as

$$\max_{c_t, A} \int_{\Phi_{t+1}} v_{t+1}(c_t, \sum_{j=1}^{N} h_j[1 + R_j(\Phi_{t+1})] | \Phi_{t+1}) \, dF_{\Phi_t}(\Phi_{t+1})$$

subject to

$$0 \leq c_t \leq w_t \quad \text{and} \quad \sum_{j=1}^{N} h_j = w_t - c_t.$$  

From Theorem 1, $U_{t+1}$ is monotone-increasing and strictly concave in $(c_t, w_{t+1})$; it follows that $v_{t+1}$ is monotone-increasing and strictly concave in $(c_t, w_{t+1})$. Thus although the consumer faces a $t$ period decision problem, the function $v_{t+1}(c_t, w_{t+1} | \Phi_{t+1})$, that is relevant for the consumption-investment decision of period $t$, has the properties of a risk averter's two-period utility of consumption-terminal wealth function. Although the consumer must solve a multiperiod problem, given $v_{t+1}$ his observed behavior in the market is indistinguishable from that of a risk-averse expected utility maximizer who has a two-period horizon.

In itself this result says little about consumer behavior. Its value derives from the fact that it can be used to provide a multiperiod setting for more detailed behavioral hypotheses usually obtained from specific two-period models. But by design the multiperiod model is based on less restrictive assumptions than most two-period models, so that adapting it to any specific two-period model requires additional assumptions. As we now show, however, these are mostly restrictions already implicit or explicit in the two-period models. Little generality is lost in going from a two-period to a multiperiod framework.

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8 But we must keep in mind that, although $v_{t+1}(c_t, w_{t+1} | \Phi_{t+1})$ is only explicitly a function of variables for periods $t$ and $t+1$, it shows the maximum expected utility of lifetime consumption, given optimal consumption-investment decisions in periods subsequent to $t$. Thus $v_{t+1}$ depends both on tastes, as expressed by the function $U_{t+1}(c_{t+1} | \Phi_{t+1})$, and on the consumption-investment opportunities that will be available in future periods.

The notion of summarizing market opportunities in a utility function should not cause concern. Indeed this is done when utility is written, as we do throughout, as a function of total consumption; then we are implicitly summarizing the consumption opportunities, in terms of goods and services and their anticipated prices, that will be available in each period. We return to this point later, when the utility function $U_{t+1}(c_{t+1} | \Phi_{t+1})$ for dollars of consumption is derived from a more basic utility function for consumption goods.

9 In particular, we have essentially assumed only that markets for consumption goods and portfolio assets are perfect and that the consumer is a risk averter in the sense that his utility function for lifetime consumption is strictly concave.
II.C.3. A multiperiod setting for two-period
two-parameter portfolio models

Given the concavity of \( v_{t+1} \), (8.5) is formally equivalent to the consumption-investment problem of a risk-averse consumer with state-dependent utilities and a two-period horizon.\(^{10}\) As such it can be used to provide a multiperiod setting for a wide variety of two-period models, such as the two-period states of the world model analyzed in Chapter 4.

But the theories of wealth allocation and capital market equilibrium most thoroughly discussed here and elsewhere in the literature of finance are the two-period two-parameter models in Chapters 6 and 7. The remainder of this section is concerned with using our model to provide a multiperiod setting for the results of these two-period models.

The two-parameter portfolio models start with the assumption that one-period returns on assets and portfolios conform to two-parameter distributions of the same general "type," in particular, symmetric stable distributions with the same value of the characteristic exponent \( \alpha \). Thus, the distribution for any asset or portfolio can be fully described once its expected value and a dispersion parameter, such as the standard deviation—in the normal case, \( \alpha = 2 \)—or the semi-interquartile range—in the general case, \( 1 < \alpha \leq 2 \)—are known. It is then shown that if investors behave as if they tried to maximize expected utility with respect to two-period utility functions \( v_{t+1}(c_t, w_{t+1}) \) that are strictly concave in \( (c_t, w_{t+1}) \), optimal portfolios are efficient in terms of the two parameters of distributions of one-period returns.\(^{11}\) The fact that optimal portfolios must be efficient then leads to a theory of risk and the relationships between expected return and risk, first for the individual consumer and then, with the additional assumptions of homogeneous expectations and riskless borrowing-lending opportunities, for the market as a whole.

But these two-period two-parameter models assume somewhat more about the utility function \( v_{t+1} \) than our multiperiod model has so far provided. In particular, in the multiperiod model the function \( v_{t+1}(c_t, w_{t+1} | \Phi_{t+1}) \), which is relevant for the consumption-investment decision of period \( t \), is strictly concave in \( (c_t, w_{t+1}) \), but utility can be a function of the state \( \Phi_{t+1} \); that is, utility can be state-dependent. Thus to provide a multiperiod setting for the two-period two-parameter models, it is sufficient to determine conditions under which \( v_{t+1} \) is independent of \( \Phi_{t+1} \).

\(^{10}\) The term "state-dependent utilities" refers to the fact that the function \( v_{t+1}(c_t, w_{t+1} | \Phi_{t+1}) \) allows the utility of a given combination \( (c_t, w_{t+1}) \) to depend on \( \Phi_{t+1} \). The term "state preference" is also used.

\(^{11}\) Recall that a portfolio is efficient only if no portfolio with the same or higher expected return has lower return dispersion.
State-dependent utilities in the derived functions $v_{t+1}$ have three possible sources. First, tastes for given bundles of consumption goods can be state-dependent. Second, as shown later in Theorem 2, utilities for given dollars of consumption depend on the available consumption goods and services and their prices, and these are elements of the state of the world. Finally, the investment opportunities available in any given future period may depend on events occurring in preceding periods, and such uncertainty about investment prospects induces state-dependent utilities. Thus the most direct way to exclude state-dependent utilities is to assume that:

1. The consumer behaves as if the consumption opportunities, in terms of goods and services and their prices, and the investment opportunities—distributions of one-period portfolio wealth relatives—that will be available in any future period can be taken as known and fixed at the beginning of any previous period.

2. The consumer’s tastes for given bundles of consumption goods and services are independent of the state of the world.\textsuperscript{11,13}

To see that (1) and (2) do indeed rule out state-dependent utilities, note that if the utility of lifetime consumption $U_{t+1}$ is independent of $\Phi_{t+1}$, that is, if we can write $U_{t+1}(C_{t+1})$ instead of $U_{t+1}(C_{t+1} \mid \Phi_{t+1})$, and if there is only one possible set of consumption goods and prices and one possible set of distributions of one-period portfolio wealth relatives that can be available in the market at period $t$, then from Equation (8.3) the maximum expected utility of lifetime consumption generated by the consumption-investment decision of period $t$ depends only on $C_{t-1}$ and $w_t$; that is, we can write $U_t(C_{t-1}, w_t)$ instead of $U_t(C_{t-1}, w_t \mid \Phi_t)$. And with these assumptions, that is, (1) and (2), on opportunities and tastes, the same result holds for all earlier periods; that is, the maximum expected utility of lifetime consumption associated with a given $(C_t, w_{t+1})$ is independent of $\Phi_{t+1}$, so that $\Phi_{t+1}$ can be dropped from $U_{t+1}(C_t, w_{t+1} \mid \Phi_{t+1})$ and thus from $v_{t+1}(c_t, w_{t+1} \mid \Phi_{t+1})$. For given wealth $w_t$, the decision problem facing the consumer at the

\textsuperscript{11} It is important to note that some such assumptions are implicit in the two-period two-parameter models themselves, because they do not allow for the effects of state-dependent utilities on the consumption-investment decision.

\textsuperscript{12} These assumptions are in fact somewhat more extreme than are strictly necessary to provide a multiperiod setting for the two-parameter model. The primary result of the two-period two-parameter models is the efficient set theorem: The optimal portfolio for a risk averter must be efficient. This result continues to hold with state-dependent utilities as long as, speaking roughly, any uncertainty about the future consumption goods and prices that will be available in the market is independent of any uncertainty about the future investment opportunities that will be available. It is not clear to us, however, that these are in fact much weaker assumptions than those in the text.
beginning of any period \( t \) can then be written as

\[
\max_{c_t, A} \int_{\Phi_{t+1}} v_{t+1}(c_t, \sum_{j=1}^{N} h_j [1 + R_j(\Phi_{t+1})]) \, dF_{\Phi_t}(\Phi_{t+1})
\]

subject to

\[
0 \leq c_t \leq w_t \quad \text{and} \quad \sum_{j=1}^{N} h_j = w_t - c_t.
\]

Because Theorem 1 applies directly to this simplified version of the multiperiod model, at any period \( t \) the function \( v_{t+1}(c_t, w_{t+1}) \) is monotone-increasing and strictly concave in \((c_t, w_{t+1})\) and is thus formally equivalent to the utility function used in the standard treatments of the two-period two-parameter portfolio models. If distributions of one-period security and portfolio wealth relatives are of the same two-parameter type, it follows directly that we have a multiperiod model in which the consumer’s behavior each period is indistinguishable from that of the consumer in the traditional two-period two-parameter portfolio models. From here it is a short step to develop a multiperiod setting for period by period application of the major results of the two-period two-parameter models of market equilibrium. In particular, all that we need are the assumptions of homogeneous expectations and riskless borrowing-lending opportunities.

But all this, of course, depends somewhat on the rather restrictive assumptions (1) and (2), which have the effect of making deterministic the evolution through time of market consumption and investment opportunities. What would happen if we allowed the investment opportunities to be available at \( t + 1 \) to depend, but not perfectly, on the opportunities available at \( t \), which in turn are somewhat uncertain in earlier periods?\(^{14}\) Or what would be the effects of allowing prices of consumption goods at \( t + 1 \) to depend, but again with some uncertainty, on the prices to be observed at \( t \)? And suppose that uncertainty about prices of consumption goods is related to the uncertainty in investment returns? All these questions would seem to take us out of context of the two-period two-parameter models, but in the current state of the art, we do not really know much about the characteristics of consumer and market equilibrium that they imply. And here, at the moment, is where things must be left.

\(^{14}\) For example, we might assume, and perhaps not unrealistically, that the levels of equilibrium expected returns on investment assets that will be available at \( t + 1 \) depend, but with some uncertainty, on the levels that will be available at \( t \), which in turn depend on those of \( t - 1 \), and so on.
II.C.4. Theorems and proofs

Thus we conclude our presentation of multiperiod consumption-investment models with a proof of Theorem 1 and an additional theorem relating the concavity of the consumer's utility function for consumption expenditures to the concavity of his underlying utility function for consumption goods and services.

Theorem 1. If \( U_{t+1}(C_t, w_{t+1} \mid \Phi_{t+1}) \) is monotone-increasing and strictly concave, henceforth m.i.s.c., in \( (C_t, w_{t+1}) \), then \( U_t(C_{t-1}, w_t \mid \Phi_t) \) is m.i.s.c. in \( (C_{t-1}, w_t) \).

Proof. The proof of the proposition relies primarily on straightforward applications of well-known properties of concave functions (see Ref. 6). We first establish Lemma 1.

Lemma 1. If \( U_{t+1}(C_t, w_{t+1} \mid \Phi_{t+1}) \) is m.i.s.c. in \( (C_t, w_{t+1}) \), the expected utility function

\[
\int_{\Phi_{t+1}} U_{t+1}(C_t, w_{t+1} \mid \Phi_{t+1}) \, dF_{\Phi_t}(\Phi_{t+1}) = \int_{\Phi_{t+1}} U_{t+1}(C_t, \sum_{j=1}^{N} h_j[1 + R_j(\Phi_{t+1})] \mid \Phi_{t+1}) \, dF_{\Phi_t}(\Phi_{t+1}) \tag{8.6}
\]

is strictly concave in \( (C_t, h_1, h_2, \ldots, h_N) \).

Proof. For any given value of \( \Phi_{t+1} \),

\[
w_{t+1} = \sum_{j=1}^{N} h_j[1 + R_j(\Phi_{t+1})]
\]

is a linear and thus concave, although not strictly concave, function of the \( h_j \). Because by assumption \( U_{t+1}(C_t, w_{t+1} \mid \Phi_{t+1}) \) is m.i.s.c. in \( (C_t, w_{t+1}) \), \( U_{t+1}(C_t, \sum_{j=1}^{N} h_j[1 + R_j(\Phi_{t+1})] \mid \Phi_{t+1}) \) is strictly concave in \( (C_t, h_1, h_2, \ldots, h_N) \). Integrating over \( \Phi_{t+1} \) in Equation (8.6) preserves this concavity.

The remainder of the proof of Theorem 1 is then as follows. Let \( \xi_t, h_t^x, \xi_t, i = 1, 2, \ldots, N \), and \( \xi_t, h_t^y, j = 1, 2, \ldots, N \), be the optimal values of

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15 If \( f(x_1, x_2, \ldots, x_N) = f(X) \) is m.i.s.c. in \( X \) and if \( x_i = g_i(y_1, y_2, \ldots, y_N) = g_i(Y) \), \( i = 1, 2, \ldots, N \), is concave, although not necessarily strictly concave, in \( Y \), then \( f(g(Y), g_1(Y), \ldots, g_N(Y) = f(G(Y)) \) is strictly concave in \( Y \). (See, for example, Ref. 5, p. 82.)
\[ c_t \text{ and the } h_j \text{ in Equation (8.4) for any two vectors } (C_{t-1}, w_t) \text{ and } (\hat{C}_{t-1}, \hat{w}_t) \text{ that differ in at least one element. For } 0 < \alpha < 1, \text{ let} \]
\[
\begin{align*}
\hat{C}_{t-1} &= \alpha C_{t-1} + (1 - \alpha) \hat{C}_{t-1}, \\
\hat{w}_t &= \alpha w_t + (1 - \alpha) \hat{w}_t, \\
\hat{c}_t &= \alpha c^*_t + (1 - \alpha) \hat{c}^*_t, \\
\hat{h}_j &= \alpha h^*_j + (1 - \alpha) \hat{h}^*_j, \quad j = 1, 2, \ldots, N.
\end{align*}
\]

To establish the concavity of \( U_i(\hat{C}_{t-1}, \hat{w}_t | \Phi_i) \), we must show that

\[
U_i(\hat{C}_{t-1}, \hat{w}_t | \Phi_i) > \alpha U_i(C_{t-1}, w_t | \Phi_i) + (1 - \alpha) U_i(\hat{C}_{t-1}, \hat{w}_t | \Phi_i). \tag{8.7}
\]

From Lemma 1, for \( 0 < \alpha < 1, \)

\[
\int_{\Phi_{i+1}} U_{i+1}(\hat{C}_{t-1}, \hat{c}_t, \sum_{j=1}^{N} \hat{h}_j[1 + R_j(\Phi_{i+1})]) | \Phi_{i+1}) \, dF_{\Phi_i}(\Phi_{i+1})
\]

\[
> \alpha \int_{\Phi_{i+1}} U_{i+1}(C_{t-1}, c^*_t, \sum_{j=1}^{N} h^*_j[1 + R_j(\Phi_{i+1})]) | \Phi_{i+1}) \, dF_{\Phi_i}(\Phi_{i+1})
\]

\[
+ (1 - \alpha) \int_{\Phi_{i+1}} U_{i+1}(\hat{C}_{t-1}, \hat{c}_t, \sum_{j=1}^{N} \hat{h}_j[1 + R_j(\Phi_{i+1})]) | \Phi_{i+1}) \, dF_{\Phi_i}(\Phi_{i+1})
\]

\[
= \alpha U_i(C_{t-1}, w_t | \phi_i) + (1 - \alpha) U_i(\hat{C}_{t-1}, \hat{w}_t | \Phi_i). \tag{8.8}
\]

Because the consumption-investment decision implied by \( \hat{c}_t, \hat{h}_j, j = 1, 2, \ldots, N, \) is feasible but not necessarily optimal for the wealth level \( \hat{w}_t, \)

\[
U_i(\hat{C}_{t-1}, \hat{w}_t | \Phi_i)
\]

\[
\geq \int_{\Phi_{i+1}} U_{i+1}(\hat{C}_{t-1}, \hat{c}_t, \sum_{j=1}^{N} \hat{h}_j[1 + R_j(\Phi_{i+1})]) | \Phi_{i+1}) \, dF_{\Phi_i}(\Phi_{i+1}),
\]

which, with Equation (8.8) implies Equation (8.7).\footnote{It is assumed that \( \hat{c}_t, \hat{h}_j, j = 1, 2, \ldots, N, \) is a feasible consumption-investment decision for the wealth level \( \hat{w}_t, \) or equivalently, that the set of feasible values of \( c_t, h_j, j = 1, 2, \ldots, N, \) is convex. But this is a weak assumption that is met, for example, when the constraints on \( c_t \) and the \( h_j \) are equations, such as \( \sum_{j=1}^{N} h_j = w_t - c_t, \) or linear inequalities, such as \( 0 \leq c_t \leq w_t \) or \( \hat{h}_j \leq h_j \leq \hat{h}_j, \) where \( \hat{h}_j \) and \( h_j \) are lower and upper bounds on the quantity invested in asset \( j. \)}
The monotonicity of \( U_t(C_{t-1}, w_t \mid \Phi_t) \) in \((C_{t-1}, w_t)\) follows straightforwardly from the monotonicity of \( U_{t+1}(C_t, w_{t+1} \mid \Phi_{t+1}) \) in \( C_t \). Thus the proposition is established.

Finally, as noted earlier (footnote 8, page 328 and also in the last section in Chapter 5), when utility is written, as we have done throughout, as a function of consumption dollars, we are implicitly summarizing the consumption opportunities, in terms of goods and services and their anticipated prices, that will be available in each period. We now show how a "cardinal" utility function for consumption dollars can be derived from a cardinal utility function for consumption commodities.

Let \( q(\Phi_i) = (q_1, q_2, \ldots, q_K(\Phi_i)) \) be the quantities of \( K(\Phi_i) \) available commodities consumed during \( i \) in state \( \Phi_i \), and let \( p(\Phi_i) = (p_1, p_2, \ldots, p_K(\Phi_i)) \) be the corresponding price vector. In any period or state one of the available consumption commodities is always "dollar gifts and bequests," which has price \$1 per unit. At the horizon \( \tau + 1 \), dollar gifts and bequests, denoted \( w_{\tau+1} \), is the only available consumption good. Let

\[
Q_r = (q(\Phi_{t-1}), \ldots, q(\Phi_1), \ldots, q(\Phi_r))
\]

represent lifetime consumption of commodities, and let \( V(Q_r, w_{\tau+1} \mid \Phi_{\tau+1}) \) be the consumer's utility of lifetime consumption, given state \( \Phi_{\tau+1} \) at \( \tau + 1 \), and where \( \Phi_{t-1} \subset \cdots \subset \Phi_1 \subset \cdots \subset \Phi_{\tau+1} \). The utility function for dollars of consumption can then be defined as

\[
U_{\tau+1}(C_{\tau+1} \mid \Phi_{\tau+1}) = \max_{Q_r} V(Q_r, w_{\tau+1} \mid \Phi_{\tau+1})
\]

\[
\text{s.t. } \quad C_{\tau+1} = (\sum_{i=1}^{K(\Phi_i)-1} p_i(\Phi_{t-1}) q_i(\Phi_{t-1}), \ldots, \sum_{i=1}^{K(\Phi_r)} p_i(\Phi_r) q_i(\Phi_r), w_{\tau+1})
\]

\[
= (c_{t-1}, \ldots, c_{\tau}, c_{\tau+1}). \quad (8.9)
\]

The role of \( \Phi_{\tau+1} \) in \( u_{\tau+1} \) is twofold. First, psychological attitudes toward current and past consumption, or "tastes," may depend on the state of the world. Second, even if tastes for consumption commodities are not state-dependent, so that \( \Phi_{\tau+1} \) can be dropped from \( V \), the utility of any stream of dollar consumption expenditures depends on the history of the set of available consumption commodities and their prices, both of which are subsumed in \( \Phi_{\tau+1} \).

A utility function \( U_{\tau+1}(C_{\tau+1} \mid \Phi_{\tau+1}) \) that has the properties required by Proposition 1 can then be obtained from \( V(Q_r, w_{\tau+1} \mid \Phi_{\tau+1}) \) as follows.

**Theorem 2.** If \( V(Q_r, w_{\tau+1} \mid \Phi_{\tau+1}) \) is m.i.s.c. in \((Q_r, w_{\tau+1})\), then \( U_{\tau+1}(C_r, w_{\tau+1} \mid \Phi_{\tau+1}) \) is m.i.s.c. in \((C_r, w_{\tau+1})\).
Proof. Let $Q^*_r$ be the optimal value of $Q_r$ in Equation (8.9) for $(C_r,w_{r+1})$, and let $\hat{Q}^*_r$ be optimal for $(\hat{C}_r,\hat{w}_{r+1})$, where the vectors $(C_r,w_{r+1})$ and $(\hat{C}_r,\hat{w}_{r+1})$ differ in at least one element. For $0 < \alpha < 1$, let

$$
(\hat{Q}_r,\hat{w}_{r+1}) = \alpha(Q^*_r,w_{r+1}) + (1 - \alpha)(\hat{Q}^*_r,\hat{w}_{r+1}),
$$

$$
(\hat{C}_r,\hat{w}_{r+1}) = \alpha(C_r,w_{r+1}) + (1 - \alpha)(\hat{C}_r,\hat{w}_{r+1}).
$$

Then the strict concavity of $V$ implies that

$$
V(\hat{Q}_r,\hat{w}_{r+1} | \Phi_{r+1}) > \alpha V(Q^*_r,w_{r+1} | \Phi_{r+1}) + (1 - \alpha) V(\hat{Q}^*_r,\hat{w}_{r+1} | \Phi_{r+1}).
$$

Or equivalently

$$
V(\hat{Q}_r,\hat{w}_{r+1} | \Phi_{r+1}) > \alpha U_{r+1}(C_r,w_{r+1} | \Phi_{r+1}) + (1 - \alpha) U_{r+1}(\hat{C}_r,\hat{w}_{r+1} | \Phi_{r+1}).
$$

Because $\hat{Q}_r$ is a feasible but not necessarily an optimal allocation of $\hat{C}_r$, an optimal allocation must have utility at least as high as that implied by $\hat{Q}_r$, so that

$$
U_{r+1}(\hat{C}_r,\hat{w}_{r+1} | \Phi_{r+1}) > \alpha U_{r+1}(C_r,w_{r+1} | \Phi_{r+1}) + (1 - \alpha) U_{r+1}(\hat{C}_r,\hat{w}_{r+1} | \Phi_{r+1}),
$$

and the concavity of $U_{r+1}$ is established.

To establish the monotonicity of $U_{r+1}$ in $C_r$, simply note that if the dollars available for consumption in any period are increased, consumption of at least one commodity can be increased without reducing consumption of any other commodity, so that utility must be increased. An optimal reallocation of consumption expenditures must do at least as well.

III. EFFICIENT CAPITAL MARKETS

Most of the models presented in this book are based on the assumption of a perfect capital market, that is, a market in which all available information is freely available to everybody, there are no transactions costs, and all market participants are price takers. Our uncertainty models of market equilibrium have in addition usually assumed homogeneous expectations; that is, market participants agree on the implications of available information for both current prices and probability distributions on future prices of individual investment assets.

Such a market has a very desirable feature. In particular, at any point in time market prices of securities provide accurate signals for resource allocation; that is, firms can make production-investment decisions, and consumers can choose among the securities that represent ownership of firms' activities under the presumption that security prices at any time "fully reflect" all available information. A market in which prices fully reflect available information is called efficient.
In keeping with the policy of this book, we only discuss here the theoretical foundations of the efficient markets model. But the model has been subjected to a substantial amount of empirical testing, and we contend that the model stands up well to the data. If true, this is important support for theoretical models based on a perfect market. Indeed we believe the available (mid-1971) evidence indicates that although the theoretical models we build in this book are abstracted substantially from real-world considerations, we nevertheless have reason to hope they might do quite well in describing actual market phenomena.

III.A. Expected Return or Fair Game Models

The definitional statement that in an efficient market prices fully reflect available information is so general that it has no testable implications. To make the model testable, the process of price formation must be specified in more detail. We must define somewhat more exactly what is meant by the phrase fully reflect.

One possibility would be to posit that equilibrium prices, or expected returns, on securities are generated by the two-parameter model in Chapter 7 and Section II in the present chapter. In general, however, the models and especially the empirical tests of capital market efficiency have not been so specific. Most of the available work is based only on the assumption that the conditions of market equilibrium can, somehow, be stated in terms of expected returns. In general terms, like the two-parameter model such theories would posit that, conditional on some relevant information set, the equilibrium expected return on a security is a function of its "risk." And different theories would differ primarily in how risk is defined.

All members of the class of such "expected return theories" can, however, be described notationally as follows:

\[ E(p_{j,t+1} | \Phi_t) = [1 + E(\tilde{R}_{j,t+1} | \Phi_t)]p_{j,t} \]

where \( E \) is the expected value operator; \( p_{j,t} \) is the price of security \( j \) at time \( t \); \( p_{j,t+1} \) is its price at \( t + 1 \), with reinvestment of any intermediate cash income from the security; \( \tilde{R}_{j,t+1} \) is the one-period percentage return; \( \Phi_t \) is a general symbol for whatever set of information is assumed to be fully reflected in the price at \( t \); and the tildes indicate that \( p_{j,t+1} \) and \( \tilde{R}_{j,t+1} \) are random variables at \( t \).

The process of price adjustment summarized in Equation (8.10) is as follows. At time \( t \) the market uses all the available information \( \Phi_t \) to assess the probability distribution of \( p_{j,t+1} \), which in turn implies an expected future price \( E(p_{j,t+1} | \Phi_t) \). This assessment of the distribution of \( p_{j,t+1} \), along with some model of equilibrium expected returns, then determines \( E(\tilde{R}_{j,t+1} | \Phi_t) \). The equilibrium expected return \( E(\tilde{R}_{j,t+1} | \Phi_t) \) then com-
bines with $E(p_{j,t+1} | \Phi_t)$ to determine the equilibrium price at $t$, $p_j$. And this is the sense in which the information $\Phi_t$ is fully reflected in the formation of $p_j$.

But we should note right off that, as simple as it is, the assumption that the conditions of market equilibrium can be stated in terms of expected returns elevates the purely mathematical concept of expected value to a status not necessarily implied by the general notion of market efficiency. The expected value is just one of many possible summary measures of a distribution of returns, and market efficiency per se, that is, the general notion that prices fully reflect available information, does not imbue it with any special importance. Thus, the results of tests based on this assumption depend to some extent on its validity as well as on the efficiency of the market. But some such assumption is the unavoidable price that one must pay to give the theory of efficient markets empirical content.

The assumptions that the conditions of market equilibrium can be stated in terms of expected returns and that equilibrium expected returns and current prices are formed on the basis of, and thus fully reflect, the information set $\Phi_t$ have a major testable implication—they rule out the possibility of trading systems based only on information in $\Phi_t$ that have expected profits or returns in excess of equilibrium expected profits or returns. Thus let

$$x_{j,t+1} = p_{j,t+1} - E(p_{j,t+1} | \Phi_t).$$

Then

$$E(x_{j,t+1} | \Phi_t) = 0,$$

which, by definition, says that the sequence $\{x_{ji}\}$ is a “fair game” with respect to the information sequence $\{\Phi_t\}$. Or equivalently, let

$$z_{j,t+1} = R_{j,t+1} - E(R_{j,t+1} | \Phi_t).$$

Then

$$E(z_{j,t+1} | \Phi_t) = 0,$$

so that the sequence $\{z_{ji}\}$ is also a fair game with respect to the information sequence $\{\Phi_t\}$.

In economic terms, $x_{j,t+1}$ is the excess market value of security $j$ at time $t + 1$: It is the difference between the observed price and the expected value of the price that was projected at $t$ on the basis of the information $\Phi_t$. And similarly, $z_{j,t+1}$ is the return at $t + 1$ in excess of the equilibrium expected return projected at $t$. Let

$$\alpha(\Phi_t) = [\alpha_1(\Phi_t), \alpha_2(\Phi_t), \ldots, \alpha_N(\Phi_t)]$$

be any trading system based on $\Phi_t$ that tells the investor the amounts $\alpha_j(\Phi_t)$ of funds available at $t$ that are to be invested in each of the $N$
available securities. The total excess market value at \( t + 1 \) that is generated by such a system is

\[
V_{t+1} = \sum_{j=1}^{N} \alpha_j(\Phi_t)[R_{j,t+1} - E(\bar{R}_{j,t+1} | \Phi_t)],
\]

which, from the fair game property in Equation (8.14), has expectation

\[
E(V_{t+1} | \Phi_t) = \sum_{j=1}^{n} \alpha_j(\Phi_t)E(\bar{R}_{j,t+1} | \Phi_t) = 0.
\]

One should not get lost in the algebra here, however. In intuitive terms, all that we have come to is the rather obvious conclusion that if all the information in \( \Phi_t \) is used by the market in assessing expected future returns and prices, there is no way an investor can use \( \Phi_t \) as the basis of a trading system with expected returns in excess of equilibrium expected returns.

We turn now to two special cases of this fair game efficient markets model, the submartingale and the random walk, that play an important role in the empirical literature.\(^7\)

### III.B. The Submartingale Model

Suppose we assume in Equation (8.10) that for all \( t \) and \( \Phi_t \)

\[
E(p_{j,t+1} | \Phi_t) \geq p_{jt},
\]

or equivalently

\[
E(\bar{R}_{j,t+1} | \Phi_t) \geq 0.
\]

(8.15)

This is a statement that the price sequence \( \{p_{jt}\} \) for security \( j \) follows a submartingale with respect to the information sequence \( \{\Phi_t\} \), which is to say nothing more than that the expected value of next period's price, as projected on the basis of the information \( \Phi_t \), is equal to or greater than the current price. If Equation (8.15) holds as an equality, so that expected returns and price changes are zero, the price sequence follows a martingale.

A submartingale in prices has one important testable implication. Consider the set of "one security and cash" mechanical trading rules, by which we mean systems that concentrate on individual securities and that define the conditions under which the investor would hold a given security, sell it short, or simply hold cash at any time \( t \). Then the assumption in

\(^7\) Although we refer to the model summarized by Equation (8.10) as the fair game model, keep in mind that the fair game properties of the model are implications of the assumptions that (1) the conditions of market equilibrium can be stated in terms of expected returns and (2) the information \( \Phi_t \) is fully utilized by the market in forming equilibrium expected returns and thus current prices.
Equation (8.15) that expected returns conditional on \( \Phi \), are nonnegative directly implies that such trading rules based only on the information in \( \Phi \), cannot have greater expected profits than a policy of always buying and holding the security during the future period in question.\(^{18}\)

III.C. The Random Walk Model

In the early treatments of the efficient markets model in the finance literature, the statement that the current price of a security fully reflects available information was assumed to imply that successive price changes, or more usually, successive one-period returns, are independent. In addition, it was usually assumed that successive changes, or returns, are identically distributed. Together the two hypotheses constitute the random walk model.\(^{19}\) Formally, the model says that

\[
f(R_{t,t+1} | \Phi_t) = f(R_t),
\]

which is the usual statement that the conditional and marginal probability distributions of an independent random variable are identical. In addition, the density function \( f \) must be the same for all \( t \).

Expression (8.16) says much more, of course, than the general expected return model summarized by Equation (8.10). For example, if we restrict Equation (8.10) by assuming that the expected return on security \( j \) is constant over time, we have

\[
E(\bar{R}_{j,t+1} | \Phi_t) = E(\bar{R}_j),
\]

\(^{18}\) Note that the expected profitability of one security and cash trading systems vis-à-vis buy-and-hold is not ruled out by the general expected return or fair game efficient markets model. The latter rules out systems with expected profits in excess of equilibrium expected returns, but because in principle it allows equilibrium expected returns to be negative, holding cash, which always has zero actual and thus expected return, may have higher expected return than holding some security.

And negative equilibrium expected returns for some securities are quite possible. For example, in the two-parameter model, a security whose returns on average move opposite to the general market is particularly valuable in reducing dispersion of portfolio returns, and so its equilibrium expected return may well be negative.

\(^{19}\) The statistically sophisticated reader will recognize that the terminology is loose. Prices follow a random walk only if price changes are independent, identically distributed; and even then we should say “random walk with drift,” because expected price changes can be nonzero. If one-period returns are independent, identically distributed, prices do not follow a random walk, because the distribution of price changes depends on the price level. But although rigorous terminology is usually desirable, our loose use of terms should not cause confusion; and our usage follows that of the efficient markets literature.
This says that the mean of the distribution of $R_{i,t+1}$ is independent of the past information, whereas the random walk model in Equation (8.16) in addition says that the entire distribution is independent of the past information.\(^\text{20}\)

It is best to regard the random walk model as an extension of the general expected return or fair game efficient markets model in the sense of making a more detailed statement about the economic environment. The fair game model just says that the conditions of market equilibrium can be stated in terms of expected returns, and thus it says little about the details of the stochastic process generating returns. A random walk arises in the context of such a model when the environment is, fortuitously, such that the evolution of investor tastes and the process generating new information combine to produce equilibria in which return distributions repeat themselves through time.

A detailed review of the empirical literature on the efficient markets model is available elsewhere [11]; thus we do not discuss the evidence here. (This is, of course, also in keeping with the policy of preceding chapters.) We wish to suggest, however, that there is much evidence in support of the position that perfect markets models, like those developed in this book, have substantial value in describing real-world economic phenomena.

REFERENCES

Much of the discussion of multiperiod consumption-investment decisions in Section II is taken from


Other works on multiperiod models are


\(^\text{20}\) The random walk model does not say, however, that past information is of no value in assessing distributions of future returns. Indeed because return distributions are assumed to be the same through time, past returns are the best source of such information. The random walk model does say, however, that the sequence, or the order, of the past returns is of no consequence in assessing distributions of future returns.
Properties of convex and concave functions used in proving Theorems 1 and 2 can be found in


Consumption-investment models involving state-dependent utilities can be found in


References for two-period two-parameter consumption-investment models and models of capital market equilibrium are at the end of Chapters 6 and 7.

The discussion of the efficient markets model in Section III draws heavily from


In addition to providing extensive references, this paper contains a detailed summary of the empirical literature. The first rigorous treatments of the efficient markets model are in