GRS Review

Karl Diether

University of Chicago

Graduate School of Business

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Empirical Implications of the ICAPM

- Consider the regressions,

$$R_{it} - R_{ft} = \alpha_i + \sum_{j=1}^{L} \beta_{ij} \left(F_{jt} \right) + \varepsilon_{it}, \; i = 1 \ldots N,$$

where,

- $L - 1$ is number of priced state variables.
- F_j is the excess return (or return from a zero-cost portfolio) on the jth factor portfolio.
- $R_i - R_f$ is the excess return on a security or portfolio.

Empirical Implications of the ICAPM

If each factor portfolio is multifactor minimum variance in a S state variable world then,

1. $\alpha_i = 0\ \forall i$
2. Some linear combination of the factor portfolios is on the minimum variance boundary.

There is actually only one implication because $\alpha_i = 0\ \forall i$ if and only if some portfolio of the right-hand side portfolios is on the minimum variance boundary.

The GRS Test

1. the GRS test is a statistical test of the hypothesis that $\alpha_i = 0\ \forall i$.
2. Equivalently, it is a test that some linear combination of the factor portfolios is on the minimum variance boundary.
3. Equivalently, it is also a test that each factor portfolio is multifactor minimum variance in a S state variable world.

GRS Test, Sharpe Ratio Representation

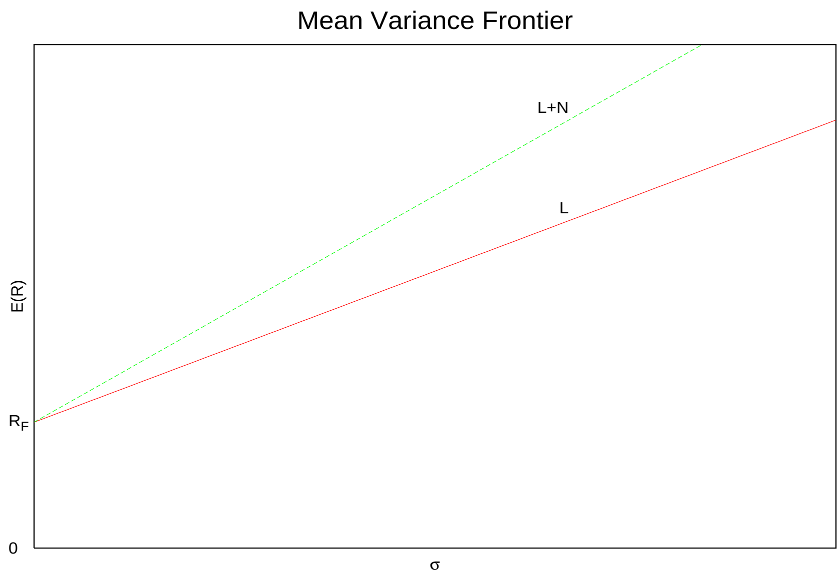
$$\left(\frac{T}{N}\frac{T-N-L}{T-L-1}\right)\left[\frac{\sqrt{1+\hat{\theta}^2_{N+L}}}{\sqrt{1+\hat{\theta}^2_L}}\right]^2 - 1 \sim F(N, T-N-L)$$

where,

1. $\hat{\theta}_{N+L}$ is the ex post maximum Sharpe ratio of the N test assets and the L factor portfolios.
2. $\hat{\theta}_L$ is the ex post maximum Sharpe ratio of the L factor portfolios.

Thus the GRS statistic determines whether $|\hat{\theta}_{N+L}|$ is statistically greater than $|\hat{\theta}_L|$.

Graphical Representation



GRS Test, $\alpha = 0$ Representation

$$\left(\frac{T}{N}\right)\left(\frac{T-N-L}{T-L-1}\right)\left[\frac{\hat{\alpha}'\hat{\Sigma}^{-1}\hat{\alpha}}{1+\bar{\mu}'\hat{\Omega}^{-1}\bar{\mu}}\right] \sim F(N, T-N-L)$$

where,

1. $\hat{\alpha}$ is a $N \times 1$ vector of estimated intercepts.
2. $\hat{\Sigma}$ is an unbiased estimate of the residual covariance matrix.
3. $\bar{\mu}$ is a $L \times 1$ vector of the factor portfolios' sample means.
4. $\hat{\Omega}$ is an unbiased estimate of the factor portfolios' covariance matrix.

if $\alpha_i = 0\ \forall i$, then the GRS statistic equals zero; the larger the αs are in absolute value the greater the GRS statistic will be.

Computing The GRS Statistic

The problem set will require you to compute a few GRS statistics. You can compute it using any software package you wish. The following software packages are commonly used:

- Excel
- A Matrix Language (i.e. MATLAB, Gauss, or Ox)

It is easier to compute a GRS statistic using a matrix language than it is using Excel. However, it is probably not worth your time to learn a matrix language just for this problem set.

GRS Recipe

Step 1: Time series regressions

Run the following regression for all N left hand side portfolios:

$$R_{it} - R_{ft} = \alpha_i + \sum_{j=1}^{L} \beta_{ij} (F_{jt}) + \varepsilon_{it}$$

If you are testing the Fama French three factor model then you would run,

$$R_{it} - R_{ft} = \alpha_i + b_i (R_{Mt} - R_{ft}) + s_i SMB_t + h_i HML_t + \varepsilon_{it}$$

GRS Recipe

Step 2: The intercept vector

Form the estimated intercepts into a $N \times 1$ vector ($\hat{\alpha}$).

$$\hat{\alpha} = \begin{bmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \vdots \\ \hat{\alpha}_N \end{bmatrix}$$

GRS Recipe

Step 3: The residual matrix

Calculate the residual for each regression,

$$\hat{\varepsilon}_{it} = (R_{it} - R_{ft}) - \hat{\alpha}_i - \sum_{j=1}^{L} \hat{\beta}_{ij} (F_{jt}).$$

For example, if you are testing the Fama French three factor model then you would compute,

$$\hat{\varepsilon}_{it} = (R_{it} - R_{ft}) - \hat{\alpha}_i - \hat{b}_i (R_{Mt} - R_{ft}) - \hat{s}_i SMB_t - \hat{h}_i HML_t.$$

GRS Recipe

Step 3: The residual matrix continued

Form the residuals into a $T \times N$ matrix (note, that now the first subscript refers to time period and the second refers to the test portfolio),

$$\hat{\varepsilon} = \begin{bmatrix} \hat{\varepsilon}_{11} & \hat{\varepsilon}_{12} & \cdots & \hat{\varepsilon}_{1N} \\ \hat{\varepsilon}_{21} & \hat{\varepsilon}_{22} & \cdots & \hat{\varepsilon}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\varepsilon}_{T1} & \hat{\varepsilon}_{T2} & \cdots & \hat{\varepsilon}_{TN} \end{bmatrix}$$

GRS Recipe

Step 4: $\hat{\Sigma} = cov(\varepsilon)$

Compute an unbiased estimate of the covariance matrix of residuals,

$$\hat{\Sigma} = \frac{\hat{\varepsilon}'\hat{\varepsilon}}{T - L - 1}$$

- $\hat{\Sigma}$ is a $N \times N$ matrix. For example, if there are ten test portfolios, then $\hat{\Sigma}$ is 10×10 matrix.
- In Excel you do not have to explicitly compute $\hat{\Sigma}$ using the above formula; you can just use the "COVAR" command. However, Excel's estimate is not unbiased. You need to multiply Excel's estimate by $T/(T - L - 1)$.

GRS Recipe

Step 5: Factor mean vector

Calculate the sample means of the factor portfolios and form a $L \times 1$ vector ($\bar{\mu}$) of sample means.

$$\bar{\mu} = \begin{bmatrix} \bar{F}_1 \\ \bar{F}_2 \\ \vdots \\ \bar{F}_L \end{bmatrix}$$

GRS Recipe

Step 6: The factor matrix

Form the factor portfolio (excess) returns into a $T \times L$ matrix (note, that now the 1st subscript refers to time period and the 2nd refers to the factor portfolio),

$$F = \begin{bmatrix} F_{11} & F_{12} & \cdots & F_{1L} \\ F_{21} & F_{22} & \cdots & F_{2L} \\ \vdots & \vdots & \ddots & \vdots \\ F_{T1} & F_{T2} & \cdots & F_{TL} \end{bmatrix}$$

GRS Recipe

Step 7: Compute an unbiased estimate of the covariance matrix of the factors (the dimension of the covariance matrix is $L \times L$),

$$\hat{\Omega} = \frac{(F - \overline{F})'(F - \overline{F})}{T - 1}$$

where,

$$\overline{F} = \begin{bmatrix} \overline{F_1} & \overline{F_2} & \cdots & \overline{F_L} \\ \overline{F_1} & \overline{F_2} & \cdots & \overline{F_L} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{F_1} & \overline{F_2} & \cdots & \overline{F_L} \end{bmatrix}$$

Note: In Excel you do not have to explicitly compute $\hat{\Omega}$ using the above formula; you can just use the “COVAR” command. However, Excel’s estimate is not unbiased. You need to multiply Excel’s estimate by $T/(T-1)$.

GRS Recipe

Step 8: Compute the GRS statistic,

$$\left(\frac{T}{N}\right)\left(\frac{T-N-L}{T-L-1}\right)\left[\frac{\hat{\alpha}'\hat{\Sigma}^{-1}\hat{\alpha}}{1+\overline{\mu}'\hat{\Omega}^{-1}\overline{\mu}}\right] \sim F(N, T-N-L)$$

Note: Both $\hat{\alpha}'\hat{\Sigma}^{-1}\hat{\alpha}$ and $\overline{\mu}'\hat{\Omega}^{-1}\overline{\mu}$ are scalars. If you do not get scalars, then you have done something wrong.

Step 9: Find the p-value of the GRS statistic. You can find the p-value in Excel using the “FDIST” command.