

NONPARAMETRIC STOCHASTIC VOLATILITY*

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We provide nonparametric methods for stochastic volatility modelling. Our methods allow for the joint evaluation of return and volatility dynamics with nonlinear drift and diffusion functions, nonlinear leverage effects, and jumps in returns and volatility with possibly state-dependent jump intensities. In the first stage, we identify spot volatility by virtue of jump-robust nonparametric estimates. Using observed prices and estimated spot volatilities, the second stage extracts the functions and parameters driving price and volatility dynamics from nonparametric estimates of the bivariate process' infinitesimal moments. We present a complete asymptotic theory under recurrence, thereby accommodating the persistence properties of volatility in finite samples.

1. Introduction. Understanding volatility is of fundamental importance for effective portfolio choice, derivative pricing, and risk management, among other issues. A successful strand of the literature on volatility estimation has focused on stochastic volatility modelling either in continuous time or in discrete time. This literature provides alternative methods to filter volatility - an inherently unobservable state variable - by using return data sampled at relatively low (generally daily) frequencies. An equally successful, but alternative, recent strand of the literature on volatility estimation has recognized the identification potential of return data sampled at intra-daily frequencies to effectively treat daily volatility (estimated by aggregating squared intra-daily returns) as an "observable" quantity, without need for filtering on the basis of low-frequency return data (for a review see Andersen, Bollerslev and Diebold, 2010 and Mykland and Zhang, 2009a). This second body of work has seldom investigated the implications of high-frequency volatility estimation for stochastic volatility modelling. The *parametric* approaches of Bollerslev and Zhou (2002), Corradi and Distaso (2006) and Todorov (2009), however, are important exceptions and promising contributions in this area.

We further bridge the gap between arguably the two main strands of the current literature on financial markets volatility by providing functional inferential methods. Specifically, we study *nonparametric* stochastic volatility modelling in continuous time using high-frequency asset price data for the purpose of spot volatility estimation.

Consider the price/variance system:

$$\begin{aligned} (1) \quad d \log(p_t) &= \mu(\sigma_t^2)dt + \sigma_t dW_t^r + dJ_t^r, \\ (2) \quad df(\sigma_t^2) &= m_{f(\cdot)}(\sigma_t^2)dt + \Lambda_{f(\cdot)}(\sigma_t^2)dW_t^\sigma + dJ_t^\sigma, \end{aligned}$$

*We thank the participants at the SoFiE Inaugural conference (New York, June 4-6, 2008), the Festschrift in honor of Peter Phillips (Singapore, July 14-15, 2008), the Far Eastern and South Asian Meetings of the Econometric Society (Singapore, July 16-18, 2008), the Hitotsubashi University's International Conference on High Frequency Data Analysis in Financial Markets (Tokyo, October 25-26), the LSE Conference on Recent Advances in High-Frequency Financial Econometrics (London, November 15, 2008), and the North American Winter Meetings of the Econometric Society (San Francisco, January 3-5, 2009) for discussions. We are especially grateful to Valentina Corradi, Fulvio Corsi, Cecilia Mancini, and Eric Renault for useful comments and suggestions. The usual disclaimers apply

Keywords and phrases: Spot variance, stochastic volatility, jumps in returns, jumps in volatility, leverage effects, risk-return trade-offs, kernel methods, recurrence

where $\{dW_t^r, dW_t^\sigma\} = \{\rho(\sigma_t^2)dW_t^1 + \sqrt{1 - \rho^2(\sigma_t^2)}dW_t^2, dW_t^1\}$ with $-1 \leq \rho(\cdot) \leq 1$, $\{W_t^1, W_t^2\}$ are standard Brownian motions, $\{J_t^r, J_t^\sigma\}$ are Poisson jump processes independent of each other and independent of $\{W_t^1, W_t^2\}$ with intensities $\lambda^r(\cdot)$ and $\lambda_{f(\cdot)}^\sigma(\cdot)$, $f(\cdot)$ is a monotone transformation, and $\mu(\cdot)$, $m_{f(\cdot)}(\cdot)$, $\Lambda_{f(\cdot)}(\cdot)$, and $\rho(\cdot)$ are generic functions satisfying suitable smoothness conditions. A similar system, but without jumps in both returns and variance, has been considered by Renò (2006, 2008) and in recent, concurrent, work by Kanaya and Kristensen (2010), Comte, Genon-Catalot and Rozenholc (2010) and, in discrete time, by Comte, Lacour and Rozenholc (2010). The absence of discontinuities in their frameworks naturally results in drastically different approaches to spot volatility filtering and nonparametric modelling of the dynamics. In light of the presence of jumps in both volatility and returns, leverage effects, and nonlinearities, the specification in Eqs. (1)-(2) nests and generalizes, as we will discuss below, a large number of popular stochastic volatility models in the extant literature.

The unobservable nature of variance leads to the two substantive aspects of the nonparametric identification procedure which we propose. In the first stage, we filter *spot variance* by localizing (in time) appropriately-chosen high-frequency estimates of *integrated variance* $\int_t^{t+\phi} \sigma_s^2 ds$ (with $\phi > 0$) which are robust to the presence of jumps. In the second stage, we use the resulting spot variance estimates to identify the parameters and functions driving return/volatility dynamics (for instance, solely focusing on the volatility equation, $\lambda_{f(\cdot)}^\sigma(\cdot)$, $m_{f(\cdot)}(\cdot)$, $\Lambda_{f(\cdot)}(\cdot)$ and, given assumptions on the jump size distribution, the moments of the volatility jumps). The second stage requires controlling the measurement error introduced by the first-stage spot volatility estimates. We provide conditions to do so.

Differently from existing work on parametric stochastic volatility modelling, we avoid imposing (possibly affine) parametric structures. Specifically, we identify the relevant functions (through estimates of the system's infinitesimal moments) using nonparametric kernel methods for jump-diffusion processes as originally proposed by Bandi and Nguyen (2003) and Johannes (2004) in more traditional frameworks, namely in the context of scalar models with observables. While our identification methods and limiting results apply more broadly, we provide asymptotic inferential theories for nonlinear generalizations of the successful stochastic variance model with exponential jumps in variance in, e.g., Duffie, Pan and Singleton (2000), and Eraker, Johannes and Polson (2003) and the log variance model with normal jumps in variance in, e.g., Jacquier, Polson and Rossi (1994) and Andersen, Benzoni and Lund (2002).

In order to present ideas in the context of a classical estimation problem, we use Nadaraya-Watson kernel estimates. However, as we illustrate below, extensions to alternative functional estimation methods may be readily derived. To this extent, we discuss the local linear and the local polynomial case explicitly.

Finally, identification does not require stationarity. Rather, it relies on recurrence, which is known to be a milder assumption than stationarity and mixing (see Bandi and Phillips (2010) for a review of identification methods for recurrent continuous-time processes). Because of the persistent behavior of daily volatility series, methods which only hinge on recurrence and do not rely on the information contained in a potentially inaccurately estimated (in finite samples) stationary density may, arguably, be particularly suitable for this problem.

2. Infinitesimal moment estimators: asymptotics. We start with the assumptions on the bivariate process.

ASSUMPTION 1. (*The return/variance system*) Define the system in Eqs. (1)-(2) on a filtered probability space $(\Omega, (\mathfrak{F}_t)_{0 \leq t \leq \infty}, \mathcal{P})$ satisfying the usual conditions. Let $f(\cdot)$ be twice continuously-differentiable. Let all of the system's functions be twice continuously-differentiable and uniformly bounded. We assume that there exists a unique strong solution and that the spot variance process is recurrent in a bounded set.

Sufficient conditions on the system's functions which guarantee existence of a strong solution and recurrence

can be found in, e.g., [Bandi and Phillips \(2010\)](#). Under recurrence, for any x in the domain of σ_t^2 and $r > 0$,

$$P_x(|\sigma_t^2 - x| < r \text{ for a sequence of times increasing to } \infty) = 1.$$

In other words, the variance process returns to open sets in its range an infinite number of times over time, thereby making consistent point-wise kernel estimation possible even in the absence of a time-invariant stationary density. Recurrent processes for which a stationary density exists converge to it and are called positive recurrent (or ergodic). They are called strictly stationary when started at the stationary density. Recurrent processes which are not endowed with a stationary density are called null recurrent. See, e.g., [Bandi and Phillips \(2010\)](#) for discussions.

The requirement in [Assumption 1](#) of uniformly bounded driving functions is a classical, often implicitly-made, assumption in the nonparametric literature when letting T grow without bound along with a vanishing sampling span. The requirement of volatility recurrence in a bounded set is less traditional. However, it is important to remark that both assumptions are far from empirically restrictive since the data - and its transformations - are clearly bounded. It is easy to show that these assumptions can handle virtually all stochastic volatility models studied in the literature by suitably modifying the driving functions outside of the domain of the data. Consider, as a relevant example, the [Heston \(1993\)](#) model. In this case, $f(\cdot)$ is the identity, there are no jumps, and $\Lambda(\sigma_t^2) = c\sqrt{\sigma_t^2}$, i.e., not uniformly bounded. Denote by $\bar{\sigma}^2$ a value greater than the maximum variance level observed in the data. Then, it is *observationally-equivalent* to consider the Heston model with $\Lambda(\cdot)$ replaced by $\bar{\Lambda}(\sigma_t^2) = c\sqrt{\min(\sigma_t^2, \bar{\sigma}^2)}$, which is uniformly bounded. The requirement of boundedness of the domain of σ_t^2 can be treated similarly. It is *observationally-equivalent* to replace $\Lambda(\cdot)$ with $\tilde{\Lambda}(\sigma_t^2)$ equal to $\Lambda(\sigma_t^2)$ when $\sigma_t^2 \leq \bar{\sigma}^2$ and to $c\sqrt{2\bar{\sigma}^2 - \sigma_t^2}$, when $\bar{\sigma}^2 < \sigma_t^2 < 2\bar{\sigma}^2$. This transformation would create a reflecting upper barrier, analogous to the reflecting lower barrier at 0 of the Heston model, which would imply recurrence in a bounded set. Also, notice that the requirement of boundedness of σ_t makes the requirement of uniformly bounded driving functions extremely mild. For example, in the Heston model this latter requirement is automatically implied by the first. Similarly, it would be satisfied by any function of the spot variance process (the price drift, for instance) under very weak regularity conditions preventing the insurgence of an asymptote at infinity. The addition of jumps is also not a problem. When the distribution of jumps is not bounded, e.g. when the jumps are exponentially or normally distributed, we can always replace it with truncated distributions differing negligibly from the original ones and, therefore, satisfying the assumption.¹ Importantly, treating the case of variance recurrence in an unbounded set, and possibly unbounded functions, in full generality solely leads to conditions on the vanishing rate of the discretization error which cannot be verified empirically. This theoretically-interesting, but empirically-uncompelling, general case is studied in the Appendix.

The persistence features of variance should be a concern when identification is conducted by heavily relying on the informational content of the variance process' stationary density. The use of identification methods which do not hinge on stationarity (but only on recurrence) is expected to lead to less distorted estimates in regions where the variance's stationary density cannot be estimated reliably and, as we will show below, a somewhat more objective representation of statistical uncertainty. In effect, the size of the point-wise (asymptotic) confidence bands of each infinitesimal moment estimate will be shown to be an inverse function of the number of visits made by the spot variance process in the local neighborhood of each spatial point at which estimation is performed. Such a number will be represented here by the chronological local time of the spot variance process, namely $\bar{L}_{\sigma^2}(T, x)$.

¹For example, we might replace the exponential distribution with parameter μ_ξ with a truncated exponential distribution with truncation parameter b and pdf $f(x) = \frac{1}{\mu_\xi} e^{-x/\mu_\xi} / (1 - e^{-b/\mu_\xi})$. Instead of having mean μ_ξ , the truncated exponential has mean $\mu_\xi \left[\frac{1 - (1 + b\mu_\xi)e^{-b/\mu_\xi}}{1 - e^{-b/\mu_\xi}} \right]$. This is arbitrarily close to μ_ξ for b sufficiently large, and similar considerations apply to the higher moments.

ASSUMPTION 2. (*The sampling scheme*) We assume availability of n discretely sampled price observations over the time span $[0, T]$. The distance between these n observations is $\Delta_{n,T} = T/n$. We also assume availability of k equispaced high-frequency price observations in each interval $[i\Delta_{n,T}, i\Delta_{n,T} + \phi]$. In the limit, $T \rightarrow \infty$, $n \rightarrow \infty$, $k \rightarrow \infty$, $\Delta_{n,T} \rightarrow 0$, and $\phi \rightarrow 0$.

For any subinterval $[i\Delta_{n,T}, i\Delta_{n,T} + \phi]$ of size ϕ , the k intra-period price observations are employed to estimate spot variance ($\hat{\sigma}_{iT/n}^2$ with $i = 1, \dots, n$). The n resulting spot variance estimates $\{\hat{\sigma}_{T/n}^2, \hat{\sigma}_{2T/n}^2, \dots, \hat{\sigma}_T^2\}$ along with the observable price observations $\{\log p_{T/n}, \log p_{2T/n}, \dots, \log p_T\}$ are then used to identify the dynamics of the bi-variate system in Eqs. (1)-(2). Asymptotic conditions on $T, n, \Delta_{n,T}, k$ and ϕ for consistency and asymptotic normality of the relevant estimates will be listed in what follows.

The functions driving the dynamics of diffusion and jump-diffusion models are known to have infinitesimal conditional moment representations which can be exploited for the purpose of nonparametric identification. Our strategy will be to identify (and discuss an asymptotic theory for) the infinitesimal conditional moments of the bi-variate system (in this section) before backing out the relevant functions from the estimated infinitesimal moments (in Section 3).

To this extent, we identify the j -th infinitesimal moment of the variance process, i.e.,

$$(3) \quad \theta^j(x) = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \mathbf{E} \left[(f(\sigma_{t+\Delta}^2) - f(\sigma_t^2))^j \mid \sigma_t^2 = x \right] \quad j = 1, \dots,$$

by virtue of

$$\hat{\theta}^j(x) = \frac{1}{\Delta_{n,T}} \frac{\sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\hat{\sigma}_{iT/n}^2 - x}{h_{n,T}} \right) \left[f(\hat{\sigma}_{(i+1)T/n}^2) - f(\hat{\sigma}_{iT/n}^2) \right]^j}{\sum_{i=1}^n \mathbf{K} \left(\frac{\hat{\sigma}_{iT/n}^2 - x}{h_{n,T}} \right)} \quad j = 1, \dots,$$

where $\mathbf{K}(\cdot)$ is a kernel function and the unobservable spot variance is replaced by suitable estimates $\hat{\sigma}_{iT/n}^2$ for $i = 1, \dots, n$. Specifically, we use a localized (in time) threshold realized variance estimator (Mancini, 2009):

$$(4) \quad \hat{\sigma}_{iT/n}^2 = \frac{1}{\phi} \sum_{j=1}^k r_{i,j}^2 \mathbf{1}_{\{r_{i,j}^2 \leq \vartheta\}}$$

where $r_{i,j} = \log p_{iT/n+j\phi/k} - \log p_{iT/n+(j-1)\phi/k}$ is the return between two consecutive intra-period observations in the interval ϕ and ϑ is a threshold such that $\vartheta \rightarrow 0$. The spot variance estimator we use ought to be robust to jumps in returns and identify spot variance (as $\phi \rightarrow 0$ with $k \rightarrow \infty$) even in the presence of price discontinuities. By eliminating discontinuities larger than the threshold ϑ , threshold realized variance is a natural way to filter jumps in prices. By adapting the results in Mancini (2009), which are for *integrated* variance, to the *spot* variance case, it can be shown that

$$(5) \quad k^{1/2} \{ \hat{\sigma}_t^2 - \sigma_t^2 \} \xrightarrow[k \rightarrow \infty, \phi \rightarrow 0, \vartheta \rightarrow 0]{} MN(0, 2\sigma_t^4) \quad \forall t > 0$$

where MN stands for mixed normal, provided $\frac{1}{\vartheta} \left(\frac{\phi}{k} \log \left(\frac{k}{\phi} \right) \right) \rightarrow 0$ (as $\phi, \vartheta \rightarrow 0$ with $k \rightarrow \infty$) and $k^{1/2} \phi^{1/2} \rightarrow 0$ (as $\phi \rightarrow 0$ with $k \rightarrow \infty$). The former condition guarantees that the threshold goes to zero more slowly than the (squared) modulus of continuity of Brownian motion, thereby only annihilating the genuine jumps, while the second condition eliminates the asymptotic bias of the estimator. A similar approach in the absence of jumps has been followed by Fan and Wang (2008), Mykland and Zhang (2008) and Kristensen (2010). Importantly,

Eq. (5) leads to a probability bound $O_p\left(\frac{1}{k^{1/2}}\right)$ on the measurement error $\{\widehat{\sigma}_t^2 - \sigma_t^2\}$ which is point-wise and, hence, insufficient for nonparametric inference on the infinitesimal moments of the system. In Appendix A, Lemma 6, we provide a uniform result.

The kernel function $\mathbf{K}(\cdot)$ satisfies the following property:

ASSUMPTION 3. *The function $\mathbf{K}(x)$ is a nonnegative, bounded, twice continuously differentiable, and symmetric kernel defined on a compact set S satisfying $\int_S \mathbf{K}(s)ds = 1$, $\int_S \mathbf{K}^2(s)ds < \infty$, and $\int_S s^2 \mathbf{K}(s)ds < \infty$. The kernel's derivative are absolutely integrable.*

Finally, we turn to the conditions on $T, n, \Delta_{n,T}, k$, the bandwidths ϕ and $h_{n,T}$, and the threshold ϑ for consistency and weak convergence of the infinitesimal moments when spot variance is estimated.

ASSUMPTION 4. *Define $\widehat{L}_{\sigma^2}(T, x) = \frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^n \mathbf{K}\left(\frac{\sigma_{iT/n}^2}{h_{n,T}}\right)$, a classical (infeasible) nonparametric estimate of the spot variance's local time. Assume $k, n, T \rightarrow \infty$, $\Delta_{n,T} \rightarrow 0$, $\phi \rightarrow 0$, $h_{n,T} \rightarrow 0$, and $\vartheta \rightarrow 0$. The following conditions will be requested, when appropriate, as specified in the statements of the Theorems (C is a constant):*

$$(4.0) \quad \frac{1}{\vartheta} \left(\frac{\phi}{k} \log \left(\frac{k}{\phi} \right) \right) \rightarrow 0,$$

$$(4.1) \quad h_{n,T} \widehat{L}_{\sigma^2}(T, x) \xrightarrow{a.s.} \infty,$$

$$(4.2) \quad \frac{\Delta_{n,T}}{h_{n,T}^2} \rightarrow 0,$$

$$(4.3) \quad h_{n,T}^5 \widehat{L}_{\sigma^2}(T, x) \xrightarrow{a.s.} C,$$

$$(4.4) \quad \frac{\Delta_{n,T} \sqrt{\widehat{L}_{\sigma^2}(T, x)}}{h_{n,T}^{3/2}} \xrightarrow{a.s.} 0,$$

$$(4.5) \quad g(n, T, k, \phi) = \frac{(\log(n))^{1/2}}{h_{n,T} k^{1/2}} + \frac{\phi^{1/2}}{h_{n,T}} \rightarrow 0,$$

$$(4.6) \quad g(n, T, k, \phi) \frac{h_{n,T}}{\Delta_{n,T}} \rightarrow 0,$$

$$(4.7) \quad g(n, T, k, \phi) \frac{1}{\Delta_{n,T}^{1/2}} \rightarrow 0,$$

$$(4.8) \quad \sqrt{h_{n,T} \widehat{L}_{\sigma^2}(T, x)} g(n, T, k, \phi) \frac{h_{n,T}}{\Delta_{n,T}} \xrightarrow{a.s.} 0,$$

$$(4.9) \quad \sqrt{h_{n,T} \widehat{L}_{\sigma^2}(T, x)} g(n, T, k, \phi) \frac{1}{\Delta_{n,T}^{1/2}} \xrightarrow{a.s.} 0.$$

Assumption 4.0 is needed to guarantee that threshold realized variance eliminates discontinuities whose order is larger than that of the modulus of continuity of Brownian motion. Assumptions 4.1 and 4.2 are necessary for consistency (in probability) of the kernel estimates of the infinitesimal moments under observability of the spot variance process. Assumption 4.6 and 4.7, along with Assumption 4.1 and 4.2, yield consistency of the infinitesimal moments when spot variance is not observable but is, instead, estimated by virtue of the estimator in Eq. (4) above (Assumption 4.6 applies to the case $j = 1$, Assumption 4.7 applies to the case $j > 1$). Assumptions 4.1, 4.3 and 4.4, guarantee weak convergence, at the optimal MSE rate $\widehat{L}_{\sigma^2}(T, x)^{-4/5}$, of the kernel estimates, again under observability of the spot variance process. Assumptions 4.8 ($j = 1$) and 4.9 ($j > 1$), along with 4.1, 4.3, and 4.4, yield weak convergence when spot variance is estimated. Assumption 4.9 suffices for the weak convergence of all infinitesimal price moments and the drift of the price equation (risk-return trade-off) since prices are observed.

REMARK 1. Call $a(T)$ the almost-sure divergence rate of the spot variance's local time. Consider the case $j > 1$ (estimation of all functions excluding the drift). To make the relevant orders explicit, it is worth rewriting the assumptions for consistency in 4.1, 4.3 (with $C = 0$) and 4.7 as follows: $h_{n,T} \sim a(T)^{-\beta}$ with $\frac{1}{5} < \beta < 1$, $\phi = o(a(T)^{-2\beta} \Delta_{n,T})$ and $o(k) = \Delta_{n,T}^{-1} \log(n) a(T)^{2\beta}$. The orders are compatible with low-frequency - daily - data (n) for the purpose of estimating the dynamics and high-frequency intra-daily data on a liquid stock (k) for the purpose of spot volatility estimation in the first stage. Since the more stringent condition 4.6 replaces condition 4.7 and $h_{n,T}/\Delta_{n,T}^{1/2} \rightarrow \infty$, the case $j = 1$ (drift estimation) requires lower (than daily) frequencies. The conditions for weak convergence (4.4 and 4.8 for $j = 1$ or 4.9 for $j > 1$) imply $n \sim T^{\frac{3}{2} + \frac{3}{2}\beta + \alpha}$ with $\alpha > 0$, and more stringent conditions on ϕ and k . See Appendix for further details.

We now turn to the asymptotics. Here and throughout the paper, Assumptions 1, 2, and 3 are assumed to hold. We will be specific about which conditions in Assumption 4 will be required for our subsequent theorems to hold. All proofs are postponed to Appendix A.

THEOREM 1. (*The chronological local time.*) Define $\widetilde{L}_{\sigma^2}(T, x) = \frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^n \mathbf{K} \left(\frac{\widehat{\sigma}_{iT/n}^2 - x}{h_{n,T}} \right)$. Given Assumption 4.0, it holds that

$$\frac{\widetilde{L}_{\sigma^2}(T, x)}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds} = 1 + O_p \left(\frac{\Delta_{n,T}}{h_{n,T}^2} \right) + O_p(g(n, T, k, \phi)).$$

Thus, if Assumptions 4.0, 4.2, and 4.5 hold, then

$$\frac{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^n \mathbf{K} \left(\frac{\widehat{\sigma}_{i\Delta_{n,T}}^2 - x}{h_{n,T}} \right)}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds} \xrightarrow{p} 1.$$

REMARK 2. When $T \rightarrow \infty$ (jointly with $\Delta_{n,T} \rightarrow 0$) and the bandwidth satisfies the listed assumptions, then $\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^n \mathbf{K} \left(\frac{\widehat{\sigma}_{i\Delta_{n,T}}^2 - x}{h_{n,T}} \right)$ is asymptotically equivalent to the spot variance's occupation density even when spot variance is estimated.

REMARK 3. The bandwidth conditions in Assumption 4 are written with respect to the infeasible local time estimator $\widehat{L}_{\sigma^2}(T, x)$. This is not problematic. Notice that, in light of Theorem 1 and its proof, we can write

$$\frac{\widetilde{L}_{\sigma^2}(T, x)}{\widehat{L}_{\sigma^2}(T, x)} = \frac{\frac{\widetilde{L}_{\sigma^2}(T, x)}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) ds}}{\frac{\widehat{L}_{\sigma^2}(T, x)}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) ds}} = \frac{1 + O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right) + O_p(g(n, T, k, \phi))}{1 + O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right)} \xrightarrow{p} 1,$$

thereby yielding asymptotic equivalence between $\widetilde{L}_{\sigma^2}(T, x)$ and $\widehat{L}_{\sigma^2}(T, x)$ if $\frac{\Delta_{n,T}}{h_{n,T}^2} \rightarrow 0$ and $g(n, T, k, \phi) \rightarrow 0$.

THEOREM 2. (**The infinitesimal moments.**) *If Assumption 4.0, 4.1, 4.2 and 4.6 (for $j = 1$) or 4.7 (for $j > 1$) hold, then*

$$\widehat{\theta}^j(x) \xrightarrow{p} \theta^j(x) \quad \forall j \geq 1.$$

If 4.0, 4.1, 4.3, 4.4 and 4.8 (for $j = 1$) or 4.9 (for $j > 1$) hold, then

$$\sqrt{h_{n,T} \widehat{L}_{\sigma^2}(T, x)} \left\{ \widehat{\theta}^j(x) - \theta^j(x) - \Gamma_{\theta^j}(x) \right\} \Rightarrow \mathbf{N}(0, \mathbf{K}_2 \theta^{2j}(x)) \quad \forall j \geq 1,$$

where, given the process' invariant measure $s(dx)$,

$$\Gamma_{\theta^j}(x) = h_{n,T}^2 \mathbf{K}_1 \left[\theta^{j'}(x) \frac{s'(x)}{s(x)} + \frac{1}{2} \theta^{j''}(x) \right].$$

REMARK 4. The look of the result is similar to that of classical findings in nonparametric kernel estimation. However, because of our assumed robustness to deviations from stationarity, the empirical chronological local time of the process replaces T in the rate of convergence whereas the invariant density of the process $s(\cdot)$ replaces the stationary probability density $p(\cdot)$ in the bias expression. The stationary case is a sub-case of this more general result for which $\widehat{L}_{\sigma^2}(T, x) \stackrel{a.s.}{\sim} T p(x)$ and $\frac{s'(x)}{s(x)} = \frac{p'(x)}{p(x)}$. Importantly, the result provides an intuitive representation of statistical uncertainty by inversely relating the width of the asymptotic confidence bands to the number of visits at the level x at which point-wise estimation is performed.

REMARK 5. (**Local linear and local polynomial estimates**) As indicated in the Introduction, while we illustrate the results by employing traditional Nadaraya-Watson kernel estimates, extensions to kernel estimators with superior asymptotic mean-squared error properties can be conducted similarly. Among other methods, $\widehat{\theta}^j(x)$ could be a local linear estimator of the form

$$\widehat{\theta}_{ll}^j(x) = \frac{1}{\Delta_{n,T}} \frac{\sum_{i=1}^{n-1} \widetilde{\mathbf{K}}_i(x, h_{n,T}) \left[f(\widehat{\sigma}_{(i+1)T/n}^2) - f(\widehat{\sigma}_{iT/n}^2) \right]^j}{\sum_{i=1}^n \widetilde{\mathbf{K}}_i(x, h_{n,T})} \quad j = 1, \dots,$$

where

$$\widetilde{\mathbf{K}}_i(x, h_{n,T}) = \frac{1}{h_{n,T}} \mathbf{K}\left(\frac{\widehat{\sigma}_{iT/n}^2 - x}{h_{n,T}}\right) \Gamma_{n,2} - (\widehat{\sigma}_{iT/n}^2 - x) \frac{1}{h_{n,T}} \mathbf{K}\left(\frac{\widehat{\sigma}_{iT/n}^2 - x}{h_{n,T}}\right) \Gamma_{n,1}$$

with

$$\Gamma_{n,s} = \sum_{i=1}^n (\widehat{\sigma}_{iT/n}^2 - x)^s \frac{1}{h_{n,T}} \mathbf{K}\left(\frac{\widehat{\sigma}_{iT/n}^2 - x}{h_{n,T}}\right)$$

for $s = 1, 2$. More generally, it could be a local polynomial estimator defined by minimizing $\{\alpha_0, \alpha_1, \dots, \alpha_p\}$ in

$$\sum_{i=1}^n \left(\frac{\left(f(\widehat{\sigma}_{(i+1)T/n}^2) - f(\widehat{\sigma}_{iT/n}^2) \right)^j}{\Delta_{n,T}} - \sum_{u=0}^p \alpha_u (\widehat{\sigma}_{iT/n}^2 - x)^u \right)^2 \mathbf{K} \left(\frac{\widehat{\sigma}_{iT/n}^2 - x}{h_{n,T}} \right)$$

where $\widehat{\theta}_{it}^j(x) = \widehat{\alpha}_0(x)$ for $p = 1$ (see, e.g., [Fan and Gijbels, 1996](#)). Local polynomial methods for diffusions without jumps are studied in [Fan and Zhang \(2003\)](#) and [Moloché \(2002\)](#). Consider, for example, local linear estimates of the conditional moments. All bandwidth conditions would be preserved. In fact, the statement in Theorem 2 would remain unchanged with the exception of an intuitive (given existing work in discrete time) modification of the asymptotic bias: $\Gamma_{\theta^j}(x)$ would become $h_{n,T}^2 \mathbf{K}_1 \left[\theta^{j'}(x) \frac{s'(x)}{s(x)} \right]$.

3. Return/volatility dynamics: asymptotics. Recent empirical work has emphasized the importance of models allowing for rapid increases in stock returns' conditional volatility (see, e.g., [Bates, 2000](#), [Duffie, Pan and Singleton, 2000](#), [Pan, 2002](#) and [Eraker, Johannes and Polson, 2003](#)). Such increases cannot be yielded by the small Gaussian changes implied by classical diffusive stochastic volatility models. Jumps in volatility provide an important means by which sudden volatility jumps translate, due to persistence in the volatility dynamics, into lasting, higher volatility levels (see Eraker et al., 2003, for discussions). In the presence of jumps in volatility, the high-order infinitesimal moments of the volatility process can be employed to learn about the intensity of the jumps and the moments of the jump size distribution as suggested, in more conventional scalar contexts with observables, by [Johannes \(2004\)](#) and studied formally by [Bandi and Nguyen \(2003\)](#).

To this end, we now turn to the identification of nonlinear generalizations of two stochastic volatility models which have drawn particular attention in recent years, namely the square-root stochastic volatility model with exponential jumps of [Duffie, Pan and Singleton \(2000\)](#) and a log-volatility model with Gaussian jumps in the spirit of [Jacquier, Polson and Rossi \(1994, 2004\)](#). It will be apparent in what follows that alternative specifications may be readily adopted provided the identification scheme is modified accordingly.

Generalized Duffie, Pan, and Singleton (2000) model. Write Eq. (2) with $f(\sigma_t^2) = \sigma_t^2$ and $dJ_t^\sigma = \xi^\sigma dN_t^\sigma$, where $\xi^\sigma \rightsquigarrow \exp(\mu_\xi)$. In Duffie et al. (2000) and Eraker et al. (2003), $m_{\sigma^2}(\sigma_t^2)$ is affine, $\Lambda_{\sigma^2}(\sigma_t^2)$ is a square-root process ($\Lambda_{\sigma^2}^2(\sigma_t^2)$ is also affine) as in [Heston \(1993\)](#), and $\lambda_{\sigma^2}(\sigma_t^2)$ (i.e., the intensity of the Poisson jump N_t^σ) is constant and, hence, independent of the state (see, also, [Andersen, Benzoni and Lund, 2002](#) for an affine stochastic volatility model with $\lambda_{\sigma^2}(\sigma_t^2) = 0$). Other papers allowing for jumps in returns, stochastic volatility, but without jumps in volatility are, for example, [Bakshi, Cao and Chen \(1997\)](#), [Bates \(2000\)](#), and [Pan \(2002\)](#). These papers find evidence for misspecification in the volatility dynamics pointing to the likely presence of discontinuities in the volatility sample path. We leave the functional form of the variance drift, diffusion, and jump intensity unspecified. Now, notice that

$$\begin{aligned} \theta^1(x) &= m_{\sigma^2}(x) + \mu_\xi(x) \lambda_{\sigma^2}(x), & \theta^3(x) &= 6\mu_\xi^3(x) \lambda_{\sigma^2}(x), \\ \theta^2(x) &= \Lambda_{\sigma^2}^2(x) + 2\mu_\xi^2(x) \lambda_{\sigma^2}(x), & \theta^4(x) &= 24\mu_\xi^4(x) \lambda_{\sigma^2}(x). \end{aligned}$$

Hence, consistent identification of the relevant functions may be conducted by computing:

$$(6) \quad \widehat{\mu}_\xi(x) = \frac{\widehat{\theta}^4(x)}{4\widehat{\theta}^3(x)},$$

$$(7) \quad \widehat{\lambda}_{\sigma^2}(x) = \frac{\widehat{\theta}^4(x)}{24\widehat{\mu}_{\xi}^4(x)},$$

$$(8) \quad \widehat{\Lambda}_{\sigma^2}^2(x) = \widehat{\theta}^2(x) - 2\widehat{\mu}_{\xi}^2(x)\widehat{\lambda}_{\sigma^2}(x),$$

$$(9) \quad \widehat{m}_{\sigma^2}(x) = \widehat{\theta}^1(x) - \widehat{\mu}_{\xi}(x)\widehat{\lambda}_{\sigma^2}(x).$$

Alternative identification methods may of course be employed. We choose here to present the main ideas by considering the most intuitive scheme.

Generalized log-variance model. Write Eq. (2) with $f(\sigma_t^2) = \log(\sigma_t^2)$ and $dJ_t^\sigma = \xi^\sigma dN_t^\sigma$, where $\xi^\sigma \rightsquigarrow \mathbf{N}(0, \sigma_\xi^2)$. As earlier, we extend the classical log-linear model by allowing for a nonlinear drift, diffusion, and intensity of the jumps. Write

$$\begin{aligned} \theta^1(x) &= m_{\log \sigma^2}(x), & \theta^4(x) &= 3\sigma_\xi^4(x)\lambda_{\log \sigma^2}(x), \\ \theta^2(x) &= \Lambda_{\log \sigma^2}^2(x) + \sigma_\xi^2(x)\lambda_{\log \sigma^2}(x), & \theta^6(x) &= 15\sigma_\xi^6(x)\lambda_{\log \sigma^2}(x). \end{aligned}$$

A potential identification method, along the lines of Johannes (2004) and Bandi and Nguyen (2003) in other contexts, is:

$$(10) \quad \widehat{\sigma}_\xi^2(x) = \frac{\widehat{\theta}^6(x)}{5\widehat{\theta}^4(x)},$$

$$(11) \quad \widehat{\lambda}_{\log \sigma^2}(x) = \frac{\widehat{\theta}^4(x)}{3\widehat{\sigma}_\xi^4(x)},$$

$$(12) \quad \widehat{\Lambda}_{\log \sigma^2}^2(x) = \widehat{\theta}^2(x) - \widehat{\sigma}_\xi^2(x)\widehat{\lambda}_{\log \sigma^2}(x),$$

$$(13) \quad \widehat{m}_{\log \sigma^2}(x) = \widehat{\theta}^1(x).$$

3.1. *Implied drift, diffusion, intensity of the jumps, and jump size: asymptotic properties.* For both models, we now discuss asymptotic inference on the functions of interest. Notice that, for added flexibility, we let even the moments of the jump sizes be functions of the latent volatility process and estimate them pointwise. If we were to assume that these moments are parameters, we could estimate them by averaging higher-order infinitesimal moments over any fixed time period \bar{T} : $\widehat{\mu}_\xi = \frac{1}{\bar{n}} \sum_{i=1}^{\bar{n}} \frac{\widehat{\theta}^4(\widehat{\sigma}_{i\bar{T}/\bar{n}}^2)}{4\widehat{\theta}^3(\widehat{\sigma}_{i\bar{T}/\bar{n}}^2)}$ and $\widehat{\sigma}_\xi^2 = \frac{1}{\bar{n}} \sum_{i=1}^{\bar{n}} \frac{\widehat{\theta}^6(\widehat{\sigma}_{i\bar{T}/\bar{n}}^2)}{5\widehat{\theta}^4(\widehat{\sigma}_{i\bar{T}/\bar{n}}^2)}$ with $\bar{T}/\bar{n} \rightarrow 0$ as $\bar{n} \rightarrow \infty$ over $\bar{T} < T$.

THEOREM 3. (Variance moments: Weak convergence) Assume 4.0, 4.1, 4.3 (with $C = 0$), 4.4, and 4.9 hold.

Generalized Duffie et al.'s model

Expected jump size:

$$\sqrt{h_{n,T}\widehat{L}_{\sigma^2}(T,x)} \{\widehat{\mu}_\xi(x) - \mu_\xi(x)\} \Rightarrow N\left(0, 20\mathbf{K}_2 \frac{\mu_\xi^2(x)}{\lambda_{\sigma^2}(x)}\right)$$

Jump intensity:

$$\sqrt{h_{n,T}\widehat{L}_{\sigma^2}(T,x)} \{\widehat{\lambda}_{\sigma^2}(x) - \lambda_{\sigma^2}(x)\} \Rightarrow N(0, 110\mathbf{K}_2\lambda_{\sigma^2}(x)).$$

Diffusive function:

$$\sqrt{h_{n,T}\widehat{L}_{\sigma^2}(T,x)} \left\{ \widehat{\Lambda}_{\sigma^2}^2(x) - \Lambda_{\sigma^2}^2(x) \right\} \Rightarrow N\left(0, 40\mathbf{K}_2\lambda_{\sigma^2}(x)\mu_{\xi}^2(x)\right).$$

Drift function (with (4.8) replacing (4.9)):

$$\sqrt{h_{n,T}\widehat{L}_{\sigma^2}(T,x)} \left\{ \widehat{m}_{\sigma^2}(x) - m_{\sigma^2}(x) \right\} \Rightarrow N\left(0, \mathbf{K}_2\left(\Lambda_{\sigma^2}^2(x) + 38\lambda_{\sigma^2}(x)\mu_{\xi}^2(x)\right)\right).$$

Generalized log-variance model

Jump standard deviation:

$$\sqrt{h_{n,T}\widehat{L}_{\sigma^2}(T,x)} \left\{ \widehat{\sigma}_{\xi}^2(x) - \sigma_{\xi}^2(x) \right\} \Rightarrow N\left(0, \frac{238}{15}\mathbf{K}_2\frac{\sigma_{\xi}^4(x)}{\lambda_{\log\sigma^2}(x)}\right)$$

Jump intensity:

$$\sqrt{h_{n,T}\widehat{L}_{\sigma^2}(T,x)} \left\{ \widehat{\lambda}_{\log\sigma^2}(x) - \lambda_{\log\sigma^2}(x) \right\} \Rightarrow N\left(0, \frac{189}{5}\mathbf{K}_2\lambda_{\log\sigma^2}(x)\right).$$

Diffusive function:

$$\sqrt{h_{n,T}\widehat{L}_{\sigma^2}(T,x)} \left\{ \widehat{\Lambda}_{\log\sigma^2}^2(x) - \Lambda_{\log\sigma^2}^2(x) \right\} \Rightarrow N\left(0, \frac{88}{15}\mathbf{K}_2\lambda_{\log\sigma^2}(x)\sigma_{\xi}^4(x)\right).$$

Drift function (with (4.8) replacing (4.9)):

$$\sqrt{h_{n,T}\widehat{L}_{\sigma^2}(T,x)} \left\{ \widehat{m}_{\log\sigma^2}(x) - m_{\log\sigma^2}(x) \right\} \Rightarrow N\left(0, \mathbf{K}_2\left(\Lambda_{\log\sigma^2}^2(x) + \lambda_{\log\sigma^2}(x)\sigma_{\xi}^2(x)\right)\right).$$

REMARK 6. In light of Theorem 3, statistical inference on all functions and parameters of interest is now straightforward given estimates of the asymptotic variances.

3.2. *Risk-return trade-offs and leverage effects: asymptotic properties.* We now focus on the full system inclusive of the return dynamics. Given spot variance estimates $\left\{\widehat{\sigma}_{iT/n}^2 \mid i = 1, \dots, n\right\}$ and infinitesimal moment estimates pertaining to the return process (which we denote by $\widehat{\theta}_r^j(\sigma^2)$ with $j = 1, 2, \dots$), the relevant functions and the features of the return jump distribution can be estimated by using an identification scheme in the spirit of those reported above.

While any jump size distribution may be assumed (by simply modifying the identification scheme reported below accordingly), here we assume Gaussian mean-zero jumps, i.e., $dJ_t^r = \psi dN_t^r$ with $\psi \rightsquigarrow \mathbf{N}(0, \sigma_{\psi}^2)$, as done in, e.g., Eraker et al. (2003). We may therefore employ

$$(14) \quad \widehat{\mu}(x) = \widehat{\theta}_r^1(x), \quad \widehat{\sigma}_{\psi}^2(x) = \frac{\widehat{\theta}_r^6(x)}{5\widehat{\theta}_r^4(x)}, \quad \widehat{\lambda}^r(x) = \frac{\widehat{\theta}_r^4(x)}{3\widehat{\sigma}_{\psi}^4(x)},$$

with

$$\widehat{\theta}_r^j(x) = \frac{1}{\Delta_{n,T}} \frac{\sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\widehat{\sigma}_{iT/n}^2 - x}{h_{n,T}}\right) (\log p_{(i+1)T/n} - \log p_{iT/n})^j}{\sum_{i=1}^n \mathbf{K}\left(\frac{\widehat{\sigma}_{iT/n}^2 - x}{h_{n,T}}\right)} \quad j = 1, \dots$$

Importantly, if $\widehat{\mu}(x)$ is a statistically increasing function of x , then a risk-return trade-off exists. Theorem 4 discusses weak convergence of $\widehat{\mu}(x)$.

THEOREM 4. (**Risk-return trade-offs: weak convergence**) *Let Assumptions 4.0, 4.1, 4.3, 4.4, and 4.9 hold, then*

$$\sqrt{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}\{\widehat{\mu}(x) - \mu(x) - \Gamma_{\mu}(x)\} \Rightarrow \mathbf{N}(0, \mathbf{K}_2\theta^{\mu}(x)),$$

with

$$\Gamma_{\mu}(x) = h_{n,T}^2 \mathbf{K}_1 \left[\mu'(x) \frac{s'(x)}{s(x)} + \frac{1}{2} \mu''(x) \right],$$

where $\theta^{\mu}(x) = x + \lambda^r(x)\mathbf{E}(\psi^2)$.

To complete the system one now has to identify the leverage function. Influential recent work on leverage estimation in parametric stochastic volatility models includes [Harvey and Shephard \(1996\)](#), [Jacquier, Polson and Rossi \(2004\)](#), and [Yu \(2005\)](#), among others. Yu (2005) also provides a thorough discussion of the existing parametric literature. Write

$$\widehat{\rho}(x) = \frac{\sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\widehat{\sigma}_{iT/n}^2 - x}{h_{n,T}} \right) (\log(p_{(i+1)T/n}) - \log(p_{iT/n})) (f(\widehat{\sigma}_{(i+1)T/n}^2) - f(\widehat{\sigma}_{iT/n}^2))}{\Delta_{n,T} \sum_{i=1}^n \mathbf{K} \left(\frac{\widehat{\sigma}_{iT/n}^2 - x}{h_{n,T}} \right) \sqrt{x \widehat{\Lambda}_{f(\cdot)}^2(x)}},$$

where $\widehat{\Lambda}_{f(\cdot)}^2(x)$ may be estimated by virtue of Eq. (8) or Eq. (12) (depending, of course, on the assumed variance model).

Theorem 5 shows that $\widehat{\rho}(x)$ identifies $\rho(x)$ consistently. The theorem assumes that the same bandwidth is used to estimate numerator and denominator of $\widehat{\rho}(x)$ and explicitly distinguishes between the linear variance case and the log variance case.

THEOREM 5. (**Leverage: weak convergence**) *If Assumption 4.0, 4.1, 4.3, 4.4, and 4.9 hold, then*

$$\sqrt{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}\{\widehat{\rho}(x) - \rho(x)\} \Rightarrow \mathbf{N}(0, \mathbf{K}_2\theta^{\rho}(x)),$$

with $\theta^{\rho}(x) = \frac{\rho^2(x)}{4\Lambda_{f(\cdot)}^4(x)} \text{Asyvar} \left(\widehat{\Lambda}_{f(\cdot)}^2(x) \right)$ and

$$\text{Asyvar} \left(\widehat{\Lambda}_{f(\cdot)}^2(x) \right) = \begin{cases} 40\mathbf{K}_2\lambda_{\sigma^2}(x)\mu_{\xi}^2(x) & \text{if } f(\cdot) = x \\ \frac{88}{15}\mathbf{K}_2\lambda_{\log\sigma^2}(x)\sigma_{\xi}^4(x) & \text{if } f(\cdot) = \log x. \end{cases}$$

4. Finite sample adjustments. Identification of the quantities which hinge on high-order moments, namely the jump intensity and the function(s) controlling the jump distribution, may be improved upon by allowing for higher first-order terms in the relevant conditional moment representations. Consider only the price process, for conciseness, but similar expansions may be conducted for spot variance. Write

$$\theta_r^4(x) = 3\lambda^r(x)\sigma_{\psi}^4(x) + \underbrace{3[\theta_r^2(x)]^2}_{\text{correction}}\Delta_{n,T} + O(\Delta_{n,T}^2)$$

and

$$\theta_r^6(x) = 15\lambda^r(x)\sigma_{\psi}^6(x) + \underbrace{15\theta_r^2(x)(3\lambda^r(x)\sigma_{\psi}^4(x))}_{\text{correction}}\Delta_{n,T} + O(\Delta_{n,T}^2)$$

$$= 15\lambda^r(x)\sigma_\psi^6(x) + 15\theta_r^2(x)(\theta_r^4(x) - 3[\theta_r^2(x)]^2)\Delta_{n,T}\Delta_{n,T} + O(\Delta_{n,T}^2).$$

Hence, if we assume σ_ψ^2 is a parameter, as in the next session:

$$\begin{aligned}\widehat{\sigma}_\psi^2 &= \frac{1}{5n} \sum_{i=1}^n \frac{\widehat{\theta}_r^6(\widetilde{\sigma}_{iT/n}^2) - 15\widehat{\theta}_r^2(\widetilde{\sigma}_{iT/n}^2) \left(\widehat{\theta}_r^4(\widetilde{\sigma}_{iT/n}^2) - 3 \left[\widehat{\theta}_r^2(\widetilde{\sigma}_{iT/n}^2) \right]^2 \Delta_{n,T} \right) \Delta_{n,T}}{\widehat{\theta}_r^4(\widetilde{\sigma}_{iT/n}^2) - 3 \left[\widehat{\theta}_r^2(\widetilde{\sigma}_{iT/n}^2) \right]^2 \Delta_{n,T}}, \\ \widehat{\lambda}^r(x) &= \frac{\widehat{\theta}_r^4(\widetilde{\sigma}_{iT/n}^2) - 3 \left[\widehat{\theta}_r^2(\widetilde{\sigma}_{iT/n}^2) \right]^2 \Delta_{n,T}}{3\widehat{\sigma}_\psi^4}.\end{aligned}$$

These additional terms are asymptotically negligible (thereby not affecting our limiting results), but may play an important role in finite samples. Because of the noisy nature of returns, they do so particularly when estimating return dynamics, as we show below by simulation.

5. Simulations. We simulate S&P 500 returns from the bi-variate system:

$$(15) \quad \log(p_{t+\Delta_t}) - \log(p_t) = b\Delta_t + \sqrt{\sigma_t^2}\Delta_t\varepsilon_t^r + \psi_t J_t^r,$$

$$(16) \quad \sigma_{t+\Delta_t}^2 - \sigma_t^2 = \kappa(\theta - \sigma_t^2)\Delta_t + \sigma_v\sqrt{\sigma_t^2}\Delta_t\varepsilon_t^\sigma + \xi_t^\sigma J_t^\sigma,$$

where $\{J_t^r, J_t^\sigma\}$ are Bernoulli random variables with constant intensities $\lambda^r\Delta_t$ and $\lambda^\sigma\Delta_t$, $\{\varepsilon_t^r, \varepsilon_t^\sigma\}$ are standard Gaussian random variables with correlation ρ , ψ_t is a mean zero Gaussian random variable with standard deviation σ_ψ , ξ_t^σ is an exponential random variable with mean μ_σ , and Δ_t is a time-discretization (one day). The parameters are those in Table III, Column 5, of Eraker et al. (2003). To illustrate the impact of measurement error, we implement our estimation procedure using the true spot variance σ_t^2 as well as a noisy spot variance measure contaminated by estimation error. Coherently with equation (5), the latter is defined as $\widehat{\sigma}_t^2 = \sigma_t^2 + \sqrt{\frac{2\sigma_t^4}{k}}\varepsilon_t$, where the ε_t 's are iid standard Normal draws and $k = 1,000$. This choice of k is consistent with the typical number of intra-day observations available for liquid financial assets. We use a second-order Gaussian kernel for all moments. The corresponding bandwidths $h_{n,T}$ are selected using the rule of thumb $h_{n,T} = 1.06\text{std}(\widehat{\sigma}_t^2)n^{-1/5}$.

Fig. 1 reports the means and the empirical 95% confidence bands of the estimates along with the true functions. The volatility dynamics are accurately estimated. The most significant deviations from the true values occur in correspondence with the intensity and the standard deviation of the price jumps. Panel (a), which is in logarithmic scale to display the true function and its unadjusted estimate on the same plot, illustrates this result. The use of first-order corrections (as described in Section 4) restores approximate unbiasedness of the price jump size and intensity estimates. Such corrections are implemented here solely for the price moments.

The impact on the estimates of the introduction of measurement error in spot variance is small, the most relevant effect being on the leverage estimator. This is a small-sample issue which is well known even for alternative approaches to nonparametric leverage estimation with high-frequency data, see e.g. Mykland and Zhang (2009b). We leave the study of a suitable small-sample correction for future research. Finally, even though our bandwidth choice is sensible, the use of smoothing sequences tailored to the divergence properties of the local time factor is expected to deliver superior performance. The design of optimal data-driven bandwidth selection procedures for infinitesimal moment estimation in the presence of jumps is an important open problem which is beyond the scopes of the present paper and is, therefore, better left for future work.

6. Conclusions. We study stochastic volatility modelling with jumps in continuous time by employing functional estimation procedures. Preliminary jump-robust estimates of spot variance are employed, in conjunction with functional estimates of the model's conditional moments, to learn about the functions and parameters

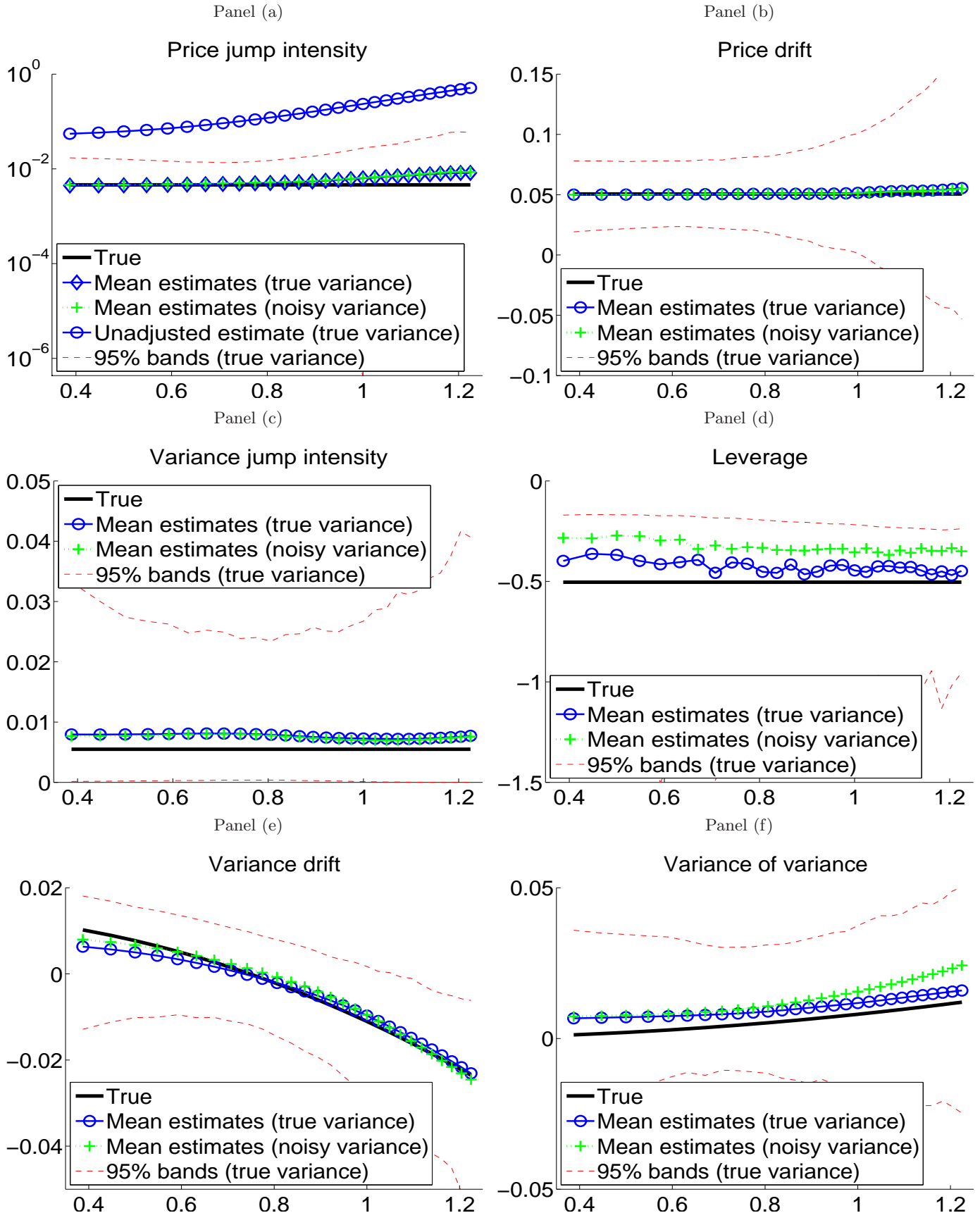


FIG 1. Displays the true functions of the system (15)-(16) and mean estimates across 1,000 replications. On the x axis we report the square root of the spot variance in daily units and percentage form. When using the true variance, the mean of the unadjusted estimator $\hat{\sigma}_\psi^2$ is 3.47, while the mean of the adjusted estimator $\hat{\sigma}_\psi^2$ is 17.72, with a 95% confidence interval given by [4.56, 33.91] (the true value is 8.94). With noisy variance these estimates are virtually the same. When using the true variance, the mean estimate of μ_ξ is 1.64, with a 95% confidence interval given by [0.782, 3.03] (the true value is 1.798). With noisy variance, the mean estimate of μ_ξ is 1.67 with a practically unchanged confidence interval.

driving the joint return/variance dynamics. The model and identification methods allow for nonlinearities in the drift and diffusion functions as well as in the intensities of the return/variance jumps. Nonlinear leverage effects are also permitted. Our limiting results control for the measurement error induced by the preliminary spot variance estimates and are derived under the weaker (than stationarity) assumption of recurrence. We show consistency and weak convergence of all the relevant functions and jump parameters of generalized (i.e., non-affine) versions of two widely-employed stochastic volatility models.

A very successful, recent literature as focused on the efficient use of intra-period price observations for the purpose of estimating variance over the period. This literature aims at being as much as possible model-free. In the same "model-free" spirit, we view this paper has an initial effort to render this literature's contributions operative in the context of continuous-time finance modelling under weak assumptions in terms of model specification and conditions needed for identification.

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APPENDIX A: PROOFS

The proofs are for the general case of driving functions which are, possibly, not uniformly bounded and a volatility process which is recurrent in an unbounded set. This is a case of theoretical interest when letting T grow without bound. The proofs, and the statements of the theorems, may however be easily re-conducted to the more familiar uniformly bounded case. Hereafter, we assume Assumption 1, 2, and 3 to hold. Assumption 5 relaxes Assumption 1 by permitting unboundedness of the system's driving functions. The symbol $A \stackrel{\mathcal{P}}{\sim} B$ signifies that A is $O_p(B)$.

ASSUMPTION 5. (a) Using Markov's inequality, we write $\max_{1 \leq i \leq n} \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} |F(\sigma_s^2)| ds \leq \sup_{s \leq T} |F(\sigma_s^2)| \Delta_{n,T} \stackrel{\mathcal{P}}{\sim} \mathbf{E} \left(\sup_{s \leq T} |F(\sigma_s^2)| \right) \Delta_{n,T} := \Delta_{n,T}^*$. The function $F(\cdot)$ satisfies the property $\frac{\mathbf{E}[\sup_{s \leq T} |f^*(\sigma_s^2)|]}{\mathbf{E}[\sup_{s \leq T} |F(\sigma_s^2)|]} = \frac{\mathbf{E}[\mathcal{M}(f^*(\sigma_s^2))]}{\mathbf{E}[\mathcal{M}(F(\sigma_s^2))]} \leq 1$, where $\mathcal{M}(F(\sigma_s^2))$ is the maximal process of $F(\sigma_s^2)$, for all relevant functionals $f^*(\sigma^2)$ of σ^2 of the type $\mu(\cdot), m(\cdot), \lambda(\cdot), \dots$ and so on in our treatment. (b) We further assume that $(\mathbf{E}[\mathcal{M}(F)])^{3/2} \Delta_{n,T} \rightarrow 0$.

REMARK 7. Assumption 5 (a) is a purely technical assumption. If the functions $\mu(\cdot), m(\cdot), \lambda(\cdot), \dots$ and the process σ_t^2 are not uniformly bounded as $T \rightarrow \infty$, then $f^*(\sigma_t^2)$ will not be uniformly bounded, in general. Thus, the condition $\max_{1 \leq i \leq n} \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} |f^*(\sigma_s^2)| ds \stackrel{\mathcal{P}}{\sim} \Delta_{n,T}$, which is a routine approximation in this literature, may not be valid as $T \rightarrow \infty$. However, $\max_{1 \leq i \leq n} \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} |f^*(\sigma_s^2)| ds \stackrel{\mathcal{P}}{\sim} \mathbf{E} \left(\sup_{s \leq T} |f^*(\sigma_s^2)| \right) \Delta_{n,T}$ will always be valid since $\mathcal{M}(f^*(\sigma_s^2)) = \sup_{s \leq T} |f^*(\sigma_s^2)|$ is the maximal process of $f^*(\sigma_s^2)$ (see, e.g., [Borkovec and Kluppelberg, 1998](#) and [Jeong and Park, 2010](#) for a recent use of the notion in continuous-time econometrics). We therefore write $\max_{1 \leq i \leq n} \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} |F(\sigma_s^2)| ds \leq \sup_{s \leq T} |F(\sigma_s^2)| \Delta_{n,T} \stackrel{\mathcal{P}}{\sim}$

$\mathbf{E} \left(\sup_{s \leq T} |F(\sigma_s^2)| \right) \Delta_{n,T} := \Delta_{n,T}^*$ for an asymptotically-dominating function $F(\cdot)$. In other words, we assume the existence of a "maximal" maximal process $\mathcal{M}(F(\sigma_s^2))$ and write, by virtue of Assumption 5 (a), $\max_{1 \leq i \leq n} \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} |f^*(\sigma_s^2)| ds \stackrel{\mathcal{P}}{\sim} \mathbf{E}(\mathcal{M}(f^*(\sigma_s^2))) \Delta_{n,T} \leq \mathbf{E}(\mathcal{M}(F(\sigma_s^2))) \Delta_{n,T} := \Delta_{n,T}^*$. Assumption 5 (b) is also technical. It simply states that the probability order of the given power of the maximal process does not dominate the vanishing rate of the distance between observations.

REMARK 8. In the discretization conditions, we will sometimes use $\Delta_{n,T}^*$ along with $\Delta_{n,T}$. The symbol $\Delta_{n,T}^*$ should always be interpreted as having the same probability order as $\mathbf{E}(\mathcal{M}(F(\sigma_s^2))) \Delta_{n,T}$. Hence, in general, $\frac{\Delta_{n,T}}{\Delta_{n,T}^*} = o_p(1)$ (or $O_p(1)$, as in the uniformly bounded case in the main text). In practise, since $\Delta_{n,T}^*$ is process specific and cannot be known, one always has to empirically treat $\Delta_{n,T}^*$ as being $\Delta_{n,T}$, the uniformly bounded case. Treating $\Delta_{n,T}^*$ as $\Delta_{n,T}$, or assuming uniform boundedness of the relevant functions is - as we have shown in the main text - not restrictive in practise. Since asymptotics are approximations of data behavior, and the data is bounded no matter how large the time span, using $\Delta_{n,T}^*$ instead of $\Delta_{n,T}$, when needed, is simply an issue of theoretical interest, one which we tackle here explicitly.

LEMMA 1. Write $L_{h_{n,T},T}(x) = \frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds$. It holds that $h_{n,T} L_{h_{n,T},T}(x) \xrightarrow{a.s.} \infty$ if, and only if, $h_{n,T} a(T) \rightarrow \infty$. Similarly, $h_{n,T} L_{h_{n,T},T}(x) \xrightarrow{a.s.} 0$ if, and only if, $h_{n,T} a(T) \rightarrow 0$, where $a(T)$ is the almost-sure rate of divergence of the spot variance's local time.

PROOF. Write $L_{h_{n,T},T}(x) = \frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds$. Since σ_t^2 is a Harris recurrent strong Markov process, Nummelin splitting in continuous time (Nummelin, 1984) implies that the increments $\zeta_n := \frac{1}{h_{n,T}} \int_{R_{n-1}}^{R_n} \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds$, where $\{R_n : n \geq 1\}$ are the regeneration times, constitute a stationary and strong mixing sequence so that $\sigma \{\zeta_k, k \leq g\}$ is independent of $\sigma \{\zeta_{g+l}, l \geq 2\}$. In other words, the increments are not independent but ζ_n is independent of ζ_{n+2} (Löcherbach and Loukianova, 2008). Denote by N_T the total number of regenerations between 0 and T and write

$$L_{h_{n,T},T}(x) = \frac{1}{h_{n,T}} \int_0^{R_0} \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds + \frac{1}{h_{n,T}} \sum_{n=1}^{N_T} \int_{R_{n-1}}^{R_n} \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds + \frac{1}{h_{n,T}} \int_{R_{N_T}}^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds.$$

Since there exists a function $a(T)$ such that $\frac{N_T}{a(T)} = O_{a.s.}(1)$ (see, e.g., Theorem 3.1 of Löcherbach and Loukianova, 2008), we may work conditionally on $\varpi \in \Omega_T = \left\{ \varpi : \underline{\alpha} \leq \limsup_{T \rightarrow \infty} \frac{N_T}{a(T)} \leq \bar{\alpha} \right\}$ for some $\underline{\alpha}$ and $\bar{\alpha}$, so that $0 < \underline{\alpha} \leq \bar{\alpha} < \infty$, as in Bandi, Corradi and Wilhelm (2010). Now,

$$L_{h_{n,T},T}(x) = \frac{1}{h_{n,T}} \sum_{n=1}^{a(T)} \int_{R_{n-1}}^{R_n} \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds + o_{a.s.}(a(T)),$$

where $a(T)$ should be interpreted as its integer part and the order term derives from the same argument as those in the proof of Theorem 5.1 of Karlsen and Tjøstheim (2001). Then,

$$L_{h_{n,T},T}(x) = \sum_{n=1}^{a(T)} \left[\left(\int_{R_{n-1}}^{R_n} \frac{1}{h_{n,T}} \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds \right) - \mu_{h_{n,T}}(x) \right] + a(T) \mu_{h_{n,T}}(x) + o_{a.s.}(a(T)),$$

where $\mu_{h_{n,T}}(x) = \mathbf{E} \left(\frac{1}{h_{n,T}} \int_{R_{n-1}}^{R_n} \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds \right)$. This implies that

$$h_{n,T} L_{h_{n,T},T}(x) = h_{n,T} \sum_{n=1}^{a(T)} \left[\left(\int_{R_{n-1}}^{R_n} \frac{1}{h_{n,T}} \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds \right) - \mu_{h_{n,T}}(x) \right] + O(h_{n,T} a(T)) + o_{a.s.}(h_{n,T} a(T)).$$

Since $\mathbf{Var} \left(\sum_{n=1}^{a(T)} \left[\left(\int_{R_{n-1}}^{R_n} \frac{1}{h_{n,T}} \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds \right) - \mu_{h_{n,T}}(x) \right] \right) = O \left(\frac{a(T)}{h_{n,T}} \right)$,

$$(17) \quad h_{n,T} L_{h_{n,T},T}(x) = O_p \left(\sqrt{a(T) h_{n,T}} \right) + O(h_{n,T} a(T)) + o_{a.s.}(h_{n,T} a(T)).$$

Thus, $h_{n,T}L_{h_{n,T},T}(x) \xrightarrow{a.s.} \infty$ only if $h_{n,T}a(T) \rightarrow \infty$. In addition, $h_{n,T}L_{h_{n,T},T}(x) \xrightarrow{a.s.} 0$ only if $h_{n,T}a(T) \rightarrow 0$. We now turn to the "if" part. Let $h_{n,T}^m = ca^{-\beta}(T)$, using the method of proof in Bandi, Corradi, and Wilhelm (2010), Proposition 1:

$$\begin{aligned}
& \limsup_T P \left(h_{n,T} \sum_{n=1}^{a(T)} \left[\left(\int_{R_{n-1}}^{R_n} \frac{1}{h_{n,T}} \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds \right) - \mu_{h_{n,T}}(x) \right] > \delta \right) \\
& \leq \frac{a(T)h_{n,T}^{2m}}{\delta^{2m}} \mathbf{E} \left(\sum_{n=1}^{a(T)} \left[\left(\int_{R_{n-1}}^{R_n} \frac{1}{h_{n,T}} \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds \right) - \mu_{h_{n,T}}(x) \right] \right)^{2m} \\
& \leq \frac{a(T)h_{n,T}^{2m}}{\delta^{2m}} a^m(T) h_{n,T}^{-m} \\
& \leq \frac{a^{m+1}(T)h_{n,T}^m}{\delta^{2m}} \\
& = c \left(\frac{a^{m+1-\beta m}(T)}{\delta^{2m}} \right).
\end{aligned}$$

Thus, the bound goes to zero if $m+1-\beta m < 0$ or $\beta > \frac{m+1}{m}$ or $\beta > 1$ for m arbitrarily large. This, however, implies that $h_{n,T}a(T) = ca^{-\beta+1}(T) \rightarrow 0$. Then,

$$h_{n,T}L_{h_{n,T},T}(x) = o_{a.s.}(a(T)h_{n,T}) + O(a(T)h_{n,T})$$

which implies that $h_{n,T}L_{h_{n,T},T}(x) \xrightarrow{a.s.} 0$ if $h_{n,T}a(T) \rightarrow 0$. Similarly,

$$\begin{aligned}
& \limsup_T P \left(\frac{1}{a(T)} \sum_{n=1}^{a(T)} \left[\left(\int_{R_{n-1}}^{R_n} \frac{1}{h_{n,T}} \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds \right) - \mu_{h_{n,T}}(x) \right] \right) \\
& \leq c \left(\frac{a^{-m+1+\beta m}(T)}{\delta^{2m}} \right),
\end{aligned}$$

In this case, the bound goes to zero if $-m+1+\beta m < 0$ or $\beta < \frac{m-1}{m}$ or $\beta < 1$ for m arbitrarily large. This, however, implies that $h_{n,T}a(T) = ca^{-\beta+1}(T) \rightarrow \infty$. Then,

$$L_{h_{n,T},T}(x) = O_{a.s.}(a(T)) + o_{a.s.}(a(T))$$

and

$$h_{n,T}L_{h_{n,T},T}(x) = O_{a.s.}(a(T)h_{n,T}) + o_{a.s.}(a(T)h_{n,T}),$$

which implies that $h_{n,T}L_{h_{n,T},T}(x) \xrightarrow{a.s.} \infty$ if $a(T)h_{n,T} \rightarrow \infty$. \square

REMARK 9. The above result makes explicit that $h_{n,T}L_{h_{n,T},T}(x) \rightarrow \infty$ is possible only when $h_{n,T}a(T) \rightarrow \infty$. More generally, the result gives a one-to-one map between conditions written in terms of empirical local time (in its continuous version) and the same conditions written with respect to the almost-sure rate of divergence of the theoretical local time. The mapping between empirical local time (in its continuous version) and empirical local time (in its discrete version) is given by Lemma 3.

LEMMA 2. Given Assumption 5,

$$\frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right) \int_{iT/n}^{(i+1)T/n} m(\sigma_s^2) ds}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds} - \frac{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) m(\sigma_s^2) ds}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds} = O_p \left(\frac{\Delta_{n,T}^*}{h_{n,T}^2} \right).$$

PROOF. Write

$$\begin{aligned} \frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) m(\sigma_s^2) ds &= \frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) m(\sigma_s^2) ds + \frac{1}{h_{n,T}} \int_0^{\Delta_{n,T}} \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) m(\sigma_s^2) ds \\ &= \frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) m(\sigma_s^2) ds + O_p \left(\frac{\Delta_{n,T}^*}{h_{n,T}} \right), \end{aligned}$$

where the order term derives from the fact that

$$\frac{1}{h_{n,T}} \int_0^{\Delta_{n,T}} \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) m(\sigma_s^2) ds \leq \frac{1}{h_{n,T}} \sup_{s \leq T} \left| \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) \right| \sup_{s \leq T} |m(\sigma_s^2)| \Delta_{n,T} \stackrel{p}{\approx} \frac{1}{h_{n,T}} \overline{\mathbf{K}} \mathbf{E}(\mathcal{M}(m(\sigma_s^2))) \Delta_{n,T} \leq \frac{1}{h_{n,T}} \overline{\mathbf{K}} \Delta_{n,T}^*,$$

for some $\overline{\mathbf{K}}$, given boundedness of the kernel function, in light of Assumption 2. Using Itô's lemma,

$$\begin{aligned} &\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left[\mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) - \mathbf{K} \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right) \right] m(\sigma_s^2) ds = \\ &\frac{1}{h_{n,T}^2} \sum_{i=1}^{n-1} \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left[\int_{i\Delta_{n,T}}^s \mathbf{K}' \left(\frac{\sigma_v^2 - x}{h_{n,T}} \right) m(\sigma_v^2) dv \right] m(\sigma_s^2) ds \\ &\frac{1}{2h_{n,T}^3} \sum_{i=1}^{n-1} \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left[\int_{i\Delta_{n,T}}^s \mathbf{K}'' \left(\frac{\sigma_v^2 - x}{h_{n,T}} \right) \Lambda^2(\sigma_v^2) dv \right] m(\sigma_s^2) ds \\ &\frac{1}{h_{n,T}^2} \sum_{i=1}^{n-1} \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left[\int_{i\Delta_{n,T}}^s \mathbf{K}' \left(\frac{\sigma_v^2 - x}{h_{n,T}} \right) \Lambda(\sigma_v^2) dW_v^\sigma \right] m(\sigma_s^2) ds \\ &\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left[\sum_{\Delta\sigma_v^2 \neq 0, i\Delta_{n,T} \leq v < s} \left(\mathbf{K} \left(\frac{\sigma_v^2 + \Delta\sigma_v^2 - x}{h_{n,T}} \right) - \mathbf{K} \left(\frac{\sigma_v^2 - x}{h_{n,T}} \right) \right) \right] m(\sigma_s^2) ds \\ &= \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4. \end{aligned}$$

Thus,

$$\begin{aligned} &\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) m(\sigma_s^2) ds - \frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right) \int_{iT/n}^{(i+1)T/n} m(\sigma_s^2) ds \\ &= \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 + O_p \left(\frac{\Delta_{n,T}^*}{h_{n,T}} \right). \end{aligned}$$

Using integration by parts (see, e.g., [Protter, 2004](#), Corollary 2, page 68):

$$\begin{aligned} \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left[\int_{i\Delta_{n,T}}^s \mathbf{K}' \left(\frac{\sigma_v^2 - x}{h_{n,T}} \right) m(\sigma_v^2) dv \right] m(\sigma_s^2) ds &= \left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \mathbf{K}' \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) m(\sigma_s^2) ds \right) \left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} m(\sigma_s^2) ds \right) \\ &\quad - \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left[\int_{i\Delta_{n,T}}^s m(\sigma_v^2) dv \right] \mathbf{K}' \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) m(\sigma_s^2) ds, \end{aligned}$$

so that

$$\begin{aligned} \Phi_1 &= \frac{1}{h_{n,T}^2} \sum_{i=1}^{n-1} \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left[\int_{i\Delta_{n,T}}^s \mathbf{K}' \left(\frac{\sigma_v^2 - x}{h_{n,T}} \right) m(\sigma_v^2) dv \right] m(\sigma_s^2) ds \\ &= \frac{1}{h_{n,T}^2} \sum_{i=1}^{n-1} \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left[\int_s^{(i+1)\Delta_{n,T}} m(\sigma_v^2) dv \right] \mathbf{K}' \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) m(\sigma_s^2) ds \end{aligned}$$

$$\stackrel{p}{\sim} \frac{\Delta_{n,T}^*}{h_{n,T}^2} \int_0^T \left| \mathbf{K}' \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) \right| |m(\sigma_s^2)| ds,$$

since, as earlier,

$$\int_s^{(i+1)\Delta_{n,T}} |m(\sigma_v^2)| dv \leq \max_{1 \leq i \leq n} \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} |m(\sigma_v^2)| dv \stackrel{p}{\sim} \mathbf{E}(\mathcal{M}(m(\sigma_s^2))) \Delta_{n,T} \leq \Delta_{n,T}^*$$

by Assumption 5. By the limit-ratio theorem (see, e.g., Revuz and Yor, 1994, Theorem 3.12), $\frac{\Phi_1}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds} = O_p \left(\frac{\Delta_{n,T}^*}{h_{n,T}} \right)$, given that the kernel's first derivative is absolutely integrable. We now turn to Φ_2 . Integrating by parts, again,

$$\begin{aligned} \Phi_2 &= \frac{1}{2h_{n,T}^3} \sum_{i=1}^{n-1} \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left[\int_{i\Delta_{n,T}}^s \mathbf{K}'' \left(\frac{\sigma_v^2 - x}{h_{n,T}} \right) \Lambda^2(\sigma_v^2) dv \right] m(\sigma_s^2) ds \\ &= \frac{1}{2h_{n,T}^3} \sum_{i=1}^{n-1} \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left[\int_s^{(i+1)\Delta_{n,T}} m(\sigma_v^2) dv \right] \mathbf{K}'' \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) \Lambda^2(\sigma_s^2) ds \\ &\stackrel{p}{\sim} \frac{\Delta_{n,T}^*}{h_{n,T}^3} \int_0^T \left| \mathbf{K}'' \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) \right| \Lambda^2(\sigma_s^2) ds, \end{aligned}$$

by Assumption 5. By the limit-ratio theorem, $\frac{\Phi_2}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds} = O_p \left(\frac{\Delta_{n,T}^*}{h_{n,T}^2} \right)$, since the kernel's second derivative is absolutely integrable. Similarly,

$$\begin{aligned} \Phi_3 &= \frac{1}{h_{n,T}^2} \sum_{i=1}^{n-1} \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left[\int_{i\Delta_{n,T}}^s \mathbf{K}' \left(\frac{\sigma_v^2 - x}{h_{n,T}} \right) \Lambda(\sigma_v^2) dW_v^\sigma \right] m(\sigma_s^2) ds \\ &= \frac{1}{h_{n,T}^2} \sum_{i=1}^{n-1} \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left[\int_s^{(i+1)\Delta_{n,T}} m(\sigma_v^2) dv \right] \mathbf{K}' \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) \Lambda(\sigma_s^2) dW_s^\sigma \\ &\stackrel{p}{\sim} \frac{\Delta_{n,T}^*}{h_{n,T}^{3/2}} \left(\frac{1}{h_{n,T}^{1/2}} \sum_{i=1}^{n-1} \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \mathbf{K}' \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) \Lambda(\sigma_s^2) dW_s^\sigma \right). \end{aligned}$$

The order of the term $\frac{1}{h_{n,T}^{1/2}} \sum_{i=1}^{n-1} \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \mathbf{K}' \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) \Lambda(\sigma_s^2) dW_s^\sigma$ is $O_p \left(\frac{1}{\sqrt{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds}} \right)$ and may be derived as

in the proof of Lemma 5. Thus,

$$\frac{\Phi_3}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds} = O_p \left(\frac{\Delta_{n,T}^*}{h_{n,T}^{3/2} \sqrt{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds}} \right).$$

Finally, using the fact that the variance process is right-continuous, writing $\hat{\sigma}_v^2$ as a point on the line segment connecting σ_v^2 and $\sigma_v^2 + \xi^\sigma$:

$$\begin{aligned} \Phi_4 &= \frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left[\sum_{\Delta\sigma_v^2 \neq 0, i\Delta_{n,T} \leq v < s} \left(\mathbf{K} \left(\frac{\sigma_v^2 + \Delta\sigma_v^2 - x}{h_{n,T}} \right) - \mathbf{K} \left(\frac{\sigma_v^2 - x}{h_{n,T}} \right) \right) \right] m(\sigma_s^2) ds \\ &= \frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left[\int_{i\Delta_{n,T}}^s \int_\xi \left(\mathbf{K} \left(\frac{\sigma_v^2 + \xi^\sigma - x}{h_{n,T}} \right) - \mathbf{K} \left(\frac{\sigma_v^2 - x}{h_{n,T}} \right) \right) v_\sigma(dv, d\xi^\sigma) \right] m(\sigma_s^2) ds \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left[\int_s^{i\Delta_{n,T}} m(\sigma_v^2) dv \right] \int \left(\mathbf{K}' \left(\frac{\widehat{\sigma}_s^2 - x}{h_{n,T}} \right) \frac{\xi^\sigma}{h_{n,T}} \right) \bar{v}_\sigma(ds, d\xi^\sigma) \\
&\quad - \frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left[\int_s^{i\Delta_{n,T}} m(\sigma_v^2) dv \right] \left(\mathbf{K}' \left(\frac{\widehat{\sigma}_s^2 - x}{h_{n,T}} \right) \frac{\mathbf{E}(\xi^\sigma)}{h_{n,T}} \right) \lambda(\sigma_s^2) ds \\
&\leq \frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left| \int_s^{i\Delta_{n,T}} m(\sigma_v^2) dv \right| \int \left| \mathbf{K}' \left(\frac{\sigma_s^2 - x}{h_{n,T}} + O_p \left(\frac{\Delta_{n,T}^{1/2}}{h_{n,T}} \right) \right) \frac{\xi^\sigma}{h_{n,T}} \right| \bar{v}_\sigma(ds, d\xi^\sigma) \\
&\quad + \frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left| \int_s^{i\Delta_{n,T}} m(\sigma_v^2) dv \right| \left(\left| \mathbf{K}' \left(\frac{\sigma_s^2 - x}{h_{n,T}} + O_p \left(\frac{\Delta_{n,T}^{1/2}}{h_{n,T}} \right) \right) \frac{\mathbf{E}(\xi^\sigma)}{h_{n,T}} \right| \right) \lambda(\sigma_s^2) ds \\
&\stackrel{p}{\leq} \frac{\Delta_{n,T}^*}{h_{n,T}^{3/2}} O_p \left(\frac{1}{h_{n,T}^{1/2}} \sum_{i=1}^{n-1} \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \int_\xi \left| \mathbf{K}' \left(\frac{\sigma_s^2 - x}{h_{n,T}} + o_p(1) \right) \right| \xi^\sigma \bar{v}_\sigma(ds, d\xi^\sigma) \right) \\
&\quad + \frac{\Delta_{n,T}^*}{h_{n,T}} O_p \left(\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left| \mathbf{K}' \left(\frac{\sigma_s^2 - x}{h_{n,T}} + o_p(1) \right) \right| \lambda(\sigma_s^2) ds \right).
\end{aligned}$$

Thus, by virtue of Lemma 4 and the ratio-limit theorem,

$$\frac{\Phi_4}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds} = O_p \left(\frac{\Delta_{n,T}^*}{h_{n,T}^{3/2} \sqrt{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds}} \right) + O_p \left(\frac{\Delta_{n,T}^*}{h_{n,T}} \right).$$

Finally,

$$\begin{aligned}
&\frac{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) m(\sigma_s^2) ds}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds} - \frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right) \int_{iT/n}^{(i+1)T/n} m(\sigma_s^2) ds}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds} \\
&= \frac{\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds} + O_p \left(\frac{\Delta_{n,T}^*}{h_{n,T} \left(\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds \right)} \right) \\
&= O_p \left(\frac{\Delta_{n,T}^*}{h_{n,T}} \right) + O_p \left(\frac{\Delta_{n,T}^*}{h_{n,T}^2} \right) + O_p \left(\frac{\Delta_{n,T}^*}{h_{n,T}^{3/2} \sqrt{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds}} \right) + O_p \left(\frac{\Delta_{n,T}^*}{h_{n,T} \left(\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds \right)} \right) \\
&= O_p \left(\frac{\Delta_{n,T}^*}{h_{n,T}^2} \right).
\end{aligned}$$

The result follows from the fact that $\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds \xrightarrow{p} \infty$ regardless of the bandwidth choice (see, e.g., Eq. (17) in the proof of Lemma 1). \square

LEMMA 3. *Given Assumption 5,*

$$\frac{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right)}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds} = 1 + O_p \left(\frac{\Delta_{n,T}}{h_{n,T}^2} \right).$$

PROOF. Straightforward from Lemma 2 by setting $m(\sigma_s^2) = 1$. Clearly, the discretization's order here is $O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right)$ rather than $O_p\left(\frac{\Delta_{n,T}^*}{h_{n,T}^2}\right)$. \square

LEMMA 4. Given Assumption 5, if $\frac{\Delta_{n,T}^*}{h_{n,T}^2} \rightarrow 0$, then

$$\sqrt{h_{n,T}} \sqrt{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) ds} \left(\frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right) \int_{iT/n}^{(i+1)T/n} \int_{\xi} \xi^{\sigma} \bar{v}_{\sigma}(ds, d\xi^{\sigma})}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) ds} \right) \Rightarrow \mathbf{N}\left(0, \mathbf{K}_2 \lambda(x) \mathbf{E}((\xi^{\sigma})^2)\right).$$

PROOF. Write $u_{iT/n, (i+1)T/n} = \frac{1}{\sqrt{h_{n,T}}} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right) \int_{iT/n}^{(i+1)T/n} \int_{\xi} \xi^{\sigma} \bar{v}_{\sigma}(ds, d\xi^{\sigma})$. For all n, T , and $h_{n,T}$, the objects $\{u_{iT/n, (i+1)T/n}, \mathfrak{S}_{i,n}, 1 \leq i \leq n, n \geq 1\}$ constitute a martingale difference array. Write

$$\mathbf{U}_{n,T} = \frac{1}{\sqrt{h_{n,T}}} \sum_{i=1}^{n-1} \int_{iT/n}^{(i+1)T/n} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right) \int_{\xi} \xi^{\sigma} \bar{v}_{\sigma}(ds, d\xi^{\sigma}) = \sum_{j=1}^{n-1} u_{jT/n, (j+1)T/n}.$$

The conditional variance of $\mathbf{U}_{n,T}$ is, for all n, T , and $h_{n,T}$,

$$\begin{aligned} \mathbf{V}_{n,T} &= \frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}^2\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right) \mathbf{E} \left\{ \left(\int_{iT/n}^{(i+1)T/n} \int_{\xi} \xi^{\sigma} \bar{v}_{\sigma}(ds, d\xi^{\sigma}) \right)^2 \middle| \mathfrak{S}_{i,n} \right\} \\ &= \frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}^2\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right) \mathbf{E}_i \left(\int_{iT/n}^{(i+1)T/n} \lambda(\sigma_s^2) \mathbf{E}((\xi^{\sigma})^2) ds \right). \end{aligned}$$

Now, write $\mathbf{E}_i \int_{iT/n}^{(i+1)T/n} \lambda(\sigma_s^2) \mathbf{E}(\xi^{\sigma 2}) ds = \mathbf{E}_i \int_{iT/n}^{(i+1)T/n} \mathbf{E}(\xi^{\sigma 2}) \left(\lambda(\sigma_s^2) - \lambda(\sigma_{iT/n}^2) \right) ds + \lambda(\sigma_{iT/n}^2) \mathbf{E}(\xi^{\sigma 2}) \Delta_{n,T} = \alpha_1 + \alpha_2$. Notice that

$$\begin{aligned} \alpha_1 &= \mathbf{E}_i \int_{iT/n}^{(i+1)T/n} \mathbf{E}(\xi^{\sigma 2}) \left(\int_{iT/n}^s \lambda'(\sigma_v^2) m(\sigma_v^2) dv \right) ds + \frac{1}{2} \mathbf{E}_i \int_{iT/n}^{(i+1)T/n} \mathbf{E}(\xi^{\sigma 2}) \left(\int_{iT/n}^s \lambda''(\sigma_v^2) \Lambda^2(\sigma_v^2) dv \right) ds \\ &\quad + \mathbf{E}_i \int_{iT/n}^{(i+1)T/n} \mathbf{E}(\xi^{\sigma 2}) \left(\int_{iT/n}^s \lambda'(\sigma_v^2) \Lambda(\sigma_v^2) dW_v \right) ds + \mathbf{E}_i \int_{iT/n}^{(i+1)T/n} \mathbf{E}(\xi^{\sigma 2}) \left(\int_{iT/n}^s \lambda'(\sigma_v^2) \int_{\xi} \xi^{\sigma 2} \bar{v}(dv, d\xi^{\sigma}) \right) ds \\ (18) \quad &+ \mathbf{E}_i \int_{iT/n}^{(i+1)T/n} \mathbf{E}(\xi^{\sigma 2}) \left(\int_{iT/n}^s \lambda'(\sigma_v^2) \lambda(\sigma_v^2) \mathbf{E}(\xi^{\sigma}) dv \right) ds \\ &= \mathbf{E}_i \int_{iT/n}^{(i+1)T/n} \mathbf{E}(\xi^{\sigma 2}) \left(\int_{iT/n}^s \lambda'(\sigma_v^2) m(\sigma_v^2) dv \right) ds + \frac{1}{2} \mathbf{E}_i \int_{iT/n}^{(i+1)T/n} \mathbf{E}(\xi^{\sigma 2}) \left(\int_{iT/n}^s \lambda''(\sigma_v^2) \Lambda^2(\sigma_v^2) dv \right) ds \\ &\quad + \mathbf{E}_i \int_{iT/n}^{(i+1)T/n} \mathbf{E}(\xi^{\sigma 2}) \left(\int_{iT/n}^s \lambda'(\sigma_v^2) \lambda(\sigma_v^2) \mathbf{E}(\xi^{\sigma}) dv \right) ds \\ &= \alpha_{11} + \alpha_{12} + \alpha_{13}. \end{aligned}$$

Given monotonicity of the expectation operator,

$$\alpha_{11} \leq \mathbf{E}(\xi^{\sigma 2}) \mathbf{E}_i \int_{iT/n}^{(i+1)T/n} \left(\int_{iT/n}^{(i+1)T/n} |\lambda'(\sigma_v^2) m(\sigma_v^2)| dv \right) ds \leq \Delta_{n,T} \mathbf{E}(\xi^{\sigma 2}) \mathbf{E}_i \left(\Delta_{n,T} \sup_{t \leq T} |\lambda'(\sigma_t^2) m(\sigma_t^2)| \right).$$

But, using the law of iterated expectations

$$\mathbf{P} \left(\mathbf{E}_i \left(\Delta_{n,T} \sup_{t \leq T} |\lambda'(\sigma_t^2) m(\sigma_t^2)| \right) > \epsilon \right) \leq \frac{1}{\epsilon} \mathbf{E} \left[\mathbf{E}_i \left(\Delta_{n,T} \sup_{t \leq T} |\lambda'(\sigma_t^2) m(\sigma_t^2)| \right) \right]$$

$$\leq \frac{1}{\epsilon} \mathbf{E} \left[\mathcal{M}(\lambda'(\sigma_v^2) m(\sigma_v^2)) \right] \Delta_{n,T}.$$

Thus, $\alpha_{11} \stackrel{p}{\sim} \Delta_{n,T}^* \Delta_{n,T}$. Similarly, $\alpha_{12} \stackrel{p}{\sim} \Delta_{n,T}^* \Delta_{n,T}$ and $\alpha_{13} \stackrel{p}{\sim} \Delta_{n,T}^* \Delta_{n,T}$. Thus, α_2 is the dominating term, yielding

$$\begin{aligned} \mathbf{V}_{n,T} &\stackrel{p}{\sim} \frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}^2 \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right) \lambda(\sigma_{iT/n}^2) \mathbf{E}(\xi^{\sigma^2}) \\ &= \frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}^2 \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right) \int_{iT/n}^{(i+1)T/n} \lambda(\sigma_s^2) \mathbf{E}(\xi^{\sigma^2}) ds + \frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}^2 \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right) \int_{iT/n}^{(i+1)T/n} \left(\lambda(\sigma_{iT/n}^2) - \lambda(\sigma_s^2) \right) \mathbf{E}(\xi^{\sigma^2}) ds. \end{aligned}$$

The lower-order terms in $\int_{iT/n}^{(i+1)T/n} \left(\lambda(\sigma_{iT/n}^2) - \lambda(\sigma_s^2) \right) \mathbf{E}(\xi^{\sigma^2}) ds$ are $\int_{iT/n}^{(i+1)T/n} \mathbf{E}(\xi^{\sigma^2}) \left(\int_{i\Delta_{n,T}}^s \lambda'(\sigma_v^2) \Lambda(\sigma_v^2) dW_v \right) ds$ and $\int_{iT/n}^{(i+1)T/n} \mathbf{E}(\xi^{\sigma^2}) \left(\int_{i\Delta_{n,T}}^s \lambda'(\sigma_v^2) \int \xi^{\sigma^2} \bar{v}(dv, d\xi^\sigma) \right) ds$ (c.f., Eq. (18)). Now notice that

$$\int_{iT/n}^{(i+1)T/n} \mathbf{E}(\xi^{\sigma^2}) \left(\int_{i\Delta_{n,T}}^s \lambda'(\sigma_v^2) \Lambda(\sigma_v^2) dW_v \right) ds \leq \mathbf{E}(\xi^{\sigma^2}) \Delta_{n,T} \max_{i\Delta_{n,T} \leq s \leq (i+1)\Delta_{n,T}} \left| \int_{i\Delta_{n,T}}^s \lambda'(\sigma_v^2) \Lambda(\sigma_v^2) dW_v \right|.$$

Burkholder-Davis-Gundy inequality (e.g., [Revuz and Yor, 1994](#), Theorem 4.1) implies that, for $f^*(\sigma_s^2) = \left(\lambda'(\sigma_v^2) \Lambda(\sigma_v^2) \right)^2$,

$$\begin{aligned} &\mathbf{E}((\xi^\sigma)^2) \Delta_{n,T} \mathbf{E} \left(\max_{i\Delta_{n,T} \leq s \leq (i+1)\Delta_{n,T}} \left| \int_{i\Delta_{n,T}}^s \lambda'(\sigma_v^2) \Lambda(\sigma_v^2) dW_v \right| \right) \\ &\leq C \mathbf{E}((\xi^\sigma)^2) \Delta_{n,T} \mathbf{E} \left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left(\lambda'(\sigma_v^2) \Lambda(\sigma_v^2) \right)^2 dv \right)^{1/2} \\ &\leq C \mathbf{E}((\xi^\sigma)^2) \Delta_{n,T} \mathbf{E} \left(\sup_{s \leq T} |f^*(\sigma_s^2)| \Delta_{n,T} \right)^{1/2} \\ &\leq C \mathbf{E}((\xi^\sigma)^2) \Delta_{n,T}^{3/2} \mathbf{E} \left(\sup_{s \leq T} |F(\sigma_s^2)| \right)^{1/2} \\ &\leq C \mathbf{E}((\xi^\sigma)^2) \Delta_{n,T}^{3/2} \left(\mathbf{E} \left(\sup_{s \leq T} |F(\sigma_s^2)| \right) \right)^{1/2} \\ &\stackrel{p}{\sim} \Delta_{n,T} (\Delta_{n,T}^*)^{1/2} \end{aligned}$$

Hence, the order of the term $\int_{iT/n}^{(i+1)T/n} \mathbf{E}(\xi^{\sigma^2}) \left(\int_{i\Delta_{n,T}}^s \lambda'(\sigma_v^2) \Lambda(\sigma_v^2) dW_v \right) ds$ is $\Delta_{n,T} (\Delta_{n,T}^*)^{1/2}$. The same applies to $\int_{iT/n}^{(i+1)T/n} \mathbf{E}((\xi^\sigma)^2) \left(\int_{i\Delta_{n,T}}^s \lambda'(\sigma_v^2) \int \xi^{\sigma^2} \bar{v}(dv, d\xi^\sigma) \right) ds$. In consequence, $\int_{iT/n}^{(i+1)T/n} \lambda(\sigma_s^2) \mathbf{E}((\xi^\sigma)^2) ds \stackrel{p}{\sim} \Delta_{n,T} \mathcal{M}(\lambda) \stackrel{p}{\sim} \Delta_{n,T}^*$ is of lower order than $\int_{iT/n}^{(i+1)T/n} \left(\lambda(\sigma_{iT/n}^2) - \lambda(\sigma_s^2) \right) \mathbf{E}(\xi^{\sigma^2}) ds$ and dominates. Note indeed that $\Delta_{n,T} (\Delta_{n,T}^*)^{1/2} = \frac{\Delta_{n,T}^* \Delta_{n,T}}{(\Delta_{n,T}^*)^{1/2}}$ and $\frac{\Delta_{n,T}}{(\Delta_{n,T}^*)^{1/2}}$ can be written as $\frac{(\Delta_{n,T})^{1/2}}{\mathbf{E}(\mathcal{M}(f^*))}$, but $\frac{(\Delta_{n,T})^{1/2}}{\mathbf{E}(\mathcal{M}(f^*))} \rightarrow 0$. Finally, by Lemma 2,

$$\begin{aligned} \mathbf{V}_{n,T} &\stackrel{p}{\sim} \frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}^2 \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right) \int_{iT/n}^{(i+1)T/n} \lambda(\sigma_s^2) \mathbf{E}((\xi^\sigma)^2) ds. \\ &\stackrel{p}{\sim} \frac{1}{h_{n,T}} \int_0^T \mathbf{K}^2 \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) \lambda(\sigma_s^2) \mathbf{E}(\xi^{\sigma^2}) ds - \frac{\Delta_{n,T}^*}{h_{n,T}^2} O_p \left(\int_0^T \left| \mathbf{K}^{2l} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) \right| |m(\sigma_s^2)| ds \right) \\ &\quad - \frac{\Delta_{n,T}^*}{h_{n,T}^2} O_p \left(\int_0^T \left| \mathbf{K}^{2l} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) \right| \Lambda^2(\sigma_s^2) ds \right) - \frac{\Delta_{n,T}^*}{h_{n,T}^{3/2}} O_p \left(\frac{1}{h_{n,T}^{1/2}} \int_0^T \left| \mathbf{K}^{2l} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) \right| \Lambda(\sigma_s^2) dW_s^\sigma \right) \end{aligned}$$

$$-\frac{\Delta_{n,T}^*}{h_{n,T}^{3/2}} O_p \left(\frac{1}{h_{n,T}^{1/2}} \int_0^T \left| \mathbf{K}^{2j} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) \right| \int \xi^\sigma \bar{v}_\sigma(ds, d\xi^\sigma) \right).$$

Now, uniformly in T and $h_{n,T}$ in the set $\mathcal{H} = [(\Delta_{n,T}^*)^{1/2}/\epsilon, \epsilon]$, for an arbitrarily small $\epsilon > 0$

$$\mathbf{V}_{n,T} \xrightarrow[\Delta_{n,T} \rightarrow 0]{p} \tilde{\mathbf{U}}$$

where $\tilde{\mathbf{U}} = \frac{1}{h_{n,T}} \int_0^T \mathbf{K}^2 \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) \lambda(\sigma_s^2) \mathbf{E}((\xi^\sigma)^2) ds$. If an appropriate negligibility condition (i.e., a conditional version of the Lindenberg condition) is satisfied, namely

$$(19) \quad \forall \epsilon > 0, \sum_{i=1}^{n-1} \mathbf{E} \left\{ u_{iT/n, (i+1)T/n}^2 \mathbf{1}_{(|u_{iT/n, (i+1)T/n}| > \epsilon)} | \mathfrak{S}_{iT/n} \right\} \xrightarrow[\Delta_{n,T} \rightarrow 0]{p} 0,$$

then, for every T and uniformly in $h_{n,T} \in \mathcal{H}$

$$(20) \quad \mathbf{U}_{n,T} \xrightarrow[\Delta_{n,T} \rightarrow 0]{} Z, \text{ with characteristic function } \mathbf{E}(\exp(-\frac{1}{2} \tilde{\mathbf{U}} t^2)),$$

which is a martingale central limit theorem ([Jacod and Shiryaev, 2003](#), Theorem VIII.3.33). Write

$$(21) \quad \begin{aligned} & \sum_{j=1}^{n-1} \mathbf{E} \left\{ u_{iT/n, (i+1)T/n}^2 \mathbf{1}_{(|u_{iT/n, (i+1)T/n}| > \epsilon)} | \mathfrak{S}_{iT/n} \right\} \\ &= \sum_{j=1}^{n-1} \mathbf{E} \left\{ u_{iT/n, (i+1)T/n}^2 | \mathfrak{S}_{iT/n} \right\} - \sum_{j=1}^{n-1} \mathbf{E} \left\{ u_{iT/n, (i+1)T/n}^2 \mathbf{1}_{(|u_{iT/n, (i+1)T/n}| \leq \epsilon)} | \mathfrak{S}_{iT/n} \right\} \\ &= \mathbf{V}_{n,T} - \sum_{j=1}^{n-1} \mathbf{E} \left\{ u_{iT/n, (i+1)T/n}^2 \mathbf{1}_{\left(\left| \mathbf{K} \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right) \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \xi^\sigma \bar{v}_\sigma(ds, d\xi^\sigma) \right| \leq \epsilon \sqrt{h_{n,T}} \right)} | \mathfrak{S}_{iT/n} \right\}, \end{aligned}$$

but $\mathbf{K} \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right) \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \xi^\sigma \bar{v}_\sigma(ds, d\xi^\sigma) = O_p(\sqrt{\Delta_{n,T}^*})$. Hence, since $\Delta_{n,T}^* \rightarrow 0$ faster than $h_{n,T}$ in \mathcal{H} , the indicator converges in probability to 1 uniformly for $h_{n,T} \in \mathcal{H}$ and, given boundedness of $\tilde{\mathbf{U}}$, the Lindenberg condition is verified. It follows that, as $\Delta_{n,T} \rightarrow 0$, uniformly in T and $h_{n,T} \in \mathcal{H}$,

$$\mathbf{U}_{n,T} \xrightarrow{d} W_{\tilde{\mathbf{U}}},$$

where W is a standard Brownian motion. By the scaling property of Brownian motion

$$\frac{\mathbf{U}_{n,T}}{\sqrt{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds}} \xrightarrow{d} W \left(\frac{\tilde{\mathbf{U}}}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds} \right).$$

Given the definition of $\mathbf{V}_{n,T}$, the ratio-limit theorem now implies that, as $\Delta_{n,T} \rightarrow 0$, $T \rightarrow \infty$, and $h_{n,T} \rightarrow 0$, provided $\frac{\Delta_{n,T}^*}{h_{n,T}^2} \rightarrow 0$ as in Lemma 2,

$$\begin{aligned} & \frac{\mathbf{V}_{n,T}}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds} \\ &= \frac{\frac{1}{h_{n,T}} \int_{-\infty}^{\infty} \mathbf{K}^2 \left(\frac{a-x}{h_{n,T}} \right) \lambda(a) \mathbf{E}((\xi^\sigma)^2) da}{\frac{1}{h_{n,T}} \int_{-\infty}^{\infty} \mathbf{K} \left(\frac{a-x}{h_{n,T}} \right) s(a) da} + o_p(1) \end{aligned}$$

$$\begin{aligned}
&= \frac{\int_{-\infty}^{\infty} \mathbf{K}^2(u) \lambda(uh_{n,T} + x) \mathbf{E}((\xi^\sigma)^2) s(uh_{n,T} + x) du}{\int_{-\infty}^{\infty} \mathbf{K}(u) s(uh_{n,T} + x) du} + o_p(1) \\
&= \left(\int_{-\infty}^{\infty} \mathbf{K}^2(u) du \right) \lambda(a) \mathbf{E}((\xi^\sigma)^2) + o_p(1),
\end{aligned}$$

where $s(\cdot)$ is the variance process' invariant measure. Hence, if $\frac{\Delta_{n,T}^*}{h_{n,T}^2} \rightarrow 0$, $\frac{\mathbf{U}_{n,T}}{\sqrt{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) ds}} \xrightarrow{\Delta_{n,T} \rightarrow 0, T \rightarrow \infty, h_{n,T} \rightarrow 0} \Rightarrow \left(\sqrt{\int_{-\infty}^{\infty} \mathbf{K}^2(u) du} \lambda(a) \mathbf{E}((\xi^\sigma)^2) \right) Z$, where Z is a standard normal random variable, by the same argument as in, e.g., [van Zanten \(2000\)](#), Theorem 4.1. \square

LEMMA 5. Given Assumption 5, if $\frac{\Delta_{n,T}^*}{h_{n,T}^2} \rightarrow 0$, then

$$\left(\sqrt{h_{n,T}} \sqrt{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) ds} \right) \frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right) \int_{iT/n}^{(i+1)T/n} \Lambda(\sigma_s^2) dW_s^\sigma}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) ds} \Rightarrow \mathbf{N}(0, \mathbf{K}_2 \Lambda^2(x))$$

PROOF. As for the proof of Lemma 4. \square

THEOREM 6 (Consistency of the infeasible estimator.). Write $\widehat{L}_{\sigma^2}(T, x) = \frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^n \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right)$. Given Assumption 5, if $\frac{\Delta_{n,T}^*}{h_{n,T}^2} \rightarrow 0$:

$$\frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right) (\sigma_{(i+1)T/n}^2 - \sigma_{iT/n}^2)}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right)} - \theta^1(x) = O_p \left(\frac{1}{\sqrt{h_{n,T} \left(\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^n \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right) \right)}} \right) + O_p \left(\frac{\Delta_{n,T}^*}{h_{n,T}^2} \right).$$

Thus, if $\frac{\Delta_{n,T}^*}{h_{n,T}^2} \rightarrow 0$ and $h_{n,T} \widehat{L}_{\sigma^2}(T, x) \xrightarrow{a.s.} \infty$, then $\frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right) (\sigma_{(i+1)T/n}^2 - \sigma_{iT/n}^2)}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right)} \xrightarrow{p} \theta^1(x)$.

PROOF. Write

$$\frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right) (\sigma_{(i+1)T/n}^2 - \sigma_{iT/n}^2)}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^n \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right)} - \theta^1(x) = \frac{\frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right) (\sigma_{(i+1)T/n}^2 - \sigma_{iT/n}^2)}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) ds}}{\frac{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^n \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right)}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) ds}} - \theta^1(x).$$

After compensating the measure $v_\sigma(ds, d\xi^\sigma)$ and writing $\bar{v}_\sigma(ds, d\xi^\sigma) = v_\sigma(ds, d\xi^\sigma) - \lambda(\sigma^2) \mathbf{E}((\xi^\sigma)^2)$ and $\bar{m}(\sigma_s^2) = m(\sigma_s^2) + \lambda(\sigma^2) \mathbf{E}(\xi^\sigma)$ instead of $v_\sigma(ds, d\xi^\sigma)$ and $m(\sigma_s^2)$, we have

$$\frac{\frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right) (\sigma_{(i+1)T/n}^2 - \sigma_{iT/n}^2)}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) ds}}{1 + O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right)} - \frac{\theta^1(x)}{1 + O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right)} - \frac{\theta^1(x) O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right)}{1 + O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right)}$$

$$\begin{aligned}
& \frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2}{h_{n,T}} \right) \int_{iT/n}^{(i+1)T/n} \overline{m}(\sigma_s^2) ds}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds} - \frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2}{h_{n,T}} \right) \int_{iT/n}^{(i+1)T/n} \Lambda(\sigma_s^2) dW_s^\sigma}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds} \\
= & \frac{1 + O_p \left(\frac{\Delta_{n,T}}{h_{n,T}^2} \right)}{1 + O_p \left(\frac{\Delta_{n,T}}{h_{n,T}^2} \right)} + \frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2}{h_{n,T}} \right) \int_{iT/n}^{(i+1)T/n} \int_\xi (\xi^\sigma) \overline{v}_\sigma(ds, d\xi^\sigma)}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds} - \frac{\theta^1(x)}{1 + O_p \left(\frac{\Delta_{n,T}}{h_{n,T}^2} \right)} - \frac{\theta^1(x) O_p \left(\frac{\Delta_{n,T}}{h_{n,T}^2} \right)}{1 + O_p \left(\frac{\Delta_{n,T}}{h_{n,T}^2} \right)} \\
& + \frac{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) \overline{m}(\sigma_s^2) ds}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds} - \frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2}{h_{n,T}} \right) \int_{iT/n}^{(i+1)T/n} \Lambda(\sigma_s^2) dW_s^\sigma}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds} + \frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2}{h_{n,T}} \right) \int_{iT/n}^{(i+1)T/n} \int_\xi (\xi^\sigma) \overline{v}_\sigma(ds, d\xi^\sigma)}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds} \\
= & \frac{1 + O_p \left(\frac{\Delta_{n,T}}{h_{n,T}^2} \right)}{1 + O_p \left(\frac{\Delta_{n,T}}{h_{n,T}^2} \right)} + \frac{1 + O_p \left(\frac{\Delta_{n,T}}{h_{n,T}^2} \right)}{1 + O_p \left(\frac{\Delta_{n,T}}{h_{n,T}^2} \right)} + \frac{1 + O_p \left(\frac{\Delta_{n,T}}{h_{n,T}^2} \right)}{1 + O_p \left(\frac{\Delta_{n,T}}{h_{n,T}^2} \right)} \\
& - \frac{\theta^1(x)}{1 + O_p \left(\frac{\Delta_{n,T}}{h_{n,T}^2} \right)} - \frac{\theta^1(x) O_p \left(\frac{\Delta_{n,T}}{h_{n,T}^2} \right)}{1 + O_p \left(\frac{\Delta_{n,T}}{h_{n,T}^2} \right)} + \frac{O_p \left(\frac{\Delta_{n,T}}{h_{n,T}^2} \right)}{1 + O_p \left(\frac{\Delta_{n,T}}{h_{n,T}^2} \right)} \\
= & \alpha + \beta + \gamma + O_p \left(\frac{\Delta_{n,T}}{h_{n,T}^2} \right).
\end{aligned}$$

Finally, as in Bandi and Nguyen (2003),

$$\alpha = \frac{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) \overline{m}(\sigma_s^2) ds}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds} - \theta^1(x) \frac{h_{n,T}^2 \left(\int s^2 \mathbf{K}(s) ds \right) \left(\frac{\partial \theta^1(\sigma^2)}{\partial \sigma^2} \frac{\partial s(\sigma^2)}{s(\sigma^2)} + \frac{1}{2} \frac{\partial \theta^1(\sigma^2)}{\partial^2 \sigma^2} \right)}{1 + O_p \left(\frac{\Delta_{n,T}}{h_{n,T}^2} \right)},$$

Also, by Lemmas 4 and 5,

$$\beta = \frac{O_p \left(\frac{1}{h_{n,T}^{1/2} \sqrt{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds}} \right)}{1 + O_p \left(\frac{\Delta_{n,T}}{h_{n,T}^2} \right)},$$

and

$$\gamma = \frac{O_p \left(\frac{1}{h_{n,T}^{1/2} \sqrt{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds}} \right)}{1 + O_p \left(\frac{\Delta_{n,T}}{h_{n,T}^2} \right)}.$$

Thus,

$$\frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2}{h_{n,T}} \right) (\sigma_{(i+1)T/n}^2 - \sigma_{iT/n}^2)}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2}{h_{n,T}} \right)} - \theta^1(x) = O_p \left(\frac{1}{h_{n,T}^{1/2} \sqrt{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds}} \right) + O_p \left(\frac{\Delta_{n,T}}{h_{n,T}^2} \right).$$

However, the last expression can be written as follows

$$O_p \left(\frac{1}{h_{n,T}^{1/2} \sqrt{\widehat{L}_{\sigma^2}(T, x)}} \right) + O_p \left(\frac{\Delta_{n,T}}{h_{n,T}^2} \right)$$

since, when $\frac{\Delta_{n,T}}{h_{n,T}^2} \rightarrow 0$, which is implied by $\frac{\Delta_{n,T}^*}{h_{n,T}^2} \rightarrow 0$, $\widehat{L}_{\sigma^2}(T, x)$ and $\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) ds$ are asymptotically equivalent (see Lemma 3). \square

THEOREM 7 (Asymptotic normality of the infeasible estimator.). *Given Assumption 5, if $h_{n,T} \widehat{L}_{\sigma^2}(T, x) \xrightarrow{a.s.} \infty$, $h_{n,T}^5 \widehat{L}_{\sigma^2}(T, x) \xrightarrow{a.s.} C$, and $\frac{\Delta_{n,T}^* \sqrt{\widehat{L}_{\sigma^2}(T, x)}}{h_{n,T}^{3/2}} \xrightarrow{a.s.} 0$,*

$$\sqrt{h_{n,T} \widehat{L}_{\sigma^2}(T, x)} \left(\frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right) (\sigma_{(i+1)T/n}^2 - \sigma_{iT/n}^2)}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right)} - \theta^1(x) - \Gamma_{\theta^1}(x) \right) \Rightarrow \mathbf{N}(0, \mathbf{K}_2 \theta_2(x)),$$

where $\Gamma_{\theta^1}(x) = h_{n,T}^2 \left(\int s^2 \mathbf{K}(s) ds \right) \left(\frac{\partial \theta^1(\sigma^2)}{\partial \sigma^2} \frac{\partial s(\sigma^2)}{s(\sigma^2)} + \frac{1}{2} \frac{\partial^2 \theta^1(\sigma^2)}{\partial^2 \sigma^2} \right)$.

PROOF. Write

$$\begin{aligned} & \sqrt{h_{n,T} \widehat{L}_{\sigma^2}(T, x)} \left(\frac{\frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right) \int_{iT/n}^{(i+1)T/n} \overline{m}(\sigma_s^2) ds}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) ds} + \frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right) \int_{iT/n}^{(i+1)T/n} \Lambda(\sigma_s^2) dW_s^\sigma}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right)}}{1 + O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right)} + \frac{\frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right) \int_{iT/n}^{(i+1)T/n} \Lambda(\sigma_s^2) dW_s^\sigma}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right)} - \frac{\theta^1(x) O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right)}{1 + O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right)} - \frac{\theta^1(x) O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right)}{1 + O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right)}}{1 + O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right)} \right) \\ &= \sqrt{h_{n,T} \widehat{L}_{\sigma^2}(T, x)} \left(\frac{\frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right) \int_{iT/n}^{(i+1)T/n} \overline{m}(\sigma_s^2) ds}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) ds} - \theta^1(x)}{1 + O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right)} - \frac{\theta^1(x) O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right)}{1 + O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right)} \right) \\ &+ \frac{\frac{1}{\sqrt{h_{n,T}}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right) \int_{iT/n}^{(i+1)T/n} \Lambda(\sigma_s^2) dW_s^\sigma}{\sqrt{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right)}} + \frac{\frac{1}{\sqrt{h_{n,T}}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right) \int_{iT/n}^{(i+1)T/n} \int (\xi^\sigma) \overline{v}_\sigma(ds, d\xi^\sigma)}{\sqrt{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right)}} \\ &= \sqrt{h_{n,T} \widehat{L}_{\sigma^2}(T, x)} \left(\frac{\frac{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) \overline{m}(\sigma_s^2) ds}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) ds} - \theta^1(x)}{1 + O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right)} - \frac{\theta^1(x) O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right)}{1 + O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right)} + \frac{O_p\left(\frac{\Delta_{n,T}^*}{h_{n,T}^2}\right)}{1 + O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right)} \right) \\ &+ \frac{\frac{1}{\sqrt{h_{n,T}}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right) \int_{iT/n}^{(i+1)T/n} \Lambda(\sigma_s^2) dW_s^\sigma}{\sqrt{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right)}} + \frac{\frac{1}{\sqrt{h_{n,T}}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right) \int_{iT/n}^{(i+1)T/n} \int (\xi^\sigma) \overline{v}_\sigma(ds, d\xi^\sigma)}{\sqrt{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right)}}. \end{aligned}$$

The result now follows from Lemma 2, Lemma 3, Lemma 4 and Lemma 5, since $\frac{\Delta_{n,T}^*}{h_{n,T}^2} \sqrt{h_{n,T} \widehat{L}_{\sigma^2}(T, x)} \rightarrow 0$ implies $\frac{\Delta_{n,T}^*}{h_{n,T}^2} \rightarrow 0$ in light of the requirement $h_{n,T} \widehat{L}_{\sigma^2}(T, x) \xrightarrow{a.s.} \infty$. \square

REMARK 10. For the classical uniformly bounded case ($\Delta_{n,T} = \Delta_{n,T}^*$), the above Theorem relaxes the bandwidth conditions provided in [Bandi and Nguyen \(2003\)](#).

THEOREM 8 (Consistency of the infeasible estimator of higher order moments.). *Let $p \geq 2$. Given Assumption 5, if $\frac{\Delta_{n,T}^*}{h_{n,T}^2} \rightarrow 0$:*

$$\frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right) (\sigma_{(i+1)T/n}^2 - \sigma_{iT/n}^2)^p}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right)} - \theta_p(x) = O_p \left(\frac{1}{h_{n,T}^{1/2} \sqrt{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^n \mathbf{K} \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right)}} \right) + O_p \left(\frac{\Delta_{n,T}^*}{h_{n,T}^2} \right).$$

Thus, if $\frac{\Delta_{n,T}^*}{h_{n,T}^2} \rightarrow 0$ and $h_{n,T} \widehat{L}_{\sigma^2}(T, x) \xrightarrow{a.s.} \infty$, then $\frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right) (\sigma_{(i+1)T/n}^2 - \sigma_{iT/n}^2)^p}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right)} \xrightarrow{p} \theta^p(x)$.

PROOF. Using Ito's lemma,

$$\begin{aligned} (\sigma_{(i+1)T/n}^2 - \sigma_{iT/n}^2)^p &= p \int_{iT/n}^{(i+1)T/n} (\sigma_{s-}^2 - \sigma_{iT/n}^2)^{p-1} m(\sigma_s^2) ds + p \int_{iT/n}^{(i+1)T/n} (\sigma_{s-}^2 - \sigma_{iT/n}^2)^{p-1} \Lambda(\sigma_s^2) dW_s^{\sigma^2} \\ &\quad + \frac{1}{2} p(p-1) \int_{iT/n}^{(i+1)T/n} (\sigma_{s-}^2 - \sigma_{iT/n}^2)^{p-2} \Lambda^2(\sigma_s^2) ds \\ (22) \quad &\quad + \sum_{\Delta \sigma_s^2 \neq 0} \left[(\sigma_{s-}^2 + \Delta \sigma_s^2 - \sigma_{iT/n}^2)^p - (\sigma_{s-}^2 - \sigma_{iT/n}^2)^p \right]. \end{aligned}$$

Also,

$$\begin{aligned} &\sum_{\Delta \sigma_s^2 \neq 0} \left[(\sigma_{s-}^2 + \Delta \sigma_s^2 - \sigma_{iT/n}^2)^p - (\sigma_{s-}^2 - \sigma_{iT/n}^2)^p \right] \\ &= \sum_{k=0}^{p-1} \binom{p}{k} \int_{iT/n}^{(i+1)T/n} \int_{\xi} (\sigma_{s-}^2 - \sigma_{iT/n}^2)^k (\xi^\sigma)^{(p-k)} \nu_{\sigma^2}(ds, d\xi^\sigma). \end{aligned}$$

Now, breaking up the summation into the term 0 and the term $\sum_{k=1}^{p-1}$, write

$$\begin{aligned} &\frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right) (\sigma_{(i+1)T/n}^2 - \sigma_{iT/n}^2)^p}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right)} \\ &= \frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right) p \int_{iT/n}^{(i+1)T/n} (\sigma_{s-}^2 - \sigma_{iT/n}^2)^{p-1} m(\sigma_s^2) ds}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right)} \\ &\quad + \frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right) p \int_{iT/n}^{(i+1)T/n} (\sigma_{s-}^2 - \sigma_{iT/n}^2)^{p-1} \Lambda(\sigma_s^2) dW_s^{\sigma^2}}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right)} \\ &\quad + \frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right) \frac{1}{2} p(p-1) \int_{iT/n}^{(i+1)T/n} (\sigma_{s-}^2 - \sigma_{iT/n}^2)^{p-2} \Lambda^2(\sigma_s^2) ds}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right)} \end{aligned}$$

$$\begin{aligned}
& + \frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right) \sum_{k=1}^{p-1} \binom{p}{k} \int_{iT/n}^{(i+1)T/n} (\sigma_{s^-}^2 - \sigma_{iT/n}^2)^k \int_{\xi} (\xi^\sigma)^{p-k} \nu(ds, d\xi^\sigma)}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right)} \\
& + \frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right) \int_{iT/n}^{(i+1)T/n} \int_{\xi} (\xi^\sigma)^p \nu(ds, d\xi^\sigma)}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right)} \\
& = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5.
\end{aligned}$$

Now we prove that $\sup_{iT/n \leq s \leq (i+1)T/n} |\sigma_{s^-}^2 - \sigma_{iT/n}^2|^p = O_p \left(\sqrt{\Delta_{n,T}^*} \right)$, $p \geq 1$. We start with

$$\begin{aligned}
& \sup_{iT/n \leq s \leq (i+1)T/n} \left| \sigma_{s^-}^2 - \sigma_{iT/n}^2 \right| \\
& = \sup_{iT/n \leq s \leq (i+1)T/n} \left(\left| \int_{iT/n}^s \bar{m}(\sigma_u^2) du + \int_{iT/n}^s \Lambda(\sigma_u) dW_u + \int_{iT/n}^s \int_{\xi} \xi^\sigma \bar{\nu}(ds, d\xi^\sigma) \right| \right) \\
& \leq \sup_{iT/n \leq s \leq (i+1)T/n} \left| \int_{iT/n}^s \bar{m}(\sigma_u^2) du \right| + \sup_{iT/n \leq s \leq (i+1)T/n} \left| \int_{iT/n}^s \Lambda(\sigma_u^2) dW_u \right| + \sup_{iT/n \leq s \leq (i+1)T/n} \left| \int_{iT/n}^s \int_{\xi} \xi^\sigma \bar{\nu}(du, d\xi^\sigma) \right|.
\end{aligned}$$

Write

$$\sup_{iT/n \leq s \leq (i+1)T/n} \left| \int_{iT/n}^s \bar{m}(\sigma_u^2) du \right| \leq \int_{iT/n}^{(i+1)T/n} |\bar{m}(\sigma_u^2)| du \stackrel{p}{\lesssim} O_p(\mathbf{E}(\mathcal{M}(\bar{m})\Delta_{n,T})) \stackrel{p}{\lesssim} \Delta_{n,T}^*.$$

Now notice that, by Burkholder-Davis-Gundy's inequality,

$$\mathbf{E} \left(\sup_{iT/n \leq s \leq (i+1)T/n} \left| \int_{iT/n}^s \Lambda(\sigma_u^2) dW_u \right| \right) \leq C \mathbf{E} \left(\int_{iT/n}^s \Lambda^2(\sigma_u^2) du \right)^{1/2} \leq C \left(\mathbf{E} \left(\sup_{s \leq T} \Lambda^2(\sigma_s^2) \Delta_{n,T} \right) \right)^{1/2} \sim (\Delta_{n,T}^*)^{1/2}$$

which implies, by Markov inequality, that the second term is $O_p((\Delta_{n,T}^*)^{1/2})$. The remaining term can be treated in the same way. We conclude that $\sup_{iT/n \leq s \leq (i+1)T/n} |\sigma_{s^-}^2 - \sigma_{iT/n}^2| = O_p((\Delta_{n,T}^*)^{1/2})$. Now consider $\sup_{iT/n \leq s \leq (i+1)T/n} |\sigma_{s^-}^2 - \sigma_{iT/n}^2|^2$. Using Ito's lemma and a similar reasoning we have:

$$\begin{aligned}
& \sup_{iT/n \leq s \leq (i+1)T/n} \left| \sigma_{s^-}^2 - \sigma_{iT/n}^2 \right|^2 \\
& = \sup_{iT/n \leq s \leq (i+1)T/n} \left(\left| \int_{iT/n}^s (\sigma_{u^-}^2 - \sigma_{iT/n}^2) m(\sigma_u^2) du + \int_{iT/n}^s (\sigma_{u^-}^2 - \sigma_{iT/n}^2) \Lambda(\sigma_u^2) dW_u + \int_{iT/n}^s \Lambda^2(\sigma_u^2) du \right. \right. \\
& \quad \left. \left. + \int_{iT/n}^s \int_{\xi} (\xi^\sigma)^2 \nu(du, d\xi^\sigma) + \int_{iT/n}^s \int_{\xi} 2(\sigma_{u^-}^2 - \sigma_{iT/n}^2) \xi^\sigma \nu(du, d\xi^\sigma) \right| \right),
\end{aligned}$$

which is, again, $O_p((\Delta_{n,T}^*)^{1/2})$ since, using the above result, the leading order is $\int_{iT/n}^s \int_{\xi} (\xi^\sigma)^2 \nu(du, d\xi^\sigma)$. Now, by induction, we obtain the same result for $\sup_{iT/n \leq s \leq (i+1)T/n} |\sigma_{s^-}^2 - \sigma_{iT/n}^2|^p$ with $p \geq 1$, since using Eq. (22) and the same previous

logic, its leading order term is $p \int_{iT/n}^{(i+1)T/n} \int_{\xi} (\xi^{\sigma})^p \bar{\nu}(ds, d\xi^{\sigma})$ which is $O_p((\Delta_{n,T}^*)^{1/2})$. Thus,

$$\begin{aligned}\alpha_1 &= \frac{O_p((\Delta_{n,T}^*)^{1/2}) \left(1 + O_p\left(\frac{\Delta_{n,T}^*}{h_{n,T}^2}\right)\right)}{1 + O_p\left(\frac{\Delta_{n,T}^*}{h_{n,T}^2}\right)}, \\ \alpha_2 &= \frac{O_p((\Delta_{n,T}^*)^{1/2}) O_p\left(\frac{1}{\sqrt{h_{n,T} \left(\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) ds\right)}}\right)}{1 + O_p\left(\frac{\Delta_{n,T}^*}{h_{n,T}^2}\right)}, \\ \alpha_3 &= \frac{O_p((\Delta_{n,T}^*)^{1/2}) \left(1 + O_p\left(\frac{\Delta_{n,T}^*}{h_{n,T}^2}\right)\right)}{1 + O_p\left(\frac{\Delta_{n,T}^*}{h_{n,T}^2}\right)}, \\ \alpha_4 &= O_p\left((\Delta_{n,T}^*)^{1/2}\right) O_p(\alpha_5).\end{aligned}$$

Finally,

$$\begin{aligned}\alpha_5 &= \frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right) \int_{iT/n}^{(i+1)T/n} \int_{\xi} \xi^{\sigma} \bar{\nu}(ds, d\xi^{\sigma})}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right)} + \frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right) \int_{iT/n}^{(i+1)T/n} \mathbf{E}(\xi^{\sigma}) \lambda(\sigma^2) ds}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right)} \\ &= \frac{\frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right) \int_{iT/n}^{(i+1)T/n} \xi^{\sigma} \bar{\nu}(ds, d\xi^{\sigma})}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) ds}}{1 + O_p\left(\frac{\Delta_{n,T}^*}{h_{n,T}^2}\right)} + \frac{\frac{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) \mathbf{E}(\xi^{\sigma}) \lambda(\sigma^2) ds}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) ds}}{1 + O_p\left(\frac{\Delta_{n,T}^*}{h_{n,T}^2}\right)} + O_p\left(\frac{\Delta_{n,T}^*}{h_{n,T}^2}\right) \\ &= \frac{O_p\left(\frac{1}{\sqrt{h_{n,T} \left(\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) ds\right)}}\right)}{1 + O_p\left(\frac{\Delta_{n,T}^*}{h_{n,T}^2}\right)} + \frac{\frac{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) \mathbf{E}(\xi^{\sigma}) \lambda(\sigma^2) ds}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) ds}}{1 + O_p\left(\frac{\Delta_{n,T}^*}{h_{n,T}^2}\right)} + O_p\left(\frac{\Delta_{n,T}^*}{h_{n,T}^2}\right)\end{aligned}$$

The proof now follows from that of previous results (see, e.g., Theorem 6). \square

THEOREM 9 (Asymptotic normality of the infeasible estimator of higher order moments.). *If $h_{n,T} \widehat{L}_{\sigma^2}(T, x) \xrightarrow{a.s.} \infty$, $h_{n,T}^5 \widehat{L}_{\sigma^2}(T, x) \xrightarrow{a.s.} C$ and $\frac{\Delta_{n,T}^* \sqrt{\widehat{L}_{\sigma^2}(T, x)}}{h_{n,T}^{3/2}} \xrightarrow{a.s.} 0$, for $p \geq 2$,*

$$\sqrt{h_{n,T} \widehat{L}_{\sigma^2}(T, x)} \left(\frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right) (\sigma_{(i+1)T/n}^2 - \sigma_{iT/n}^2)^p}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^n \mathbf{K}\left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}}\right)} - \theta^p(x) - \Gamma_{\theta^p}(x) \right) \Rightarrow \mathbf{N}(0, \mathbf{K}_2 \theta_{2p}(x)),$$

where $\Gamma_{\theta^p}(x) = h_{n,T}^2 \left(\int s^2 \mathbf{K}(s) ds \right) \left(\frac{\partial \theta^p(\sigma^2)}{\partial \sigma^2} \frac{\partial s(\sigma^2)}{\partial \sigma^2} + \frac{1}{2} \frac{\partial^2 \theta^p(\sigma^2)}{\partial^2 \sigma^2} \right)$

PROOF. Given the method of proof of Theorem 8, the result follows from the same arguments as for Theorem 7. It is sufficient to notice that the (sufficient) condition for a vanishing discretization error for the terms α_1 , α_2 , α_3 , and α_4 is $\Delta_{n,T}^* h_{n,T} \widehat{L}_{\sigma^2}(T, x) \xrightarrow{a.s.} 0$, and that α_5 is the leading order. Notice, however, that the condition $\frac{\Delta_{n,T}^* \sqrt{\widehat{L}_{\sigma^2}(T, x)}}{h_{n,T}^{3/2}} \xrightarrow{a.s.} 0$ can be

rewritten as follows:

$$(23) \quad \frac{\Delta_{n,T}^* \widehat{L}_{\sigma^2}(T, x) h_{n,T}}{h_{n,T}^{5/2} \widehat{L}_{n,T}^{1/2}(x)} \xrightarrow{a.s.} 0.$$

Now, the vanishing-bias condition $h_{n,T}^5 \widehat{L}_{\sigma^2}(T, x) \xrightarrow{a.s.} C$ implies that $\Delta_{n,T}^* \widehat{L}_{\sigma^2}(T, x) h_{n,T} \xrightarrow{a.s.} 0$ (given Eq. (23)) regardless of whether $C = 0$ or $C \neq 0$. \square

LEMMA 6. Assume $\widehat{\sigma}_{iT/n}^2$ is given by Eq. (4). If, as $n, k \rightarrow \infty$, we have $\frac{1}{\phi} \left(\frac{\phi}{k} \log \left(\frac{k}{\phi} \right) \mathcal{M}^2(\sigma^2) + \left(\frac{\phi}{k} \right)^2 \mathcal{M}^2(\mu) \right) \xrightarrow{a.s.} 0$, then

$$\begin{aligned} & \max_{1 \leq i \leq n} \left| \widehat{\sigma}_{iT/n}^2 - \sigma_{iT/n}^2 \right| = O_p \left(\left(\frac{\phi}{k} \right)^2 \mathcal{M}(\mu^2) \mathbf{E}(\mathcal{M}(\lambda)) \right) \\ & + O_p \left(\left(\frac{\phi}{k} \right)^{3/2} \mathbf{E}(\mathcal{M}^{1/2}(\sigma^2)) \mathcal{M}(\mu) \mathbf{E}(\mathcal{M}(\lambda)) \right) + O_p \left(\left(\frac{\phi}{k} \right)^{3/2} (\mathbf{E}(\mathcal{M}(\sigma^2) \mathcal{M}(\mu^2)))^{1/2} \mathbf{E}(\mathcal{M}(\lambda)) \right) \\ & + O_p \left(\left(\frac{\phi}{k} \right) \mathcal{M}(\sigma^2) \mathbf{E}(\mathcal{M}(\lambda)) \right) + O_p \left(\left(\frac{\phi}{k} \right) (\mathbf{E}(\mathcal{M}(\sigma^4)))^{1/2} \mathbf{E}(\mathcal{M}(\lambda)) \right) \\ & + O_p \left(\frac{\phi}{k} \mathcal{M}(\mu^2) \right) + O_p \left(\left(\frac{\phi}{k} \right)^{1/2} \mathcal{M}^{1/2}(\mu^2) (\mathbf{E} \mathcal{M}(\sigma^2))^{1/2} \right) \\ & + O_p \left(\frac{(\mathbf{E} \mathcal{M}(\sigma^4))^{1/2} \sqrt{\log n}}{\sqrt{k}} \right) + O_p \left(\left(\frac{\phi}{k} \right)^{1/2} \frac{\mathcal{M}^{*1/2}(\sigma^4, \mu^4) \sqrt{\log n}}{\sqrt{k}} \right) + O_p \left(\mathcal{M}^{*1/2}(\Lambda^2, \lambda) \phi^{1/2} + \mathcal{M}(m) \phi \right) \\ & = \Theta(n, T, k, \phi), \end{aligned}$$

where $\mathcal{M}^{*k}(f, g) = \max(\mathbf{E} \mathcal{M}^k(f), \mathbf{E} \mathcal{M}^k(g))$.

PROOF. Write the price process as $\log p = \log \tilde{p} + J$, where \tilde{p} is the continuous price component and J is the jump component. Without loss of generality, for notational convenience, let $\phi < \Delta_{n,T}$.² Write $\tilde{p}_j = \tilde{p}_{t_j}$ where $t_j = \left[\frac{j}{k} \right] \frac{T}{n} + \left(\frac{j}{k} - \left[\frac{j-1}{k} \right] \right) \phi$, $j = 1, \dots, nk$. Write

$$\widetilde{\sigma}_{iT/n}^2 := \frac{1}{\phi} \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} (\log \tilde{p}_{j+1} - \log \tilde{p}_j)^2,$$

which is the spot volatility estimator applied to the continuous component of the price process. Using Ito's Lemma,

$$\begin{aligned} & \max_{1 \leq i \leq n} \left| \widetilde{\sigma}_{iT/n}^2 - \sigma_{iT/n}^2 \right| \\ & \leq \max_{1 \leq i \leq n} \left| \frac{2}{\phi} \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \mu(\sigma_v^2) dv \right) \mu(\sigma_s^2) ds \right| + \max_{1 \leq i \leq n} \left| \frac{2}{\phi} \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \sigma_v dW_v \right) \mu(\sigma_s^2) ds \right| \\ & + \max_{1 \leq i \leq n} \left| \frac{2}{\phi} \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \sigma_v dW_v \right) \sigma_s dW_s \right| + \max_{1 \leq i \leq n} \left| \frac{2}{\phi} \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \mu(\sigma_v^2) dv \right) \sigma_s dW_s \right| \\ & + \max_{1 \leq i \leq n} \left| \frac{1}{\phi} \int_{iT/n}^{iT/n+\phi} \sigma_s^2 ds - \sigma_{iT/n}^2 \right|. \end{aligned}$$

²Also, the condition $\phi = o(\Delta_{n,T})$ is required for estimation of the dynamics, as implied, for example, by conditions 4.6 and 4.7 in Assumption 4.

We begin with the first component:

$$\begin{aligned}
\frac{2}{\phi} \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \mu(\sigma_v^2) dv \right) \mu(\sigma_s^2) ds &\leq \frac{2}{\phi} \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s |\mu(\sigma_v^2)| dv \right) |\mu(\sigma_s^2)| ds \\
&\leq \frac{2}{\phi} \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^{t_{j+1}} |\mu(\sigma_v^2)| dv \right) |\mu(\sigma_s^2)| ds \\
&\leq \frac{2}{\phi} \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \left(\int_{t_j}^{t_{j+1}} |\mu(\sigma_s^2)| ds \right)^2 \\
\text{(by Jensen's inequality)} &\leq \frac{2}{k} \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \left(\int_{t_j}^{t_{j+1}} \mu^2(\sigma_s^2) ds \right) \\
&= \frac{2}{k} \int_{iT/n}^{iT/n+\phi} \mu^2(\sigma_s^2) ds \\
(24) &\leq \frac{2\phi\mathcal{M}(\mu^2)}{k},
\end{aligned}$$

since $\int_{iT/n}^{iT/n+\phi} \mu^2(\sigma_s^2) ds \leq \max_{0 \leq t \leq T} |\mu^2(\sigma_s^2)| \phi = \mathcal{M}(\mu^2)\phi$. The second term can be bounded, using Cauchy-Schwartz inequality, as follows:

$$\begin{aligned}
&\frac{2}{\phi} \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \sigma_v dW_v \right) \mu(\sigma_s^2) ds \\
&\leq \frac{2}{\phi} \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \left(\max_{s \in [t_j, t_{j+1}]} \left| \int_{t_j}^s \sigma_v dW_v \right| \right) \int_{t_j}^{t_{j+1}} |\mu(\sigma_s^2)| ds \\
&\leq \frac{2}{\phi} \left(\sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \left(\int_{t_j}^{t_{j+1}} |\mu_s| ds \right)^2 \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \left(\max_{s \in [t_j, t_{j+1}]} \left| \int_{t_j}^s \sigma_v dW_v \right| \right)^2 \right)^{1/2}.
\end{aligned}$$

Now, fix $\epsilon > 0$. From Markov's inequality we have:

$$\Pr \left(\left(\max_{s \in [t_j, t_{j+1}]} \left| \int_{t_j}^s \sigma_v dW_v \right| \right)^2 > \epsilon \right) \leq \frac{1}{\epsilon} \mathbf{E} \left(\left(\max_{s \in [t_j, t_{j+1}]} \left| \int_{t_j}^s \sigma_v dW_v \right| \right)^2 \right).$$

The probability order can now be found using Burkholder-Davis-Gundy inequality, since

$$\sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \mathbf{E} \left(\left(\max_{s \in [t_j, t_{j+1}]} \left| \int_{t_j}^s \sigma_v dW_v \right| \right)^2 \right) \leq C \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \mathbf{E} \left(\int_{t_j}^{t_{j+1}} \sigma_u^2 du \right) \leq C \mathbf{E} \left(\int_{iT/n}^{iT/n+\phi} \sigma_u^2 du \right) \leq C\phi \mathbf{E} \mathcal{M}(\sigma^2)$$

for some C . Thus, using the steps leading to Eq. (24), the probability order of the second term becomes

$$\frac{2}{\phi} \left(\frac{\phi^3 \mathcal{M}(\mu^2) \mathbf{E} \mathcal{M}(\sigma^2)}{k} \right)^{1/2} = 2 \left(\frac{\phi}{k} \right)^{1/2} \mathcal{M}^{1/2}(\mu^2) (\mathbf{E} \mathcal{M}(\sigma^2))^{1/2}.$$

Now we turn to the third term. Write

$$\Phi_{i,n,k}^1(b) = \sum_{j=1}^{nk-2} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \sigma_v dW_v \right) \sigma_s dW_s + \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \int_{t_{nk-2}}^{t_{nk-2}+b\phi/k} \left(\int_{t_{nk-2}}^s \sigma_v dW_v \right) \sigma_s dW_s,$$

with $0 \leq b \leq 1$. Now, for any two events A and B ,

$$\Pr(A) = \Pr(A \cap B) + \Pr(A \cap \bar{B}) \leq \Pr(A \cap B) + P(\bar{B}).$$

Thus, for every i such that $1 \leq i \leq n$,

$$\begin{aligned} \Pr\left(\max_{1 \leq i \leq n} |\Phi_{i,n,k}^1(b)| \geq c\right) &\leq \Pr\left(\max_{1 \leq i \leq n} |\Phi_{i,n,k}^1(b)| \geq c, [\Phi_{i,n,k}^1]_{b=1} \leq \beta_{n,k,\phi}\right) + \Pr([\Phi_{i,n,k}^1]_{b=1} > \beta_{n,k,\phi}) \\ &\leq 2n \exp\left\{-\frac{c^2}{2\beta_{n,k,\phi}}\right\} + \Pr([\Phi_{i,n,k}^1]_{b=1} > \beta_{n,k,\phi}), \end{aligned}$$

for a generic positive sequence $\beta_{n,k,\phi}$, where the second inequality derives from a Bernstein-type inequality for continuous martingales (see, e.g., [Dzhaparidze and Van Zanten, 2001](#), Eq. (1.5)) and Boole inequality applied to the max term. Using Doob's representation,

$$\begin{aligned} [\Phi_{i,n,k}^1]_{b=1} &= \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \left\{ \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \sigma_v dW_v \right)^2 \sigma_s^2 ds \right\} \\ &= \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \left\{ \int_{t_j}^{t_{j+1}} \left[\left(\int_{t_j}^s \sigma_v^2 dv \right) + 2 \left(\int_{t_j}^s \left(\int_{t_j}^u \sigma_v dW_v \right) \sigma_u dW_u \right) \right] \sigma_s^2 ds \right\} \\ &\leq \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \left(\int_{t_j}^{t_{j+1}} \left(\int_{t_j}^{t_{j+1}} \sigma_v^2 dv \right) \sigma_s^2 ds \right) + 2 \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \left(\int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \left(\int_{t_j}^u \sigma_v dW_v \right) \sigma_u dW_u \right) \sigma_s^2 ds \right) \\ &\leq \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \left(\int_{t_j}^{t_{j+1}} \sigma_s^2 ds \right)^2 + 2 \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \left(\int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \left(\int_{t_j}^u \sigma_v dW_v \right) \sigma_u dW_u \right) \sigma_s^2 ds \right) \\ &\leq \frac{\phi}{k} \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \left(\int_{t_j}^{t_{j+1}} \sigma_s^4 ds \right) + 2 \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \left(\int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \left(\int_{t_j}^u \sigma_v dW_v \right) \sigma_u dW_u \right) \sigma_s^2 ds \right) \\ &= \frac{\phi}{k} \int_{iT/n}^{iT/n+\phi} \sigma_s^4 ds + 2 \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \left(\int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \left(\int_{t_j}^u \sigma_v dW_v \right) \sigma_u dW_u \right) \sigma_s^2 ds \right). \end{aligned}$$

Thus, given Markov's inequality

$$\begin{aligned} \Pr([\Phi_{i,n,k}^1]_{b=1} > \beta_{n,k,\phi}) &\leq \Pr\left(\left(\frac{\phi}{k} \int_{iT/n}^{iT/n+\phi} \sigma_s^4 ds + 2 \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \left(\int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \left(\int_{t_j}^u \sigma_v dW_v \right) \sigma_u dW_u \right) \sigma_s^2 ds \right)\right) > \beta_{n,k,\phi}\right) \\ &\leq \frac{1}{\beta_{n,k,\phi}} \left(\underbrace{\frac{\phi}{k} \mathbf{E} \left(\int_{iT/n}^{iT/n+\phi} \sigma_s^4 ds \right)}_{\alpha} + \underbrace{2 \mathbf{E} \left(\sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \left(\int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \left(\int_{t_j}^u \sigma_v dW_v \right) \sigma_u dW_u \right) \sigma_s^2 ds \right)\right)}_{2\beta} \right). \end{aligned}$$

Now notice that

$$\alpha \leq \frac{\phi^2 \mathbf{E} \mathcal{M}(\sigma^4)}{k},$$

since $\int_{iT/n}^{iT/n+\phi} \sigma_s^4 ds \leq \max_{0 \leq t \leq T} |\sigma_s^4| \phi = \mathcal{M}(\sigma^4) \phi$. Now we turn to the term β . Denoting the continuous martingale $\int_{t_j}^t \sigma_v dW_v$ by \widetilde{M}_t and integrating by parts, we obtain

$$\beta = \mathbf{E} \left(\sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \left(\int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \widetilde{M}_u \sigma_u dW_u \right) \sigma_s^2 ds \right) \right)$$

$$\begin{aligned}
&= \mathbf{E} \left(- \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \left(\int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \sigma_v^2 dv \right) \widetilde{M}_s \sigma_s dW_s \right) \right) + \mathbf{E} \left(\sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \left(\int_{t_j}^{t_{j+1}} \sigma_v^2 dv \right) \left(\int_{t_j}^{t_{j+1}} \widetilde{M}_s d\widetilde{M}_s \right) \right) \\
&= \mathbf{E} \left(\sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \left(\int_{t_j}^{t_{j+1}} \sigma_v^2 dv \right) \left(\int_{t_j}^{t_{j+1}} \widetilde{M}_s d\widetilde{M}_s \right) \right).
\end{aligned}$$

Now, by Jensen's inequality, Ito isometry, and Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned}
&\mathbf{E} \left(\sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \left(\int_{t_j}^{t_{j+1}} \sigma_v^2 dv \right) \left(\int_{t_j}^{t_{j+1}} \widetilde{M}_s d\widetilde{M}_s \right) \right) \\
&\leq \frac{1}{2} \left(\sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \left(\mathbf{E} \left(\int_{t_j}^{t_{j+1}} \sigma_v^2 dv \right)^2 + \mathbf{E} \left(\int_{t_j}^{t_{j+1}} \widetilde{M}_s d\widetilde{M}_s \right)^2 \right) \right) \\
&\leq \frac{1}{2} \frac{\phi}{k} \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \mathbf{E} \left(\int_{t_j}^{t_{j+1}} \sigma_v^4 dv \right) + \frac{1}{2} \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \mathbf{E} \left(\int_{t_j}^{t_{j+1}} \widetilde{M}_s^2 \sigma_s^2 ds \right) \\
&\leq \frac{1}{2} \frac{\phi}{k} \mathbf{E} \left(\int_{iT/n}^{iT/n+\phi} \sigma_s^4 ds \right) + \frac{1}{2} \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \mathbf{E} \left(\left(\int_{t_j}^{t_{j+1}} \sigma_s^2 ds \right) \left(\left(\max_{s \in [t_j, t_{j+1}]} \left| \int_{t_j}^s \sigma_u dW_u \right| \right)^2 \right) \right) \\
&\leq \frac{1}{2} \frac{\phi}{k} \mathbf{E} \left(\int_{iT/n}^{iT/n+\phi} \sigma_s^4 ds \right) + \frac{1}{4} \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \mathbf{E} \left(\left(\int_{t_j}^{t_{j+1}} \sigma_s^2 ds \right)^2 + \left(\left(\max_{s \in [t_j, t_{j+1}]} \left| \int_{t_j}^s \sigma_u dW_u \right| \right)^2 \right)^2 \right) \\
&\leq \frac{3}{4} \frac{\phi}{k} \mathbf{E} \left(\int_{iT/n}^{iT/n+\phi} \sigma_s^4 ds \right) + \frac{1}{4} \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \mathbf{E} \left(\left(\max_{s \in [t_j, t_{j+1}]} \left| \int_{t_j}^s \sigma_u dW_u \right| \right)^4 \right) \\
&\leq \frac{3}{4} \frac{\phi}{k} \mathbf{E} \left(\int_{iT/n}^{iT/n+\phi} \sigma_s^4 ds \right) + C \frac{1}{4} \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \mathbf{E} \left(\int_{t_j}^{t_{j+1}} \sigma_s^2 ds \right)^2 \\
(25) \quad &\leq C_1 \frac{\phi}{k} \mathbf{E} \left(\int_{iT/n}^{iT/n+\phi} \sigma_s^4 ds \right).
\end{aligned}$$

In sum,

$$\beta \leq C_1 \frac{\phi^2 \mathbf{E}(\mathcal{M}(\sigma^4))}{k},$$

and

$$\Pr([\Phi_{i,n,k}^1]_{b=1} > \beta_{n,k,\phi}) \leq C_2 \frac{\phi^2 \mathbf{E}(\mathcal{M}(\sigma^4))}{k \beta_{n,k,\phi}}.$$

Thus, for every $\beta_{n,k,\phi} > 0$,

$$\Pr \left(\max_{1 \leq i \leq n} |\Phi_{i,n,k}^1(b)| \geq c \right) \leq 2n \exp \left\{ -\frac{c^2}{2\beta_{n,k,\phi}} \right\} + C_2 \frac{\phi^2 \mathbf{E}(\mathcal{M}(\sigma^4))}{k \beta_{n,k,\phi}},$$

which also implies, for a sequence $C_{n,k}$,

$$\Pr \left(\frac{1}{\phi} \frac{\max_{1 \leq i \leq n} |\Phi_{i,n,k}^1(b)|}{C_{n,k}} \geq c \right) = \Pr \left(\max_{1 \leq i \leq n} |\Phi_{i,n,k}^1(b)| \geq c \phi C_{n,k} \right) \leq 2n \exp \left\{ -\frac{c^2 \phi^2 C_{n,k}^2}{2\beta_{n,k,\phi}} \right\} + C_2 \frac{\phi^2 \mathbf{E}(\mathcal{M}(\sigma^4))}{k \beta_{n,k,\phi}}.$$

Set, now, $\beta_{n,k,\phi} = c \frac{\phi^2 \mathbf{E}(\mathcal{M}(\sigma^4))}{k}$, and $C_{n,k} = (\mathbf{E} \mathcal{M}(\sigma^4))^{1/2} \sqrt{\log(n)}/\sqrt{k}$. Then,

$$\Pr \left(\frac{1}{\phi} \frac{\max_{1 \leq i \leq n} |\Phi_{i,n,k}^1(b)|}{C_{n,k}} \geq c \right) \leq 2n^{1-c/2} + \frac{C_2}{c}$$

which proves that $\frac{1}{\phi} \max_{1 \leq i \leq n} |\Phi_{i,n,k}^1(b)|$ is $O_p(C_{n,k}) = O_p\left((\mathbf{E}\mathcal{M}(\sigma^4))^{1/2} \sqrt{\log(n)}/\sqrt{k}\right)$.

As for the fourth term, write

$$\Phi_{i,n,k}^2(b) = \sum_{j=1}^{nk-2} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \mu(\sigma_v^2) dv \right) \sigma_s dW_s + \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \int_{t_{nk-2}}^{t_{nk-2}+b\phi/k} \left(\int_{t_{nk-2}}^s \mu(\sigma_v^2) dv \right) \sigma_s dW_s.$$

As before,

$$\Pr\left(\max_{1 \leq i \leq n} |\Phi_{i,n,k}^2(b)| \geq c\right) \leq 2n \exp\left\{-\frac{c^2}{2\beta_{n,k,\phi}}\right\} + \Pr([\Phi_{i,n,k}^2]_{b=1} > \beta_{n,k,\phi}).$$

But,

$$\begin{aligned} [\Phi_{i,n,k}^2]_{b=1} &= \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \left\{ \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \mu(\sigma_v^2) dv \right)^2 \sigma_s^2 ds \right\} \\ &\leq \frac{\phi}{k} \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \left(\int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \mu^2(\sigma_v^2) dv \right) \sigma_s^2 ds \right) \\ &\leq \frac{1}{2} \frac{\phi}{k} \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \left\{ \left(\int_{t_j}^{t_{j+1}} \mu^2(\sigma_s^2) ds \right)^2 + \left(\int_{t_j}^{t_{j+1}} \sigma_s^2 ds \right)^2 \right\} \\ &\leq \frac{1}{2} \left(\frac{\phi}{k} \right)^2 \sum_{j=1}^{nk-1} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} \left\{ \left(\int_{t_j}^{t_{j+1}} \mu^4(\sigma_s^2) ds \right) + \left(\int_{t_j}^{t_{j+1}} \sigma_s^4 ds \right) \right\} \\ &= \frac{1}{2} \left(\frac{\phi}{k} \right)^2 \left(\int_{iT/n}^{iT/n+\phi} \mu^4(\sigma_s^2) ds + \int_{iT/n}^{iT/n+\phi} \sigma_s^4 ds \right). \end{aligned}$$

Thus, using Markov's inequality

$$\begin{aligned} \Pr([\Phi_{i,n,k}^2]_{b=1} > \beta_{n,k,\phi}) &\leq \Pr\left(\frac{1}{2} \left(\frac{\phi}{k} \right)^2 \left(\int_{iT/n}^{iT/n+\phi} \mu^4(\sigma_s^2) ds + \int_{iT/n}^{iT/n+\phi} \sigma_s^4 ds \right) > \beta_{n,k,\phi}\right) \\ &\leq \frac{1}{2} \frac{\phi^2}{k^2 \beta_{n,k,\phi}} \mathbf{E} \left(\int_{iT/n}^{iT/n+\phi} \mu^4(\sigma_s^2) ds + \int_{iT/n}^{iT/n+\phi} \sigma_s^4 ds \right) \\ &\leq \frac{1}{2} \frac{\phi^3 \mathbf{E}(\mathcal{M}(\sigma^4))}{k^2 \beta_{n,k,\phi}} + \frac{1}{2} \frac{\phi^3 \mathbf{E}(\mathcal{M}(\mu^4))}{k^2 \beta_{n,k,\phi}}, \end{aligned}$$

which implies, for a given sequence $C_{n,k}$,

$$\Pr\left(\frac{1}{\phi} \max_{1 \leq i \leq n} |\Phi_{i,n,k}^2(b)| \geq c\right) \leq 2n \exp\left\{-\frac{c^2 \phi^2 C_{n,k}^2}{2\beta_{n,k,\phi}}\right\} + \frac{1}{2} \frac{\phi^3 \mathcal{M}^*(\sigma^4, \mu^4)}{k^2 \beta_{n,k,\phi}},$$

where $\mathcal{M}^*(\sigma^4, \mu^4) = \max(\mathbf{E}(\mathcal{M}(\sigma^4)), \mathbf{E}(\mathcal{M}(\mu^4)))$. Now set $\beta_{n,k,\phi} = c \frac{\phi^3 \mathcal{M}^*(\sigma^4, \mu^4)}{k^2}$ and $C_{n,k} = \mathcal{M}^*(\sigma^4, \mu^4) \sqrt{\log(n)} \phi/k$, then

$$\Pr\left(\frac{1}{\phi} \max_{1 \leq i \leq n} |\Phi_{i,n,k}^2(b)| \geq c\right) \leq 2n^{1-c/2} + \frac{1}{2c},$$

which implies that $\frac{1}{\phi} \max_{1 \leq i \leq n} |\Phi_{i,n,k}^1(b)|$ is $O_p(C_{n,k}) = O_p\left(\left(\frac{\phi^{1/2}}{k^{1/2}}\right) \frac{\mathcal{M}^{*1/2}(\sigma^4, \mu^4) \sqrt{\log n}}{\sqrt{k}}\right)$.

We now come to the last term:

$$\left| \frac{1}{\phi} \int_{iT/n}^{iT/n+\phi} \sigma_s^2 ds - \sigma_{iT/n}^2 \right| \leq \sup_{iT/n \leq s \leq iT/n+\phi} |\sigma_s^2 - \sigma_{iT/n}^2|$$

$$\begin{aligned}
&= \sup_{iT/n \leq s \leq iT/n + \phi} \left(\left| \int_{iT/n}^s m(\sigma_s^2) ds + \int_{iT/n}^s \Lambda(\sigma_s^2) dW_s + \int_{iT/n}^s \xi^\sigma \nu(ds, d\xi^\sigma) \right| \right) \\
&\leq \sup_{iT/n \leq s \leq iT/n + \phi} \left| \int_{iT/n}^s m(\sigma_s^2) ds \right| + \sup_{iT/n \leq s \leq iT/n + \phi} \left| \int_{iT/n}^s \Lambda(\sigma_s^2) dW_s \right| + \sup_{iT/n \leq s \leq iT/n + \phi} \left| \int_{iT/n}^s \xi^\sigma \nu(ds, d\xi^\sigma) \right|.
\end{aligned}$$

Notice that

$$\sup_{iT/n \leq s \leq iT/n + \phi} \left| \int_{iT/n}^s m(\sigma_s^2) ds \right| \leq \sup_{iT/n \leq s \leq iT/n + \phi} \int_{iT/n}^s |m(\sigma_s^2)| ds = \int_{iT/n}^{iT/n + \phi} |m(\sigma_s^2)| ds \leq \sup_{s \leq T} |m(\sigma_s^2)| \phi.$$

By Burkholder-Davis-Gundy's inequality

$$\mathbf{E} \left(\sup_{iT/n \leq s \leq iT/n + \phi} \left| \int_{iT/n}^s \Lambda(\sigma_s^2) dW_s \right| \right) \leq C \mathbf{E} \left(\int_{iT/n}^{iT/n + \phi} \Lambda^2(\sigma_s^2) ds \right)^{1/2} \leq C \mathbf{E} \left(\sup_{s \leq T} \Lambda^2(\sigma_s^2) \phi \right)^{1/2},$$

for some constant C . Hence,

$$\sup_{iT/n \leq s \leq iT/n + \phi} \left| \int_{iT/n}^s \Lambda(\sigma_s^2) dW_s \right| = O_p \left(\sup_{s \leq T} \Lambda^2(\sigma_s^2) \phi \right)^{1/2}.$$

Similarly,

$$\sup_{iT/n \leq s \leq iT/n + \phi} \left| \int_{iT/n}^s \xi^\sigma \nu(ds, d\xi^\sigma) \right| = O_p \left(\sup_{s \leq T} \lambda(\sigma_s^2) \phi \right)^{1/2}.$$

Thus,

$$\max_{1 \leq i \leq n} \sup_{iT/n \leq s \leq iT/n + \phi} \left| \sigma_s^2 - \sigma_{iT/n}^2 \right| = O_p \left(\mathcal{M}^{*1/2}(\Lambda^2, \lambda) \phi^{1/2} + \mathcal{M}(m) \phi \right).$$

In sum,

$$\begin{aligned}
\max_{1 \leq i \leq n} \left| \tilde{\sigma}_{iT/n}^2 - \sigma_{iT/n}^2 \right| &= O_P \left(\frac{\phi^{1/2} \mathcal{M}^{*1/2}(\sigma^4, \mu^4) \log n}{k^{1/2} \sqrt{k}} \right) + O_p \left(\frac{\mathcal{M}^{1/2}(\sigma^4) \log n}{\sqrt{k}} \right) + \\
&+ O_p \left(\left(\frac{\phi}{k} \right) \mathcal{M}(\mu^2) \right) + O_p \left(\left(\frac{\phi}{k} \right)^{1/2} \mathcal{M}^{1/2}(\mu^2) (\mathbf{E} \mathcal{M}(\sigma^2))^{1/2} \right) \\
&+ O_p(\mathcal{M}^{*1/2}(\Lambda^2, \lambda) \phi^{1/2} + \mathcal{M}^{1/2}(m) \phi).
\end{aligned}$$

Finally, using the results in Mancini (2009) we have that, provided $\lim_{k \rightarrow \infty, \phi \rightarrow 0, \vartheta \rightarrow 0} \frac{1}{\vartheta} \left(\frac{\phi}{k} \log \left(\frac{k}{\phi} \right) \mathcal{M}^2(\sigma^2) + \left(\frac{\phi}{k} \right)^2 \mathcal{M}^2(\mu) \right) \xrightarrow{a.s.} 0$, the difference between $\tilde{\sigma}_{iT/n}^2$ and $\tilde{\sigma}_{iT/n}^2$ satisfies

$$\max_{1 \leq i \leq n} \left| \tilde{\sigma}_{iT/n}^2 - \tilde{\sigma}_{iT/n}^2 \right| = \max_{1 \leq i \leq n} \left| \frac{1}{\phi} \sum_{\{(\log \tilde{p}_{j+1} - \log \tilde{p}_j)^2 > \vartheta\}} \mathbf{1}_{\{i < \frac{j}{k} < i+1\}} (\log \tilde{p}_{j+1} - \log \tilde{p}_j)^2 \right| = \frac{1}{\phi} \max_{1 \leq i \leq n} (N_i) O_p \left((\log \tilde{p}_{j+1} - \log \tilde{p}_j)^2 \right),$$

with N_i defining the number of Poisson jumps over $1 \leq i \leq n$. Now notice that

$$(\log \tilde{p}_{j+1} - \log \tilde{p}_j)^2 = 2 \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \mu(\sigma_v^2) dv \right) \mu(\sigma_s^2) ds + 2 \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \sigma_v dW_v \right) \mu(\sigma_s^2) ds$$

$$+2 \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \sigma_v dW_v \right) \sigma_s dW_s + 2 \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \mu(\sigma_v^2) dv \right) \sigma_s dW_s + \int_{t_j}^{t_{j+1}} \sigma_s^2 ds,$$

where

$$\begin{aligned} 2 \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \mu(\sigma_v^2) dv \right) \mu(\sigma_s^2) ds &\leq 2 \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^{t_{j+1}} |\mu(\sigma_v^2)| dv \right) |\mu(\sigma_s^2)| ds \leq \frac{\phi}{k} \int_{t_j}^{t_{j+1}} \mu^2(\sigma_s^2) ds = O_p \left(\left(\frac{\phi}{k} \right)^2 \mathcal{M}(\mu^2) \right), \\ 2 \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \sigma_v dW_v \right) \mu(\sigma_s^2) ds &\leq \sup_{t_j \leq s \leq t_{j+1}} \left| \int_{t_j}^s \sigma_v dW_v \right| \int_{t_j}^{t_{j+1}} |\mu(\sigma_s^2)| ds = O_p \left(\left(\frac{\phi}{k} \right)^{3/2} \mathbf{E}(\mathcal{M}^{1/2}(\sigma^2)) \mathcal{M}(\mu) \right), \\ \int_{t_j}^{t_{j+1}} \sigma_s^2 ds &= O_p \left(\left(\frac{\phi}{k} \right) \mathcal{M}(\sigma^2) \right). \end{aligned}$$

We turn to $2 \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \mu(\sigma_v^2) dv \right) \sigma_s dW_s$. By Ito isometry and Jensen's inequality,

$$\begin{aligned} \mathbf{E} \left(2 \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \mu(\sigma_v^2) dv \right) \sigma_s dW_s \right)^2 &= \mathbf{E} \left(2 \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \mu(\sigma_v^2) dv \right)^2 \sigma_s^2 ds \right) \leq \mathbf{E} \left(4 \frac{\phi}{k} \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^{t_{j+1}} \mu^2(\sigma_v^2) dv \right) \sigma_s^2 ds \right) \\ &\leq 4 \left(\frac{\phi}{k} \right)^3 \mathbf{E}(\mathcal{M}(\sigma^2) \mathcal{M}(\mu^2)). \end{aligned}$$

Thus,

$$2 \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \mu(\sigma_v^2) dv \right) \sigma_s dW_s \leq O_p \left(\left(\frac{\phi}{k} \right)^{3/2} (\mathbf{E}(\mathcal{M}(\sigma^2) \mathcal{M}(\mu^2)))^{1/2} \right).$$

Finally, consider $2 \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \sigma_v dW_v \right) \sigma_s dW_s$. By Ito isometry and Doob's decomposition,

$$\begin{aligned} &\mathbf{E} \left(2 \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \sigma_v dW_v \right) \sigma_s dW_s \right)^2 \\ &= \mathbf{E} \left(2 \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \sigma_v dW_v \right)^2 \sigma_s^2 ds \right) \\ &= \mathbf{E} \left(2 \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \sigma_v^2 dv \right) \sigma_s^2 ds \right) + \mathbf{E} \left(4 \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \left(\int_{t_j}^u \sigma_v dW_v \right) \sigma_u dW_u \right) \sigma_s^2 ds \right) \\ &= 2 \frac{\phi}{k} \mathbf{E} \left(\int_{t_j}^{t_{j+1}} \sigma_s^4 ds \right) + \mathbf{E} \left(4 \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \left(\int_{t_j}^u \sigma_v dW_v \right) \sigma_u dW_u \right) \sigma_s^2 ds \right) \\ &= b_1 + b_2 \end{aligned}$$

Now writing, as earlier, $\int_{t_j}^t \sigma_v dW_v$ by \widetilde{M}_t and integrating by parts

$$b_2 \leq \frac{\phi}{k} \mathbf{E} \left(\int_{t_j}^{t_{j+1}} \sigma_s^4 ds \right)$$

using the same steps above. Thus,

$$2 \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s \sigma_v dW_v \right) \sigma_s dW_s = O_p \left(\left(\frac{\phi}{k} \right) (\mathbf{E}(\mathcal{M}(\sigma^4)))^{1/2} \right).$$

and

$$\begin{aligned} (\log \tilde{p}_{j+1} - \log \tilde{p}_j)^2 &= O_p \left(\left(\frac{\phi}{k} \right)^2 \mathcal{M}(\mu^2) \right) + O_p \left(\left(\frac{\phi}{k} \right)^{3/2} \mathbf{E}(\mathcal{M}^{1/2}(\sigma^2)) \mathcal{M}(\mu) \right) + O_p \left(\left(\frac{\phi}{k} \right)^{3/2} (\mathbf{E}(\mathcal{M}(\sigma^2) \mathcal{M}(\mu^2)))^{1/2} \right) \\ &\quad + O_p \left(\left(\frac{\phi}{k} \right) \mathcal{M}(\sigma^2) \right) + O_p \left(\left(\frac{\phi}{k} \right) (\mathbf{E}(\mathcal{M}(\sigma^4)))^{1/2} \right). \end{aligned}$$

To conclude,

$$\Pr(N_i \geq \epsilon) \leq \frac{1}{\epsilon} \mathbf{E}[N_i] = \frac{\mathbf{E} \left[\int_{iT/n}^{iT/n+\phi} \lambda(\sigma_s^2) ds \right]}{\epsilon} \leq \frac{\mathbf{E}[\mathcal{M}(\lambda)]}{\epsilon} \phi.$$

Hence, $\max_{1 \leq i \leq n} |N_i| = O_p(\mathbf{E}(\mathcal{M}(\lambda)) \phi)$ and

$$\begin{aligned} \max_{1 \leq i \leq n} \left| \hat{\sigma}_{iT/n}^2 - \tilde{\sigma}_{iT/n}^2 \right| &= O_p \left(\left(\frac{\phi}{k} \right)^2 \mathcal{M}(\mu^2) \mathbf{E}(\mathcal{M}(\lambda)) \right) + O_p \left(\left(\frac{\phi}{k} \right)^{3/2} \mathbf{E}(\mathcal{M}^{1/2}(\sigma^2)) \mathcal{M}(\mu) \mathbf{E}(\mathcal{M}(\lambda)) \right) \\ &\quad + O_p \left(\left(\frac{\phi}{k} \right)^{3/2} (\mathbf{E}(\mathcal{M}(\sigma^2) \mathcal{M}(\mu^2)))^{1/2} \mathbf{E}(\mathcal{M}(\lambda)) \right) + O_p \left(\left(\frac{\phi}{k} \right) \mathcal{M}(\sigma^2) \mathbf{E}(\mathcal{M}(\lambda)) \right) \\ &\quad + O_p \left(\left(\frac{\phi}{k} \right) (\mathbf{E}(\mathcal{M}(\sigma^4)))^{1/2} \mathbf{E}(\mathcal{M}(\lambda)) \right), \end{aligned}$$

which completes the proof. \square

REMARK 11. In the case in which the driving functions and σ_t^2 are uniformly bounded the statement of Lemma 6 simplifies and $\Theta(n, T, k, \phi) = \frac{(\log n)^{1/2}}{k^{1/2}} \phi^{1/2}$.

THEOREM 10 (Local time estimation with estimated spot volatility - Consistency). Define $\tilde{L}_{\sigma^2}(T, x) = \frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^n \mathbf{K} \left(\frac{\hat{\sigma}_{i\Delta_{n,T}}^2 - x}{h_{n,T}} \right)$. We have,

$$\frac{\tilde{L}_{\sigma^2}(T, x)}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds} = 1 + O_p \left(\frac{\Delta_{n,T}}{h_{n,T}^2} \right) + O_p(g(n, T, k, \phi)),$$

where $g(n, T, k, \phi) = \frac{1}{h_{n,T}} \Theta(n, T, k, \phi)$.

PROOF. Using the mean-value theorem, letting $\tilde{\sigma}_{i\Delta_{n,T}}^2$ be a value on the line segment connecting $\hat{\sigma}_{i\Delta_{n,T}}^2$ and $\sigma_{i\Delta_{n,T}}^2$, write

$$\begin{aligned} \tilde{L}_{\sigma^2}(T, x) &= \frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^n \mathbf{K} \left(\frac{\hat{\sigma}_{i\Delta_{n,T}}^2 - x}{h_{n,T}} \right) \\ &= \frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^n \mathbf{K} \left(\frac{\sigma_{i\Delta_{n,T}}^2 + \tilde{\sigma}_{i\Delta_{n,T}}^2 - \sigma_{i\Delta_{n,T}}^2 - x}{h_{n,T}} \right) \\ &= \frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^n \mathbf{K} \left(\frac{\sigma_{i\Delta_{n,T}}^2 - x}{h_{n,T}} \right) + \frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^n \left(\frac{\tilde{\sigma}_{i\Delta_{n,T}}^2 - \sigma_{i\Delta_{n,T}}^2}{h_{n,T}} \right) \mathbf{K}' \left(\frac{\tilde{\sigma}_{i\Delta_{n,T}}^2 - x}{h_{n,T}} \right) \end{aligned}$$

Thus,

$$\tilde{L}_{\sigma^2}(T, x) - \frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^n \mathbf{K} \left(\frac{\sigma_{i\Delta_{n,T}}^2 - x}{h_{n,T}} \right) \leq \max_{1 \leq i \leq n} \left| \frac{\tilde{\sigma}_{i\Delta_{n,T}}^2 - \sigma_{i\Delta_{n,T}}^2}{h_{n,T}} \right| \frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^n \left| \mathbf{K}' \left(\frac{\sigma_{i\Delta_{n,T}}^2 + O_p(\Theta(n, T, k, \phi)) - x}{h_{n,T}} \right) \right|$$

Now, using Lemma 2 and 6:

$$\frac{\tilde{L}_{\sigma^2}(T, x)}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds} - 1 = O_p \left(\frac{\Delta_{n,T}}{h_{n,T}^2} \right) + O_p(g(n, T, k, \phi)) O_p(1),$$

where $g(n, T, k, \phi) = \frac{1}{h_{n,T}} \Theta(n, T, k, \phi)$. □

THEOREM 11 (Consistency and weak convergence of the feasible estimator.). 1. Given Assumption 5, for $p = 1$:

$$\begin{aligned} & \frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\hat{\sigma}_{iT/n}^2 - x}{h_{n,T}} \right) (\hat{\sigma}_{(i+1)T/n}^2 - \hat{\sigma}_{iT/n}^2)}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\hat{\sigma}_{iT/n}^2 - x}{h_{n,T}} \right)} - \frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right) (\sigma_{(i+1)T/n}^2 - \sigma_{iT/n}^2)}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right)} \\ &= O_p \left(\frac{\Delta_{n,T}^*}{h_{n,T}^2} \right) + O_p \left(\frac{g(n, T, k, \phi) h_{n,T}}{\Delta_{n,T}} \right). \end{aligned}$$

Thus,

$$\hat{\theta}^1 \xrightarrow{P} \theta^1$$

if $\frac{\Delta_{n,T}^*}{h_{n,T}^2} \rightarrow 0$, $h_{n,T} \hat{L}_{\sigma^2}(T, x) \xrightarrow{a.s.} \infty$ and $\frac{g(n, T, k, \phi) h_{n,T}}{\Delta_{n,T}} \rightarrow 0$. Also,

$$\sqrt{\hat{L}_{\sigma^2}(T, x) h_{n,T}} \left(\hat{\theta}^1(x) - \theta^1(x) - \Gamma_{\theta^1}(x) \right) \Rightarrow \mathbf{N}(0, \mathbf{K}_2 \theta^2(x)),$$

if $\frac{\Delta_{n,T}^* \sqrt{\hat{L}_{\sigma^2}(T, x)}}{h_{n,T}^{3/2}} \rightarrow 0$, $h_{n,T} \hat{L}_{\sigma^2}(T, x) \xrightarrow{a.s.} \infty$, $h_{n,T}^5 \hat{L}_{\sigma^2}(T, x) \xrightarrow{a.s.} C$ and $\sqrt{h_{n,T} \hat{L}_{\sigma^2}(T, x)} \frac{g(n, T, k, \phi) h_{n,T}}{\Delta_{n,T}} \xrightarrow{a.s.} 0$.

2. Given Assumption 5, for $p > 1$:

$$\begin{aligned} & \frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\hat{\sigma}_{iT/n}^2 - x}{h_{n,T}} \right) (\hat{\sigma}_{(i+1)T/n}^2 - \hat{\sigma}_{iT/n}^2)^p}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\hat{\sigma}_{iT/n}^2 - x}{h_{n,T}} \right)} - \frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right) (\sigma_{(i+1)T/n}^2 - \sigma_{iT/n}^2)^p}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K} \left(\frac{\sigma_{iT/n}^2 - x}{h_{n,T}} \right)} \\ &= O_p \left(\frac{\Delta_{n,T}^*}{h_{n,T}^2} \right) + O_p \left(\frac{g(n, T, k, \phi) \Delta_{n,T}^{*1/2}}{\Delta_{n,T}} \right), \end{aligned}$$

Thus,

$$\hat{\theta}^p \xrightarrow{P} \theta^p$$

if $\frac{\Delta_{n,T}^*}{h_{n,T}^2} \rightarrow 0$, $h_{n,T} \hat{L}_{\sigma^2}(T, x) \xrightarrow{a.s.} \infty$ and $\frac{g(n, T, k, \phi) \Delta_{n,T}^{*1/2}}{\Delta_{n,T}} \rightarrow 0$. Also,

$$\sqrt{\hat{L}_{\sigma^2}(T, x) h_{n,T}} \left(\hat{\theta}^p(x) - \theta^p(x) - \Gamma_{\theta^p}(x) \right) \Rightarrow \mathbf{N}(0, \mathbf{K}_2 \theta^{2p}(x)),$$

if $\frac{\Delta_{n,T}^* \sqrt{\hat{L}_{\sigma^2}(T, x)}}{h_{n,T}^{3/2}} \rightarrow 0$, $h_{n,T} \hat{L}_{\sigma^2}(T, x) \xrightarrow{a.s.} \infty$, $h_{n,T}^5 \hat{L}_{\sigma^2}(T, x) \xrightarrow{a.s.} C$ and $\sqrt{h_{n,T} \hat{L}_{\sigma^2}(T, x)} \frac{g(n, T, k, \phi) \Delta_{n,T}^{*1/2}}{\Delta_{n,T}} \xrightarrow{a.s.} 0$.

PROOF. Write

$$\hat{\mathbf{K}}_i = \mathbf{K} \left(\frac{\hat{\sigma}_{i\Delta_{n,T}}^2 - x}{h_{n,T}} \right), \quad \mathbf{K}_i = \mathbf{K} \left(\frac{\sigma_{i\Delta_{n,T}}^2 - x}{h_{n,T}} \right),$$

and $\mathbf{M}_{n,k,T} = \max_{1 \leq i \leq n} \left| (\hat{\sigma}_{iT/n}^2 - \sigma_{iT/n}^2) \right| = g(n, T, k, \phi) h_{n,T}$. We have

$$\hat{\theta}_p(x) - \frac{\sum_{i=1}^{n-1} \mathbf{K}_i \left[f(\sigma_{(i+1)T/n}^2) - f(\sigma_{iT/n}^2) \right]^p}{\Delta_{n,T} \sum_{i=1}^n \mathbf{K}_i} =$$

$$\begin{aligned}
&= \frac{\sum_{i=1}^{n-1} \widehat{\mathbf{K}}_i \left[f(\widehat{\sigma}_{(i+1)T/n}^2) - f(\widehat{\sigma}_{iT/n}^2) \right]^p}{\Delta_{n,T} \sum_{i=1}^n \widehat{\mathbf{K}}_i} - \frac{\sum_{i=1}^{n-1} \mathbf{K}_i \left[f(\widehat{\sigma}_{(i+1)T/n}^2) - f(\widehat{\sigma}_{iT/n}^2) \right]^p}{\Delta_{n,T} \sum_{i=1}^n \widehat{\mathbf{K}}_i} \\
&\quad \underbrace{\hspace{15em}}_{(a)} \\
&+ \frac{\sum_{i=1}^{n-1} \mathbf{K}_i \left[f(\widehat{\sigma}_{(i+1)T/n}^2) - f(\widehat{\sigma}_{iT/n}^2) \right]^p}{\Delta_{n,T} \sum_{i=1}^n \widehat{\mathbf{K}}_i} - \frac{\sum_{i=1}^{n-1} \mathbf{K}_i \left[f(\sigma_{(i+1)T/n}^2) - f(\sigma_{iT/n}^2) \right]^p}{\Delta_{n,T} \sum_{i=1}^n \widehat{\mathbf{K}}_i} \\
&\quad \underbrace{\hspace{15em}}_{(b)} \\
&+ \frac{\sum_{i=1}^{n-1} \mathbf{K}_i \left[f(\sigma_{(i+1)T/n}^2) - f(\sigma_{iT/n}^2) \right]^p}{\Delta_{n,T} \sum_{i=1}^n \widehat{\mathbf{K}}_i} - \frac{\sum_{i=1}^{n-1} \mathbf{K}_i \left[f(\sigma_{(i+1)T/n}^2) - f(\sigma_{iT/n}^2) \right]^p}{\Delta_{n,T} \sum_{i=1}^n \mathbf{K}_i} \\
&\quad \underbrace{\hspace{15em}}_{(c)}.
\end{aligned}$$

First we notice that, by the mean-value theorem,

$$(26) \quad \max_{1 \leq i \leq n} \left| f(\widehat{\sigma}_{iT/n}^2) - f(\sigma_{iT/n}^2) \right| = O_p(\mathbf{M}_{n,k,T}) = O_p(g(n, T, k, \phi)h_{n,T}).$$

We further notice that, as in Lemma 4,

$$\max_{1 \leq i \leq n} \left| \left\{ f(\sigma_{(i+1)T/n}^2) - f(\sigma_{iT/n}^2) \right\}^p \right| = O_p\left(\Delta_{n,T}^{*1/2}\right).$$

Thus, we have, by the mean-value theorem,

$$(a) = \frac{\sum_{i=1}^{n-1} (\widehat{\mathbf{K}}_i - \mathbf{K}_i) \left[f(\widehat{\sigma}_{(i+1)T/n}^2) - f(\widehat{\sigma}_{iT/n}^2) \right]^p}{\Delta_{n,T} \sum_{i=1}^n \widehat{\mathbf{K}}_i} = \frac{O_p\left(\frac{\mathbf{M}_{n,k,T}}{h_{n,T}}\right) \sum_{i=1}^{n-1} \mathbf{K}'\left(\frac{\sigma_{iT/n}^2 + O_p(\mathbf{M}_{n,k,T}) - x}{h_{n,T}}\right) \left[f(\widehat{\sigma}_{(i+1)T/n}^2) - f(\widehat{\sigma}_{iT/n}^2) \right]^p}{\Delta_{n,T} \sum_{i=1}^n \widehat{\mathbf{K}}_i}.$$

Note that

$$\begin{aligned}
&\left[f(\widehat{\sigma}_{(i+1)T/n}^2) - f(\widehat{\sigma}_{iT/n}^2) \right]^p \\
&= \left[f(\widehat{\sigma}_{(i+1)T/n}^2) - f(\sigma_{(i+1)T/n}^2) + f(\sigma_{(i+1)T/n}^2) - f(\widehat{\sigma}_{iT/n}^2) + f(\sigma_{(i+1)T/n}^2) - f(\sigma_{iT/n}^2) \right]^p \\
&\leq \max_{1 \leq i \leq n-1} \left| \left[f(\widehat{\sigma}_{(i+1)T/n}^2) - f(\sigma_{(i+1)T/n}^2) + f(\sigma_{(i+1)T/n}^2) - f(\widehat{\sigma}_{iT/n}^2) + f(\sigma_{(i+1)T/n}^2) - f(\sigma_{iT/n}^2) \right]^p \right| \\
&\leq \sum_{k=0}^p \binom{p}{k} \max_{1 \leq i \leq n-1} \left| \left(f(\widehat{\sigma}_{(i+1)T/n}^2) - f(\sigma_{(i+1)T/n}^2) + f(\widehat{\sigma}_{iT/n}^2) - f(\sigma_{iT/n}^2) \right)^k \right| \max_{1 \leq i \leq n-1} \left| \left(f(\sigma_{(i+1)T/n}^2) - f(\sigma_{iT/n}^2) \right)^{p-k} \right| \\
(27) \quad &O_p\left(\mathbf{M}_{n,k,T}^p + \Delta_{n,T}^{*1/2}\right).
\end{aligned}$$

Using Theorem 10, the bound becomes

$$(a) = \frac{O_p\left(\frac{\mathbf{M}_{n,k,T}}{h_{n,T}}\right) O_p(\mathbf{M}_{n,k,T}^p + \Delta_{n,T}^{*1/2}) \frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}'\left(\frac{\sigma_{iT/n}^2 + O_p(\mathbf{M}_{n,k,T}) - x}{h_{n,T}}\right)}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^n \widehat{\mathbf{K}}_i}$$

$$\begin{aligned}
& O_p \left(\frac{\mathbf{M}_{n,k,T}}{\Delta_{n,T} h_{n,T}} \right) O_p(\mathbf{M}_{n,k,T}^p + \Delta_{n,T}^{*1/2}) \left(\frac{\frac{1}{h_{n,T}} \int_0^T |\mathbf{K}'\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) ds|}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) ds} + O_p \left(\frac{\Delta_{n,T}}{h_{n,T}^2} \right) \right) \\
&= \frac{\quad}{1 + O_p \left(\frac{\Delta_{n,T}}{h_{n,T}^2} \right) + O_p(g(n, T, k, \phi))} \\
&= O_p \left(\frac{\mathbf{M}_{n,k,T}}{\Delta_{n,T} h_{n,T}} \left(\mathbf{M}_{n,k,T}^p + \Delta_{n,T}^{*1/2} \right) \right).
\end{aligned}$$

We now turn to (b). Write

$$(b) = \frac{\sum_{i=1}^{n-1} \mathbf{K}_i \left[\underbrace{\left(f(\widehat{\sigma}_{(i+1)T/n}^2) - f(\widehat{\sigma}_{iT/n}^2) \right)^p}_a - \underbrace{\left(f(\sigma_{(i+1)T/n}^2) - f(\sigma_{iT/n}^2) \right)^p}_b \right]}{\Delta_{n,T} \sum_{i=1}^{n-1} \widehat{\mathbf{K}}_i}.$$

Since $a^p - b^p = (a - b) \left(\sum_{k=1}^p a^{p-k} b^{k-1} \right)$, then

$$\begin{aligned}
& \max_{1 \leq i \leq n-1} \left| \left(f(\widehat{\sigma}_{(i+1)T/n}^2) - f(\widehat{\sigma}_{iT/n}^2) \right)^p - \left(f(\sigma_{(i+1)T/n}^2) - f(\sigma_{iT/n}^2) \right)^p \right| \\
&= \max_{1 \leq i \leq n-1} \left| \left(\left(f(\widehat{\sigma}_{(i+1)T/n}^2) - f(\widehat{\sigma}_{iT/n}^2) \right) - \left(f(\sigma_{(i+1)T/n}^2) - f(\sigma_{iT/n}^2) \right) \right) \cdot \right. \\
&\quad \left. \left(\sum_{k=1}^p \left(f(\widehat{\sigma}_{(i+1)T/n}^2) - f(\widehat{\sigma}_{iT/n}^2) \right)^{p-k} \left(f(\sigma_{(i+1)T/n}^2) - f(\sigma_{iT/n}^2) \right)^{k-1} \right) \right| \\
&= O_p(\mathbf{M}_{n,k,T}) \max_{1 \leq i \leq n-1} \left| \sum_{k=1}^p \left(f(\widehat{\sigma}_{(i+1)T/n}^2) - f(\widehat{\sigma}_{iT/n}^2) \right)^{p-k} \left(f(\sigma_{(i+1)T/n}^2) - f(\sigma_{iT/n}^2) \right)^{k-1} \right| \\
&= O_p(\mathbf{M}_{n,k,T}) \sum_{k=1}^p \max_{1 \leq i \leq n-1} \left| \left(f(\widehat{\sigma}_{(i+1)T/n}^2) - f(\widehat{\sigma}_{iT/n}^2) \right)^{p-k} \right| \max_{1 \leq i \leq n-1} \left| \left(f(\sigma_{(i+1)T/n}^2) - f(\sigma_{iT/n}^2) \right)^{k-1} \right|.
\end{aligned}$$

Thus, if $p = 1$, the bound is

$$O_p(\mathbf{M}_{n,k,T}).$$

If $p > 1$, the bound is

$$\begin{aligned}
& O_p(\mathbf{M}_{n,k,T}) \sum_{k=1}^p \max_{1 \leq i \leq n-1} \left| \left(f(\widehat{\sigma}_{(i+1)T/n}^2) - f(\widehat{\sigma}_{iT/n}^2) \right)^{p-k} \right| \max_{1 \leq i \leq n-1} \left| \left(f(\sigma_{(i+1)T/n}^2) - f(\sigma_{iT/n}^2) \right)^{k-1} \right| \\
&= O_p(\mathbf{M}_{n,k,T}) O_p \left(\max_{1 \leq i \leq n-1} \left| \left(f(\widehat{\sigma}_{(i+1)T/n}^2) - f(\widehat{\sigma}_{iT/n}^2) \right)^{p-1} \right| + \Delta_{n,T}^{*1/2} \right) \\
&= O_p(\mathbf{M}_{n,k,T}) O_p \left(\mathbf{M}_{n,k,T}^{p-1} + \Delta_{n,T}^{*1/2} \right)
\end{aligned}$$

using Eq. (27). Thus, if $p = 1$,

$$(b) = \frac{\frac{O_p(\mathbf{M}_{n,k,T})}{\Delta_{n,T}} \left(\frac{\Delta_{n,T} \sum_{i=1}^{n-1} \mathbf{K}_i}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) ds} \right)}{1 + O_p \left(\frac{\Delta_{n,T}}{h_{n,T}^2} \right) + O_p(g(n, T, k, \phi))} = O_p \left(\frac{\mathbf{M}_{n,k,T}}{\Delta_{n,T}} \right).$$

If $p > 1$

$$(b) = \frac{\frac{O_p(\mathbf{M}_{n,k,T}^p + \mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2})}{\Delta_{n,T}} \left(\frac{\Delta_{n,T} \sum_{i=1}^{n-1} \mathbf{K}_i}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) ds} \right)}{1 + O_p \left(\frac{\Delta_{n,T}}{h_{n,T}^2} \right) + O_p(g(n, T, k, \phi))} = O_p \left(\frac{\mathbf{M}_{n,k,T}^p + \mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2}}{\Delta_{n,T}} \right).$$

Finally, for (c) we have

$$\begin{aligned}
(c) &= \frac{\sum_{i=1}^{n-1} \mathbf{K}_i \left[f(\sigma_{(i+1)T/n}^2) - f(\sigma_{iT/n}^2) \right]^p}{\Delta_{n,T} \sum_{i=1}^n \mathbf{K}_i} \times \frac{\Delta_{n,T} \left(\sum_{i=1}^{n-1} \widehat{\mathbf{K}}_i - \sum_{i=1}^{n-1} \mathbf{K}_i \right)}{\Delta_{n,T} \sum_{i=1}^n \widehat{\mathbf{K}}_i} \\
&= O_p(1) \times O_p \left(\frac{\max_{1 \leq i \leq n} \left| \frac{\widehat{\sigma}_{i\Delta_{n,T}}^2 - \sigma_{i\Delta_{n,T}}^2}{h_{n,T}} \right| \frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^n \left| \mathbf{K}' \left(\frac{\sigma_{i\Delta_{n,T}}^2 - x}{h_{n,T}} \right) \right|}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^n \widehat{\mathbf{K}}_i} \right) \\
&= O_p(1) O_p \left(\frac{\mathbf{M}_{n,k,T}}{h_{n,T}} \right).
\end{aligned}$$

Now, if $p = 1$,

$$(a) + (b) + (c) = O_p \left(\frac{\mathbf{M}_{n,k,T}}{\Delta_{n,T} h_{n,T}} \left(\mathbf{M}_{n,k,T} + \Delta_{n,T}^{*1/2} \right) \right) + O_p \left(\frac{\mathbf{M}_{n,k,T}}{\Delta_{n,T}} \right) + O_p \left(\frac{\mathbf{M}_{n,k,T}}{h_{n,T}} \right).$$

Note that $\frac{\Delta_{n,T}^*}{h_{n,T}^2} \rightarrow 0$, hence $\frac{\Delta_{n,T}}{h_{n,T}^2} \rightarrow 0$. This implies that, $\frac{\mathbf{M}_{n,k,T}}{h_{n,T}} = \frac{\mathbf{M}_{n,k,T} \Delta_{n,T}}{\Delta_{n,T} h_{n,T}} = o_p \left(\frac{\mathbf{M}_{n,k,T}}{\Delta_{n,T}} \right)$. In addition, $\frac{\mathbf{M}_{n,k,T}^2}{\Delta_{n,T} h_{n,T}} = \frac{\mathbf{M}_{n,k,T}^2 \Delta_{n,T}}{\Delta_{n,T}^2 h_{n,T}} = o_p \left(\frac{\mathbf{M}_{n,k,T}^2}{\Delta_{n,T}^2} \right) = o_p \left(\frac{\mathbf{M}_{n,k,T}}{\Delta_{n,T}} \right)$. Finally, $\frac{\mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2}}{\Delta_{n,T} h_{n,T}} = o_p \left(\frac{\mathbf{M}_{n,k,T}}{\Delta_{n,T}} \right)$. In sum,

$$(a) + (b) + (c) = O_p \left(\frac{\mathbf{M}_{n,k,T}}{\Delta_{n,T}} \right).$$

Consistency now follows from Theorem 8. The same theorem implies asymptotic normality provided $\sqrt{h_{n,T} \widehat{L}_{n,T}(x)} \left(\frac{\mathbf{M}_{n,k,T}}{\Delta_{n,T}} \right) \xrightarrow{a.s.} 0$. If $p > 1$,

$$(a) + (b) + (c) = O_p \left(\frac{\mathbf{M}_{n,k,T}^{p+1}}{\Delta_{n,T} h_{n,T}} + \frac{\mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2}}{\Delta_{n,T} h_{n,T}} \right) + O_p \left(\frac{\mathbf{M}_{n,k,T}^p}{\Delta_{n,T}} + \frac{\mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2}}{\Delta_{n,T}} \right) + O_p \left(\frac{\mathbf{M}_{n,k,T}}{h_{n,T}} \right),$$

Now notice that, since $\frac{\Delta_{n,T}^{*1/2}}{\Delta_{n,T}} \rightarrow \infty$, $\frac{\mathbf{M}_{n,k,T}}{h_{n,T}} = \frac{\mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2}}{\Delta_{n,T} h_{n,T}} \frac{\Delta_{n,T}}{\Delta_{n,T}^{*1/2}} = o_p \left(\frac{\mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2}}{\Delta_{n,T} h_{n,T}} \right)$. Similarly, $\frac{\mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2}}{\Delta_{n,T}} = o_p \left(\frac{\mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2}}{\Delta_{n,T} h_{n,T}} \right)$. In addition,

$$(28) \quad \frac{\mathbf{M}_{n,k,T}^p}{\Delta_{n,T}} = \frac{\mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2}}{\Delta_{n,T} h_{n,T}} \frac{\mathbf{M}_{n,k,T}^{p-1}}{\Delta_{n,T}^{*1/2}} = \frac{\mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2}}{\Delta_{n,T} h_{n,T}} \frac{\mathbf{M}_{n,k,T}^{p-1} \Delta_{n,T}^{*1/2}}{\Delta_{n,T} h_{n,T}} \frac{h_{n,T}^2 \Delta_{n,T}}{\Delta_{n,T}^*} = o_p \left(\left(\frac{\mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2}}{\Delta_{n,T} h_{n,T}} \right)^2 \right).$$

Similarly, using Eq. (28),

$$\frac{\mathbf{M}_{n,k,T}^{p+1}}{\Delta_{n,T} h_{n,T}} = \frac{\mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2}}{\Delta_{n,T} h_{n,T}} \frac{\mathbf{M}_{n,k,T}^p}{\Delta_{n,T}^{*1/2} \Delta_{n,T}^{*1/2}} = o_p \left(\left(\frac{\mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2}}{\Delta_{n,T} h_{n,T}} \right)^3 \right).$$

Thus,

$$(a) + (b) + (c) = O_p \left(\frac{\mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2}}{\Delta_{n,T} h_{n,T}} \right).$$

Consistency now follows from Theorem 8. The same theorem implies asymptotic normality provided $\sqrt{h_{n,T} \widehat{L}_{n,T}(x)} \left(\frac{\mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2}}{\Delta_{n,T} h_{n,T}} \right) \xrightarrow{a.s.} 0$. □

PROOF OF THEOREM 3. For conciseness, we focus on two cases for the generalized Duffie et al.'s specification ($f(\sigma^2) = \sigma^2$ and $\xi \rightsquigarrow \exp(\mu_\xi)$). The estimation error induced by the spot variance estimates is handled as in proof of Theorem 11.

$$\widehat{\mu}_\xi(x) - \mu_\xi(x) \stackrel{d}{=} \frac{(\widehat{\theta}^4(x) - \theta^4(x))}{4\theta^3(x)} - \frac{\theta^4(x) (\widehat{\theta}^3(x) - \theta^3(x))}{4(\theta^3(x))^2}.$$

Thus,

$$\sqrt{h_{n,T} \widehat{L}_{\sigma^2}(T, x)} \{\widehat{\mu}_\xi(x) - \mu_\xi(x)\} \Rightarrow N\left(0, \frac{\theta^8(x)}{16(\theta^3(x))^2} + \frac{(\theta^4(x))^2 \theta^6(x)}{16(\theta^3(x))^4} - 2 \frac{\theta^4(x) \theta^7(x)}{16\theta^3(x) (\theta^3(x))^2}\right).$$

Similarly,

$$\begin{aligned} & \widehat{\lambda}(x) - \lambda(x) \\ & \stackrel{d}{=} \frac{(\widehat{\theta}^4(x) - \theta^4(x))}{24\mu^4(x)} - \frac{4\theta^4(x) (\widehat{\mu}_\xi(x) - \mu_\xi(x))}{24\mu^5(x)} \\ & = \frac{(\widehat{\theta}^4(x) - \theta^4(x))}{24\mu^4(x)} - \frac{4\theta^4(x)}{24\mu^5(x)} \left\{ \frac{(\widehat{\theta}^4(x) - \theta^4(x))}{4\theta^3(x)} - \frac{\theta^4(x) (\widehat{\theta}^3(x) - \theta^3(x))}{4(\theta^3(x))^2} \right\} \\ & = \left(\frac{4(24)\mu^5(x)\theta^3(x) - 24\mu^4(x)}{4(24)^2\mu^9(x)\theta^3(x)} \right) (\widehat{\theta}^4(x) - \theta^4(x)) + \frac{4(\theta^4(x))^2}{24\mu^5(x)4(\theta^3(x))^2} (\widehat{\theta}^3(x) - \theta^3(x)) \end{aligned}$$

Thus,

$$\begin{aligned} & \sqrt{h_{n,T} \widehat{L}_{\sigma^2}(T, x)} \{\widehat{\lambda}(x) - \lambda(x)\} \\ & \Rightarrow N\left(0, \left(\frac{4(24)\mu^5(x)\theta^3(x) - 24\mu^4(x)}{4(24)^2\mu^9(x)\theta^3(x)} \right)^2 \theta^8(x) + \left(\frac{4(\theta^4(x))^2}{24\mu^5(x)4(\theta^3(x))^2} \right)^2 \theta^6(x) \right. \\ & \quad \left. + 2 \left(\frac{4(24)\mu^5(x)\theta^3(x) - 24\mu^4(x)}{4(24)^2\mu^9(x)\theta^3(x)} \right) \left(\frac{4(\theta^4(x))^2}{24\mu^5(x)4(\theta^3(x))^2} \right) \theta^7(x) \right). \end{aligned}$$

Replacing the infinitesimal moments in the theoretical variances with their definitions leads to the stated result. \square

PROOF OF THEOREM 4. The same method of proof as for Theorem 11 leads to the result. Notice that, since prices are observed, only terms (a) and (c) are to be treated leading to the stated conditions - on the vanishing discretization error - for weak convergence. \square

THEOREM 12. Given Assumption 5, if $h_{n,T} \widehat{L}_{\sigma^2}(T, x) \xrightarrow{\text{a.s.}} 0$, $\frac{\Delta_{n,T}^*}{\Delta_{n,T}} \xrightarrow{\text{a.s.}} \infty$, $\frac{h_{n,T} \widehat{L}_{\sigma^2}(T, x)}{\Delta_{n,T}} \xrightarrow{\text{a.s.}} C$ and $\sqrt{\frac{h_{n,T} \widehat{L}_{\sigma^2}(T, x)}{\Delta_{n,T}}} \left(\frac{g(n, T, k, \phi) h_{n,T} \Delta_{n,T}^{*1/2}}{\Delta_{n,T}} \right) \xrightarrow{\text{a.s.}} 0$, then

$$\sqrt{h_{n,T} \widehat{L}_{\sigma^2}(T, x) / \Delta_{n,T}} \{\widehat{C}(x) - z(x) - \Gamma_C(x)\} \Rightarrow \mathbf{N}(0, \mathbf{K}_2 \theta^C(x)),$$

with

$$\theta^C(x) = \Lambda^2(x) x (1 + \rho^2(x)) + x \lambda_\sigma(x) \mathbf{E}[(\xi^\sigma)^2] + \Lambda^2(x) \lambda_r(x) \mathbf{E}[\psi^2] + \lambda_\sigma(x) \mathbf{E}[(\xi^\sigma)^2] \lambda_r(x) \mathbf{E}[\psi^2]$$

Moreover, if $h_{n,T} \widehat{L}_{\sigma^2}(T, x) \xrightarrow{\text{a.s.}} \infty$, $h_{n,T}^5 \widehat{L}_{\sigma^2}(T, x) \xrightarrow{\text{a.s.}} 0$, $\frac{\Delta_{n,T}^* \sqrt{\widehat{L}_{\sigma^2}(T, x)}}{h_{n,T}^{3/2}} \xrightarrow{\text{a.s.}} 0$ and $\sqrt{h_{n,T} \widehat{L}_{\sigma^2}(T, x)} \left(\frac{g(n, T, k, \phi) \Delta_{n,T}^{*1/2}}{\Delta_{n,T}} \right) \xrightarrow{\text{a.s.}} 0$, then

$$\sqrt{h_{n,T} \widehat{L}_{\sigma^2}(T, x)} \{\widehat{\rho}(x) - \rho(x)\} \Rightarrow \mathbf{N}\left(0, \mathbf{K}_2 \frac{\rho^2(x)}{4\Lambda^4(x)} \lambda_\sigma(x) \mathbf{V}(x)\right),$$

where

$$\mathbf{V}(x) = \begin{cases} 40\mathbf{K}_2\lambda_{\sigma^2}(x)\mu_\xi^2(x) & \text{if } f(\cdot) = x \\ \frac{88}{15}\mathbf{K}_2\lambda_{\log \sigma^2}(x)\sigma_\xi^4(x) & \text{if } f(\cdot) = \log x. \end{cases}$$

PROOF. For conciseness, as earlier, we only consider the case $f(\sigma^2) = \sigma^2$ and $\xi^\sigma \rightsquigarrow \exp(\mu_\xi)$. The case $f(\sigma^2) = \log \sigma^2$ with $\xi^\sigma \rightsquigarrow \mathbf{N}(0, \sigma_\xi^2)$ can, of course, be handled analogously. We start by separating measurement error as in the proof of Theorem 11 and write

$$\begin{aligned} & \widehat{C}(x) - \frac{\sum_{i=1}^{n-1} \mathbf{K}_i (\log(p_{(i+1)T/n}) - \log(p_{iT/n})) (\sigma_{(i+1)T/n}^2 - \sigma_{iT/n}^2)}{\Delta_{n,T} \sum_{i=1}^n \mathbf{K}_i} \\ = & \frac{\sum_{i=1}^{n-1} \widehat{\mathbf{K}}_i (\log(p_{(i+1)T/n}) - \log(p_{iT/n})) (\widehat{\sigma}_{(i+1)T/n}^2 - \widehat{\sigma}_{iT/n}^2)}{\Delta_{n,T} \sum_{i=1}^n \widehat{\mathbf{K}}_i} - \frac{\sum_{i=1}^{n-1} \mathbf{K}_i (\log(p_{(i+1)T/n}) - \log(p_{iT/n})) (\widehat{\sigma}_{(i+1)T/n}^2 - \widehat{\sigma}_{iT/n}^2)}{\Delta_{n,T} \sum_{i=1}^n \widehat{\mathbf{K}}_i} \\ & + \frac{\sum_{i=1}^{n-1} \mathbf{K}_i (\log(p_{(i+1)T/n}) - \log(p_{iT/n})) (\widehat{\sigma}_{(i+1)T/n}^2 - \widehat{\sigma}_{iT/n}^2)}{\Delta_{n,T} \sum_{i=1}^n \widehat{\mathbf{K}}_i} - \frac{\sum_{i=1}^{n-1} \mathbf{K}_i (\log(p_{(i+1)T/n}) - \log(p_{iT/n})) (\sigma_{(i+1)T/n}^2 - \sigma_{iT/n}^2)}{\Delta_{n,T} \sum_{i=1}^n \widehat{\mathbf{K}}_i} \\ & + \frac{\sum_{i=1}^{n-1} \mathbf{K}_i (\log(p_{(i+1)T/n}) - \log(p_{iT/n})) (\sigma_{(i+1)T/n}^2 - \sigma_{iT/n}^2)}{\Delta_{n,T} \sum_{i=1}^n \widehat{\mathbf{K}}_i} - \frac{\sum_{i=1}^{n-1} \mathbf{K}_i (\log(p_{(i+1)T/n}) - \log(p_{iT/n})) (\sigma_{(i+1)T/n}^2 - \sigma_{iT/n}^2)}{\Delta_{n,T} \sum_{i=1}^n \mathbf{K}_i} \\ = & (a) + (b) + (c). \end{aligned}$$

We focus on (a) first.

$$\begin{aligned} (a) &= \frac{\sum_{i=1}^{n-1} (\widehat{\mathbf{K}}_i - \mathbf{K}_i) (\log(p_{(i+1)T/n}) - \log(p_{iT/n})) (\widehat{\sigma}_{(i+1)T/n}^2 - \sigma_{(i+1)T/n}^2 + \sigma_{(i+1)T/n}^2 - \sigma_{iT/n}^2 + \sigma_{iT/n}^2 - \widehat{\sigma}_{iT/n}^2)}{\Delta_{n,T} \sum_{i=1}^n \widehat{\mathbf{K}}_i} \\ &= O_p(\mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2} + \Delta_{n,T}^*) \frac{\max_{1 \leq i \leq n} \left| \frac{\widehat{\sigma}_{i\Delta_{n,T}}^2 - \sigma_{i\Delta_{n,T}}^2}{h_{n,T}} \right| \sum_{i=1}^{n-1} \left| \mathbf{K}' \left(\frac{\sigma_{iT/n}^2 + O_p(\mathbf{M}_{n,k,T}) - x}{h_{n,T}} \right) \right|}{\Delta_{n,T} \sum_{i=1}^n \widehat{\mathbf{K}}_i} \\ &= O_p \left(\frac{\mathbf{M}_{n,k,T} (\mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2} + \Delta_{n,T}^*)}{\Delta_{n,T} h_{n,T}} \right) \frac{\frac{1}{h_{n,T}} \int_0^T \left| \mathbf{K}' \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) \right| ds}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K} \left(\frac{\sigma_s^2 - x}{h_{n,T}} \right) ds} + O_p \left(\frac{\Delta_{n,T}}{h_{n,T}^2} \right) \\ &= O_p \left(\frac{\mathbf{M}_{n,k,T}^2 \Delta_{n,T}^{*1/2}}{\Delta_{n,T} h_{n,T}} + \frac{\mathbf{M}_{n,k,T} \Delta_{n,T}^*}{\Delta_{n,T} h_{n,T}} \right), \end{aligned}$$

if $\frac{\Delta_{n,T}}{h_{n,T}^2} \rightarrow 0$ and $g(n, T, k, \phi) \rightarrow 0$. We now turn to (b).

$$(b) = \frac{\sum_{i=1}^{n-1} \mathbf{K}_i (\log(p_{(i+1)T/n}) - \log(p_{iT/n})) \left[(\widehat{\sigma}_{(i+1)T/n}^2 - \widehat{\sigma}_{iT/n}^2) - (\sigma_{(i+1)T/n}^2 - \sigma_{iT/n}^2) \right]}{\Delta_{n,T} \sum_{i=1}^n \widehat{\mathbf{K}}_i}$$

$$= O_p \left(\frac{\mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2}}{\Delta_{n,T}} \right) \frac{\Delta_{n,T} \sum_{i=1}^{n-1} \mathbf{K}_i}{\Delta_{n,T} \sum_{i=1}^n \widehat{\mathbf{K}}_i} = O_p \left(\frac{\mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2}}{\Delta_{n,T}} \right),$$

if $\frac{\Delta_{n,T}}{h_{n,T}^2} \rightarrow 0$ and $g(n, T, k, \phi) \rightarrow 0$. Finally,

$$\begin{aligned} (c) &= \frac{\sum_{i=1}^{n-1} \mathbf{K}_i (\log(p_{(i+1)T/n}) - \log(p_{iT/n})) (\sigma_{(i+1)T/n}^2 - \sigma_{iT/n}^2)}{\Delta_{n,T} \sum_{i=1}^n \mathbf{K}_i} \times \frac{\Delta_{n,T} \left(\sum_{i=1}^n \widehat{\mathbf{K}}_i - \sum_{i=1}^n \mathbf{K}_i \right)}{\Delta_{n,T} \sum_{i=1}^n \widehat{\mathbf{K}}_i} \\ &= O_p \left(\frac{\max_{1 \leq i \leq n} \left| \frac{\widehat{\sigma}_{i\Delta_{n,T}}^2 - \sigma_{i\Delta_{n,T}}^2}{h_{n,T}} \right| \frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^n \left| \mathbf{K}' \left(\frac{\sigma_{i\Delta_{n,T}}^2 - x}{h_{n,T}} \right) \right|}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^n \widehat{\mathbf{K}}_i} \right) \\ &= O_p \left(\frac{\mathbf{M}_{n,k,T}}{h_{n,T}} \right), \end{aligned}$$

if $\frac{\Delta_{n,T}}{h_{n,T}^2} \rightarrow 0$ and $g(n, T, k, \phi) \rightarrow 0$. We notice that $\frac{\mathbf{M}_{n,k,T}}{h_{n,T}} = \frac{\mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2}}{\Delta_{n,T}} \frac{\Delta_{n,T}}{\Delta_{n,T}^{*1/2} h_{n,T}} = \frac{\mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2}}{\Delta_{n,T}} \frac{\Delta_{n,T}^{1/2}}{h_{n,T} \mathcal{M}^{1/2}(\mathcal{M}(f(\sigma_s^2)))} = o_p \left(\frac{\mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2}}{\Delta_{n,T}} \right)$ since $\frac{\Delta_{n,T}^*}{h_{n,T}^2} \rightarrow 0$ (and, of course, $\frac{\Delta_{n,T}}{h_{n,T}^2} \rightarrow 0$). Also, $\frac{\mathbf{M}_{n,k,T} \Delta_{n,T}^*}{\Delta_{n,T} h_{n,T}} = \frac{\mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2}}{\Delta_{n,T}} \frac{\Delta_{n,T}^{1/2}}{h_{n,T}} = o_p \left(\frac{\mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2}}{\Delta_{n,T}} \right)$. Thus,

$$\frac{\mathbf{M}_{n,k,T}^2 \Delta_{n,T}^{*1/2}}{\Delta_{n,T} h_{n,T}} = \frac{\mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2}}{\Delta_{n,T}} \frac{\mathbf{M}_{n,k,T}}{h_{n,T}} = \frac{\mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2}}{\Delta_{n,T}} \frac{\mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2}}{\Delta_{n,T}} \frac{\Delta_{n,T}}{\Delta_{n,T}^{*1/2} h_{n,T}} = o_p \left(\left(\frac{\mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2}}{\Delta_{n,T}} \right)^2 \right),$$

leading to

$$(a) + (b) + (c) = O_p \left(\frac{\mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2}}{\Delta_{n,T}} \right).$$

We can now turn to the estimator based on the true spot variance process. Ito's lemma yields

$$\begin{aligned} & (\log(p_{(i+1)T/n}) - \log(p_{iT/n})) (\sigma_{(i+1)T/n}^2 - \sigma_{iT/n}^2) \\ &= \int_{iT/n}^{(i+1)T/n} (\sigma_{s-}^2 - \sigma_{iT/n}^2) \mu(\sigma_{s-}^2) ds + \int_{iT/n}^{(i+1)T/n} (\sigma_{s-}^2 - \sigma_{iT/n}^2) \sigma_s dW_s^r \\ &+ \int_{iT/n}^{(i+1)T/n} \int (\sigma_{s-}^2 - \sigma_{iT/n}^2) \psi v_r(ds, d\psi) \\ &+ \int_{iT/n}^{(i+1)T/n} (\log(p_{s-}) - \log(p_{iT/n})) m(\sigma_s^2) ds \\ &+ \int_{iT/n}^{(i+1)T/n} (\log(p_{s-}) - \log(p_{iT/n})) \Lambda(\sigma_s^2) dW_s^\sigma \\ &+ \int_{iT/n}^{(i+1)T/n} \int (\log(p_{s-}) - \log(p_{iT/n})) \xi^\sigma v_\sigma(ds, d\xi^\sigma) \\ &+ \sum_{i\Delta_{n,T} \leq s \leq (i+1)\Delta_{n,T}} (\Delta \log(p_s) \Delta \sigma_s^2) + \int_{iT/n}^{(i+1)T/n} \rho(\sigma_s^2) \sigma_s \Lambda(\sigma_s^2) ds \\ (29) \quad &= \Theta_1 + \Theta_2 + \Theta_3 + \Theta_4 + \Theta_5 + \Theta_6 + \Theta_7 + \Theta_8, \end{aligned}$$

where $\Delta \log p_s = \log p_s - \log p_{s-}$ and $\Delta \sigma_s^2 = \sigma_s^2 - \sigma_{s-}^2$. For convenience, in what follows we compensate the random measures $v_r(ds, d\psi)$, $v_\sigma(ds, d\xi^\sigma)$ and, consequently, write $m(\sigma_s^2) + \lambda_\sigma(\sigma_s^2)\mathbf{E}[\xi^\sigma]$ in place of $m(\sigma_s^2)$ and $\tilde{\mu}(\sigma_s^2) = \mu(\sigma_s^2) + \lambda_r(\sigma_s^2)\mathbf{E}[\psi]$ in place of $\mu(\sigma_s^2)$. Consider all of the terms in turn. As earlier in Lemma 2,

$$\frac{\sum_{i=1}^{n-1} \mathbf{K}_i \Theta_8}{\Delta_{n,T} \sum_{i=1}^n \mathbf{K}_i} = \frac{\int_0^T \mathbf{K}_s \rho(\sigma_s^2) \sigma_s \Lambda(\sigma_s^2) ds}{\int_0^T \mathbf{K}_s ds} + O_p\left(\frac{\Delta_{n,T}^*}{h_{n,T}^2}\right).$$

$$1 + O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right).$$

In addition,

$$\frac{\sum_{i=1}^{n-1} \mathbf{K}_i \Theta_1}{\Delta_{n,T} \sum_{i=1}^n \mathbf{K}_i} = O_p(\Delta_{n,T}^{*1/2}) \left(1 + O_p\left(\frac{\Delta_{n,T}^*}{h_{n,T}^2}\right)\right),$$

$$\frac{\sum_{i=1}^{n-1} \mathbf{K}_i \Theta_4}{\Delta_{n,T} \sum_{i=1}^n \mathbf{K}_i} = O_p(\Delta_{n,T}^{*1/2}) \left(1 + O_p\left(\frac{\Delta_{n,T}^*}{h_{n,T}^2}\right)\right).$$

Notice also that,

$$\frac{\sum_{i=1}^{n-1} \mathbf{K}_i \Theta_7}{\Delta_{n,T} \sum_{i=1}^n \mathbf{K}_i} = 0 \quad a.s.$$

since, by independence of the jumps, the probability of common jumps is zero (see, e.g., [Cont and Tankov, 2004](#), Proposition 5.3). Now write

$$\mathbf{R}_1 = \underbrace{\frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}_i \Theta_2}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^n \mathbf{K}_i}}_{\mathbf{R}_{1,a}} + \underbrace{\frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}_i \Theta_5}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^n \mathbf{K}_i}}_{\mathbf{R}_{1,b}}$$

and

$$\mathbf{R}_2 = \underbrace{\frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}_i \Theta_3}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^n \mathbf{K}_i}}_{\mathbf{R}_{2,a}} + \underbrace{\frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}_i \Theta_6}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^n \mathbf{K}_i}}_{\mathbf{R}_{2,b}}.$$

Start with $\mathbf{R}_{1,b}$. Express the conditional variance of the standardized numerator $\tilde{\mathbf{R}}_{1,b}^{num} = \left(\sqrt{\frac{h_{n,T}}{\Delta_{n,T}}}\right) \frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}_i \Theta_5$ as

$$\left(\tilde{\mathbf{R}}_{1,b}^{num}\right)^2 = \frac{1}{h_{n,T} \Delta_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}_i^2 \mathbf{E}_i \left\{ \int_{i \frac{T}{n}}^{(i+1) \frac{T}{n}} \left(\log p_{s-} - \log p_{i \frac{T}{n}}\right)^2 \Lambda^2(\sigma_s^2) ds \right\}.$$

Now, write $\Delta_i \log p_s = \log p_{s-} - \log p_{i \frac{T}{n}}$. It's formula gives

$$\begin{aligned} (\Delta_i \log p_s)^2 &= 2 \int_{iT/n}^s (\Delta_i \log p_u) \tilde{\mu}(\sigma_u^2) du + 2 \int_{iT/n}^s (\Delta_i \log p_u) \sigma_u dW_u^r + \int_{iT/n}^s \int \psi^2 \bar{v}_r(du, d\psi) \\ &\quad + 2 \int_{iT/n}^s \int (\Delta_i \log p_u) \psi \bar{v}_r(du, d\psi) + \int_{iT/n}^s \sigma_u^2 du + \int_{iT/n}^s \lambda_r(\sigma_u^2) \mathbf{E}(\psi^2) du, \end{aligned}$$

and, as a consequence,

$$\begin{aligned}
\int_{iT/n}^{(i+1)T/n} (\log p_{s-} - \log p_{i\frac{T}{n}})^2 ds &= 2 \int_{iT/n}^{(i+1)T/n} \left(\int_{iT/n}^s (\Delta_i \log p_u) \tilde{\mu}(\sigma_u^2) du \right) ds \\
&+ 2 \int_{iT/n}^{(i+1)T/n} \left(\int_{iT/n}^s (\Delta_i \log p_u) \sigma_u dW_u^r \right) ds \\
&+ \int_{iT/n}^{(i+1)T/n} \left(\int_{iT/n}^s \int \psi^2 \bar{v}_r(du, d\psi) \right) ds \\
&+ 2 \int_{iT/n}^{(i+1)T/n} \left(\int_{iT/n}^s \int (\Delta_i \log p_u) \psi \bar{v}_r(du, d\psi) \right) ds \\
&+ \int_{iT/n}^{(i+1)T/n} \left(\int_{iT/n}^{(i+1)T/n} \left\{ (\sigma_u^2 + \lambda_r(\sigma_u^2) \mathbf{E}(\psi^2)) - (\sigma_{i\frac{T}{n}}^2 + \lambda_r(\sigma_{i\frac{T}{n}}^2) \mathbf{E}(\psi^2)) \right\} du \right) ds \\
&+ \frac{1}{2} \Delta_{n,T}^2 (\sigma_{iT/n}^2 + \lambda_r(\sigma_{iT/n}^2) \mathbf{E}(\psi^2)).
\end{aligned}$$

Noting that

$$\begin{aligned}
\int_{iT/n}^{(i+1)T/n} (\log p_{s-} - \log p_{iT/n})^2 \Lambda^2(\sigma_s^2) ds &= \Lambda^2(\sigma_{i\frac{T}{n}}^2) \int_{iT/n}^{(i+1)T/n} (\log p_{s-} - \log p_{iT/n})^2 ds \\
&+ \int_{iT/n}^{(i+1)T/n} (\log p_{s-} - \log p_{iT/n})^2 (\Lambda^2(\sigma_s^2) - \Lambda^2(\sigma_{iT/n}^2)) ds
\end{aligned}$$

we have

$$\begin{aligned}
&\int_{i\frac{T}{n}}^{(i+1)\frac{T}{n}} (\log p_{s-} - \log p_{iT/n})^2 \Lambda^2(\sigma_s^2) ds \\
&= 2\Lambda^2(\sigma_{iT/n}^2) \int_{iT/n}^{(i+1)T/n} \left(\int_{iT/n}^s (\Delta_i \log p_u) \tilde{\mu}(\sigma_u^2) du \right) ds \\
&+ 2\Lambda^2(\sigma_{iT/n}^2) \int_{iT/n}^{(i+1)T/n} \left(\int_{iT/n}^s (\Delta_i \log p_u) \sigma_u dW_u^r \right) ds \\
&+ \Lambda^2(\sigma_{iT/n}^2) \int_{iT/n}^{(i+1)T/n} \left(\int_{iT/n}^s \int \psi^2 \bar{v}_r(du, d\psi) \right) ds \\
&+ 2\Lambda^2(\sigma_{iT/n}^2) \int_{iT/n}^{(i+1)T/n} \left(\int_{iT/n}^s \int (\Delta_i \log p_u) \psi \bar{v}_r(du, d\psi) \right) ds \\
&+ \frac{1}{2} \Lambda^2(\sigma_{iT/n}^2) \int_{iT/n}^{(i+1)T/n} \int_{iT/n}^{(i+1)T/n} \left(\left\{ (\sigma_u^2 + \lambda_r(\sigma_u^2) \mathbf{E}(\psi^2)) - (\sigma_{iT/n}^2 + \lambda_r(\sigma_{iT/n}^2) \mathbf{E}(\psi^2)) \right\} du \right) ds \\
&+ \frac{1}{2} \Delta_{n,T}^2 (\sigma_{iT/n}^2 + \lambda_r(\sigma_{iT/n}^2) \mathbf{E}(\psi^2)) \Lambda^2(\sigma_{iT/n}^2) \\
&+ \Lambda^2(\sigma_{iT/n}^2) \int_{i\frac{T}{n}}^{(i+1)\frac{T}{n}} \left((\log p_{s-} - \log p_{iT/n})^2 (\Lambda^2(\sigma_{s-}^2) - \Lambda^2(\sigma_{iT/n}^2)) \right) ds \\
&= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7.
\end{aligned}$$

Now write

$$\left(\tilde{\mathbf{R}}_{1,b}^{num} \right)^2$$

$$\begin{aligned}
&= \frac{1}{h_{n,T} \Delta_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}_i^2 \mathbf{E}_i \{ \alpha_6 \} \\
&= \frac{1}{h_{n,T} \Delta_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}_i^2 \mathbf{E}_i \left\{ \frac{1}{2} \Delta_{n,T}^2 (\sigma_{iT/n}^2 + \lambda_r(\sigma_{iT/n}^2) \mathbf{E}(\psi^2)) \Lambda^2(\sigma_{i\frac{T}{n}}^2) \right\} \\
&= \frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n-1} \frac{1}{2} \mathbf{K}_i^2 (\sigma_{iT/n}^2 + \lambda_r(\sigma_{iT/n}^2) \mathbf{E}(\psi^2)) \Lambda^2(\sigma_{i\frac{T}{n}}^2) \\
&\xrightarrow[p_{n,T} \rightarrow 0]{} \frac{1}{2} \frac{1}{h_{n,T}} \int_0^T \mathbf{K}^2 \left(\frac{\sigma_s^2 - \sigma^2}{h_{n,T}} \right) \Lambda^2(\sigma_s^2) [\sigma_s^2 + \lambda_r(\sigma_s^2) \mathbf{E}(\psi^2)] ds,
\end{aligned}$$

using the same methods of proof leading to the results in Lemma 2 and Lemma 4 if $\frac{\Delta_{n,T}^*}{h_{n,T}^2} \rightarrow 0$. We now turn to α_3 .

Integrating by parts, note that

$$\begin{aligned}
\int_{iT/n}^{(i+1)T/n} \left(\int_{iT/n}^s \int \psi^2 \bar{v}_r(du, d\psi) \right) ds &= - \int_{iT/n}^{(i+1)T/n} \left(\int_{iT/n}^s du \right) \int \psi^2 \bar{v}_r(ds, d\psi) + \left(\int_{iT/n}^{(i+1)T/n} du \right) \int_{iT/n}^{(i+1)T/n} \int \psi^2 \bar{v}_r(ds, d\psi) \\
&= \int_{iT/n}^{(i+1)T/n} ((i+1)T/n - s) \int \psi^2 \bar{v}_r(ds, d\psi).
\end{aligned}$$

Hence, $\mathbf{E}_i \{ \alpha_3 \} = 0$. Similarly, $\mathbf{E}_i \{ \alpha_2 \} = \mathbf{E}_i \{ \alpha_4 \} = 0$. Consider

$$\begin{aligned}
\frac{1}{2} \alpha_1 &= \Lambda^2(\sigma_{i\frac{T}{n}}^2) \int_{iT/n}^{(i+1)T/n} \left(\int_{iT/n}^s (\Delta_i \log p_u) \tilde{\mu}(\sigma_u^2) du \right) ds \\
&= \Lambda^2(\sigma_{i\frac{T}{n}}^2) \int_{iT/n}^{(i+1)T/n} \left(\int_{iT/n}^s \left(\int_{iT/n}^u \tilde{\mu}(\sigma_v^2) dv \right) \tilde{\mu}(\sigma_u^2) du \right) ds \\
&\quad + \Lambda^2(\sigma_{i\frac{T}{n}}^2) \int_{iT/n}^{(i+1)T/n} \left(\int_{iT/n}^s \left(\int_{iT/n}^u \sigma_v dW_v^r \right) \tilde{\mu}(\sigma_u^2) du \right) ds \\
&\quad + \Lambda^2(\sigma_{i\frac{T}{n}}^2) \int_{iT/n}^{(i+1)T/n} \left(\int_{iT/n}^s \left(\int_{iT/n}^u \int \psi \bar{v}_r(dv, d\psi) \right) \tilde{\mu}(\sigma_u^2) du \right) ds \\
&= \alpha_{11} + \alpha_{12} + \alpha_{13}.
\end{aligned}$$

Notice that

$$\begin{aligned}
\mathbf{E}_i(\alpha_{11}) &\leq \mathbf{E}_i \Lambda^2(\sigma_{i\frac{T}{n}}^2) \int_{iT/n}^{(i+1)T/n} \left(\int_{iT/n}^s \left(\int_{iT/n}^u |\tilde{\mu}(\sigma_v^2)| dv \right) |\tilde{\mu}(\sigma_u^2)| du \right) ds \\
&\leq \mathbf{E}_i \Lambda^2(\sigma_{i\frac{T}{n}}^2) \Delta_{n,T} \left(\int_{iT/n}^{(i+1)T/n} |\tilde{\mu}(\sigma_s^2)| ds \right)^2 \\
&\leq \mathbf{E}_i \Delta_{n,T}^2 \Lambda^2(\sigma_{iT/n}^2) \left(\int_{iT/n}^{(i+1)T/n} \tilde{\mu}^2(\sigma_s^2) ds \right) \stackrel{p}{\approx} \Delta_{n,T}^2 \Delta_{n,T}^*.
\end{aligned}$$

By Holder's inequality, Jensen's inequality, and Burkholder, Davis, Gundy's inequality,

$$\begin{aligned}
\mathbf{E}_i(\alpha_{12}) &\leq \mathbf{E}_i \Lambda^2(\sigma_{iT/n}^2) \int_{iT/n}^{(i+1)T/n} \mathbf{E} \left(\int_{iT/n}^s |\tilde{\mu}(\sigma_u^2)| du \right) \left(\max_{u < s} \left| \int_{iT/n}^u \sigma_v dW_v^r \right| \right) ds \\
&\leq \mathbf{E}_i \Lambda^2(\sigma_{iT/n}^2) \int_{iT/n}^{(i+1)T/n} \sqrt{\mathbf{E} \left(\int_{iT/n}^s |\tilde{\mu}(\sigma_u^2)| du \right)^2 \mathbf{E} \left(\max_{u < s} \left| \int_{iT/n}^u \sigma_v dW_v^r \right| \right)^2} ds
\end{aligned}$$

$$\begin{aligned}
&\leq C \mathbf{E}_i \Lambda^2(\sigma_{iT/n}^2) \int_{iT/n}^{(i+1)T/n} \sqrt{\Delta_{n,T} \mathbf{E} \left(\int_{iT/n}^{(i+1)T/n} |\tilde{\mu}(\sigma_u^2)|^2 du \right) \mathbf{E} \left(\int_{iT/n}^{(i+1)T/n} \sigma_v^2 dv \right)} ds \\
&\leq C \mathbf{E}_i \Lambda^2(\sigma_{iT/n}^2) \Delta_{n,T}^{3/2} \left(\mathbf{E} \left(\int_{iT/n}^{(i+1)T/n} |\tilde{\mu}(\sigma_u^2)|^2 du \right) \right)^{1/2} \left(\mathbf{E} \left(\int_{iT/n}^{(i+1)T/n} \sigma_v^2 dv \right) \right)^{1/2} \stackrel{p}{\sim} \Delta_{n,T}^{3/2} \Delta_{n,T}^* = \Delta_{n,T}^2 \frac{\Delta_{n,T}^*}{\Delta_{n,T}^{1/2}},
\end{aligned}$$

where $\frac{\Delta_{n,T}^*}{\Delta_{n,T}^{1/2}} = \mathbf{E} \left(\mathcal{M}(F) \Delta_{n,T}^{1/2} \right) \rightarrow 0$ as implied by Assumption 5(b). Finally, focusing on α_{13} ,

$$\begin{aligned}
\mathbf{E}_i(\alpha_{13}) &= \mathbf{E}_i \Lambda^2(\sigma_{iT/n}^2) \int_{iT/n}^{(i+1)T/n} \mathbf{E} \left(\int_{iT/n}^s |\tilde{\mu}(\sigma_u^2)| du \right) \left(\max_{u < s} \left| \int_{iT/n}^u \int \psi \bar{v}_r(dv, d\psi) \right| \right) ds \\
&\leq \mathbf{E}_i \Lambda^2(\sigma_{iT/n}^2) \int_{iT/n}^{(i+1)T/n} \sqrt{\mathbf{E} \left(\int_{iT/n}^s |\tilde{\mu}(\sigma_u^2)| du \right)^2 \mathbf{E} \left(\max_{u < s} \left| \int_{iT/n}^u \int \psi \bar{v}_r(dv, d\psi) \right| \right)^2} ds \\
&\leq \mathbf{E}_i \Lambda^2(\sigma_{iT/n}^2) \int_{iT/n}^{(i+1)T/n} \sqrt{\Delta_{n,T} \mathbf{E} \left(\int_{iT/n}^{(i+1)T/n} |\tilde{\mu}(\sigma_u^2)|^2 du \right) \mathbf{E} \left(\int_{iT/n}^{(i+1)T/n} \mathbf{E}(\psi^2) \lambda(\sigma_v^2) dv \right)} ds \\
&\leq \mathbf{E}_i \Lambda^2(\sigma_{iT/n}^2) \Delta_{n,T}^{3/2} \left(\mathbf{E} \left(\int_{iT/n}^{(i+1)T/n} |\tilde{\mu}(\sigma_u^2)|^2 du \right) \right)^{1/2} \left(\mathbf{E} \left(\int_{iT/n}^{(i+1)T/n} \mathbf{E}(\psi^2) \lambda(\sigma_v^2) dv \right) \right)^{1/2} \\
&\stackrel{p}{\sim} \Delta_{n,T}^{3/2} \Delta_{n,T}^* = \Delta_{n,T}^2 \frac{\Delta_{n,T}^*}{\Delta_{n,T}^{1/2}}.
\end{aligned}$$

Thus, α_1 is of higher order than α_6 . We now turn to α_5 . Write

$$\mathbf{E}_i(\alpha_5) = \mathbf{E}_i \left(\Lambda^2(\sigma_{iT/n}^2) \int_{iT/n}^{(i+1)T/n} \int_{iT/n}^{(i+1)T/n} \left(\left\{ (\sigma_u^2 + \lambda_r(\sigma_u^2) \mathbf{E}(\psi^2)) - (\sigma_{iT/n}^2 + \lambda_r(\sigma_{iT/n}^2) \mathbf{E}(\psi^2)) \right\} du \right) ds \right) \stackrel{p}{\sim} \mathbf{E}_i \left(\Delta_{n,T}^2 \Delta_{n,T}^{*1/2} \right),$$

which is, again, of higher order than α_6 . The last term, α_7 , is of order $\Delta_{n,T} \Delta_{n,T}^{*3/2} = \Delta_{n,T}^2 \frac{\Delta_{n,T}^{*3/2}}{\Delta_{n,T}} = o_p(\Delta_{n,T}^2)$ since $\frac{\Delta_{n,T}^{*3/2}}{\Delta_{n,T}} = (\mathbf{E}(\mathcal{M}(F)))^{3/2} \Delta_{n,T}^{1/2} \rightarrow 0$ given Assumption 5(b).

Finally, note that $\left\{ \int_{iT/n}^{(i+1)T/n} (\log p_{s-} - \log p_{iT/n}) \Lambda(\sigma_{s-}^2) dW_s^\sigma, \mathfrak{F}_{iT/n} \right\}$ is a martingale difference array. Using the same method as in the proof of Lemma 4, as $\Delta_{n,T} \rightarrow 0, T \rightarrow \infty$ and $h_{n,T} \rightarrow 0$, with $\frac{\Delta_{n,T}^*}{h_{n,T}^2} \rightarrow 0$, we have:

$$\left(\frac{\sqrt{h_{n,T} \widehat{L}_{\sigma^2}(T, x)}}{\Delta_{n,T}} \right) \frac{\mathbf{R}_{1,b}^{num}}{\widehat{L}_{n,T}(\sigma^2)} \Rightarrow \left(\sqrt{\mathbf{K}_2 \frac{1}{2} (x + \lambda_r(x) \mathbf{E}[\psi^2]) \Lambda^2(x)} \right) Z,$$

and, similarly,

$$\left(\frac{\sqrt{h_{n,T} \widehat{L}_{\sigma^2}(T, x)}}{\Delta_{n,T}} \right) \frac{\mathbf{R}_{1,a}^{num}}{\widehat{L}_{n,T}(\sigma^2)} \Rightarrow \left(\sqrt{\mathbf{K}_2 \frac{1}{2} (\lambda_\sigma(x) \mathbf{E}[\xi^2] + \Lambda^2(x)) x} \right) Z.$$

In addition, the asymptotic covariance term can be expressed as

$$\begin{aligned}
&asycov \left(\sqrt{\frac{h_{n,T} \widehat{L}_{\sigma^2}(T, x)}{\Delta_{n,T}}} \frac{\mathbf{R}_{1,b}^{num}}{\widehat{L}_{\sigma^2}(T, x)}, \sqrt{\frac{h_{n,T} \widehat{L}_{\sigma^2}(T, x)}{\Delta_{n,T}}} \frac{\mathbf{R}_{1,a}^{num}}{\widehat{L}_{\sigma^2}(T, x)} \right) \\
&\stackrel{p}{\underset{\Delta_{n,T} \rightarrow 0, T \rightarrow \infty, h_{n,T} \rightarrow 0}{\rightarrow}} \frac{1}{2} \mathbf{K}_2 \rho^2(x) x \Lambda^2(x).
\end{aligned}$$

This result yields

$$\sqrt{\frac{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}{\Delta_{n,T}}}\mathbf{R}_1 \Rightarrow \left(\sqrt{\mathbf{K}_2 \begin{pmatrix} \frac{1}{2}\lambda_\sigma(x)\mathbf{E}[\xi^2]x \\ +\frac{1}{2}\lambda_r(x)\mathbf{E}[\psi^2]\Lambda^2(x) \\ + (1+\rho^2(x))x\Lambda^2(x) \end{pmatrix}} \right) Z.$$

Using the same reasoning (combined with the independence of the jumps), we obtain:

$$\sqrt{\frac{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}{\Delta_{n,T}}}\mathbf{R}_2 \Rightarrow \left(\sqrt{\frac{\mathbf{K}_2}{2} \begin{pmatrix} (\Lambda^2(x) + \lambda_\sigma(x)\mathbf{E}[\xi^2])\lambda_r(x)\mathbf{E}[\psi^2] \\ + (x + \lambda_r(x)\mathbf{E}[\psi^2])\lambda_\sigma(x)\mathbf{E}[\xi^2] \end{pmatrix}} \right) Z.$$

Now write

$$\begin{aligned} & \sqrt{\frac{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}{\Delta_{n,T}}} \left\{ \frac{\sum_{i=1}^{n-1} \mathbf{K}_i \Theta_1}{\Delta_{n,T} \sum_{i=1}^n \mathbf{K}_i} + \frac{\sum_{i=1}^{n-1} \mathbf{K}_i \Theta_4}{\Delta_{n,T} \sum_{i=1}^n \mathbf{K}_i} + \frac{\sum_{i=1}^{n-1} \mathbf{K}_i \Theta_7}{\Delta_{n,T} \sum_{i=1}^n \mathbf{K}_i} + \mathbf{R}_1 + \mathbf{R}_2 + O_p \left(\frac{\mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2}}{\Delta_{n,T}} \right) \right\} \\ &= \sqrt{\frac{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}{\Delta_{n,T}}} \left\{ O_p(\Delta_{n,T}^{*1/2}) \left(1 + O_p \left(\frac{\Delta_{n,T}^*}{h_{n,T}^2} \right) \right) + \mathbf{R}_1 + \mathbf{R}_2 + O_p \left(\frac{\mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2}}{\Delta_{n,T}} \right) \right\} \\ &= \sqrt{\frac{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}{\Delta_{n,T}}} O_p(\Delta_{n,T}^{*1/2}) \left(1 + O_p \left(\frac{\Delta_{n,T}^*}{h_{n,T}^2} \right) \right) + \sqrt{\frac{h_{n,T}\widehat{L}_{n,T}(\sigma^2)}{\Delta_{n,T}}} \{ \mathbf{R}_1 + \mathbf{R}_2 \} + O_p \left(\sqrt{\frac{h_{n,T}\widehat{L}_{n,T}(\sigma^2)}{\Delta_{n,T}}} \left(\frac{\mathbf{M}_{n,k,T} \Delta_{n,T}^{*1/2}}{\Delta_{n,T}} \right) \right). \end{aligned}$$

Finally, consider the bias term

$$\begin{aligned} & \frac{\sum_{i=1}^{n-1} \mathbf{K}_i \Theta_8}{\Delta_{n,T} \sum_{i=1}^n \mathbf{K}_i} - \rho(x)\sqrt{x}\Lambda(x) \\ &= \frac{\sum_{i=1}^{n-1} \mathbf{K}_i \int_{iT/n}^{(i+1)T/n} (\rho(\sigma_s^2)\sigma_s - \Lambda(\sigma_s^2)) ds}{\Delta_{n,T} \sum_{i=1}^n \mathbf{K}_i} - \rho(x)\sqrt{x}\Lambda(x) \\ &= \frac{\int_0^T \mathbf{K}_s \rho(\sigma_s^2)\sigma_s \Lambda(\sigma_s^2) ds}{\int_0^T \mathbf{K}_s ds} - \rho(x)\sqrt{x}\Lambda(x) + O_p \left(\frac{\Delta_{n,T}^*}{h_{n,T}^2} \right) - \frac{(\rho(x)\sqrt{x}\Lambda(x)) O_p \left(\frac{\Delta_{n,T}^*}{h_{n,T}^2} \right)}{1 + O_p \left(\frac{\Delta_{n,T}^*}{h_{n,T}^2} \right)} \\ &= \frac{\Gamma_C(\sigma^2)}{1 + O_p \left(\frac{\Delta_{n,T}^*}{h_{n,T}^2} \right)} + \frac{O_p \left(\frac{\Delta_{n,T}^*}{h_{n,T}^2} \right)}{1 + O_p \left(\frac{\Delta_{n,T}^*}{h_{n,T}^2} \right)}. \end{aligned}$$

Define $z(\cdot) = \sqrt{\cdot}\rho(\cdot)\Lambda(\cdot)$. An application of the ratio-limit theorem combined with a second-order Taylor expansion leads to

$$\Gamma_C(x) = h_{n,T}^2 \left(\frac{1}{2} z''(x) + z'(x) \frac{s'(x)}{s(x)} \right).$$

Thus,

$$\sqrt{h_{n,T}\widehat{L}_{\sigma^2}(T,x)/\Delta_{n,T}} \left\{ \widehat{C}(x) - z(x) - \Gamma_C(x) \right\} \Rightarrow \mathbf{N}(0, \mathbf{K}_2 \theta^C(x)),$$

with

$$\begin{aligned} \theta^C(x) &= \Lambda^2(x)x(1+\rho^2(x)) + x\lambda_\sigma(x)\mathbf{E}[\xi^2] \\ &\quad + \Lambda^2(x)\lambda_r(x)\mathbf{E}[\psi^2] + \lambda_\sigma(x)\mathbf{E}[\xi^2]\lambda_r(x)\mathbf{E}[\psi^2], \end{aligned}$$

if $h_{n,T}\widehat{L}_{\sigma^2}(T,x)\frac{\Delta_{n,T}^*}{\Delta_{n,T}} \xrightarrow{a.s.} 0$, $\frac{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}{\Delta_{n,T}} \xrightarrow{a.s.} \infty$, $\frac{h_{n,T}^5\widehat{L}_{\sigma^2}(T,x)}{\Delta_{n,T}} \xrightarrow{a.s.} C$, $\frac{\Delta_{n,T}^*}{h_{n,T}^2}\sqrt{\frac{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}{\Delta_{n,T}}} \xrightarrow{a.s.} 0$, $\frac{\Delta_{n,T}^*}{h_{n,T}^2} \rightarrow 0$ and $\sqrt{\frac{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}{\Delta_{n,T}}}\left(\frac{g(n,T,k,\phi)h_{n,T}\Delta_{n,T}^{*1/2}}{\Delta_{n,T}}\right) \xrightarrow{a.s.} 0$. Given $h_{n,T}\widehat{L}_{\sigma^2}(T,x)\frac{\Delta_{n,T}^*}{\Delta_{n,T}} \xrightarrow{a.s.} 0$, however, $\frac{h_{n,T}^5\widehat{L}_{\sigma^2}(T,x)}{\Delta_{n,T}} = \frac{h_{n,T}^5\widehat{L}_{\sigma^2}(T,x)\Delta_{n,T}^*}{\Delta_{n,T}\Delta_{n,T}^*} = O_{a.s.}(1)$ implies $\frac{h_{n,T}^4}{\Delta_{n,T}^*} = O_{a.s.}\left(\left(h_{n,T}\widehat{L}_{\sigma^2}(T,x)\frac{\Delta_{n,T}^*}{\Delta_{n,T}}\right)^{-1+\delta}\right)$ for some $1 > \delta > 0$ if $C = 0$. Thus, $\frac{\Delta_{n,T}^*}{h_{n,T}^4} \rightarrow 0$ and the conditions

$$\frac{\Delta_{n,T}^*}{h_{n,T}^2}\sqrt{\frac{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}{\Delta_{n,T}}} = \frac{\Delta_{n,T}^*}{h_{n,T}^2\Delta_{n,T}^{*1/2}}\sqrt{\frac{h_{n,T}\widehat{L}_{\sigma^2}(T,x)\Delta_{n,T}^*}{\Delta_{n,T}}} = \underbrace{\frac{\Delta_{n,T}^{*1/2}}{h_{n,T}^2}}_{\rightarrow 0}\underbrace{\sqrt{\frac{h_{n,T}\widehat{L}_{\sigma^2}(T,x)\Delta_{n,T}^*}{\Delta_{n,T}}}}_{\rightarrow 0} \rightarrow 0$$

and $\frac{\Delta_{n,T}^*}{h_{n,T}^2} \rightarrow 0$ are automatically satisfied. Now, by the delta method:

$$\begin{aligned} \widehat{\rho}(x) - \rho(x) &= \frac{\widehat{C}(x)}{\sqrt{x\widehat{\Lambda}(x)}} - \frac{C(x)}{\sqrt{x\Lambda(x)}} \\ &\underset{p}{\sim} \frac{\widehat{C}(x) - C(x)}{\sqrt{x\Lambda(x)}} - \frac{C(x)(\widehat{\Lambda}(x) - \Lambda(x))}{2\sqrt{x\Lambda^3(x)}} + O_p\left(\frac{\mathbf{M}_{n,k,T}\Delta_{n,T}^{*1/2}}{\Delta_{n,T}}\right) + O_p\left(\frac{\mathbf{M}_{n,k,T}\Delta_{n,T}^{*1/2}}{\Delta_{n,T}h_{n,T}}\right) \\ &\underset{p}{\sim} \frac{O(h_{n,T}^2) + O_p\left(\sqrt{\frac{\Delta_{n,T}}{h_{n,T}\widehat{L}_{\sigma^2}(T,\sigma^2)}}\right) + O_p(\Delta_{n,T}^{*1/2})\left(1 + O_p\left(\frac{\Delta_{n,T}^*}{h_{n,T}^2}\right)\right) + O_p\left(\frac{\Delta_{n,T}^*}{h_{n,T}^2}\right)}{\sigma\Lambda(\sigma^2)} \\ &\quad - \rho(x)\frac{O(h_{n,T}^2) + O_p\left(\sqrt{\frac{1}{h_{n,T}\widehat{L}_{\sigma^2}(T,\sigma^2)}}\right) + O_p\left(\frac{\Delta_{n,T}^*}{h_{n,T}^2}\right)}{2\Lambda^2(\sigma^2)} + O_p\left(\frac{\mathbf{M}_{n,k,T}\Delta_{n,T}^{*1/2}}{\Delta_{n,T}h_{n,T}}\right) \end{aligned}$$

Hence, if $h_{n,T}\widehat{L}_{\sigma^2}(T,x) \rightarrow \infty$, $h_{n,T}\widehat{L}_{\sigma^2}(T,x)\Delta_{n,T}^* \rightarrow 0$, $\frac{\Delta_{n,T}^*}{h_{n,T}^2}\sqrt{h_{n,T}\widehat{L}_{\sigma^2}(T,x)} \rightarrow 0$, the asymptotic distribution of the leverage function is driven by $\widehat{\Lambda}^2(\sigma^2) - \Lambda^2(\sigma^2)$ and has variance

$$\begin{aligned} \text{Asyvar}(\widehat{\rho}(x)) &= \frac{\rho^2(\sigma^2)}{4\Lambda^4(\sigma)} \left(\text{Asyvar}\left(\widehat{\Lambda}^2(x)\right)\right) \\ &= \frac{\rho^2(\sigma^2)}{4\Lambda^4(\sigma)} \left(\frac{40\mathbf{K}_2\lambda_{\sigma^2}(x)\mu_{\xi}^2(x)}{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}\right) + o_p(1). \end{aligned}$$

Finally, under condition $h_{n,T}^5\widehat{L}_{\sigma^2}(T,x) \xrightarrow{a.s.} 0$, $\widehat{\rho}(\sigma^2) - \rho(\sigma^2)$ has a vanishing asymptotic bias. Now notice, as in the proof of Theorem 9, that the condition

$$\frac{\Delta_{n,T}^*}{h_{n,T}^2}\sqrt{h_{n,T}\widehat{L}_{\sigma^2}(T,x)} = \frac{\Delta_{n,T}^*\sqrt{\widehat{L}_{\sigma^2}(T,x)}}{h_{n,T}^{3/2}} \xrightarrow{a.s.} 0$$

can be rewritten as follows

$$(30) \quad \frac{\Delta_{n,T}^*\widehat{L}_{\sigma^2}(T,x)h_{n,T}}{h_{n,T}^{5/2}\widehat{L}_{n,T}^{1/2}(x)} \rightarrow 0.$$

The vanishing-bias condition $h_{n,T}^5\widehat{L}_{\sigma^2}(T,x) \xrightarrow{a.s.} 0$ implies that $\Delta_{n,T}^*\widehat{L}_{\sigma^2}(T,x)h_{n,T} \rightarrow 0$. Thus, we can dispense with the condition $h_{n,T}\widehat{L}_{\sigma^2}(T,x)\Delta_{n,T}^* \rightarrow 0$ and the proof is complete. \square

PROOF AND FURTHER DISCUSSION OF REMARK 1. First, we focus on the conditions that are needed for the estimation of the dynamics. Three conditions were presented for weak convergence: 1) $h_{n,T} \widehat{L}_{\sigma^2}(T, x) \xrightarrow{a.s.} \infty$, 2) $h_{n,T}^5 \widehat{L}_{\sigma^2}(T, x) \xrightarrow{a.s.} 0$, and 3) $\frac{\Delta_{n,T} \sqrt{\widehat{L}_{\sigma^2}(T, x)}}{h_{n,T}^{3/2}} \xrightarrow{a.s.} 0$. Given Lemma 3, if $\frac{\Delta_{n,T}}{h_{n,T}^2} \rightarrow 0$, then $\widehat{L}_{\sigma^2}(T, x) = O_{a.s.}(L_{h_{n,T}, T}(x))$, where $L_{h_{n,T}, T}(x)$ is defined in Lemma 1 (it is clear that the probability result in Lemma 3 extends to an almost-sure statement). Furthermore, in light of Lemma 1, $L_{h_{n,T}, T}(x) \xrightarrow{a.s.} \infty$ for all $h_{n,T}$. This implies that 1') $h_{n,T} L_{h_{n,T}, T}(x) \xrightarrow{a.s.} \infty$, 2') $h_{n,T}^5 L_{h_{n,T}, T}(x) \xrightarrow{a.s.} 0$, and 3') $\frac{\Delta_{n,T} \sqrt{L_{h_{n,T}, T}(x)}}{h_{n,T}^{3/2}} \xrightarrow{a.s.} 0$ yield 1), 2), and 3) since 1') and 3') guarantee $\frac{\Delta_{n,T}}{h_{n,T}^2} \rightarrow 0$. Now, if $a(T)h_{n,T} \rightarrow \infty$, where $a(T)$ is the almost-sure rate of variance process' local time, then $L_{h_{n,T}, T}(x) = O_{a.s.}(a(T))$. We could, therefore, write 1'') $h_{n,T} a(T) \rightarrow \infty$, 2'') $h_{n,T}^5 a(T) \rightarrow 0$, and 3'') $\frac{\Delta_{n,T} \sqrt{a(T)}}{h_{n,T}^{3/2}} \rightarrow 0$.

Now notice that, for 1'') and 2'') to be satisfied, $h_{n,T} \sim a(T)^{-\beta}$ with $\frac{1}{5} < \beta < 1$. However, 3'') also has to be satisfied. Write

$$(31) \quad \frac{\Delta_{n,T} \sqrt{a(T)}}{a(T)^{-\frac{3}{2}\beta}} = \frac{T}{n} a(T)^{\frac{1}{2} + \frac{3}{2}\beta} = o(1)$$

Since $T \succ a(T)$, where \succ signifies larger in order, a sufficient condition for Eq. (31) to be satisfied is

$$(32) \quad n \succ T^{\frac{3}{2} + \frac{3}{2}\beta}.$$

The first part of the weak convergence condition for $j = 1$ reads $\frac{h_{n,T}^{1/2} a^{1/2}(T) (\log(n))^{1/2}}{\Delta_{n,T} k^{1/2}} \rightarrow 0$ which implies

$$(33) \quad \frac{a^{(1-\beta)}(T) (\log(n))}{\Delta_{n,T}^2} \prec k.$$

The second part reads $\frac{h_{n,T}^{1/2} a^{1/2}(T)}{\Delta_{n,T}} \phi^{1/2} \rightarrow 0$ which requires

$$(34) \quad \phi \prec \Delta_{n,T}^2 a^{(\beta-1)}(T).$$

Similarly, the first part of the weak convergence condition for $j > 1$ reads $\frac{a^{1/2}(T) (\log(n))^{1/2}}{h_{n,T}^{1/2} \Delta_{n,T}^{1/2} k^{1/2}} \rightarrow 0$ which implies

$$(35) \quad \frac{a^{(1+\beta)}(T) (\log(n))}{\Delta_{n,T}} \prec k.$$

The second part reads $\frac{a^{1/2}(T)}{h_{n,T}^{1/2} \Delta_{n,T}^{1/2}} \phi^{1/2} \rightarrow 0$ which requires

$$(36) \quad \phi \prec \Delta_{n,T} a^{-\beta-1}(T).$$

Similar considerations lead to the orders stated in Remark 1 for consistency in the case $j > 1$, while in the case $j = 1$, using condition 4.6, we get:

$$(37) \quad \phi \prec \Delta_{n,T}^2$$

and

$$(38) \quad k \succ \frac{\log(n)}{\Delta_{n,T}^2}$$

To gain a feel for the different orders, let us now assume that $T = 20$ (years) and $n = 250 \times T$ (daily observations). Assume, conservatively, that $a(T) = T$ and consider the orders for consistency first. For $j > 1$, setting $\beta = \frac{1}{5}$, which is MSE

optimal, we obtain $\phi < 0.0012$ and $k > 7,057$. Since, for a liquid stock, roughly 25,000 transactions over a 6.5 hour trading day are not uncommon, the orders may be compatible with the use of ϕ equal to a fraction of a day ($\phi^{day} = 1/250 = 0.004$ so that 6.5 hours correspond to $\phi = 0.0011$) and a typical number of high-frequency observations, even for less liquid stocks. Notice that $n = 250 \times 20$ (the use of daily data to estimate the dynamics) is also compatible with the bound in Eq. (32).

For $j = 1$, which is needed for the drift only, setting $\beta = \frac{1}{5}$, we obtain $\phi < 1.6 \cdot 10^{-5}$ and $k > 532,524$. This is clearly not feasible since over a very small fraction of a day one cannot hope to utilize so many observations. Setting, however, $n = 26 \times 20$ (effectively amounting to bi-weekly observations over 20 years of data) one obtains $\phi < 0.0014$ and $k > 4,227$, thereby implying that high-frequency observations over a fraction of a day may be used for spot variance estimation in this context.

Now we turn to the orders for weak convergence which are necessarily more stringent. For $j > 1$, in the case of daily observations ($n = 250 \times T$), we have $\phi < 1.1 \cdot 10^{-4}$ (roughly an hour) and $k > 77,530$. These conditions are harder to meet for typical situations, but we must remember that these conditions give orders, rather than actual numbers. In the case of bi-weekly observations ($n = 26 \times T$), we have $\phi < 0.001$ and $k > 5,920$, again compatible with high-frequency observations over a fraction of a day. For $j = 1$, similar computations show that the orders are consistent with the use of monthly frequencies to estimate the dynamics, and a fraction of a day of high-frequency data, as before, for the evaluation of the spot variance estimates.

These are back-of-the-envelope calculations based on (1) asymptotic orders and (2) typical data sets. While they should be taken with caution, they are suggestive of the identification potential of a combination of high-frequency and low-frequency data to estimate spot variance in the first stage and the dynamics in the second stage. Naturally, asymptotic normality imposes more stringent conditions than consistency. When computing confidence bands, finite sample improvements may be obtained by bootstrapping them or by simulating based on a suitable parametric model. \square

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