Nonparametric identification in stochastic volatility models

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Joint work with Roberto Reno' (Universita' di Siena)

Background papers - the first two are available on our web sites

- "Nonparametric stochastic volatility"
- "Nonparametric leverage effects"
- "Infinitesimal cross-moments and return/volatility co-jumps"

Outline

- Motivation - a benchmark nonlinear stochastic volatility model with independent return/volatility jumps (SVM-J)
- A theory of high-frequency spot variance estimation
    - *Examples*
- Spot variance dynamics: identification in a nonlinear SVM-J
    - *Empirical work*
- Allowing for co-jumps - a notion of "generalized leverage"
- Allowing for co-jumps - identification of the full system
    - *Empirical work*

Motivation

A benchmark nonlinear stochastic volatility model with independent jumps (SVM-J)

$$
\begin{aligned}
d\log(p_t) &= \mu(\sigma_t^2)dt + \sigma_t dW_t^r + dJ_t^r, \\
df(\sigma_t^2) &= m_{f(.)}(\sigma_t^2)dt + \Lambda_{f(.)}(\sigma_t^2)dW_t^\sigma + dJ_t^\sigma, \\
\{dW_t^r, dW_t^\sigma\} &= \{\rho(\sigma_t^2)dW_t^1 + \sqrt{1-\rho^2(\sigma_t^2)}dW_t^2, dW_t^1\},
\end{aligned}
$$

where

- $\left\{W_t^1, W_t^2\right\}$ are independent Brownian motions
- $-1 \leq \rho(.) \leq 1$
- $\{J_t^r, J_t^\sigma\}$ are Poisson jump processes (with intensities $\lambda^r(.)$ and $\lambda_{f(.)}^\sigma(.)$) independent of each other and independent of $\left\{W_t^1, W_t^2\right\}$
- $\mu(.)$, $m_{f(.)}(.)$, and $\Lambda_{f(.)}(.)$ are functions satisfying mild smoothness conditions

Motivation - continued

A benchmark nonlinear stochastic volatility model with independent jumps (SVM-J)

$$\begin{aligned} d\log(p_t) &= \mu(\sigma_t^2)dt + \sigma_t dW_t^r + dJ_t^r, \\ df(\sigma_t^2) &= m_{f(.)}(\sigma_t^2)dt + \Lambda_{f(.)}(\sigma_t^2)dW_t^\sigma + dJ_t^\sigma, \\ \{dW_t^r, dW_t^\sigma\} &= \{\rho(\sigma_t^2)dW_t^1 + \sqrt{1-\rho^2(\sigma_t^2)}dW_t^2, dW_t^1\} \end{aligned}$$

- Estimation is generally conducted by using *parametric* filtering methods relying on daily (or lower frequency) stock returns.
- A recent literature has emphasized the importance of high-frequency (intra-daily) stock returns for identifying (daily) integrated variance, thereby effectively treating volatility as an "observable" quantity with no need for filtering using low-frequency returns (e.g., Andersen et al., 2003, and Barndorff-Nielsen and Shephard, 2002).

Our approach

$$\begin{aligned} d\log(p_t) &= \mu(\sigma_t^2)dt + \sigma_t dW_t^r + dJ_t^r, \\ df(\sigma_t^2) &= m_{f(.)}(\sigma_t^2)dt + \Lambda_{f(.)}(\sigma_t^2)dW_t^\sigma + dJ_t^\sigma, \\ \{dW_t^r, dW_t^\sigma\} &= \{\rho(\sigma_t^2)dW_t^1 + \sqrt{1-\rho^2(\sigma_t^2)}dW_t^2, dW_t^1\} \end{aligned}$$

- We use current advances in high-frequency volatility estimation to derive a theory of spot variance (σ_t^2) estimation using high-frequency asset price data.
  - We allow for market microstructure noise
- We model the dynamics of spot variance by using nonparametric methods.
  - Identification is conducted under recurrence
  - Consistency and asymptotic normality of all functions and parameters driving return and variance dynamics is shown
  - Conditions under which the measurement error induced by the preliminary spot variance estimates $\widehat{\sigma}_t^2$ is asymptotically negligible are provided

A theory of high-frequency spot variance estimation

- Write
$$\widehat{\sigma}_t^2 = \frac{\widehat{V}_{t,t+\phi}}{\phi} = \frac{f(r_1, r_2, ..., r_k)}{\phi},$$
where $\widehat{V}_{t,t+\phi}$ is an *integrated variance estimator* (over a time interval $\phi \longrightarrow 0$) constructed using k intra-daily return observations (with $k \longrightarrow \infty$).
- We show that
$$\phi^{\beta} k^{\alpha} \left\{ \widehat{\sigma}_t^2 - \sigma_t^2 \right\} \underset{k \to \infty, \phi \to 0}{\Rightarrow} MN\left(0, a\left(\sigma_t^4\right)^{\eta} + b\right)$$
is satisfied, *under additional assumptions and for appropriate choices of* β, α, a, η, *and* b, by virtually all popular integrated (for a fixed $\overline{\phi}$) variance estimators $\widehat{V}_{t,t+\overline{\phi}}$ recently proposed in the literature provided
$\phi^{\beta} k^{\alpha} \longrightarrow \infty$ and $k^{\alpha} \phi^{\beta} \left(\phi \log\left(\frac{1}{\phi}\right)\right)^{1/2} \longrightarrow 0.$
- Depending on $\widehat{V}_{t,t+\phi}$, we may allow for microstructure noise or jumps in returns.

A classical dgp for intra-daily returns

Assume availability of $k+1$ price observations in each interval $[t, t+\phi]$. The intra-daily price formation mechanism is defined as:

$$\log(p_j)^* \;=\; \log(p_j) + \eta_j, \quad j = 0, ..., k$$

or, in terms of continuously-compounded returns,

$$\underbrace{\log(p_j)^* - \log(p_{(j-1)})^*}_{r^*_{j\delta}} = \underbrace{\log(p_j) - \log(p_{(j-1)})}_{r_{j\delta}} + \underbrace{\eta_j - \eta_{(j-1)}}_{\varepsilon_{j\delta}}, \quad j = 1, ..., k$$

where $\log(p)$ denotes the *unobservable* equilibrium price and η denotes *unobservable* market microstructure noise.

1. The noise is independent of the true price process and IID mean zero with second moment $\mathbf{E}(\eta^2) = \sigma^2_\eta$ (or, equivalently, $\mathbf{E}(\varepsilon^2) = 2\sigma^2_\eta$).
2. The equilibrium price follows the SDE above (with or without jumps).
3. The equilibrium price's spot variance follows the SDE above (with jumps).

Example 1: Realized variance

(Andersen et al., 2003) - No noise, no jumps in returns

$$\widehat{\sigma}_t^2 = \frac{\widehat{V}_{t,t+\phi}}{\phi} = \frac{\sum_{j=1}^{k} \left(r_{j\delta}^* \right)^2}{\phi}$$

Then, if $k^{\alpha} \phi^{\beta} \left(\phi \log \left(\frac{1}{\phi} \right) \right)^{1/2} \rightarrow 0$,

$$\phi^{\beta} k^{\alpha} \left\{ \widehat{\sigma}_t^2 - \sigma_t^2 \right\} \underset{k \rightarrow \infty, \phi \rightarrow 0}{\Rightarrow} MN \left(0, a \left(\sigma_t^4 \right)^{\eta} + b \right),$$

with $\alpha = \frac{1}{2}$, $\beta = 0$, $a = 2$, $\eta = 1$, and $b = 0$.

- **The rate of convergence is** $k^{1/2}$.

Example 2: Multipower variation

(Barndorff-Nielsen and Shephard, 2004, 2006) - No noise, jumps in returns

$$\widehat{\sigma}_t^2 = \frac{\widehat{V}_{t,t+\phi}}{\phi} = \frac{\left(\mu_{2/z}\right)^{-z} \sum_{j=1}^{k-z} \left| r_{j\delta}^* \right|^{2/z} \left| r_{(j+1)\delta}^* \right|^{2/z} \cdots \left| r_{(j+z)\delta}^* \right|^{2/z}}{\phi},$$

where $\mu_k = E(|Z^k|)$ and Z is standard normal.

Then, if $k^\alpha \phi^\beta \left(\phi \log\left(\frac{1}{\phi}\right)\right)^{1/2} \rightarrow 0$ and $\frac{k^{\frac{1}{2}}}{\phi}\left(\frac{\phi}{k}\log\left(\frac{k}{\phi}\right)\right)^{(z-1)/z} \rightarrow 0$,

$$\phi^\beta k^\alpha \left\{\widehat{\sigma}_t^2 - \sigma_t^2\right\} \underset{k\rightarrow\infty,\phi\rightarrow 0}{\Rightarrow} MN\left(0, a\left(\sigma_t^4\right)^\eta + b\right),$$

with $\alpha = \frac{1}{2}$, $\beta = 0$, $a \neq 0$, $\eta = 1$, and $b = 0$.

- **Again, the rate of convergence is** $k^{1/2}$.

Example 3: The two-scale estimator

(Zhang et al., 2005) - Noise, No jumps in returns

Consider q non-overlapping sub-grids of the original grid of $k+1$ arrival times. Define q realized variance measures on these subgrids $\widehat{V}^{(i)}$ $i = 1, \ldots., q$. Now write,

$$\widehat{\sigma}_t^2 = \frac{\widehat{V}_{t,t+\phi}}{\phi} = \frac{\frac{\sum_{i=1}^{q} \widehat{V}^{(i)}}{q} - \left(\frac{k-q+1}{q}\right)\widehat{\mathbf{E}}(\varepsilon^2)}{\phi}.$$

Then, if $\phi^\beta k^\alpha \to \infty$ and $k^\alpha \phi^\beta \left(\phi \log\left(\frac{1}{\phi}\right)\right)^{1/2} \to 0$,

$$\phi^\beta k^\alpha \left\{\widehat{\sigma}_t^2 - \sigma_t^2\right\} \underset{k\to\infty,\phi\to 0}{\Rightarrow} MN\left(0, a\left(\sigma_t^4\right)^\eta + b\right).$$

- *If* $q = \tau k^{2/3}$, *then* $\beta = 1$, $\alpha = \frac{1}{6}$, $a = 0$, *and* $b = \left(\frac{8}{\tau^2}\right)\left(\mathbf{E}(\varepsilon^2)\right)^2$.
- *If* $q^o = \tau\left(\frac{k}{\phi}\right)^{2/3}$ *with* $\tau = \left(\frac{12\left(\mathbf{E}(\varepsilon^2)\right)^2}{\sigma_t^4}\right)^{1/3}$, *then* $\beta = \frac{1}{3}$, $\alpha = \frac{1}{6}$, $a = 2\left(12\left(\mathbf{E}(\varepsilon^2)\right)^2\right)^{1/3}$, $\eta = \frac{2}{3}$, *and* $b = 0$. **Optimal rate:** $k^{1/10}$.

Example 4: Realized kernels

(Barndorff-Nielsen et al., 2008) - Noise, no jumps in returns

Consider a kernel function $g(x)$ on $[0,1]$ satisfying $g(0)=1$ and $g(1)=0$. Write

$$\widehat{\sigma}_t^2 = \frac{\widehat{V}_{t,t+\phi}}{\phi} = \frac{\widehat{\gamma}_0 + \sum_{s=1}^{q} w_s(\widehat{\gamma}_s + \widehat{\gamma}_{-s})}{\phi},$$

where $\widehat{\gamma}_s = \sum_{j=1}^{k} r_j^* r_{j-s}^*$ with $s=-q,...,q$ and $w_s = g\left(\frac{s-1}{q}\right)$.

Then, if $\phi^\beta k^\alpha \to \infty$ and $k^\alpha \phi^\beta \left(\phi \log\left(\frac{1}{\phi}\right)\right)^{1/2} \to 0$,

$$\phi^\beta k^\alpha \left\{\widehat{\sigma}_t^2 - \sigma_t^2\right\} \underset{k\to\infty,\phi\to 0}{\Rightarrow} MN\left(0, a\left(\sigma_t^4\right)^\eta + b\right).$$

Assume $g'(0)=0$ *and* $g'(1)=0$. *If* $q^o = \tau\left(\frac{k}{\phi_{n,T}}\right)^{1/2}$ *with*

$\tau = \left\{\frac{g_\bullet^{1,1} + \sqrt{(g_\bullet^{1,1})^2 + 3g_\bullet^{0,0}g_\bullet^{2,2}}}{g_\bullet^{0,0}}\right\}^{1/2}\left(\frac{\mathbf{E}(\varepsilon^2)}{\sigma_{iT/n}^2}\right)^{1/2}$, *then* $\beta=\frac{1}{4}$, $\alpha=\frac{1}{4}$, $a\neq 0$, $\eta=\frac{3}{4}$, *and* $b=0$. **Optimal rate:** $k^{1/6}$.

Spot variance dynamics: identification in a nonlinear SVM-J

$$
\begin{aligned}
r_{t,t+dt} &= d\log(p_t) = \mu(\sigma_t^2)dt + \sigma_t dW_t^r + dJ_t^r, \\
df(\sigma_t^2) &= m_{f(.)}(\sigma_t^2)dt + \Lambda_{f(.)}(\sigma_t^2)dW_t^\sigma + dJ_t^\sigma, \\
\{dW_t^r, dW_t^\sigma\} &= \{\rho(\sigma_t^2)dW_t^1 + \sqrt{1-\rho^2(\sigma_t^2)}dW_t^2, dW_t^1\}
\end{aligned}
\qquad (1)
$$

1. **Generalized Duffie, Pan, and Singleton (2000) model.** Write Eq. (1) with $f(\sigma_t^2) = \sigma_t^2$ and $dJ_t^\sigma = \xi^\sigma dN_t^\sigma$, where $\xi^\sigma \sim \exp(\mu_\xi)$.
2. **Generalized log-variance model.** Write Eq. (1) with $f(\sigma_t^2) = \log(\sigma_t^2)$ and $dJ_t^\sigma = \xi^\sigma dN_t^\sigma$, where $\xi^\sigma \sim \mathbf{N}(0, \sigma_\xi^2)$.

Generalized Duffie et al.'s model

An identification scheme in the spirit of Johannes (2004) and Bandi and Nguyen (2003)

$$d\sigma_t^2 = m_{\sigma^2}(\sigma_t^2)dt + \Lambda_{\sigma^2}(\sigma_t^2)dW_t^{\sigma} + \xi^{\sigma} dN_t^{\sigma} \text{ with } \xi^{\sigma} \sim \exp(\mu_{\xi})$$

Write

$$\left\{ \begin{array}{l} \theta^1(x) = m_{\sigma^2}(x) + \mu_{\xi}\lambda_{\sigma^2}(x), \\ \theta^2(x) = \Lambda^2_{\sigma^2}(x) + 2\mu_{\xi}^2\lambda_{\sigma^2}(x), \\ \theta^3(x) = 6\mu_{\xi}^3\lambda_{\sigma^2}(x), \\ \theta^4(x) = 24\mu_{\xi}^4\lambda_{\sigma^2}(x) \end{array} \right\},$$

where the infinitesimal moments $\theta^j(x)$ are defined as follows:

$$\theta^j(x) = \lim_{\Delta \to 0} \frac{1}{\Delta} \mathbf{E}\left[\left(\sigma^2_{t+\Delta} - \sigma_t^2\right)^j | \sigma_t^2 = x \right] \qquad j = 1, \ldots.$$

Generalized Duffie et al.'s model

An identification scheme (continued)

Write

$$\left\{\begin{array}{l} \widehat{\mu}_{\xi} = \frac{1}{n}\sum_{i=1}^{n} \frac{\widehat{\theta}^{4}(\widehat{\sigma}^{2}_{iT/n})}{4\widehat{\theta}^{3}(\widehat{\sigma}^{2}_{iT/n})}, \\ \widehat{\lambda}_{\sigma^2}(x) = \frac{\widehat{\theta}^{4}(x)}{24\widehat{\mu}^{4}_{\xi}}, \\ \widehat{\Lambda}^{2}_{\sigma^2}(x) = \widehat{\theta}^{2}(x) - 2\widehat{\lambda}_{\sigma^2}(x)\widehat{\mu}^{2}_{\xi}, \\ \widehat{m}_{\sigma^2}(x) = \widehat{\theta}^{1}(x) - \widehat{\lambda}_{\sigma^2}(x)\widehat{\mu}_{\xi} \end{array}\right\},$$

where the infinitesimal moments are estimated as follows:

$$\widehat{\theta}^{j}(x) = \frac{1}{\Delta_{n,T}} \frac{\sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\widehat{\sigma}^{2}_{iT/n} - x}{h_{n,T}}\right)\left[\widehat{\sigma}^{2}_{(i+1)T/n} - \widehat{\sigma}^{2}_{iT/n}\right]^{j}}{\sum_{i=1}^{n} \mathbf{K}\left(\frac{\widehat{\sigma}^{2}_{iT/n} - x}{h_{n,T}}\right)} \qquad j = 1, \ldots,$$

with $\widehat{\sigma}^{2}_{iT/n} = \frac{\widehat{V}_{iT/n}}{\phi_{n,T}}$

Infinitesimal moment estimation - consistency

Under recurrence (c.f., Bandi and Phillips, 2003, Bandi and Nguyen, 2003)

Theorem. *If* $k, n, T \to \infty$ *and* $h_{n,T}, \phi_{n,T} \to 0$ *so that*

$$\begin{aligned}
\lim_{n,T\to\infty} h_{n,T} v(T) &= \infty, \\
\lim_{n,T\to\infty} \frac{v(T)}{h_{n,T}} \left(\Delta_{n,T} \log \frac{1}{\Delta_{n,T}}\right)^{1/2} &= 0, \\
\lim_{k,n,T\to\infty} \frac{T v(T)^{-1} \log(n)}{\Delta_{n,T} h_{n,T} k^{\alpha} \phi_{n,T}^{\beta}} + \frac{T v(T)^{-1}}{\Delta_{n,T} h_{n,T}} \left(\phi_{n,T} \log\left(\frac{1}{\phi_{n,T}}\right)\right)^{1/2} &= 0,
\end{aligned}$$

with $\alpha \in (0, \frac{1}{2}]$ *and* $\beta = [0, 1]$, *then,*

$$\widehat{\theta}^{j}(x) \xrightarrow{p} \theta^{j}(x) \qquad j \geq 1,$$

where $\overline{L}_{\sigma^2}(T, x) \propto v(T)$ *and* $\overline{L}_{\sigma^2}(T, x)$ *is the spot variance's local time.*

Infinitesimal moment estimation - weak convergence

Under recurrence

Theorem (continued). If $\lim_{n,T\to\infty} h_{n,T} v(T) = \infty$, $\lim_{n,T\to\infty} h^5_{n,T} v(T) = C_3$, $\lim_{n,T\to\infty} \frac{v(T)}{h_{n,T}} \left(\Delta_{n,T} \log \frac{1}{\Delta_{n,T}}\right)^{1/2} = 0$, and $\lim_{k,n,T\to\infty} \frac{T^{3/2} v(T)^{-1} \log(n)}{\Delta_{n,T} h^{1/2}_{n,T} k^{\alpha} \phi^{\beta}_{n,T}} + \frac{T^{3/2} v(T)^{-1}}{\Delta_{n,T} h^{1/2}_{n,T}} \left(\phi_{n,T} \log\left(\frac{1}{\phi_{n,T}}\right)\right)^{1/2} = 0$, *then,*

$$\sqrt{h_{n,T}\widehat{\overline{L}}_{\sigma^2}(T,x)} \left\{\widehat{\theta}^j(x) - \theta^j(x) - \Gamma_{\theta^j}(x)\right\} \Rightarrow \mathbf{N}\left(0, \mathbf{K}_2 \theta^{2j}(x)\right), \qquad \forall j \geq 1$$

with

$$\Gamma_{\theta^j}(x) = h^2_{n,T} \mathbf{K}_1 \left[\theta^{j'}(x) \frac{s'(x)}{s(x)} + \frac{1}{2} \theta^{j''}(x)\right],$$

where $s(dx)$ is the process' invariant measure and C_3 is a constant.

Some specific volatility moments

The volatility jump component

Let $k, n, T \to \infty$, $\phi, h \to 0$ and let the previous assumptions be satisfied.

The intensity of the jumps:

$$\sqrt{h_{n,T}\widehat{\overline{L}}_{\sigma^2}(T,x)}\left\{\widehat{\lambda}_{\sigma^2}(x)-\lambda_{\sigma^2}(x)\right\}\Rightarrow \mathbf{N}\left(0, \mathbf{K}_2\frac{\lambda_{\sigma^2}(x)\mathbf{E}\left(\left(\xi^{\sigma}\right)^8\right)}{(24)^2\mu_{\xi}^8}\right)$$

The jump size:

$$\left(\Upsilon(T)\right)^{-1/2}\overline{T}\left\{\widehat{\mu}_{\xi}-\mu_{\xi}\right\}\Rightarrow \mathbf{N}(0,1),$$

where

$$\Upsilon(T)=\int_{-\infty}^{\infty}\lambda_{\sigma^2}(x)\left(\frac{\overline{L}^2_{\sigma^2}(\overline{T},x)}{\overline{L}_{\sigma^2}(T,x)}\right)\mathbf{E}\left(\left(\frac{1}{4\theta^3(x)}\left(\xi^{\sigma}\right)^4-\frac{\theta^4(x)}{4\left(\theta^3(x)\right)^2}\left(\xi^{\sigma}\right)^3\right)^2\right)dx.$$

The joint return/variance dynamics

Nonlinear leverage effects

Write

$$\widehat{\rho}(\sigma^2) = \frac{\widehat{\theta}_{1,1}(\sigma^2)}{\sqrt{\sigma^2 \widehat{\Lambda}^2_{\sigma^2}(\sigma^2)}},$$

where $\widehat{\theta}_{1,1}(\sigma^2)$ is the first infinitesimal cross-moment between spot variance and returns (more on cross-moments later). Then,

$$\sqrt{h_{n,T}\widehat{\overline{L}}_{\sigma^2}(T,\sigma^2)}\left\{\widehat{\rho}(\sigma^2) - \rho(\sigma^2)\right\} \Rightarrow \mathbf{N}\left(0, \mathbf{K}_2 \theta_\rho(\sigma^2)\right)$$

with $\theta_\rho(\sigma^2) = \frac{\rho^2(\sigma^2)}{4\Lambda^4_{\sigma^2}(\sigma^2)} Asyvar\left(\widehat{\Lambda}^2_{\sigma^2}(\sigma^2)\right)$ and

$$Asyvar\left(\widehat{\Lambda}^2_{\sigma^2}(\sigma^2)\right) = \lambda_{\sigma^2}(x)\mathbf{E}\left(\left((\xi^\sigma)^2 - \frac{1}{12\mu^2_\xi}(\xi^\sigma)^4\right)^2\right).$$

Empirical work

Spiders from January 2, 1998 to March 31, 2006

$$\begin{aligned} d\log(p_t) &= \mu(\sigma_t^2)dt + \sigma_t dW_t^r + \xi^r dN_t^r \text{ with } \xi^r \sim \mathbf{N}(0,\sigma_\xi^2) \\ d\sigma_t^2 &= m_{\sigma^2}(\sigma_t^2)dt + \Lambda_{\sigma^2}(\sigma_t^2)dW_t^\sigma + \xi^\sigma dN_t^\sigma \text{ with } \xi^\sigma \sim \exp(\mu_\xi) \end{aligned}$$

Variance dynamics

- We use a robustified (to noise) version of bipower variation - staggered bipower variation
- We estimate the volatility functions by using the identification method described earlier (again, under exponential jumps)

Return dynamics and joint dynamics

- We estimate the return functions by using a similar moment-based method (under Gaussian jumps in returns, however)
- We estimate functional leverage as discussed earlier

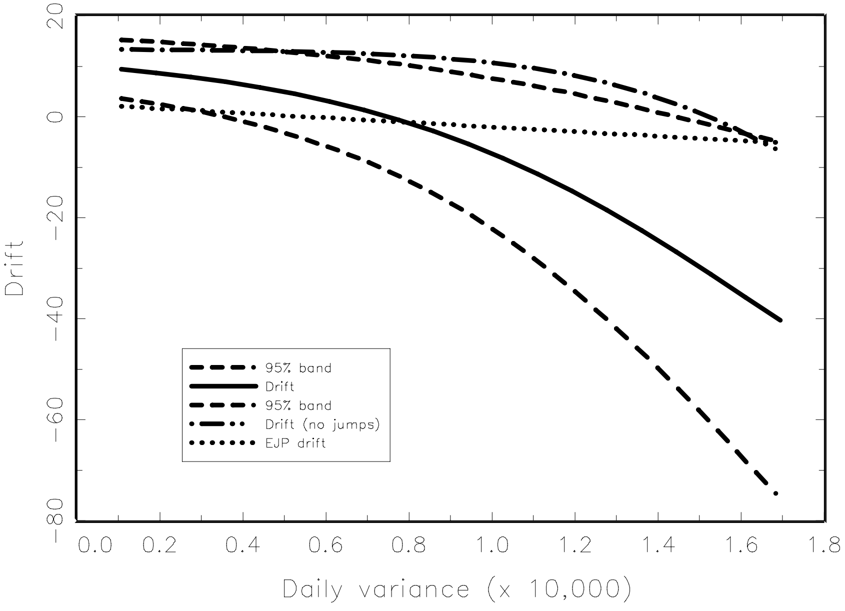


Figure 1(a)
Variance drift estimates

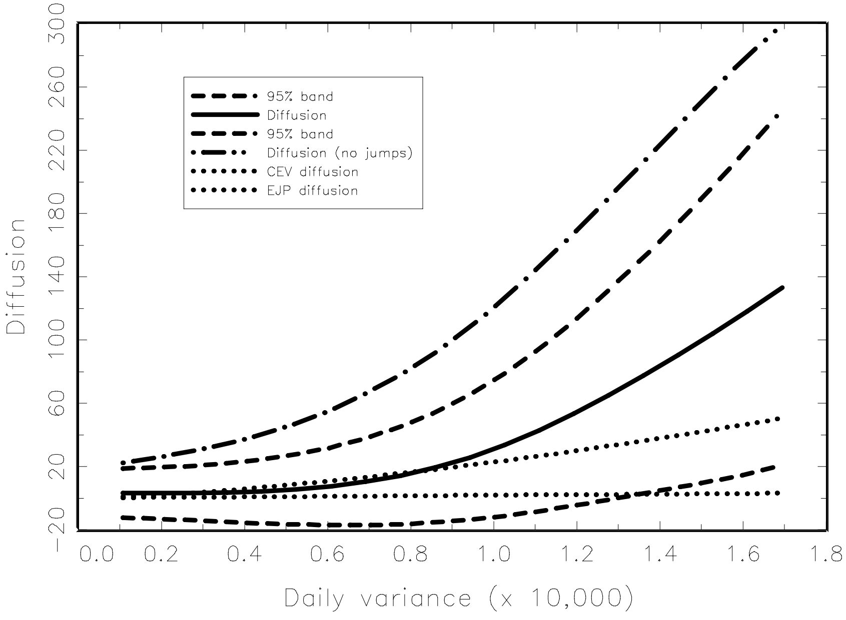


Figure 1(b)
Variance diffusion estimates

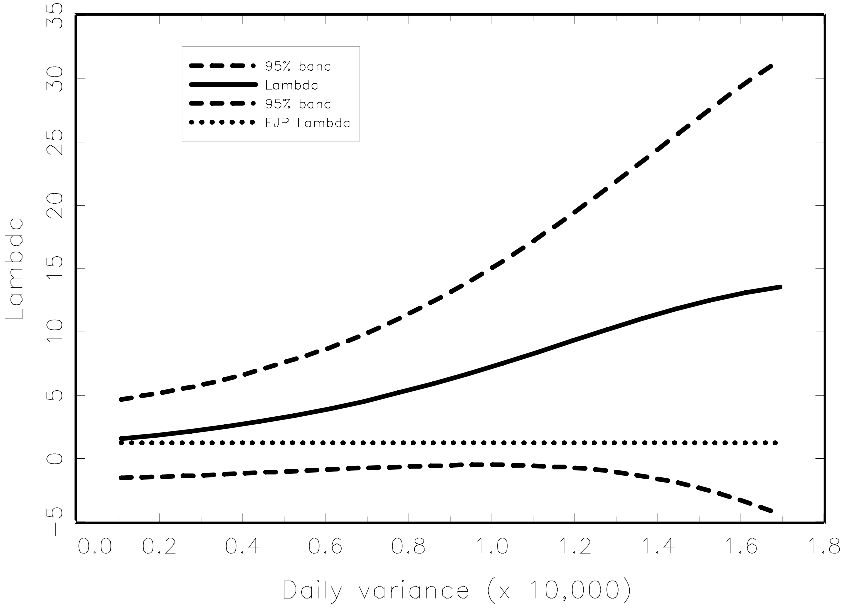


Figure 1(c)
Variance jump intensity

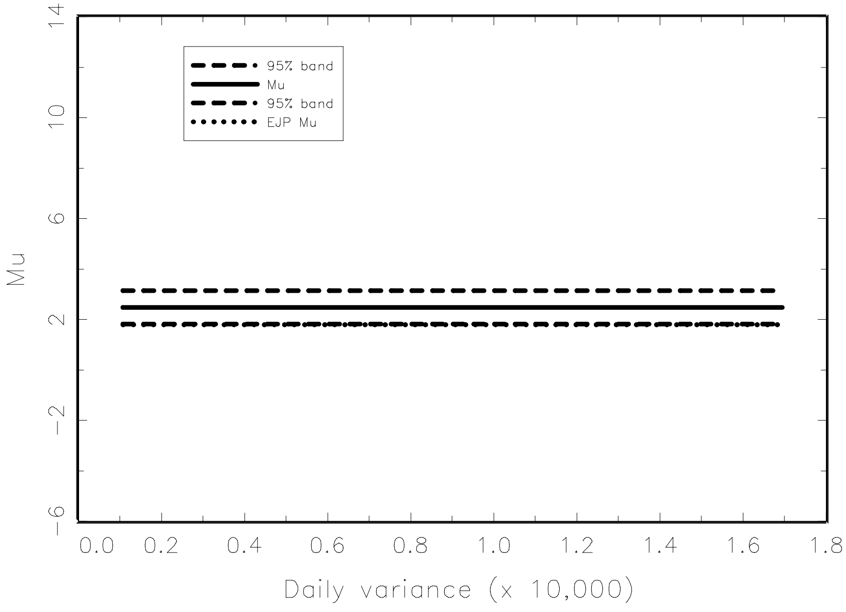


Figure 1(d)
Variance expected jump size

Figure 2(a)
Price drift estimates

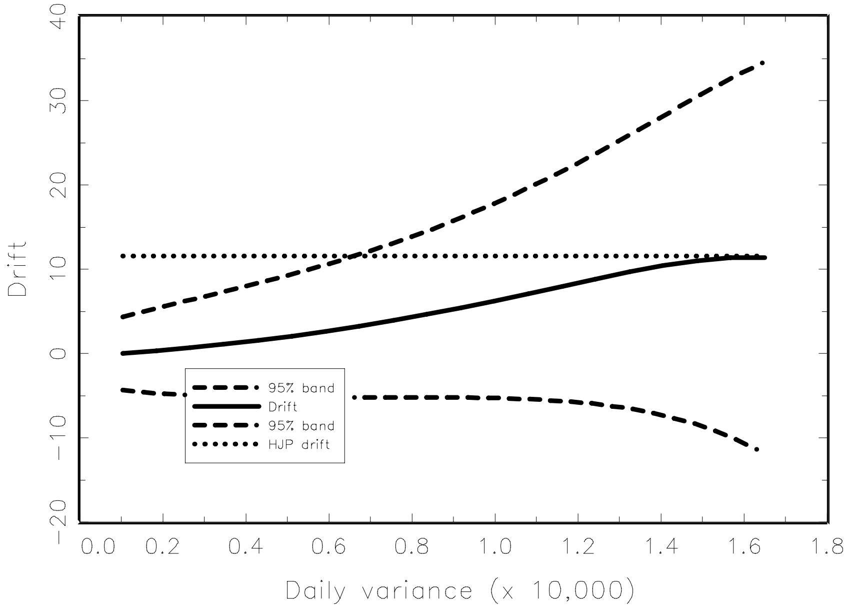


Figure 2(b)
Leverage estimates

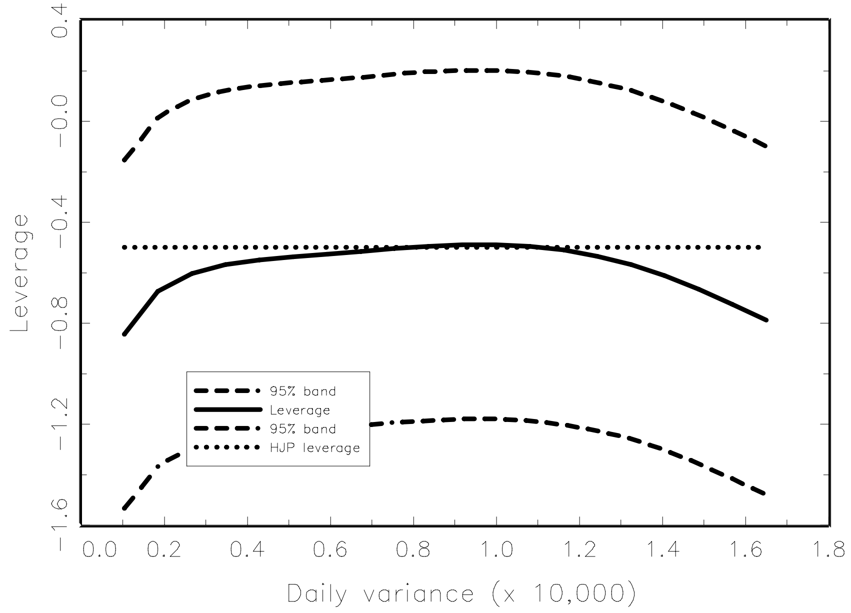


Figure 2(c)
Price jump intensity

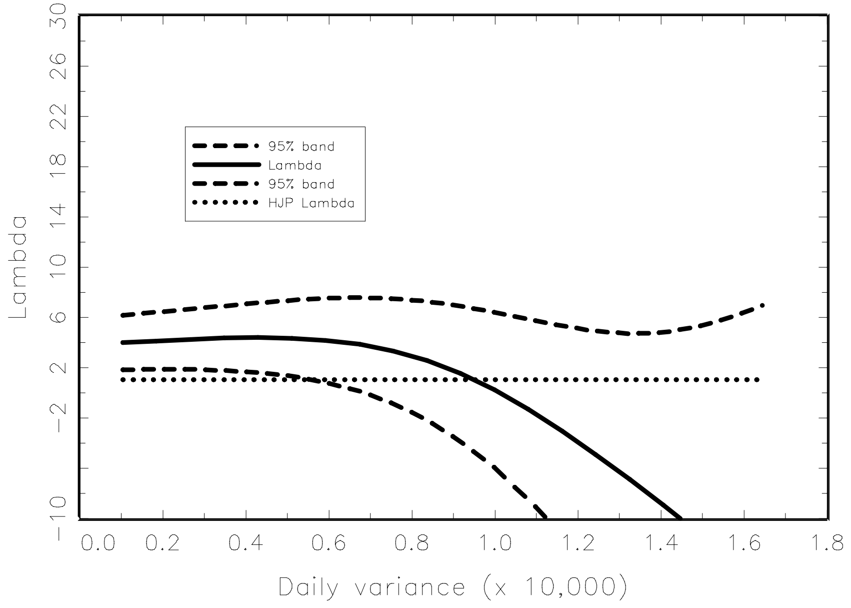
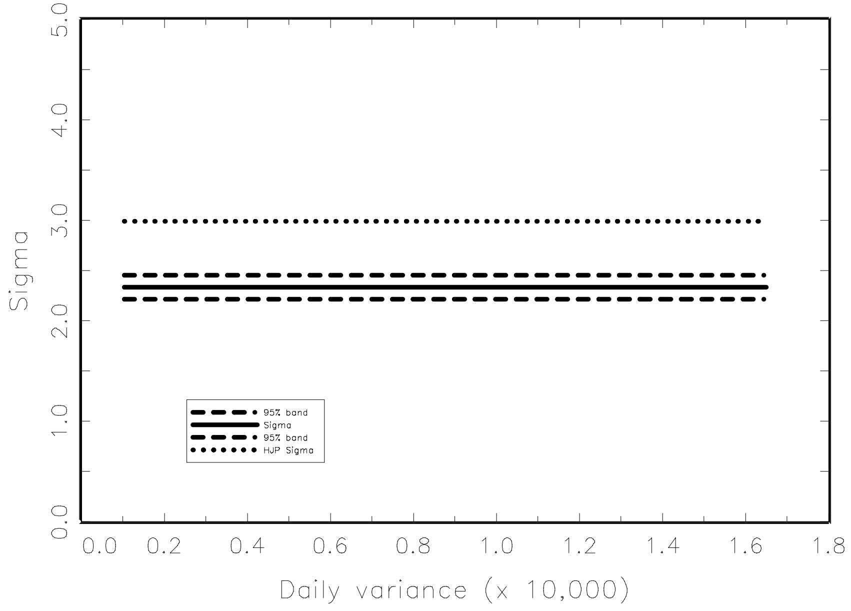


Figure 2(d)
Price jump standard deviation



Allowing for co-jumps

$$
\begin{aligned}
d\log(p_t) &= \mu(\sigma_t^2)dt + \sigma_t dW_t^r + \xi_1^r dN_t^r + \underbrace{\xi_2^r dN_t}_{co-jump}, \\
df(\sigma_t^2) &= m(\sigma_t^2)dt + \Lambda(\sigma_t^2)dW_t^\sigma + \xi_1^\sigma dN_t^\sigma + \underbrace{\xi_2^\sigma dN_t}_{co-jump}, \\
\{dW_t^r, dW_t^\sigma\} &= \{\rho(\sigma_t^2)dW_t^1 + \sqrt{1-\rho^2(\sigma_t^2)}dW_t^2, dW_t^1\}
\end{aligned}
$$

- The Poisson processes N_t^r, N_t^σ, and N_t are independent of each other and independent of the driving Brownian motions W_t^1 and W_t^2.
- The co-jump components have conditional intensity $\lambda_{r,\sigma^2}(\sigma^2)$.

Infinitesimal cross-moments

Define

$$\theta_{p_1,p_2}(\sigma^2) = \lim_{\Delta\to 0} \frac{\mathbf{E}\left[(\log(p_{t+\Delta}) - \log(p_t))^{p_1} \left(f(\sigma^2_{t+\Delta}) - f(\sigma^2_t)\right)^{p_2} |\sigma^2\right]}{\Delta}.$$

In terms of the underlying model components:

$$\theta_{0,p_2}(\sigma^2) = \begin{cases} m(\sigma^2) + \lambda_{\sigma^2}(\sigma^2)\mathbf{E}[\xi_1^\sigma] + \lambda_{r,\sigma^2}(\sigma^2)\mathbf{E}[\xi_2^\sigma] & p_2 = 1 \\ \Lambda^2(\sigma^2) + \lambda_{\sigma^2}(\sigma^2)\mathbf{E}[(\xi_1^\sigma)^2] + \lambda_{r,\sigma^2}(\sigma^2)\mathbf{E}[(\xi_2^\sigma)^2] & p_2 = 2 \\ \lambda_{\sigma^2}(\sigma^2)\mathbf{E}[(\xi_1^\sigma)^{p_2}] + \lambda_{r,\sigma^2}(\sigma^2)\mathbf{E}[(\xi_2^\sigma)^{p_2}] & p_2 \geq 3 \end{cases},$$

$$\theta_{p_1,p_2}(\sigma^2) = \begin{cases} \rho(\sigma^2)\Lambda(\sigma^2)\sigma + \lambda_{r,\sigma^2}(\sigma^2)\mathbf{E}[(\xi_2^r)(\xi_2^\sigma)] & p_1 = p_2 = 1 \\ \lambda_{r,\sigma^2}(\sigma^2)\mathbf{E}[(\xi_2^r)^{p_1}(\xi_2^\sigma)^{p_2}] & \begin{array}{l} p_1 \geq p_2 \geq 1 \\ (p_1 > p_2 \\ \mathit{if}\ p_2 = 1) \end{array} \end{cases}.$$

A notion of generalized leverage

Write

$$\frac{\theta_{1,1}(\sigma^2)}{\sqrt{\Lambda^2(\sigma^2)\sigma^2}} = \underbrace{\rho(\sigma^2)}_{\text{Brownian leverage}} + \underbrace{\frac{\lambda_{r,\sigma^2}(\sigma^2)\mathbf{E}[(\xi_2^r)\,(\xi_2^\sigma)]}{\sqrt{\Lambda^2(\sigma^2)\sigma^2}}}_{\text{Jump leverage}}.$$

- Allows for dependence on the state of the economy (as summarized by spot variance)
- Depends on the classical (conditional) correlation between the driving continuous shocks
- Depends also on the (conditional) correlation between the co-jump processes

Cross-moment estimates

Define

$$\widehat{\theta}_{p_1,p_2}(\sigma^2) = \frac{\sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\widehat{\sigma}^2_{iT/n}-\sigma^2}{h_{n,T}}\right)\left(\log(p_{(i+1)T/n})-\log(p_{iT/n})\right)^{p_1}\left(f(\widehat{\sigma}^2_{(i+1)T/n})-f(\widehat{\sigma}^2_{iT/n})\right)^{p_2}}{\Delta_{n,T}\sum_{i=1}^{n} \mathbf{K}\left(\frac{\widehat{\sigma}^2_{iT/n}-\sigma^2}{h_{n,T}}\right)}.$$

- $\widehat{\sigma}^2_{iT/n} = \frac{\widehat{V}_{iT/n}}{\phi_{n,T}}$ is, as earlier, a high-frequency spot variance estimate.
- We generalize previous approaches by choosing p_1 and p_2 appropriately. Earlier, $p_2 = 0$ (for identification of the price moments), $p_1 = 0$ (for identification of the variance moments), and $p_1 = p_2 = 1$ for identification of $\rho(\sigma^2)$.
- The extra leverage component requires alternative choices of p_1 and p_2 (i.e., $p_1 \geq p_2 \geq 1$).
- In the presence of co-jumps, $\widehat{\theta}_{p_1,p_2}(\sigma^2)$ converges to the true cross-moment at speed $\sqrt{h_{n,T}\widehat{\overline{L}}_{\sigma^2}(T,\sigma^2)}$ for all p_1 and p_2.

Generalized log-variance model

Write

$$\begin{aligned} d\log(p_t) &= \mu(\sigma_t^2)dt + \sigma_t dW_t^r + \xi^r\left(dN_t^r + dN_t\right), \\ d\log(\sigma_t^2) &= m(\sigma_t^2)dt + \Lambda(\sigma_t^2)dW_t^\sigma + \xi^\sigma\left(dN_t^\sigma + dN_t\right), \\ \{dW_t^r, dW_t^\sigma\} &= \{\rho(\sigma_t^2)dW_t^1 + \sqrt{1-\rho^2(\sigma_t^2)}dW_t^2, dW_t^1\}, \end{aligned}$$

with

$$\begin{pmatrix} \xi^r \\ \xi^\sigma \end{pmatrix} \sim \mathbf{N}(0, \Sigma_J) \text{ and } \Sigma_J = \begin{pmatrix} \sigma_{J,r}^2 & \bullet \\ \rho_J \sigma_{J,r} \sigma_{J,\sigma} & \sigma_{J,\sigma}^2 \end{pmatrix}.$$

Note - Σ_J can be a function of the variance process ($\Sigma_J(\sigma^2)$).

Identification

A possible scheme

Price moments: $$\left\{ \begin{array}{l} \theta_{1,0}(x) = \mu(x) \\ \theta_{2,0}(x) = x + (\lambda_r(x) + \lambda_{r,\sigma}(x))\sigma_{J,r}^2 \\ \theta_{4,0}(x) = 3(\lambda_r(x) + \lambda_{r,\sigma}(x))\sigma_{J,r}^4 \\ \theta_{6,0}(x) = 15(\lambda_r(x) + \lambda_{r,\sigma}(x))\sigma_{J,r}^6 \end{array} \right\},$$

They identify $\mu(x)$, $\sigma_{J,r}^2$, and $\{\lambda_r(x) + \lambda_{r,\sigma}(x)\}$.

Variance moments: $$\left\{ \begin{array}{l} \theta_{0,1}(x) = m(x) \\ \theta_{0,2}(x) = \Lambda^2(x) + (\lambda_\sigma(x) + \lambda_{r,\sigma}(x))\sigma_{J,\sigma}^2 \\ \theta_{0,4}(x) = 3(\lambda_\sigma(x) + \lambda_{r,\sigma}(x))\sigma_{J,\sigma}^4 \\ \theta_{0,6}(x) = 15(\lambda_\sigma(x) + \lambda_{r,\sigma}(x))\sigma_{J,\sigma}^6 \end{array} \right\},$$

They identify $m(x)$, $\Lambda(x)$, $\sigma_{J,\sigma}^2$, and $\{\lambda_\sigma(x) + \lambda_{r,\sigma}(x)\}$.

Cross-moments: $$\left\{ \begin{array}{l} \theta_{1,1}(x) = \rho(x)\sqrt{x}\Lambda(x) + \lambda_{r,\sigma^2}(x)\rho_J\sigma_{J,r}\sigma_{J,\sigma} \\ \theta_{2,2}(x) = \lambda_{r,\sigma^2}(x)\sigma_{J,r}^2\sigma_{J,\sigma}^2(1 + 2\rho_J^2) \\ \theta_{3,1}(x) = 3\lambda_{r,\sigma^2}(x)\rho_J\sigma_{J,r}^3\sigma_{J,\sigma} \end{array} \right\},$$

They identify $\rho(x)$, ρ_J, and $\lambda_{r,\sigma^2}(x)$.

The features of the co-jumps and generalized leverage

The co-jump intensity:

$$\widehat{\lambda}_{r,\sigma^2}(\sigma^2) = \frac{1}{2}\left(\Pi(\sigma^2) + \sqrt{\Pi(\sigma^2)^2 - 8\Omega(\sigma^2)^2}\right),$$

where

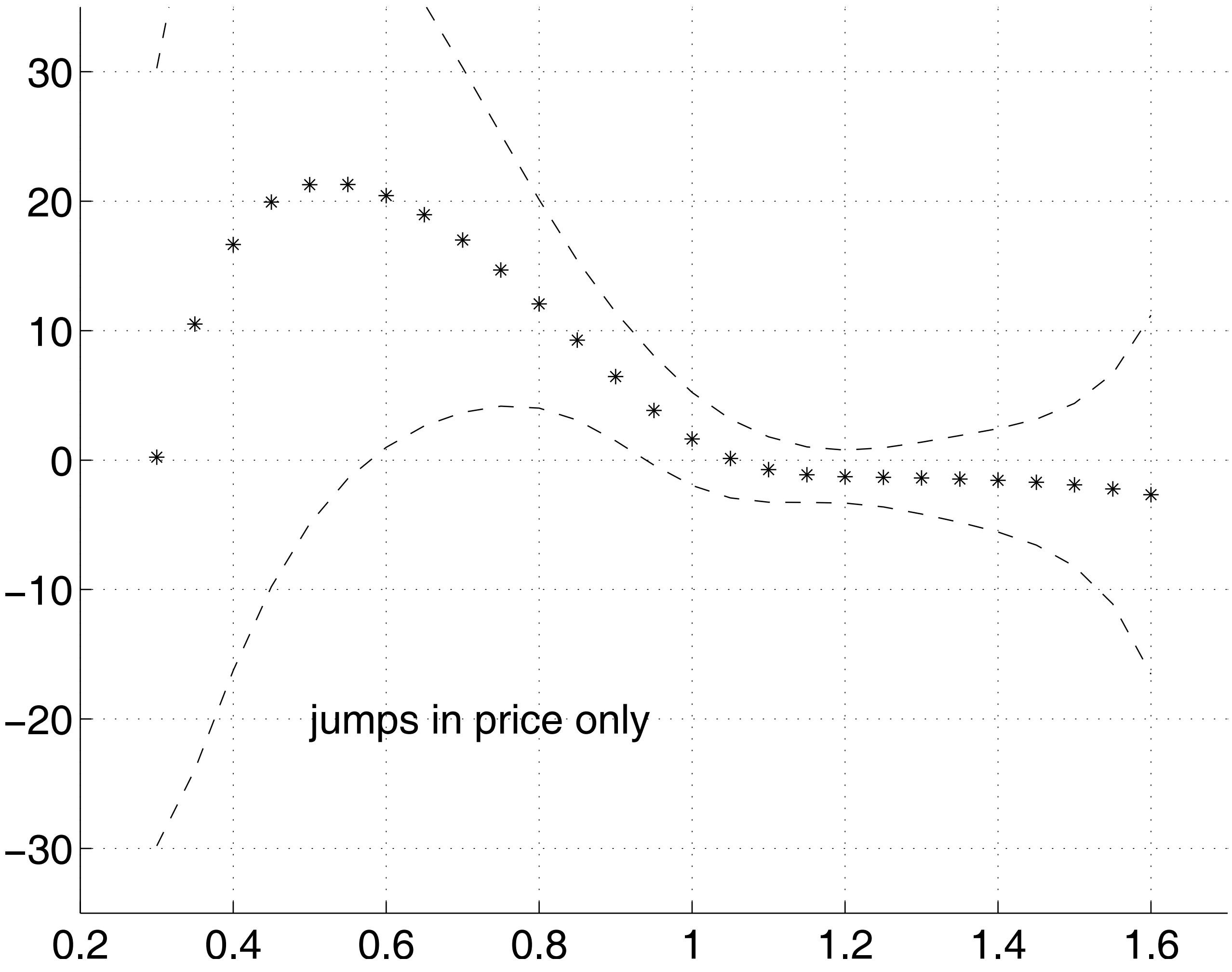
$$\Omega(\sigma^2) = \frac{1}{3}\frac{\widehat{\theta}_{3,1}(\sigma^2) + \widehat{\theta}_{1,3}(\sigma^2)}{\widehat{\sigma}_{J,r}\widehat{\sigma}_{J,\sigma}(\widehat{\sigma}^2_{J,r} + \widehat{\sigma}^2_{J,\sigma})} \text{ and } \Pi = \frac{\widehat{\theta}_{2,2}(\sigma^2)}{\widehat{\sigma}^2_{J,r}\widehat{\sigma}^2_{J,\sigma}}.$$

The co-jump correlation:

$$\widehat{\rho}_J(\sigma^2) = \frac{\Omega(\sigma^2)}{\widehat{\lambda}_{r,\sigma^2}(\sigma^2)}.$$

The Brownian correlation:

$$\underbrace{\widehat{\rho}(\sigma^2)}_{\text{Brownian leverage}} = \frac{\widehat{\theta}_{1,1}(\sigma^2)}{\sigma\widehat{\Lambda}(\sigma^2)} - \underbrace{\frac{\widehat{\lambda}_{r,\sigma^2}(\sigma^2)\widehat{\rho}_J\widehat{\sigma}_{J,r}\widehat{\sigma}_{J,\sigma}}{\sigma\widehat{\Lambda}(\sigma^2)}}_{\text{Jump leverage}}.$$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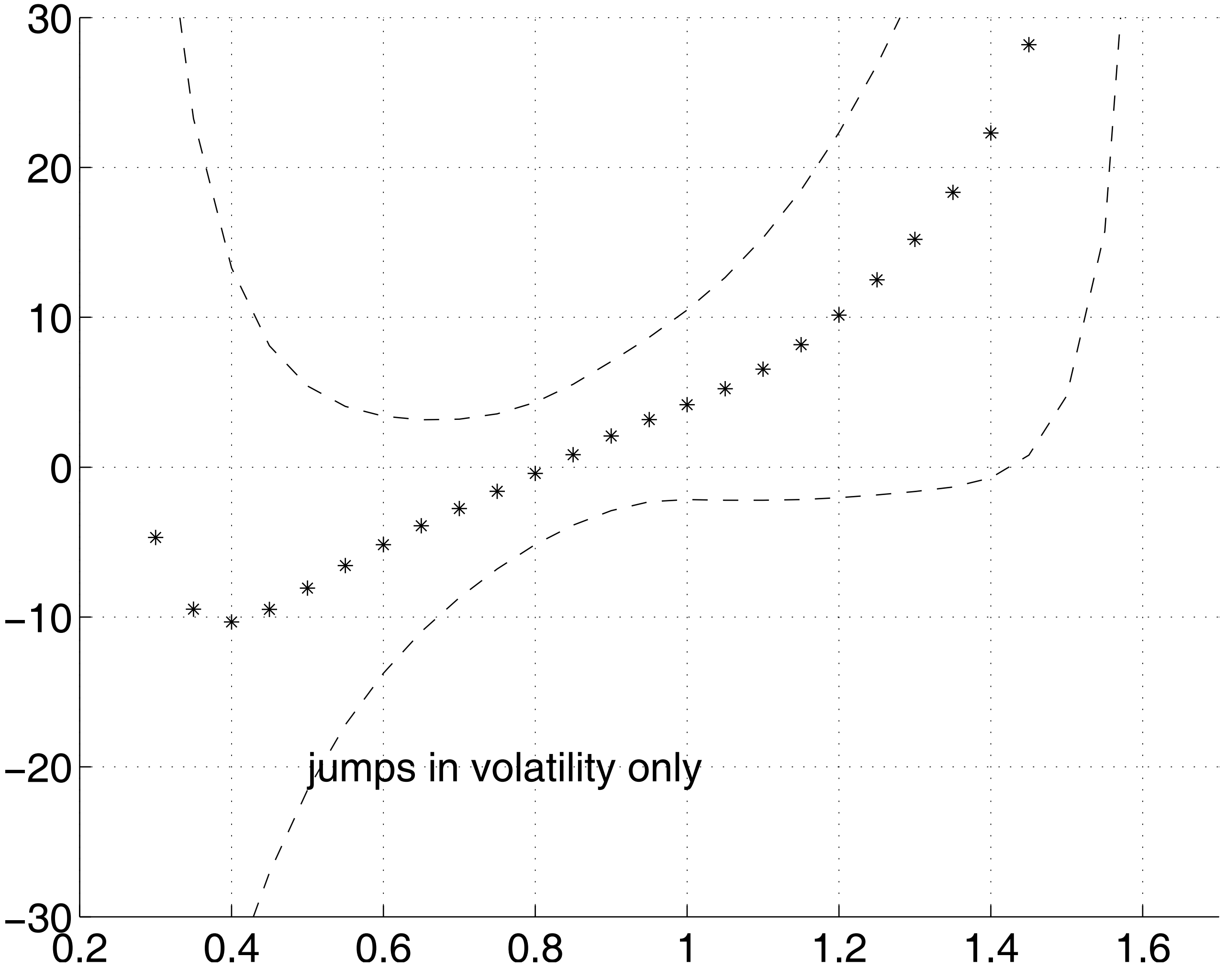
jumps in price only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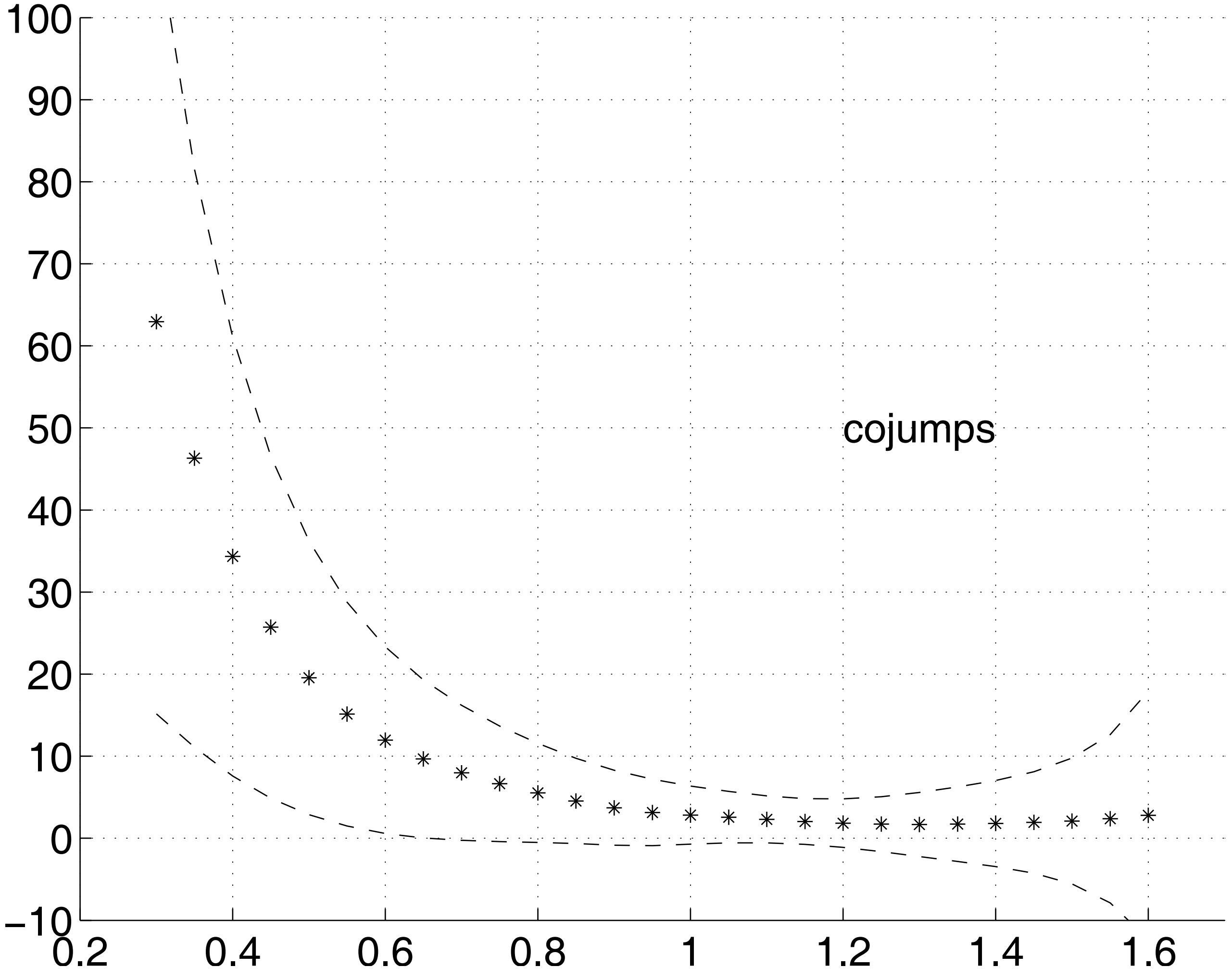
jumps in volatility only



cojumps
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90
80
70
60
50
40
30
20
10
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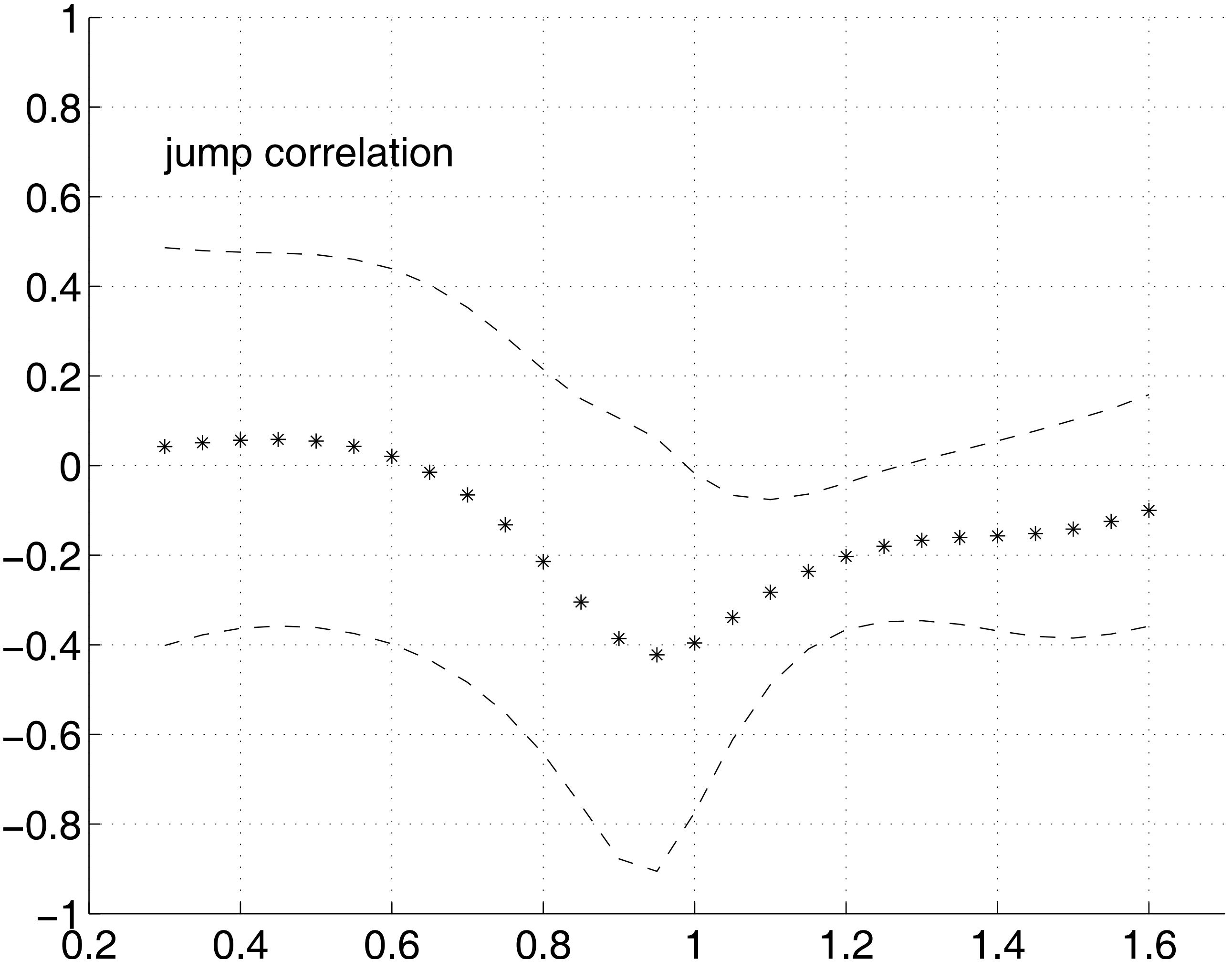
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6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1
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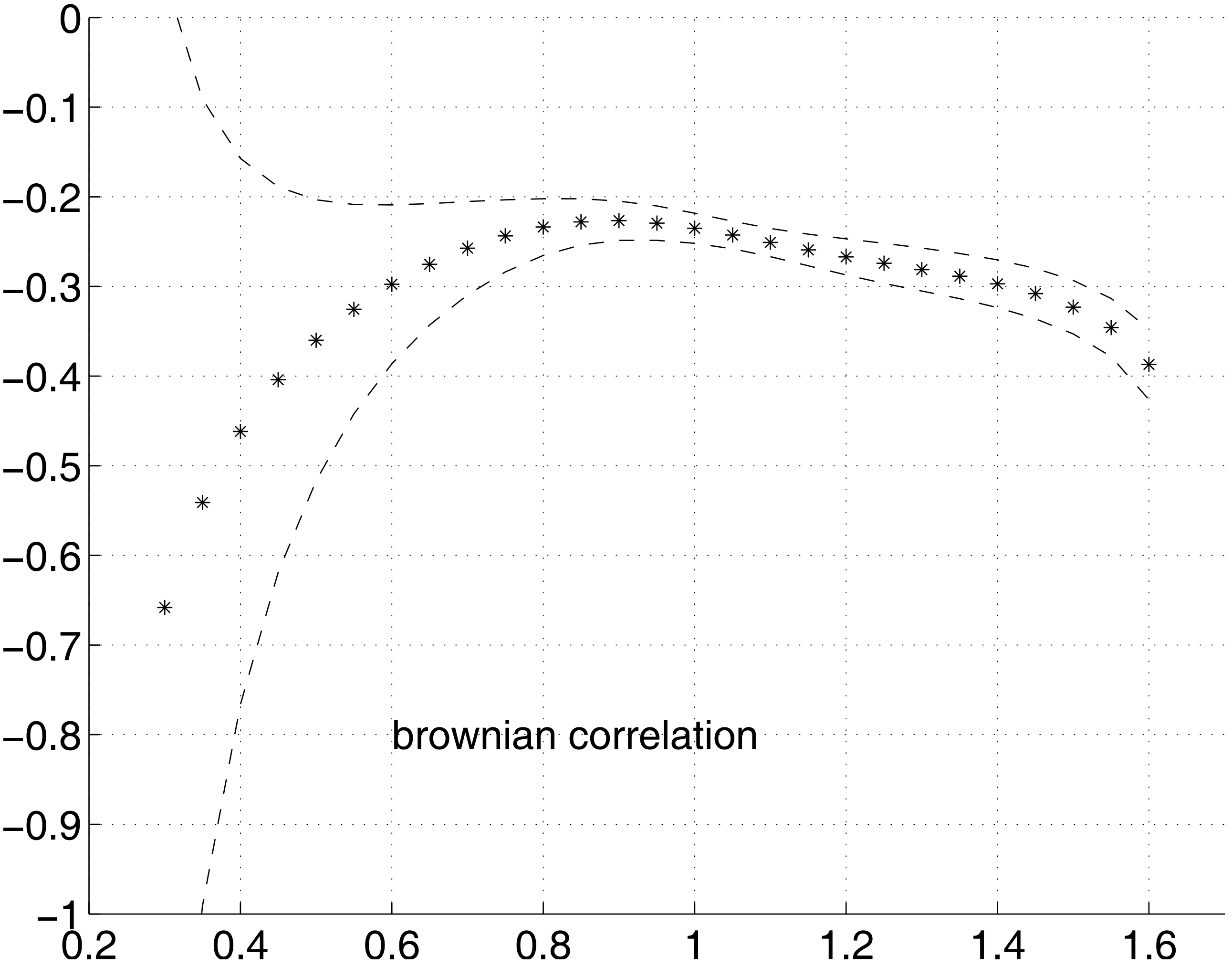
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brownian correlation

Conclusions

1. We presented novel nonparametric identification methods for the infinitesimal moment functionals of nonlinear stochastic volatility models with jumps in returns and in variance.
2. We emphasized that consistency and weak convergence of all relevant functions and parameters (including a generalized notion of leverage) may be achieved under mild statistical assumptions, i.e., recurrence.
3. As a necessary preliminary input, we discussed a theory of spot variance estimation using noise-contaminated high-frequency asset price data. This theory gives us a way to control (asymptotically) the estimation error induced by the preliminary variance estimates.