

Realized covariation, realized beta, and microstructure noise

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Abstract

The presence of market microstructure frictions renders realized covariance and beta estimates obtained by virtue of high-frequency asset price data inconsistent estimates of the underlying quadratic covariations and betas. We characterize the finite-sample properties of these estimates and provide a methodology to estimate quadratic covariations and betas optimally on the basis of a mean-squared error criterion.

JEL Classification: G12, C14, C22

1 Introduction

Multidimensional high-frequency return data can be used to identify the quadratic covariations and betas of the underlying stock prices. Specifically, identification obtains by averaging an “asymptotically large” number of high-frequency return data over the period (Andersen et al. (2004) and Barndorff-Nielsen and Shephard (2004) (BN-S, hereafter)). While the current theoretical approaches are motivated by the recent availability of an almost continuous record of quotes and transaction prices for a variety of financial assets, they largely abstract from the presence of market microstructure effects in the recorded prices (see Martens (2004) for a recent discussion of applied work on this issue). However, this recent literature is aware of the distortions that market microstructure contaminations induce. BN-S (2004), for example, recognize that “...market microstructure effects (e.g., discreteness of prices, bid/ask bounce, irregular trading, etc.) mean that there is a mismatch between asset pricing theory based on semimartingales and the data at very fine time intervals...” The interested reader is also referred to the discussions in the review paper by Andersen et al. (2002).

The “mismatch” between recorded prices and efficient prices induced by the presence of market microstructure effects renders conventional estimates of quadratic covariation and beta through sample moments of high-frequency return data inconsistent. This note provides a formal discussion of the implications of market microstructure effects on the finite sample properties of covariance and beta estimates obtained by virtue of high-frequency return data. We show that microstructure contaminations generate a bias/variance trade-off. High sampling frequencies induce accumulation of microstructure noise and, therefore, a substantial bias. Low sampling frequencies determine imprecise estimates. We characterize the resulting bias/variance trade-off and arrive at an optimal (in a MSE sense) sampling frequency for quadratic covariation and beta estimates. Finally, we discuss empirical implementation of our procedure.

This note is organized as follows. Section 2 describes the price formation mechanism. Section 3 studies optimal sampling for realized covariation. Section 4 studies optimal sampling for realized beta. Section 5 provides an application of our methods. Section 6 concludes. Proofs and technical details are in the Appendix.

2 The price formation mechanism

Consider a k -vector of logarithmic price processes $\tilde{\mathbf{p}}_{ih}$ and a fixed time period h (a trading day, for instance). Let the number of periods in the sample be equal to n . Write the observed price vector value at the end of the i -th period as

$$\tilde{\mathbf{p}}_{ih} = \mathbf{p}_{ih} + \boldsymbol{\eta}_{ih} \quad i = 1, 2, \dots, n, \quad (1)$$

where \mathbf{p}_{ih} is the k -vector of efficient prices, i.e., the equilibrium prices that would prevail in the absence of market microstructure effects, and $\boldsymbol{\eta}_{ih}$ denotes a k -vector of microstructure noise contaminations. Denote by $\tilde{p}_{(l)ih}$ the l th component of the observed price vector. The l th components of the efficient price vector and microstructure noise vector are defined analogously, namely $p_{(l)ih}$ and $\eta_{(l)ih}$. We can now divide each period h into M sub-periods and define the observed high-frequency continuously-compounded return vector as

$$\tilde{\mathbf{r}}_{j,i} = \tilde{\mathbf{p}}_{(i-1)h+j\delta} - \tilde{\mathbf{p}}_{(i-1)h+(j-1)\delta} \quad j = 1, 2, \dots, M, \quad (2)$$

where $\delta = h/M$ is the time distance between adjacent logarithmic prices or, equivalently, the time horizon over which the high-frequency continuously-compounded returns are computed. Hence, $\tilde{\mathbf{r}}_{j,i}$ is the j -th intra-period return over the i -th period. More precisely,

$$\tilde{\mathbf{r}}_{j,i} = \mathbf{r}_{j,i} + \boldsymbol{\varepsilon}_{j,i}, \quad (3)$$

where $\mathbf{r}_{j,i}$ and $\boldsymbol{\varepsilon}_{j,i} = \boldsymbol{\eta}_{(i-1)h+j\delta} - \boldsymbol{\eta}_{(i-1)h+(j-1)\delta}$ have natural interpretations in terms of efficient return vector and microstructure contamination in the return data. Both the efficient return $\mathbf{r}_{j,i}$ and the microstructure noise contamination $\boldsymbol{\varepsilon}_{j,i}$ are unobserved. The econometrician only observes $\tilde{\mathbf{r}}_{j,i}$.

Assumption 1 (The price process.)

- (1) *The logarithmic price process k -vector \mathbf{p}_t is a continuous stochastic volatility vector local martingale. Specifically,*

$$\mathbf{p}_t = \mathbf{m}_t, \quad (4)$$

where $\mathbf{m}_t = \int_0^t \Theta(s) d\mathbf{W}_s$, with $\{\mathbf{W}_t : t \geq 0\}$ denoting a standard vector Brownian motion with dimension q .

- (2) *The instantaneous volatility process $\Theta(s)$ has elements that are all càdlàg. Define the l th row of the matrix $\Theta(s)$ as $(\sigma_{(l)1}(s), \sigma_{(l)2}(s), \dots, \sigma_{(l)q}(s))$.*

- (3) *Define the instantaneous covariance*

$$\Sigma(s) = \Theta(s)\Theta^\perp(s) \quad (5)$$

with generic element $\Sigma_{(u)(l)}(s)$ with $1 \leq l, u \leq k$ and assume that

$$\int_0^t \Sigma_{(l)(u)}(s) ds < \infty \quad (6)$$

for all $t < \infty$.

The quadratic covariation process between the l th price process and the u th price process between 0 and h can be expressed as

$$C_{(u)(l)} = \int_0^h \Sigma_{(u)(l)}(s) ds = \int_0^h \sum_{b=1}^q \sigma_{(u)b}(s) \sigma_{(l)b}(s) ds. \quad (7)$$

Thus,

$$V_{(u)} = \int_0^h \Sigma_{(u)}(s) ds = \int_0^h \sum_{b=1}^q \sigma_{(u)b}^2(s) ds \quad (8)$$

is the quadratic variation of $p_{(u)}$ between 0 and h . Clearly, $C_{(u)(u)} = V_{(u)}$.

Assumption 2. (The Microstructure Noise.)

- (1) *The microstructure noise vectors in the price process, namely the $\boldsymbol{\eta}'_j s$, have mean zero and are covariance-stationary with joint density $f_M(\cdot)$ depending on the sampling frequency M .*
- (2) *The variance of $\boldsymbol{\varepsilon}_j = \boldsymbol{\eta}_j - \boldsymbol{\eta}_{j-1}$ is $O(1)$ for all M .*
- (3) *The frictions $\eta'_{(l)j} s$ are independent of the price processes $p_{(u)j}$ for all M and all $1 \leq u, l \leq k$.*

As required by classical microstructure theory (O'Hara (1995)), the efficient return vector process evolves continuously in time as a stochastic volatility vector local martingale (Assumption 1(1)). The instantaneous volatility can display jumps, diurnal effects, high-persistence (possibly of the long-memory type), and nonstationarities (Assumption 1(2)). The microstructure noise components in the recorded prices are potentially dependent (Assumption 2(1)). Their dependence permits to capture first-order negative autocorrelations as induced by bid-ask bounce effects as well as higher order autocorrelations as determined by clustering in order flows, for instance. Importantly, while the variances of the equilibrium returns vanish with δ going to zero, the variances of the noise returns do not vanish (Assumption 2(2)). Provided sampling does not occur between price updates, price discreteness alone provides a compelling reason for this property of our assumed model. Bandi and Russell (2004) provide further discussions of these assumptions.

Andersen et al. (2004) and BN-S (2004) suggest to estimate the quadratic covariation matrix for the i th period in the sample, i.e., $\int_{(i-1)h}^{ih} \Sigma(s) ds$, by taking the outer-product of the observed high-frequency returns over the period, namely

$$\widehat{C}_i = \sum_{j=1}^M \widetilde{\mathbf{r}}_{j,i} \widetilde{\mathbf{r}}_{j,i}^\perp. \quad (9)$$

However, increasing the sampling frequency asymptotically over h by letting $M \rightarrow \infty$ does not deliver a consistent estimate of the object of interest when microstructure noise plays a role, i.e.,

$$\sum_{j=1}^M \widetilde{\mathbf{r}}_{j,i} \widetilde{\mathbf{r}}_{j,i}^\perp \xrightarrow[M \rightarrow \infty]{p} \int_{(i-1)h}^{ih} \Sigma(s) ds. \quad (10)$$

The presence of noise determines a bias/variance trade-off. High sampling frequencies induce accumulation of microstructure noise and, as a consequence, a substantial bias. Low sampling frequencies deliver imprecise estimates. The bias/variance trade-off can be characterized in the form of a conditional, on the underlying volatility paths, MSE for all frequencies $\delta = h/M$. The MSE can be minimized to determine the optimal sampling frequency or, equivalently, the optimal number of equispaced observations M^* to be used to define the estimator in Eq. (9) over any fixed time span h .

3 The conditional MSE of the realized covariation

Theorem 1 contains the conditional MSE of the realized covariation process between any pair of logarithmic prices $\tilde{p}_{(u)}$ and $\tilde{p}_{(l)}$ with $1 \leq l, u \leq k$. In what follows the symbol \mathbf{E}_M denotes expectation conditional on the volatility paths of the corresponding price processes. The subscript M makes the dependence of the expectation on the sampling frequency explicit. Without loss of generality, we set $i = 1$ and focus on the first period in the sample.

Theorem 1. (The MSE of the covariation process.) *Assume Assumption 1 and Assumption 2 are satisfied, then*

$$\begin{aligned} & \mathbf{E}_M \left(\sum_{j=1}^M \tilde{r}_{(u)j} \tilde{r}_{(l)j} - C_{(u)(l)} \right)^2 \\ = & \mathbf{E}_M(\varepsilon_{(l)}^2) \int_0^h \Sigma_{(u)}(s) ds + 2\mathbf{E}_M(\varepsilon_{(l)}\varepsilon_{(u)}) \int_0^h \Sigma_{(u)(l)}(s) ds + \mathbf{E}_M(\varepsilon_{(u)}^2) \int_0^h \Sigma_{(l)}(s) ds \\ & + M\mathbf{E}_M(\varepsilon_{(u)}^2\varepsilon_{(l)}^2) + 2 \sum_{j=1}^M (M-j) \mathbf{E}_M(\varepsilon_{(u)}\varepsilon_{(l)}\varepsilon_{(u)-j}\varepsilon_{(l)-j}) + \frac{h}{M}Q_{(u)(l)} + o(1), \end{aligned} \quad (11)$$

where

$$Q_{(u)(l)} = \int_0^h \Sigma_{(u)}(s)\Sigma_{(l)}(s) + \Sigma_{(u)(l)}^2(s) ds. \quad (12)$$

Proof. *See Appendix.*

At each frequency $\delta = h/M$ the conditional MSE can be written as a function of the frequency-dependent moments of the noise components and the moments of the underlying efficient price processes. Estimates of both moments for the purpose of MSE evaluation can be obtained by adapting the estimation method described by Bandi and Russell (2004), Section 5.1, in the context of quadratic variation estimation. The optimal (in an MSE sense) sampling frequency can then be characterized as

$$M^* = \arg \min_M \mathbf{E}_M \left(\sum_{j=1}^M \tilde{r}_{(u)j} \tilde{r}_{(l)j} - C_{(u)(l)} \right)^2. \quad (13)$$

If

$$\eta_{(u)} \perp\!\!\!\perp \eta_{(u)-j}, \eta_{(u)} \perp\!\!\!\perp \eta_{(l)-j} \quad \forall l, u, j \neq 0, \quad (14)$$

then the MSE simplifies. We call the assumption in Eq. (14) *MA(1) assumption*. The *MA(1)* assumption can be a good approximation in decentralized markets where traders arrive in a random fashion with idiosyncratic price setting behavior. For example, it can be a good approximation in the case of exchange rates as well as in the case of equities when considering transaction prices or even quotes posted on multiple exchanges. The interested reader is referred to Bandi and Russell (2005a,b) for discussions.

Corollary to Theorem 1. (The MSE of the covariation process in the MA(1) case.)
Assume Assumption 1, Assumption 2, and the condition in Eq. (14) are satisfied, then

$$\begin{aligned}
& \mathbf{E}_M \left(\sum_{j=1}^M \tilde{r}_{(u)j} \tilde{r}_{(l)j} - C_{(u)(l)} \right)^2 \\
= & M \mathbf{E} \left(\varepsilon_{(u)}^2 \varepsilon_{(l)}^2 \right) + 2(M-1) \mathbf{E}(\varepsilon_{(u)} \varepsilon_{(l)} \varepsilon_{(u)-1} \varepsilon_{(l)-1}) + (M^2 - 3M + 2) \mathbf{E}^2(\varepsilon_{(l)} \varepsilon_{(u)}) \\
& + \mathbf{E}(\varepsilon_{(l)}^2) \int_0^h \Sigma_{(u)}(s) ds + 2 \mathbf{E}(\varepsilon_{(l)} \varepsilon_{(u)}) \int_0^h \Sigma_{(u)(l)}(s) ds + \mathbf{E}(\varepsilon_{(u)}^2) \int_0^h \Sigma_{(l)}(s) ds \\
& + \frac{h}{M} Q_{(u)(l)} + o(1). \tag{15}
\end{aligned}$$

(ii) If M^* is large, then

$$M^* \approx \left(\frac{Q_{(u)(l)}}{2 \mathbf{E}^2(\varepsilon_{(l)} \varepsilon_{(u)})} \right)^{1/3}. \tag{16}$$

In the MA(1) case, the moments of the noise and, as a consequence, the MSE do not need to be computed on a grid of plausible frequencies M . Instead, consistent estimates of the relevant noise moments can be obtained by sampling at the highest available frequency. Theorem 2 contains the relevant result.

Theorem 2. Assume Assumption 1, Assumption 2, and the condition in Eq. (14) are satisfied. If

$$\mathbf{E} \left(\eta_{(u)}^{2(a+c)} \eta_{(l)}^{2(b+d)} \right) < \infty, \tag{17}$$

then

$$\frac{1}{M-1} \sum_{j=2}^M \tilde{r}_{(u)j}^a \tilde{r}_{(l)j}^b \tilde{r}_{(u)j-1}^c \tilde{r}_{(l)j-1}^d \xrightarrow[M \rightarrow \infty]{p} \mathbf{E} \left(\varepsilon_{(u)}^a \varepsilon_{(l)}^b \varepsilon_{(u)-1}^c \varepsilon_{(l)-1}^d \right) \tag{18}$$

for $a = 0, 1, 2, b = 0, 1, 2, c = 0, 1, d = 0, 1$.

Proof. The result follows from a simple modification of the proof of Theorem 2 in Bandi and Russell (2004).

As opposed to the noise return moments, preliminary estimates of the efficient price moments for the purpose of MSE evaluation can be obtained by estimating $C_{(u)(l)}$ and $Q_{(u)(l)}$ (for $1 \leq u, l \leq k$) by virtue of sample moments constructed using continuously-compounded returns sampled at low frequencies. It is standard to use 15 or 20-minute frequencies. Thus,

$$\widehat{C}_{(u)(l)} = \sum_{j=1}^{\overline{M}} \tilde{r}_{(u)j} \tilde{r}_{(l)j} \quad 1 \leq u, l \leq k \tag{19}$$

and

$$\widehat{Q}_{(u)(l)} = \frac{\overline{M}}{h} \sum_{j=1}^{\overline{M}} \tilde{r}_{(u)j}^2 \tilde{r}_{(l)j}^2 - \frac{\overline{M}}{h} \sum_{j=1}^{\overline{M}-1} \tilde{r}_{(u)j} \tilde{r}_{(l)j} \tilde{r}_{(u)j+1} \tilde{r}_{(l)j+1} \quad 1 \leq u, l \leq k \tag{20}$$

with $\bar{\delta} = \frac{h}{M} = 15$ or 20 minutes. The first estimator is obvious. The second estimator has been recently suggested by BN-S (2004).

4 The conditional MSE of the realized beta

Define the realized beta of $\tilde{p}_{(l)j}$ and $\tilde{p}_{(u)j}$ as

$$\hat{\beta}_{(l)(u)} = \frac{\sum_{j=1}^M \tilde{r}_{(u)j} \tilde{r}_{(l)j}}{\sum_{j=1}^M \tilde{r}_{(u)j}^2} \quad (21)$$

In the presence of market microstructure noise, the realized beta is an inconsistent estimate of the true beta over the period, namely $\beta_{(l)(u)} = \frac{\int_0^h \Sigma_{(u)(l)}(s) ds}{\int_0^h \Sigma_{(u)}(s) ds}$. Theorem 2 presents the conditional MSE of the realized beta.

Theorem 3. (The MSE of the realized beta.) *Assume Assumption 1 and Assumption 2 are satisfied, then*

$$\begin{aligned} & \mathbf{E}_M \left(\frac{\sum_{j=1}^M \tilde{r}_{(u)j} \tilde{r}_{(l)j}}{\sum_{j=1}^M \tilde{r}_{(u)j}^2} - \frac{\int_0^h \Sigma_{(u)(l)}(s) ds}{\int_0^h \Sigma_{(u)}(s) ds} \right)^2 \\ &= \left(\frac{1}{\int_0^h \Sigma_{(u)}(s) ds} \right)^2 \underbrace{\mathbf{E}_M \left(\sum_{j=1}^M \tilde{r}_{(u)j} \tilde{r}_{(l)j} - \int_0^h \Sigma_{(u)(l)}(s) ds \right)^2}_{\mathbf{MSE}(C_{(u)(l)})} \\ & \quad + \frac{\left(\int_0^h \Sigma_{(u)(l)}(s) ds \right)^2}{\left(\int_0^h \Sigma_{(u)}(s) ds \right)^4} \underbrace{\mathbf{E}_M \left(\sum_{j=1}^M \tilde{r}_{(u)j}^2 - \int_0^h \Sigma_{(u)}(s) ds \right)^2}_{\mathbf{MSE}(V_{(u)})} \\ & \quad - 2 \frac{\int_0^h \Sigma_{(u)(l)}(s) ds}{\left(\int_0^h \Sigma_{(u)}(s) ds \right)^3} \underbrace{\mathbf{E}_M \left(\left(\sum_{j=1}^M \tilde{r}_{(u)j}^2 - \int_0^h \Sigma_{(u)}(s) ds \right) \left(\sum_{j=1}^M \tilde{r}_{(u)j} \tilde{r}_{(l)j} - \int_0^h \Sigma_{(u)(l)}(s) ds \right) \right)}_{\mathbf{MSE}(C_{(u)(l)}, V_{(u)})} \\ & \quad + o_p(1), \end{aligned} \quad (22)$$

where $\mathbf{MSE}(C_{(u)(l)})$ is given by the result in Theorem 1, $\mathbf{MSE}(V_{(u)}) = \mathbf{MSE}(C_{(u)(u)})$, and

$$\begin{aligned} & \mathbf{MSE}(C_{(u)(l)}, V_{(u)}) \\ &= M \mathbf{E}_M \left(\varepsilon_{(u)}^3 \varepsilon_{(l)} \right) + \sum_{g=1}^{h/\delta} \left(\frac{h}{\delta} - j \right) \mathbf{E}_M \left(\varepsilon_{(u)}^2 \varepsilon_{(u)-j} \varepsilon_{(l)-j} \right) + \sum_{g=1}^{h/\delta} \left(\frac{h}{\delta} - j \right) \mathbf{E}_M \left(\varepsilon_{(u)} \varepsilon_{(l)} \varepsilon_{(u)-j}^2 \right) \\ & \quad + 2 \mathbf{E}_M \left(\varepsilon_{(u)} \varepsilon_{(l)} \right) \int_0^h \Sigma_{(u)}(s) ds + 2 \mathbf{E}_M \left(\varepsilon_{(u)}^2 \right) \int_0^h \Sigma_{(u)(l)}(s) ds \\ & \quad + 2 \frac{h}{M} \int_0^h \Sigma_{(u)}(s) \Sigma_{(u)(l)}(s) ds + o(1). \end{aligned} \quad (23)$$

Proof. See Appendix.

As in the case of the MSE in Eq. (11) above, empirical evaluation of the MSE in Eq. (22) requires estimation of the moments of the noise on a grid of frequencies as in Bandi and Russell (2004). The problem simplifies when the condition in Eq. (14) is satisfied.

Corollary to Theorem 3. (The MSE of the realized beta in the MA(1) case.) Assume Assumption 1, Assumption 2, and the condition in Eq. (14) are satisfied, then

$$\begin{aligned}
& \mathbf{E}_M \left(\frac{\sum_{j=1}^M \tilde{r}_{(u)j} \tilde{r}_{(l)j}}{\sum_{j=1}^M \tilde{r}_{(u)j}^2} - \frac{\int_0^h \Sigma_{(u)(l)}(s) ds}{\int_0^h \Sigma_{(u)}(s) ds} \right)^2 \\
&= \left(\frac{1}{\int_0^h \Sigma_{(u)}(s) ds} \right)^2 \underbrace{\mathbf{E}_M \left(\sum_{j=1}^M \tilde{r}_{(u)j} \tilde{r}_{(l)j} - \int_0^h \Sigma_{(u)(l)}(s) ds \right)^2}_{\mathbf{MSE}(C_{(u)(l)})} \\
&+ \frac{\left(\int_0^h \Sigma_{(u)(l)}(s) ds \right)^2}{\left(\int_0^h \Sigma_{(u)}(s) ds \right)^4} \underbrace{\mathbf{E}_M \left(\sum_{j=1}^M \tilde{r}_{(u)j}^2 - \int_0^h \Sigma_{(u)}(s) ds \right)^2}_{\mathbf{MSE}(V_{(u)})} \\
&- 2 \frac{\int_0^h \Sigma_{(u)(l)}(s) ds}{\left(\int_0^h \Sigma_{(u)}(s) ds \right)^3} \underbrace{\mathbf{E}_M \left(\left(\sum_{j=1}^M \tilde{r}_{(u)j}^2 - \int_0^h \Sigma_{(u)}(s) ds \right) \left(\sum_{j=1}^M \tilde{r}_{(u)j} \tilde{r}_{(l)j} - \int_0^h \Sigma_{(u)(l)}(s) ds \right) \right)}_{\mathbf{MSE}(C_{(u)(l)}, V_{(u)})} \\
&+ o_p(1), \tag{24}
\end{aligned}$$

where $\mathbf{MSE}(C_{(u)(l)})$ is given by the result in the Corollary to Theorem 1, $\mathbf{MSE}(V_{(u)}) = \mathbf{MSE}(C_{(u)(u)})$ and

$$\begin{aligned}
& \mathbf{MSE}(C_{(u)(l)}, V_{(u)}) \\
&= M \mathbf{E} \left(\varepsilon_{(u)}^3 \varepsilon_{(l)} \right) + (M-1) \left(\mathbf{E} \left(\varepsilon_{(u)}^2 \varepsilon_{(u)-1} \varepsilon_{(l)-1} \right) + \mathbf{E}_M \left(\varepsilon_{(u)} \varepsilon_{(l)} \varepsilon_{(u)-1}^2 \right) \right) \\
&+ (M^2 - 3M + 2) \left(\mathbf{E}(\varepsilon_{(u)}^2) \mathbf{E}(\varepsilon_{(u)} \varepsilon_{(l)}) \right) \\
&+ 2 \mathbf{E}(\varepsilon_{(u)} \varepsilon_{(l)}) \int_0^h \Sigma_{(u)}(s) ds + 2 \mathbf{E}_M(\varepsilon_{(u)}^2) \int_0^h \Sigma_{(u)(l)}(s) ds \\
&+ 2 \frac{h}{M} \int_0^h \Sigma_{(u)}(s) \Sigma_{(u)(l)}(s) ds + o(1). \tag{25}
\end{aligned}$$

In the MA(1) case consistent estimation of the constant (across frequencies) moments of the noise can be conducted as in Section 3, Theorem 2. As for the moments of the efficient price process, their evaluation can be conducted as in Eq. (19) and Eq. (20) above.

5 The case of XOM and SBC

5.1 The data

This section applies our methods to SBC communication (SBC) and EXXON Corporation (XOM). The data come from the TAQ data set and cover the period from January 1993 to June 2004. We

use mid-values of quotes posted on two exchanges, the NYSE and the MIDWEST, between 9:55 am to 4 pm. We remove quotes whose associated mid-value changes and/or recorded spreads are either larger than 10% or equal to zero. We also remove trading days which do not have quote updates after 3:15 pm, possibly due to holidays, and/or have fewer than 50 quote updates after applying the previous filter.

5.2 Optimal sampling intervals and covariation forecasts

We compute optimal daily frequencies by using the approximate method in the Corollary to Theorem 1, Point (ii). Both $\mathbf{E}(\varepsilon_{(l)}\varepsilon_{(u)})$ and $Q_{(u)(l)}$ are calculated using equally-spaced continuously-compounded returns constructed on the basis of the previous tick method. In light of Theorem 2, the former is computed by sampling at high-frequencies. Specifically, we sample at a frequency equal to the maximum duration of SBC and XOM. The latter is calculated by using the estimator in Eq. (20) and 20-minute returns. Figure 1 reports the time series of daily (approximate) optimal sampling intervals for the covariation between SBC and XOM. We notice substantial time variation as well as decreases in the optimal frequencies over time.

The covariance estimates are computed using the approximate optimal frequencies whenever these frequencies are lower than twice the maximum duration of SBC and XOM. If this is not the case, we use the lower bound to sample returns. To account for non-synchronous trading, we compute the covariance estimates by using leads and lags. Given an optimal daily M^* we calculate

$$\tilde{C}_{(u)(l)} = \sum_{j=1}^{M^*} \tilde{r}_{(u)j} \sum_{s=-K}^K \tilde{r}_{(l)j-s}. \quad (26)$$

The theoretical justification behind the adjustment in Eq. (26) is well-known (see, for example, Cohen et al. (1983)). Assume \tilde{r} is a local martingale efficient return. Then, $\tilde{C}_{(u)(l)}$ is virtually unbiased for the true covariation over the period provided K is large enough. If K is small, then lack of price updates for either stock is bound to induce biases. In this application we set K equal to 2.

We compare one-day ahead covariance forecasts based on realized covariances constructed from fixed 15-minute intervals and our optimal method. The forecasts are based on an ARFIMA(2, d , 2) model. The d estimates are equal to 0.45 and 0.48, respectively. Figure 2 reports our results for the last year of our data. From Figure 1 we see that the optimal sampling interval over this period of time is well under 15 minutes averaging about 1.2 minutes. The forecasted optimal series appears far less volatile than the forecasted 15-minute series. This is driven by the fact that the optimally-sampled covariances are more precisely estimated which translates into more precise forecasts.

6 Conclusions

This note proposes an MSE-based optimal sampling theory designed to reduce the effect of market microstructure noise contaminations in recorded asset prices for the purpose of quadratic covariation and beta estimation. Our discussion complements recent work on high-frequency estimation of quadratic covariations and betas in the frictionless case (Andersen et al. (2004) and BN-S (2004)) and provides robustness to the existing estimators. We believe our framework will prove useful in the practise of portfolio choice and risk management through high-frequency asset price data. Bandi et al. (2005) explore the utility benefit of MSE-based optimal sampling in the context of volatility-timing and asset allocation.

7 Appendix

Proof of Theorem 1. Recall that $\frac{h}{\delta} = M$. We expand the conditional MSE of $\widehat{C}_{(u)(l)}$ and obtain

$$\begin{aligned}
& \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} (r_{(u)j} + \varepsilon_{(u)j})(r_{(l)j} + \varepsilon_{(l)j}) - \int_0^h \Sigma_{(u)(l)}(s) ds \right)^2 \\
&= \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} (r_{(u)j}r_{(l)j} + r_{(u)j}\varepsilon_{(l)j} + \varepsilon_{(u)j}r_{(l)j} + \varepsilon_{(u)j}\varepsilon_{(l)j}) - \int_0^h \Sigma_{(u)(l)}(s) ds \right)^2 \\
&= \underbrace{\mathbf{E}_M \left(\sum_{j=1}^{h/\delta} r_{(u)j}r_{(l)j} - \int_0^h \Sigma_{(u)(l)}(s) ds \right)^2}_A + \underbrace{\mathbf{E}_M \left(\sum_{j=1}^{h/\delta} r_{(u)j}\varepsilon_{(l)j} + \varepsilon_{(u)j}r_{(l)j} + \varepsilon_{(u)j}\varepsilon_{(l)j} \right)^2}_B \\
&\quad + \underbrace{2\mathbf{E}_M \left(\left(\sum_{j=1}^{h/\delta} r_{(u)j}r_{(l)j} - \int_0^h \Sigma_{(u)(l)}(s) ds \right) \left(\sum_{j=1}^{h/\delta} r_{(u)j}\varepsilon_{(l)j} + \varepsilon_{(u)j}r_{(l)j} + \varepsilon_{(u)j}\varepsilon_{(l)j} \right) \right)}_C.
\end{aligned}$$

We start with B .

$$\begin{aligned}
B &= \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} \sum_{g=1}^{h/\delta} (r_{(u)j}\varepsilon_{(l)j} + \varepsilon_{(u)j}r_{(l)j} + \varepsilon_{(u)j}\varepsilon_{(l)j})(r_{(u)g}\varepsilon_{(l)g} + \varepsilon_{(u)g}r_{(l)g} + \varepsilon_{(u)g}\varepsilon_{(l)g}) \right) \\
&= \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} \sum_{g=1}^{h/\delta} \varepsilon_{(u)j}\varepsilon_{(l)j}\varepsilon_{(u)g}\varepsilon_{(l)g} \right) + \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} \sum_{g=1}^{h/\delta} \varepsilon_{(u)j}\varepsilon_{(l)j}r_{(u)g}\varepsilon_{(l)g} \right) + \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} \sum_{g=1}^{h/\delta} \varepsilon_{(u)j}\varepsilon_{(l)j}\varepsilon_{(u)g}r_{(l)g} \right) \\
&\quad + \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} \sum_{g=1}^{h/\delta} r_{(u)j}\varepsilon_{(l)j}r_{(u)g}\varepsilon_{(l)g} \right) + \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} \sum_{g=1}^{h/\delta} r_{(u)j}\varepsilon_{(l)j}\varepsilon_{(u)g}r_{(l)g} \right) + \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} \sum_{g=1}^{h/\delta} r_{(u)j}\varepsilon_{(l)j}\varepsilon_{(u)g}\varepsilon_{(l)g} \right) \\
&\quad + \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} \sum_{g=1}^{h/\delta} \varepsilon_{(u)j}r_{(l)j}r_{(u)g}\varepsilon_{(l)g} \right) + \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} \sum_{g=1}^{h/\delta} \varepsilon_{(u)j}r_{(l)j}\varepsilon_{(u)g}r_{(l)g} \right) + \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} \sum_{g=1}^{h/\delta} \varepsilon_{(u)j}r_{(l)j}\varepsilon_{(u)g}\varepsilon_{(l)g} \right) \\
&= \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} \sum_{g=1}^{h/\delta} \varepsilon_{(u)j}\varepsilon_{(l)j}\varepsilon_{(u)g}\varepsilon_{(l)g} \right) + \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} \sum_{g=1}^{h/\delta} r_{(u)j}\varepsilon_{(l)j}r_{(u)g}\varepsilon_{(l)g} \right) \\
&\quad + 2\mathbf{E}_M \left(\sum_{j=1}^{h/\delta} \sum_{g=1}^{h/\delta} \varepsilon_{(u)j}r_{(l)j}r_{(u)g}\varepsilon_{(l)g} \right) + \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} \sum_{g=1}^{h/\delta} \varepsilon_{(u)j}r_{(l)j}\varepsilon_{(u)g}r_{(l)g} \right) \\
&= \sum_{j=1}^{h/\delta} \mathbf{E}_M(\varepsilon_{(u)j}^2\varepsilon_{(l)j}^2) + 2 \sum_{g=1}^{h/\delta} \sum_{j < g} \mathbf{E}_M(\varepsilon_{(u)j}\varepsilon_{(l)j}\varepsilon_{(u)g}\varepsilon_{(l)g}) + \sum_{j=1}^{h/\delta} \mathbf{E}_M(r_{(u)j}^2\varepsilon_{(l)j}^2) \\
&\quad + 2 \sum_{j=1}^{h/\delta} \mathbf{E}_M(r_{(l)j}r_{(u)j}\varepsilon_{(l)j}\varepsilon_{(u)j}) + \sum_{j=1}^{h/\delta} \mathbf{E}_M(r_{(l)j}^2\varepsilon_{(u)j}^2) \\
&= \frac{h}{\delta} \mathbf{E}_M(\varepsilon_{(u)}^2\varepsilon_{(l)}^2) + 2 \sum_{j=1}^{h/\delta} \left(\frac{h}{\delta} - j \right) \mathbf{E}_M(\varepsilon_{(u)}\varepsilon_{(l)}\varepsilon_{(u)-j}\varepsilon_{(l)-j}) + \\
&\quad + \mathbf{E}_M(\varepsilon_{(l)}^2)V_{(u)} + \mathbf{E}_M(\varepsilon_{(u)}^2)V_{(l)} + 2\mathbf{E}_M(\varepsilon_{(l)}\varepsilon_{(u)})C_{(u)(l)}
\end{aligned}$$

Now consider C .

$$\begin{aligned}
C &= 2\mathbf{E}_M \left(\left(\sum_{j=1}^{h/\delta} r_{(u)j} r_{(l)j} - \int_0^h \Sigma_{(u)(l)}(s) ds \right) \left(\sum_{j=1}^{h/\delta} r_{(u)j} \varepsilon_{(l)j} + \varepsilon_{(u)j} r_{(l)j} + \varepsilon_{(u)j} \varepsilon_{(l)j} \right) \right) \\
&= 2\mathbf{E}_M \left(\sum_{j=1}^{h/\delta} r_{(u)j} r_{(l)j} \right) \mathbf{E}_M \left(\sum_{g=1}^{h/\delta} \varepsilon_{(u)g} \varepsilon_{(l)g} \right) - 2\mathbf{E}_M \left(\left(\int_0^h \Sigma_{(u)(l)}(s) ds \right) \left(\sum_{j=1}^{h/\delta} r_{(u)j} \varepsilon_{(l)j} + \varepsilon_{(u)j} r_{(l)j} + \varepsilon_{(u)j} \varepsilon_{(l)j} \right) \right) \\
&= 2\frac{h}{\delta} C_{(u)(l)} \mathbf{E}_M(\varepsilon_{(u)} \varepsilon_{(l)}) - 2\frac{h}{\delta} C_{(u)(l)} \mathbf{E}_M(\varepsilon_{(u)} \varepsilon_{(l)}) = 0.
\end{aligned}$$

We now turn to A . Write

$$\begin{aligned}
A &= \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} r_{(u)j} r_{(l)j} - \int_0^h \Sigma_{(u)(l)}(s) ds \right)^2 \\
&= \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} r_{(u)j} r_{(l)j} \right)^2 - 2\mathbf{E}_M \left(\sum_{j=1}^{h/\delta} r_{(u)j} r_{(l)j} \right) \left(\int_0^h \Sigma_{(u)(l)}(s) ds \right) + \left(\int_0^h \Sigma_{(u)(l)}(s) ds \right)^2 \\
&= \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} \sum_{g=1}^{h/\delta} r_{(u)j} r_{(l)j} r_{(u)g} r_{(l)g} \right) - \left(\int_0^h \Sigma_{(u)(l)}(s) ds \right)^2 \\
&= \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} r_{(u)j}^2 r_{(l)j}^2 \right) + 2 \sum_{g=1}^{h/\delta} \sum_{j < g} \mathbf{E}_M (r_{(u)j} r_{(l)j} r_{(u)g} r_{(l)g}) - \left(\int_0^h \Sigma_{(u)(l)}(s) ds \right)^2 \\
&= \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} r_{(u)j}^2 r_{(l)j}^2 \right) + 2 \sum_{g=1}^{h/\delta} \sum_{j < g} \mathbf{E}_M (r_{(u)j} r_{(l)j}) \mathbf{E}_M (r_{(u)g} r_{(l)g}) - \left(\int_0^h \Sigma_{(u)(l)}(s) ds \right)^2 \\
&= \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} r_{(u)j}^2 r_{(l)j}^2 \right) + 2 \sum_{g=1}^{h/\delta} \sum_{j < g} \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(u)(l)}(s) ds \right) \left(\int_{(g-1)\delta}^{g\delta} \Sigma_{(u)(l)}(s) ds \right) - \left(\int_0^h \Sigma_{(u)(l)}(s) ds \right)^2 \\
&= \sum_{j=1}^{h/\delta} \mathbf{E}_M (r_{(u)j}^2 r_{(l)j}^2) - \sum_{j=1}^{h/\delta} \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(u)(l)}(s) ds \right)^2 \\
&= \sum_{j=1}^{h/\delta} \mathbf{V}_M (r_{(u)j} r_{(l)j}) + \sum_{j=1}^{h/\delta} (\mathbf{E}_M (r_{(u)j} r_{(l)j}))^2 - \sum_{j=1}^{h/\delta} \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(u)(l)}(s) ds \right)^2 \\
&= \sum_{j=1}^{h/\delta} \mathbf{V}_M (r_{(u)j} r_{(l)j})
\end{aligned}$$

Finally,

$$\begin{aligned}
& \sum_{j=1}^{h/\delta} \mathbf{V}_M (r_{(u)j} r_{(l)j}) \\
&= \sum_{j=1}^{h/\delta} \mathbf{E}_M (\mathbf{V}_M (r_{(u)j} r_{(l)j} | r_{(l)j})) + \mathbf{V}_M (\mathbf{E}_M (r_{(u)j} r_{(l)j} | r_{(l)j})) \\
&= \sum_{j=1}^{h/\delta} \mathbf{E}_M (r_{(l)j}^2 \mathbf{V}_M (r_{(u)j} | r_{(l)j})) + \mathbf{V}_M (r_{(l)j} \mathbf{E}_M (r_{(u)j} | r_{(l)j}))
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{h/\delta} \mathbf{E}_M \left(r_{(l)j}^2 \left(\left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(u)}(s) ds \right) - \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(u)(l)}(s) ds \right)^2 \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(l)}(s) ds \right)^{-1} \right) \right) \\
&\quad + \sum_{j=1}^{h/\delta} \mathbf{V}_M \left(r_{(l)j}^2 \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(u)(l)}(s) ds \right) \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(l)}(s) ds \right)^{-1} \right) \\
&= \sum_{j=1}^{h/\delta} \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(l)}(s) ds \right) \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(u)}(s) ds \right) - \sum_{j=1}^{h/\delta} \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(u)(l)}(s) ds \right)^2 \\
&\quad + \sum_{j=1}^{h/\delta} \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(u)(l)}(s) ds \right)^2 \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(l)}(s) ds \right)^{-2} \mathbf{V}_M(r_{(l)j}^2) \\
&= \sum_{j=1}^{h/\delta} \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(l)}(s) ds \right) \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(u)}(s) ds \right) - \sum_{j=1}^{h/\delta} \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(u)(l)}(s) ds \right)^2 \\
&\quad + 2 \sum_{j=1}^{h/\delta} \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(u)(l)}(s) ds \right)^2 \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(l)}(s) ds \right)^{-2} \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(l)}(s) ds \right)^2 \\
&= \sum_{j=1}^{h/\delta} \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(l)}(s) ds \right) \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(u)}(s) ds \right) + \sum_{j=1}^{h/\delta} \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(u)(l)}(s) ds \right)^2 \\
&= \frac{h}{M} \left(\frac{M}{h} \sum_{j=1}^{h/\delta} \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(l)}(s) ds \right) \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(u)}(s) ds \right) \right) + \frac{h}{M} \left(\frac{M}{h} \sum_{j=1}^{h/\delta} \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(u)(l)}(s) ds \right)^2 \right) \\
&= \frac{h}{M} \int_0^h \left(\Sigma_{(l)}(s) \Sigma_{(u)}(s) + \Sigma_{(u)(l)}^2(s) \right) ds + o(1).
\end{aligned}$$

This proves the stated result. ■

Proof of Theorem 4. Write

$$\begin{aligned}
&\frac{\sum_{j=1}^{h/\delta} \tilde{r}_{(u)j} \tilde{r}_{(l)j}}{\sum_{j=1}^{h/\delta} \tilde{r}_{(u)j}^2} - \frac{\int_0^h \Sigma_{(u)(l)}(s) ds}{\int_0^h \Sigma_{(u)}(s) ds} \\
&= \frac{1}{\int_0^h \Sigma_{(u)}(s) ds} \left(\sum_{j=1}^{h/\delta} \tilde{r}_{(u)j} \tilde{r}_{(l)j} - \int_0^h \Sigma_{(u)(l)}(s) ds \right) \\
&\quad - \frac{\int_0^h \Sigma_{(u)(l)}(s) ds}{\left(\int_0^h \Sigma_{(u)}(s) ds \right)^2} \left(\sum_{j=1}^{h/\delta} \tilde{r}_{(u)j}^2 - \int_0^h \Sigma_{(u)}(s) ds \right) + o_p(1).
\end{aligned}$$

Hence, neglecting the order term $o_p(1)$,

$$\begin{aligned}
&\mathbf{E}_M \left(\frac{\sum_{j=1}^{h/\delta} \tilde{r}_{(u)j} \tilde{r}_{(l)j}}{\sum_{j=1}^{h/\delta} \tilde{r}_{(u)j}^2} - \frac{\int_0^h \Sigma_{(u)(l)}(s) ds}{\int_0^h \Sigma_{(u)}(s) ds} \right)^2 \\
&= \left(\frac{1}{\int_0^h \Sigma_{(u)}(s) ds} \right)^2 \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} \tilde{r}_{(u)j} \tilde{r}_{(l)j} - \int_0^h \Sigma_{(u)(l)}(s) ds \right)^2
\end{aligned}$$

$$\begin{aligned}
& + \frac{\left(\int_0^h \Sigma_{(u)(l)}(s) ds\right)^2}{\left(\int_0^h \Sigma_{(u)}(s) ds\right)^4} \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} \tilde{r}_{(u)j}^2 - \int_0^h \Sigma_{(u)}(s) ds \right)^2 \\
& - 2 \frac{\int_0^h \Sigma_{(u)(l)}(s) ds}{\left(\int_0^h \Sigma_{(u)}(s) ds\right)^3} \mathbf{E}_M \left(\left(\sum_{j=1}^{h/\delta} \tilde{r}_{(u)j}^2 - \int_0^h \Sigma_{(u)}(s) ds \right) \left(\sum_{j=1}^{h/\delta} \tilde{r}_{(u)j} \tilde{r}_{(l)j} - \int_0^h \Sigma_{(u)(l)}(s) ds \right) \right).
\end{aligned}$$

The first and the second term have obvious expressions given Theorem 1. We focus on the third term.

$$\begin{aligned}
& \mathbf{E}_M \left(\left(\sum_{j=1}^{h/\delta} \tilde{r}_{(u)j}^2 - \int_0^h \Sigma_{(u)}(s) ds \right) \left(\sum_{j=1}^{h/\delta} \tilde{r}_{(u)j} \tilde{r}_{(l)j} - \int_0^h \Sigma_{(u)(l)}(s) ds \right) \right) \\
& = \mathbf{E}_M \left(\left(\sum_{j=1}^{h/\delta} (r_{(u)j} + \varepsilon_{(u)j})^2 - \int_0^h \Sigma_{(u)}(s) ds \right) \left(\sum_{j=1}^{h/\delta} (r_{(u)j} + \varepsilon_{(u)j})(r_{(l)j} + \varepsilon_{(l)j}) - \int_0^h \Sigma_{(u)(l)}(s) ds \right) \right) \\
& = \mathbf{E}_M \left(\begin{aligned} & \left(\sum_{j=1}^{h/\delta} (r_{(u)j}^2 + \varepsilon_{(u)j}^2 + 2r_{(u)j}\varepsilon_{(u)j}) - \int_0^h \Sigma_{(u)}(s) ds \right) \\ & \left(\sum_{j=1}^{h/\delta} (r_{(u)j}r_{(l)j} + r_{(u)j}\varepsilon_{(l)j} + \varepsilon_{(u)j}r_{(l)j} + \varepsilon_{(u)j}\varepsilon_{(l)j}) - \int_0^h \Sigma_{(u)(l)}(s) ds \right) \end{aligned} \right) \\
& = \underbrace{\mathbf{E}_M \left(\left(\sum_{j=1}^{h/\delta} r_{(u)j}^2 - \int_0^h \Sigma_{(u)}(s) ds \right) \left(\sum_{j=1}^{h/\delta} r_{(u)j}r_{(l)j} - \int_0^h \Sigma_{(u)(l)}(s) ds \right) \right)}_A \\
& + \underbrace{\mathbf{E}_M \left(\left(\sum_{j=1}^{h/\delta} r_{(u)j}^2 - \int_0^h \Sigma_{(u)}(s) ds \right) \left(\sum_{j=1}^{h/\delta} (r_{(u)j}\varepsilon_{(l)j} + \varepsilon_{(u)j}r_{(l)j} + \varepsilon_{(u)j}\varepsilon_{(l)j}) \right) \right)}_B \\
& + \underbrace{\mathbf{E}_M \left(\sum_{j=1}^{h/\delta} \varepsilon_{(u)j}^2 + 2r_{(u)j}\varepsilon_{(u)j} \right) \left(\sum_{j=1}^{h/\delta} (r_{(u)j}r_{(l)j} + r_{(u)j}\varepsilon_{(l)j} + \varepsilon_{(u)j}r_{(l)j} + \varepsilon_{(u)j}\varepsilon_{(l)j}) - \int_0^h \Sigma_{(u)(l)}(s) ds \right)}_C
\end{aligned}$$

First, we analyze B .

$$\begin{aligned}
& \mathbf{E}_M \left(\left(\sum_{j=1}^{h/\delta} r_{(u)j}^2 - \int_0^h \Sigma_{(u)}(s) ds \right) \left(\sum_{j=1}^{h/\delta} (r_{(u)j}\varepsilon_{(l)j} + \varepsilon_{(u)j}r_{(l)j} + \varepsilon_{(u)j}\varepsilon_{(l)j}) \right) \right) \\
& = \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} r_{(u)j}^2 \sum_{g=1}^{h/\delta} r_{(u)g}\varepsilon_{(l)g} + \sum_{j=1}^{h/\delta} r_{(u)j}^2 \sum_{g=1}^{h/\delta} \varepsilon_{(u)g}r_{(l)g} + \sum_{j=1}^{h/\delta} r_{(u)j}^2 \sum_{g=1}^{h/\delta} \varepsilon_{(u)g}\varepsilon_{(l)g} \right) \\
& \quad - \left(\int_0^h \Sigma_{(u)}(s) ds \right) \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} r_{(u)j}\varepsilon_{(l)j} + \varepsilon_{(u)j}r_{(l)j} + \varepsilon_{(u)j}\varepsilon_{(l)j} \right) \\
& = \left(\int_0^h \Sigma_{(u)}(s) ds \right) M \mathbf{E}_M (\varepsilon_{(u)}\varepsilon_{(l)}) - \left(\int_0^h \Sigma_{(u)}(s) ds \right) M \mathbf{E}_M (\varepsilon_{(u)}\varepsilon_{(l)}) = 0.
\end{aligned}$$

We now analyze C .

$$\mathbf{E}_M \left(\sum_{j=1}^{h/\delta} \varepsilon_{(u)j}^2 + 2r_{(u)j}\varepsilon_{(u)j} \right) \left(\sum_{j=1}^{h/\delta} (r_{(u)j}r_{(l)j} + r_{(u)j}\varepsilon_{(l)j} + \varepsilon_{(u)j}r_{(l)j} + \varepsilon_{(u)j}\varepsilon_{(l)j}) - \int_0^h \Sigma_{(u)(l)}(s) ds \right)$$

$$\begin{aligned}
&= -\mathbf{E}_M \left(\left(\sum_{j=1}^{h/\delta} \varepsilon_{(u)j}^2 + 2r_{(u)j}\varepsilon_{(u)j} \right) \int_0^h \Sigma_{(u)(l)}(s) ds \right) \\
&\quad + \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} \varepsilon_{(u)j}^2 + 2r_{(u)j}\varepsilon_{(u)j} \right) \left(\sum_{j=1}^{h/\delta} (r_{(u)j}r_{(l)j} + r_{(u)j}\varepsilon_{(l)j} + \varepsilon_{(u)j}r_{(l)j} + \varepsilon_{(u)j}\varepsilon_{(l)j}) \right) \\
&= -M\mathbf{E}_M(\varepsilon_{(u)}^2) \int_0^h \Sigma_{(u)(l)}(s) ds + \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} \varepsilon_{(u)j}^2 \right) \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} r_{(u)j}r_{(l)j} \right) + \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} \varepsilon_{(u)j}^2 \sum_{g=1}^{h/\delta} \varepsilon_{(u)g}\varepsilon_{(l)g} \right) \\
&\quad + 2\mathbf{E}_M \left(\sum_{j=1}^{h/\delta} r_{(u)j}\varepsilon_{(u)j} \sum_{j=1}^{h/\delta} r_{(u)j}\varepsilon_{(l)j} \right) + 2\mathbf{E}_M \left(\sum_{j=1}^{h/\delta} r_{(u)j}\varepsilon_{(u)j} \sum_{j=1}^{h/\delta} \varepsilon_{(u)j}r_{(l)j} \right) \\
&= -M\mathbf{E}_M(\varepsilon_{(u)}^2) \int_0^h \Sigma_{(u)(l)}(s) ds + M\mathbf{E}_M(\varepsilon_{(u)}^2) \int_0^h \Sigma_{(u)(l)}(s) ds \\
&\quad + \sum_{j=1}^{h/\delta} \mathbf{E}_M \left(\varepsilon_{(u)j}^3 \varepsilon_{(l)j} \right) + \mathbf{E}_M \left(\sum_{g=1}^{h/\delta} \sum_{j<g} \varepsilon_{(u)j}^2 \varepsilon_{(u)g} \varepsilon_{(l)g} \right) + \mathbf{E}_M \left(\sum_{g=1}^{h/\delta} \sum_{j<g} \varepsilon_{(u)g}^2 \varepsilon_{(u)j} \varepsilon_{(l)j} \right) \\
&\quad + 2\mathbf{E}_M \left(\sum_{j=1}^{h/\delta} r_{(u)j}^2 \varepsilon_{(u)j} \varepsilon_{(l)j} + \sum_{g=1}^{h/\delta} \sum_{j<g} r_{(u)j} \varepsilon_{(u)j} r_{(u)g} \varepsilon_{(l)g} + \sum_{g=1}^{h/\delta} \sum_{j<g} r_{(u)g} \varepsilon_{(u)g} r_{(u)j} \varepsilon_{(l)j} \right) \\
&\quad + 2\mathbf{E}_M \left(\sum_{j=1}^{h/\delta} \varepsilon_{(u)j}^2 r_{(l)j} r_{(u)j} + \sum_{g=1}^{h/\delta} \sum_{j<g} r_{(u)j} \varepsilon_{(u)j} r_{(l)g} \varepsilon_{(u)g} + \sum_{g=1}^{h/\delta} \sum_{j<g} r_{(u)g} \varepsilon_{(u)g} r_{(l)j} \varepsilon_{(u)j} \right).
\end{aligned}$$

Hence,

$$\begin{aligned}
C &= 2\mathbf{E}_M(\varepsilon_{(u)}\varepsilon_{(l)}) \int_0^h \Sigma_{(u)}(s) ds + 2\mathbf{E}_M(\varepsilon_{(u)}^2) \int_0^h \Sigma_{(u)(l)}(s) ds \\
&\quad + M\mathbf{E}_M \left(\varepsilon_{(u)}^3 \varepsilon_{(l)} \right) + \sum_{g=1}^{h/\delta} \sum_{j<g} \mathbf{E}_M \left(\varepsilon_{(u)j}^2 \varepsilon_{(u)g} \varepsilon_{(l)g} \right) + \sum_{g=1}^{h/\delta} \sum_{j<g} \mathbf{E}_M \left(\varepsilon_{(u)g}^2 \varepsilon_{(u)j} \varepsilon_{(l)j} \right) \\
&= 2\mathbf{E}_M(\varepsilon_{(u)}\varepsilon_{(l)}) \int_0^h \Sigma_{(u)}(s) ds + 2\mathbf{E}_M(\varepsilon_{(u)}^2) \int_0^h \Sigma_{(u)(l)}(s) ds \\
&\quad + M\mathbf{E}_M \left(\varepsilon_{(u)}^3 \varepsilon_{(l)} \right) + \sum_{g=1}^{h/\delta} \left(\frac{h}{\delta} - j \right) \mathbf{E}_M \left(\varepsilon_{(u)}^2 \varepsilon_{(u)-j} \varepsilon_{(l)-j} \right) + \sum_{g=1}^{h/\delta} \left(\frac{h}{\delta} - j \right) \mathbf{E}_M \left(\varepsilon_{(u)} \varepsilon_{(l)} \varepsilon_{(u)-j}^2 \right).
\end{aligned}$$

We now turn to A .

$$\begin{aligned}
&\mathbf{E}_M \left(\left(\sum_{j=1}^{h/\delta} r_{(u)j}^2 - \int_0^h \Sigma_{(u)}(s) ds \right) \left(\sum_{j=1}^{h/\delta} r_{(u)j}r_{(l)j} - \int_0^h \Sigma_{(u)(l)}(s) ds \right) \right) \\
&= \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} r_{(u)j}^2 \sum_{j=1}^{h/\delta} r_{(u)j}r_{(l)j} - \sum_{j=1}^{h/\delta} r_{(u)j}^2 \int_0^h \Sigma_{(u)(l)}(s) ds \right) \\
&\quad - \left(\int_0^h \Sigma_{(u)}(s) ds \right) \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} r_{(u)j}r_{(l)j} \right) + \int_0^h \Sigma_{(u)}(s) ds \int_0^h \Sigma_{(u)(l)}(s) ds \\
&= \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} r_{(u)j}^2 \sum_{j=1}^{h/\delta} r_{(u)j}r_{(l)j} \right) - \left(\int_0^h \Sigma_{(u)}(s) ds \right) \left(\int_0^h \Sigma_{(u)(l)}(s) ds \right)
\end{aligned}$$

$$\begin{aligned}
&= \mathbf{E}_M \left(\sum_{j=1}^{h/\delta} r_{(u)j}^2 r_{(u)j} r_{(l)j} \right) + \sum_{g=1}^{h/\delta} \sum_{j < g} \mathbf{E}_M \left(r_{(u)j}^2 r_{(u)g} r_{(l)g} \right) + \sum_{g=1}^{h/\delta} \sum_{j < g} \mathbf{E}_M \left(r_{(u)g}^2 r_{(u)j} r_{(l)j} \right) \\
&\quad - \left(\int_0^h \Sigma_{(u)}(s) ds \right) \left(\int_0^h \Sigma_{(u)(l)}(s) ds \right) \\
&= \sum_{j=1}^{h/\delta} \mathbf{E}_M \left(r_{(u)j}^2 r_{(u)j} r_{(l)j} \right) - \sum_{j=1}^{h/\delta} \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(u)(l)}(s) ds \right) \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(u)}(s) ds \right).
\end{aligned}$$

Now write

$$\begin{aligned}
&\sum_{j=1}^{h/\delta} \mathbf{E}_M \left(r_{(u)j}^2 r_{(u)j} r_{(l)j} \right) - \sum_{j=1}^{h/\delta} \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(u)(l)}(s) ds \right) \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(u)}(s) ds \right) \\
&= \sum_{j=1}^{h/\delta} \mathbf{Cov}_M \left(r_{(u)j}^2, r_{(u)j} r_{(l)j} \right) + \sum_{j=1}^{h/\delta} \mathbf{E}_M \left(r_{(u)j}^2 \right) \mathbf{E}_M \left(r_{(u)j} r_{(l)j} \right) \\
&\quad - \sum_{j=1}^{h/\delta} \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(u)(l)}(s) ds \right) \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(u)}(s) ds \right) \\
&= \sum_{j=1}^{h/\delta} \mathbf{Cov}_M \left(r_{(u)j}^2, r_{(u)j} r_{(l)j} \right) \\
&= \sum_{j=1}^{h/\delta} \mathbf{E}_M \left(\mathbf{Cov}_M \left(r_{(u)j}^2, r_{(u)j} r_{(l)j} \mid r_{(u)} \right) \right) + \sum_{j=1}^{h/\delta} \mathbf{Cov}_M \left(\mathbf{E} \left(r_{(u)j}^2 \mid r_{(u)} \right), \mathbf{E} \left(r_{(u)j} r_{(l)j} \mid r_{(u)} \right) \right) \\
&= \sum_{j=1}^{h/\delta} \mathbf{E}_M \left(\mathbf{E}_M \left(r_{(u)j}^2 r_{(u)j} r_{(l)j} \mid r_{(u)} \right) - \mathbf{E}_M \left(r_{(u)j}^2 \mid r_{(u)} \right) \mathbf{E}_M \left(r_{(u)j} r_{(l)j} \mid r_{(u)} \right) \right) + \\
&\quad + \sum_{j=1}^{h/\delta} \mathbf{Cov}_M \left(\mathbf{E} \left(r_{(u)j}^2 \mid r_{(u)} \right), \mathbf{E} \left(r_{(u)j} r_{(l)j} \mid r_{(u)} \right) \right) \\
&= \sum_{j=1}^{h/\delta} \mathbf{E}_M \left(r_{(u)j}^3 \mathbf{E}_M \left(r_{(l)j} \mid r_{(u)} \right) - r_{(u)j}^3 \mathbf{E}_M \left(r_{(l)j} \mid r_{(u)} \right) \right) + \sum_{j=1}^{h/\delta} \mathbf{Cov}_M \left(r_{(u)j}^2, r_{(u)j} \mathbf{E} \left(r_{(l)j} \mid r_{(u)} \right) \right) \\
&= \sum_{j=1}^{h/\delta} \mathbf{Cov}_M \left(r_{(u)j}^2, r_{(u)j} \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(u)(l)}(s) ds \right) \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(u)}(s) ds \right)^{-1} \right) \\
&= \sum_{j=1}^{h/\delta} \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(u)(l)}(s) ds \right) \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(u)}(s) ds \right)^{-1} \mathbf{Cov}_M \left(r_{(u)j}^2, r_{(u)j} \right) \\
&= 2 \sum_{j=1}^{h/\delta} \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(u)(l)}(s) ds \right) \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(u)}(s) ds \right)^{-1} \left(\int_{(j-1)\delta}^{j\delta} \Sigma_{(u)}(s) ds \right)^2 \\
&= 2 \frac{h}{M} \int_0^h \Sigma_{(u)}(s) \Sigma_{(u)(l)}(s) ds + o(1).
\end{aligned}$$

This proves the stated result. ■

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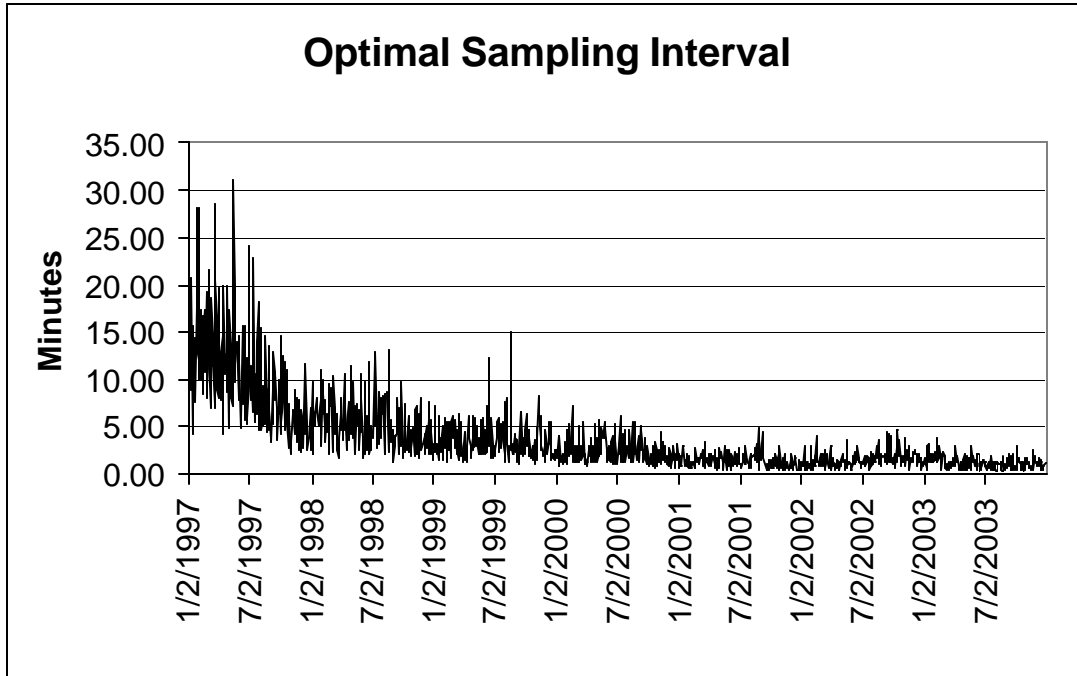


Figure 1. Optimal sampling intervals for realized covariance.

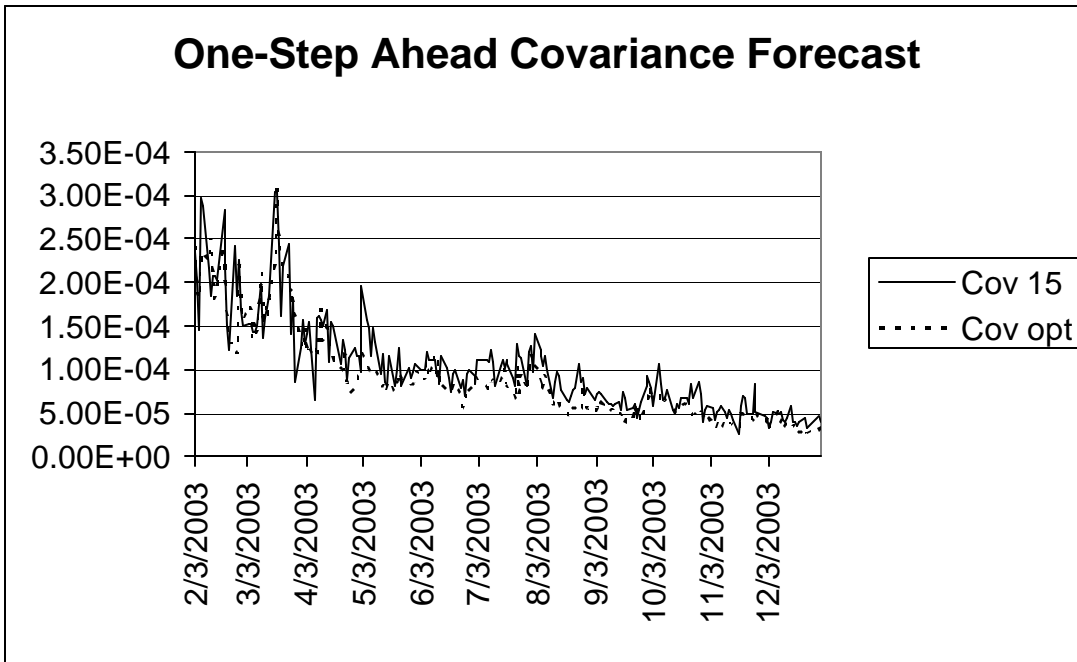


Figure 2. Comparison of one-day ahead covariance forecasts obtained by using 15-minute and optimally-sampled covariance estimates.