An Equilibrium Model of Investment Under Uncertainty*

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Abstract

This paper analyzes the optimal investment decisions of heterogeneous firms in a competitive, uncertain environment. We characterize firms’ optimal investment strategy explicitly, and derive a closed form solution for firm value. We show that in the strategic equilibrium real option premia are significant. As a result firms delay investment, choosing optimally not to undertake some positive NPV projects. The model predicts that firm returns vary over the business cycle, with returns negatively skewed during expansions but positively skewed in recessions.

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1 Introduction

Using real options theory to account for the value of future development is now standard in finance among both academics and practitioners. It provides a heuristic, intuitively appealing explanation of the real world’s observed deviations from neoclassical, Tobin’s Q theory of investment. The most notable of these deviations are market values that exceed book values, often greatly, and investment thresholds that may significantly exceed zero net present value. Real options theory, however, completely ignores the role of competition. The theory’s conclusion that delaying investment can result in excess profits seems to be at odds with what we know about competition. Competition drives down profit opportunities, a fact known even before Cournot modeled the effect in 1838.

This paper presents an equilibrium model integrating the two disparate branches of the economics literature—real options theory and the theory of competition. Recent papers, most notably Grenadier (2002), have used the Cournot intuition to argue that competition erodes real option values and reduces investment delays.1 These results are difficult to reconcile, however, with important sectors of the economy. Option premia are significantly positive and firms delay investment in some highly competitive industries. Titman (1985) illustrates this with a simple example: empty lots in city centers. Real estate is a highly competitive industry with many players, yet owners sometimes choose not to build on property that could certainly be developed profitably. Instead, as Titman argues, the value of a lot derives from the option to develop, and owners sometimes delay building believing that the lot can be developed more profitably later.

The analysis presented in this paper shows that in a competitive industry firms can actually deviate more from neoclassical behavior than the standard real options analysis predicts. In particular, firms may delay irreversible investment longer, and invest only at significantly positive option premia.

Grenadier’s results—that competition erodes option values and pushes firms back to the zero NPV investment rule—and similar results found in Leahy (1993) and Kogan (2001), are a consequence of the type of industry they model, one in which the production technology is linear and incremental.2 In these papers firms may add capacity in arbitrarily small

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1 Several other papers have also considered the effects of competitive interactions on real options. Smets (1991), Grenadier (1996), Garlappi (1999), and Lambrecht and Perraudin (2002) have considered duopolistic settings; Leahy (1993) perfect competition; in addition to Grenadier (2002) Spatt and Sterbenz (1985) and Williams (1993) have considered the intermediate cases. In all of these papers competition leads agents to exercise options earlier than would a strategic monopolist who accounts for the price impact of her own option exercise strategy.

2 Similar results that also depend on the choice of an irreversible linear incremental production technology
increments without suffering any adjustment costs. Option values and investment behavior in the real estate market example, and in many other industries, differ from Grenadier’s predictions because undertaking investment often entails opportunity costs, and because firms vary in scope and size. We show in this paper that in industries in which opportunity costs and heterogeneity are important real option values are significant, investment decisions are delayed, and investment is lumpy.

Since Hotelling’s (1929) seminal paper it has been well understood in economics that demand side heterogeneity can reduce competition. The whole concept of horizontal product differentiation is predicated on the idea that variations in tastes allow firms to segment the market, compete less, and extract more of the consumer surplus. Heterogeneity can also provide a natural ordering to agents’ actions. We do not all buy new computers, or cars, at the same time at least in part because those of us with older, obsolete models are more likely near term buyers than those with newer, contemporary models.

This paper shows that supply side heterogeneity can reduce competition as well. When heterogeneity extends to costs or profitability it is a wedge that breaks the idea of “perfect competition,” even when firms are perfectly competitive. In the real estate example considered previously, a large number of firms compete vigorously, yet heterogeneity prevents them from all competing directly over any investment opportunity. When the owner of the empty lot considers putting up a thirty-story office tower she is not competing with the owner of the fifteen-story apartment complex next door. The opportunity costs to the owner of the fifteen-story apartment complex, which include walking away from the existing building, effectively preclude her from competing with the owner of the empty lot. The owner of the fifteen-story apartment complex however still possesses a valuable option to compete over investment opportunities in the future. If the city grows there may be demand in the future for a new sixty-story office tower. If at that time all the empty lots are developed the owner of the fifteen-story apartment complex may have low enough opportunity costs to compete.

The fact that supply side heterogeneity reduces competition is not suprising. What is suprising... is the heterogeneity itself. The rest of the literature... While supply side heterogeneity reduces competition, it itself arises endogenously as a consequence of the fact that heterogeneous firms compete less. Firms naturally make investment decisions that differentiate them from others in a manner that reduces the amount of competition they face. The natural, stable cross-sectional distribution of firms is the one that minimizes intertemporal competition between firms. The degree of cross-sectional dispersion depends

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may also be found in Williams (1993) and Caballero and Pindyck (1996).
on the economic primitives. Greater cross-sectional variation in firms is associated with a) high uncertainty regarding future demand, b) a high average demand growth, c) low cost-to-scale (i.e., close to linear) for developing capacity, and d) a low discount rate.

We develop a model in which firms incur opportunity as well as direct investment costs to altering capacity, and face aggregate uncertainty regarding demand for their output, the price of which is determined endogenously and is a function of firms’ investment decisions. We find that even with an infinite number of competitive heterogeneous firms, option premia are significant and firms optimally delay irreversible investment, choosing not to undertake some positive NPV projects.

The Cournot intuition that competition should drive firms to invest earlier is compelling, so our result that competition can lead firms to delay investment longer is somewhat surprising. This result derives from the endogenously determined equilibrium price of firms’ output. We show that the price of firms’ output is negatively skewed because aggregate industry capacity responds asymmetrically to changing demand. Firms can add capacity quickly in response to rising demand, but cannot adjust capacity as quickly to falling demand due to investment irreversibility. As a result, increasing supply attenuates positive demand shocks, which are only partially translated into prices, while negative demand shocks are translated into prices more fully. As Dixit (1999) shows, negative skewness leads firms to delay investment. Firms have an incentive to delay investment when large drops in the price of firms’ output are more likely. This additional incentive to delay investment, in conjunction with the significantly positive option premia, results in investment delays even longer than predicted by standard, partial equilibrium real options models.

Because these large drops in the price of firms’ output result from firms adding capacity they are, ipso facto, most likely when firms choose to develop. In fact, firms always add capacity expecting prices to fall. That is, while capacity added immediately prior to large price drops might look, ex post, like overbuilding, its development was in fact ex ante optimal. We analytically construct forward curves for the price of firms’ output to explore such implications of negative skewness in greater detail.

We also derive closed form expressions for firm value as a function of the price of firms’ output and current aggregate industry capacity. This allows us to make predictions about stock returns. Stock returns are a combination of the returns to firms’ ongoing projects and their growth options. Away from historic highs in the price of firms’ output, aggregate capacity is unlikely to increase so the return to firms’ output exhibits little skew. This

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3Leahy (1993) also noted that a competitive firm’s free entry threshold can resemble a monopolist’s option-value threshold.
translates into positive skew in stock returns because the option component of firm value is convex in the price of firm output. When the price of firms’ output is high the effect of increasing aggregate capacity dominates and stock returns exhibit negative skew. This provides a hitherto untested empirical prediction: skewness in stock returns should vary over the business cycle. Stock returns should be negatively skewed during expansions, but positively skewed in recessions.

The remainder of the paper is organized as follows. Section 2 introduces the basic model. Section 3 demonstrates the equilibrium strategy. The section begins by discussing the intuition and the general form of the strategy before characterizing the strategy explicitly. Section 4 discusses implications of the analysis, including properties of the equilibrium price process. We also compare the equilibrium exercise strategy to the standard, partial equilibrium strategy in detail. Section 5 concludes.

2 The Model

In this model the defining characteristic of a “firm” is ownership of productive capital. A firm’s ongoing assets, or installed “capacity,” costlessly produce a good (or service) flow. A firm is able to produce a flow of the good in proportion to its capacity. This good may then be sold in a competitive market at the instantaneous price $P_t$ that clears the market. The total instantaneous cash flow to a site with capacity $q$, excluding development costs, is therefore simply $qP_t$.

Following the literature we assume that the market clearing price for firms’ output satisfies an inverse demand function of a constant elasticity form,

$$P_t = X_t Q_t^{-\alpha},$$

where $Q_t$ is the instantaneous aggregate supply of the good, $X_t$ is a multiplicative demand shock, and $\alpha$ is the price elasticity of demand.\(^4\) This formulation is equivalent to assuming that prices are set by market clearing and demand is time varying but has constant elasticity with respect to price. That is, demand is given by

$$D_t = X_t^\alpha P_t^{-\alpha},$$

where $X_t^\alpha$ is stochastic and may be thought of as demand in a world in which the good has

\(^4\)Because projects produce the good in proportion to their capacities $Q_t$ will be use to denote both the instantaneous aggregate supply of the good and the aggregate capacity to produce the good.
unit price. The multiplicative demand shock is assumed to evolve as a geometric Brownian motion. That is,
\[ dX_t = \mu X_t \, dt + \sigma X_t \, dz_t, \]
where \( \mu \) and \( \sigma \) are constant.

At any time a firm may increase capacity by developing. Development, which may be undertaken repeatedly, entails two costs, the investment cost, which is the direct cost of development, and an opportunity cost.\(^5\)

The investment is the cash outlay required to undertake the new project. For example, when a personal computer manufacturer retools a production line for a new model it incurs costs. Likewise, when the owner of a small tenement in Manhattan decides to redevelop her property she incurs the direct construction costs of building a new office tower.

The direct cost of development depends on the scale of the undertaking. Continuing the previous examples, it is more expensive to set up a production line to produce a million computers a year than it is to set it up to produce a hundred thousand computers; likewise, it is more expensive to build a sixty-story office tower than it is to build a thirty-story office tower. We model this cost of investment as Cobb-Douglas with increasing costs-to-scale. That is, the direct cost of developing capacity \( q^* \) is \( q^{\gamma} \) where \( \gamma > 1 \).

Opportunity costs result because the new investment damages the firm’s ongoing business. In the examples above, for instance, undertaking the new project entails a significant loss in value of the ongoing assets. When the computer manufacturer introduces the new model it effectively kills demand for the old model. The manufacturer must consider this lost revenue in her investment decision. The owner of the Manhattan tenement is in much the same position. Before putting up the office tower she must first raze the tenement, foregoing future rents.

The opportunity cost to undertaking investment is largely independent of the scale of the undertaking. The computer manufacturer kills demand for the old model whether it produces a hundred thousand units or a million units of the new model, and the tenement owner must raze the existing building irregardless of the size of the new office tower.

The opportunity cost is modeled here as a fractional loss of the value of projects currently in place. We assume for convenience that the fractional loss from adjustment is one, \( i.e. \), that development entails abandonment of the ongoing project. This assumption simpli-\(^5\)

\(^5\)While our focus is on the role of competition, allowing for firms that incur opportunity as well as investment costs to increase capacity repeatedly is itself a departure from the standard real options literature, with important implications for optimal investment choice. Retaining development rights leads to higher option values, and to firms developing sooner but to lower capacities than they would if they were only able to develop once. See, for example, Williams (1997).
fies the analysis, but with modification the analysis presented in this paper applies to any other choice.\(^6\)

We assume, for the sake of simplicity, that cash flows are valued in a risk-neutral framework, discounted at a constant risk-free rate \(r\). Firms are then priced at the expected value of future revenues, less investment costs, all discounted appropriately for the time value of money. Firms are assumed to maximize value. That is, firms choose their investment strategies to maximizes the current expected value of all future cash flows, including development costs, discounted appropriately.

Finally, we would like to capture the fact that firms vary greatly in scope and size. We model heterogeneity in this dimension. The initial heterogeneity is a sort of natural variation in firm size: firm size initially follows Zipf’s law, with firms distributed uniformly with respect to log-capacity between the smallest and largest firms in the industry.\(^7\) Firm size then evolves endogenously as a result of firms’ equilibrium investment decisions.

3 Determination of the Equilibrium Strategy

Firms must account for the actions of their competitors when determining their optimal investment strategies. Firms produce a good that they sell in a competitive market at the market clearing price, which is determined by supply and demand. Demand is exogenous but supply is endogenous, resulting from firms’ investment decisions.\(^8\) Firms consequently must invest accounting for the investment strategy of other firms in the industry and the impact of other firms’ investment decisions on prices.

The equilibrium concept employed in this paper is competitive equilibrium with rational expectations. An equilibrium consists of a consistent 1) price process for the industry good, and 2) set of investment strategies for all firms in the industry. That is, in equilibrium firms’ investment decisions are both 1) consistent with the evolution of the price process, and 2) optimal given the price process. As a firm has no incentive to deviate from its investment strategy readers more comfortable with Nash equilibrium may choose to interpret the equilibrium using that concept.

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\(^6\)Assuming that development entails abandonment makes solving a fixed-point problem that arises in the course of the analysis particularly simple.

\(^7\)Because opportunity costs are proportional to the scale of a firm’s ongoing project this is really an assumption of heterogeneous costs to adjusting capacity. That is, while firms are homogenous in the cost of investing, they differ with respect to the opportunity cost to undertaking investment.

\(^8\)Supply is determined purely by firms’ investment decisions because in the model firms never idle capacity; as they are small (price takers) and able to produce the good costlessly, they operate at full capacity.
To demonstrate an equilibrium strategy we must establish two elements. The first is to hypothesized an investment strategy for firms and determine the price process that results if all firms adhere to the strategy. The second is to show that given the resultant price process the hypothesized strategy is optimal. While the goal here is establishing an equilibrium strategy, and the methodology here does not rule out the possible existence of other equilibria, we will establish a limited uniqueness. In particular, the equilibrium presented here will be the unique equilibrium for which the evolution of the log-price process only depends on its level relative to the historical maximum, \( i.e., \) for which \( d \ln P_t \) is a homogenous, degree-zero function of \( P_t \) and \( P_0 \).

Finally, before proceeding to demonstrate the two elements required to establish an equilibrium strategy it is useful to note that any equilibrium strategy will not be symmetric. That is, even if firms are \( ex \ ante \) identical there is no equilibrium in which homogenous agents make identical investment decisions, \( i.e., \) of the type prevalent in the literature, because no such equilibrium exists (irrespective of the particular equilibrium concept). Symmetric behavior is always suboptimal, and a firm will invariably have incentives to deviate from the behavior of other similar firms. This is formalized in the following proposition. The proofs of all propositions are, in an effort to avoid excessive digression, left for the appendix.

**Proposition 3.1** There is no symmetric equilibrium.

### 3.1 The Price of Firms’ Output

We now turn our attention to establishing the first element of demonstrating an equilibrium strategy, hypothesizing a strategy and determining the resultant price process.

**Proposition 3.2** Suppose firms’ log-capacities are distributed uniformly on \( (\ln q_m, \ln q_M) \) and that each firm follows the strategy of developing to \( q_M/q_m \) times its existing capacity \( q \) whenever the price of firms’ output reaches \( (q/q_m)^{\gamma-1} P_0 \), where \( P_0 \), the initial “historical maximum” for the price of firms’ output, is assumed to be at least as great as \( P_0 \), the initial price of firms’ output. Then the price of firms’ output evolves according to

\[
\ln P_t = \ln P_0 + \ln X_t - \frac{1}{1+\alpha(\gamma-1)} \ln X_t
\]  

\(^9\)Exogenously imposing price-dependent investment behavior on firms maps strategies to price processes. The optimal investment behavior of an unconstrained firm maps price processes to strategies. An equilibrium strategy is a fixed point in the composition mapping from strategies to strategies.
where \( \ln X_s \) is a standard Brownian motion and \( \overline{X}_t \) is the greater of \( \overline{P}_0/P_0 \) and the maximum of \( X_s \) up to time \( t \).\(^{10}\)

The first term on the right hand side of equation (2) calibrates the level of prices at time zero. The second term expresses the direct effect of demand on prices, and is directly proportional to the current level of the multiplicative demand shock. The third term expresses the effect of supply, and aggregate capacity is related to the highest level demand has reached previously. Not surprisingly, this supply effect is greater when the cost to scale of building is low or demand is inelastic with respect to prices.

The log-price process follows a “partially reflected” geometric Brownian motion. Below historic highs in the price of firms’ output capacity is fixed, so the instantaneous evolution of the price process is the same as the evolution of the multiplicative demand shock. Below price highs the only term that changes in the right hand side of equation (2) in response to a demand shock is \( \ln X_t \), so prices change in exactly the same way as demand. At historic highs, however, positive demand shocks result in a supply response that mitigates the effect on prices. Because \( X_t = \overline{X}_t \) at these times the \( -\frac{1}{1+\alpha(y-1)} \ln \overline{X}_t \) term in equation (2) disappears, canceling \( \frac{1}{1+\alpha(y-1)} \ln X_t \) of the \( \ln X_t \) term, and positive demand shocks are translated into upward movements in the log-price process attenuated by the factor \( \frac{\alpha(y-1)}{1+\alpha(y-1)} \), which is less than one.

In Figure 1, below, this partially reflected geometric Brownian equilibrium price process is shown for a particular realization of the evolution of the multiplicative demand shock. Also plotted, for the same realization of the demand shock and starting at the same initial price level, are the standard geometric Brownian motion typical of partial equilibrium models (top path), and a reflected geometric Brownian motion typical of linear, incremental investment models (bottom path, reflecting barrier at one). The “degree” of reflection in the equilibrium price process depends on the cost-to-scale of adding new capacity, and on the elasticity of prices with respect to supply.

### 3.2 A Firm’s Optimal Investment Strategy

Now we consider an arbitrary firm with existing capacity \( q \) that takes the price process from Proposition 3.2 as given. The firm chooses an investment strategy to maximize the expected value of all future cash flows, accounting for development costs. Adding capacity entails

\(^{10}\)That the price process evolves in a manner independent of the width of the distribution of log-capacities is quite remarkable. This independence results from modeling opportunity costs as abandonment of ongoing assets. If we choose to model opportunity costs as some other fractional loss \( f \) of the value of assets in place, we would need to replace \( (y-1) \) with \( (1 + \ln \kappa (1 + \frac{1-L}{\kappa}))^{-1} (y-1) \), where \( \kappa = q_M/q_m \).
adjustment costs, so firms will not find it optimal to adjust capacity continuously. A firm optimally determines discreet times $\tau_i$, for $i = 1, 2, \ldots$, at which to increase to capacities $q_i$. Letting $V(q, P_t, \overline{P}_t)$ denote the value of a firm with capacity $q$ when price of firms’ output is $P_t$ and the all-time high in the price is $\overline{P}_t$, the value of a firm may then be written as

$$V(q, P_t, \overline{P}_t) = \max_{\{((\tau_i, q_i))\}_{i=1}^{\infty}} E_t \left[ \int_t^{\infty} e^{-r(s-t)} q_s P_s ds - \sum_{i=1}^{\infty} e^{-r(\tau_i-t)} q_i^\gamma \right]$$

where $q_s \equiv q_i$ if $s \in [\tau_i, \tau_{i+1})$.

Following the standard methodology, the first step in determining a firm’s optimal investment strategy is to decompose firm value into two pieces, the expected value of cash flows received prior to the time at which the next development occurs and the current discounted expected value of the firm at the time of development. That is, firm value is given by

$$V(q, P_t, \overline{P}_t) = E_t \left[ \int_t^{\overline{P}_q} e^{-r(s-t)} q P_s ds + e^{-r(\tau_q-t)} W(q, P^*_q) \right],$$

Figure 1: **Price Paths**

The top path is a standard, partial equilibrium, geometric Brownian price process. The bottom path follows reflected geometric Brownian motion. The middle path is the equilibrium price process, and follows "partially reflected" geometric Brownian motion. The degree of reflection depends on the cost-to-scale of adding new capacity and the price elasticity of supply.
where $P^*_q$ denotes the price level at which the firm next optimally increases capacity, $	au_{P^*_q} = \min\{s \geq t | P_s = P^*_q \}$ denotes the first passage of the price process to this exercise boundary, and $W(q, P^*_q) = V(q, P^*_q)$, which we will refer to as the “value-at-max” function, is shorthand for the value of a firm with capacity $q$ when the price of firms’ output is at the historical maximum $P^*_q$. We have implicitly used the fact that firms will only choose to develop at new price maxima because the price process is Markovian below the maxima.

Because firms will only choose to add new capacity at price maxima it will be sufficient to restrict our attention to these times, in which case we may again decompose firm value into two pieces: the expected value of cash flows received up until the date of the next development, and the present expected value of receiving the firm at that date. That is, we can write firm value as

$$W(q, P_t) = E_t \left[ \int_t^{\tau_{P^*_q}} e^{-r(s-t)} q P_s ds \right] + E_t \left[ e^{-r(\tau_{P^*_q} - t)} \right] W(q, P^*_q)$$

(5)

where we have used the fact that $P^*_q$ is non-stochastic to take $W(q, P^*_q)$ out of the expectation.

Explicit calculation of the right hand side of equation (5) is complicated by the fact that the price of firms’ output is not an Itô process, which prevents direct application of the standard, continuous time machinery. Nevertheless, standard techniques may be applied indirectly to yield a closed form solution, given in the following proposition.

**Proposition 3.3** The value of a firm with existing capacity $q$ when the unit price of the firm’s output is at an all time high $P_t$ is given by

$$W(q, P_t) = \Pi q P_t + \left( \frac{P_t}{P^*_q} \right)^\eta (W(q, P^*_q) - \Pi q P^*_q)$$

(6)

where $\Pi = \frac{\pi}{1 + \frac{1}{\alpha \gamma} (\beta - 1)}$ and $\eta \equiv \left( 1 + \frac{1}{\alpha (\gamma - 1)} \right) \beta$.

The first term on the right hand side of equation (6) is the value of the future cash flows to the firm’s existing assets, or the firm’s “intrinsic value.” The second term, the value of the firm’s future development opportunities or “option value,” is the firm value in excess of the intrinsic value of assets in place on the date of redevelopment, times the state price for the development date. Currently both firm value on the date of redevelopment and the date at which this development is optimally undertaken are unknown, and depend on the development strategy employed by the firm.
Explicitly determining these values, and the firm’s optimal investment strategy, may be accomplished using standard techniques. The functional form of the dependency of firm value on the output price can be determined by simple inspection of equation (6). Letting

\[ a_q = \left( W(q, P_q^*) - \Pi q P_q^* \right) / P_q^* \]

we have that

\[ W(q, P) = \Pi q P + a_q P^\eta. \]  

(7)

This, in conjunction with the fact that a firm chooses its redevelopment strategy to maximize firm value, defines a free boundary problem. The solution to this problem is the firm’s optimal strategy.

The problem may be solved using standard procedures. Optimality and feasibility concerns require value matching and smooth pasting conditions hold at the time of development, \( i.e., \)

\[ W_1(q, P_q^*) \quad = \quad W_2(q_q^*, P_q^*) - q_q^{*\gamma} \]  

(8)

\[ W_1^*(q, P_q^*) \quad = \quad W_2^*(q_q^*, P_q^*) \]  

(9)

where \( q_q^* \) denotes the optimal capacity to which a firm with existing capacity \( q \) develops when prices reach \( P_q^* \), and \( W_1 \) and \( W_2 \) are the value-at-max functions before and after development, respectively.

Solving these explicitly is complicated by the fact that firm value is a “compound option.” Development rights are retained at exercise, so post development firm value still includes a component due to the right to develop further in the future. The fact that firms retain development rights, however, itself implies a boundary condition that may be used to determine the optimal development strategy explicitly. The explicit strategy is given in the following proposition.

**Proposition 3.4** Suppose the price of firms’ output follows the partially reflected geometric Brownian process given by equation (2), \( i.e., \)

\[ P_t = P_0 X_t X_t(1+\sigma^{1/(\gamma-1)}) \]

where \( d \ln X_t = \mu dt + \sigma dz_t \). Then the optimal strategy for a firm with existing capacity \( q \) is to redevelop to capacity \( q_q^* = q_1^* \cdot q \) when prices reach \( P_q^* = q^{(\gamma-1)} P_1^* \), where

\[ P_1^* = \frac{\eta}{(\eta - 1) \Pi q_1^* - 1}. \]

(10)
\( q_1^* \) is the unique root of

\[
(y \eta - \gamma - \eta)x^{(y \eta - \gamma - \eta + 1)} - (y \eta - \gamma)x^{(y \eta - \gamma)} + \eta = 0
\]

in the interval \( (1 + \frac{\eta - 1}{y \eta - \gamma - \eta + 1}, 1 + \frac{\eta}{y \eta - \gamma - \eta}) \), and

\[
\eta = \left(1 + \frac{1}{\alpha(y - 1)}\right) \beta
\]

\[
\Pi = \frac{x}{1 + \alpha(y - 1)/\beta - 1}
\]

\[
\beta = \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2} - \left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)}.
\]

Propositions 3.2 and 3.4 establish both elements required to demonstrate an equilibrium strategy. In Proposition 3.4, firms optimally follow the strategy of developing to \( q_1^* \) times their existing capacity so the biggest firm is \( q_1^* \) times as big as the smallest, i.e., \( q_1^M = q_1^* q_1^m \) or \( q_1^* = q_1^M / q_1^m \).\footnote{In an abuse of notation, made for the sake of convenience, we have used \( q_1^* \), defined as the capacity to which a firm with unit existing capacity will optimally redevelop, to also denote a unitless multiplier. The meaning will always be clear from the context.} This strategy is consistent with the strategy hypothesis of Proposition 3.2, which generated the price process assumed in Proposition 3.4. The price process given in Proposition 3.2 and the strategy given in Proposition 3.4 consequently define an equilibrium.

Note that the optimal equilibrium development multiple \( q_1^* \), essentially a measure of heterogeneity in the economy, is not arbitrary, but a fixed constant that depends on the primatives of the economy. It is useful to think about how this parameter varies with the with the economic primatives. Figure 2 shows the development multiples dependency on interest rates, the growth rate and volatility of the multiplicative demand multiple, and the cost to scale of developing capacity.

While we were primarily concerned with identifying an equilibrium, the methodology employed does imply a limited uniqueness. In particular, the equilibrium presented here is the unique equilibrium for which the price process has some “nice” properties. The exact nature of these properties is clarified by introducing the following concept. Think of the ratio \( P_t / \overline{P}_t \) as a proxy for the business cycle. When this ratio is close to one then near term increases in aggregate industry capacity are likely, and we identify these times with expansions. Similarly, when this ratio is significantly less than one then near term increases in aggregate industry capacity are unlikely, and we identify these times with recessions. Using this concept, the equilibrium presented in this paper is the unique equilibrium for

Figure 2: **Comparative Statics for the Optimal Development Multiple**

The optimal development density multiple, \( q^*_1 \), as a function of the economy’s primitives. The optimal development multiple is a) decreasing in the interest rate (top right), as higher rates decrease the marginal value of development, b) increasing in demand growth for the industry good (top right), as higher growth rates reduce the likelihood of weak future demand encouraging greater investment, c) increasing in uncertainty regarding future demand (bottom left), as higher uncertainty incentivizing firms to delay investment, and d) decreasing, not surprisingly, in the cost to scale of development (bottom right). Default parameter values are \( \mu = .05, \sigma = .18, \alpha = 1, \) and \( \gamma = 1.5 \).

which the evolution of the log-price is independent of its current level, except through possible dependence through the business cycle proxy. The following propositions presents this result formally.

**Proposition 3.5** The strategy given in Proposition 3.4 is the unique equilibrium strategy for which the instantaneous return distribution for the price of firms’ output depends only on the relative level of the price to the historical maximum, i.e., for which \( d \ln P_t \) is a homogenous, degree-zero function of \( P_t \) and \( \overline{P}_t \).

Finally, Proposition 3.4 also allows for the equilibrium valuation of firms. The value of an arbitrary firm is given explicitly in the following proposition.
Proposition 3.6 The value of a firm with existing capacity $q$ when the price of the firm’s output is $P_t$ and the historical maximum for prices is $\overline{P}_t \leq P^*_q$ is given by

$$V(q, P_t, \overline{P}_t) = \pi qP_t - \left( \frac{P_t}{\overline{P}_t} \right)^\beta (\pi - \Pi) q \overline{P}_t + \left( \frac{P_t}{\overline{P}_t} \right)^\eta \frac{A q^\gamma}{P_t}$$

(12)

where the parameters are those given in Proposition 3.4.

The first term in equation (12), $\pi qP_t$, is the value of the cash flows from the current project, ignoring supply effects on prices. The second term, $\left( \frac{P_t}{\overline{P}_t} \right)^\beta (\Pi - \pi) q \overline{P}_t$, corrects for these supply effects, and is negative reflecting the fact that new capacity puts downward pressure on the price of firms’ output. The last term, $\frac{A q^\gamma}{P_t} P_t$, is the option value of the project, which derives from the firm’s ability to increase capacity in the future.

4 Implications

The preceding analysis yields several positive implications that we will explore in this section. These concern 1) properties of the equilibrium price of firms’ output, 2) the evolution of firm value, i.e., stock returns, 3) equilibrium real option premia, and 4) firms’ optimal investment strategy, especially as it compares to the predictions of the standard, partial equilibrium analysis.

With respect to the price of firms’ output, the model predicts negative skewness, resulting from endogenous increases in capacity. Because capacity is added at price maxima, this skewness is especially pronounced when prices are high. In fact, at maxima in the price of firms’ output we always expect prices to drop in the short term, and for some parameterizations to remain low far into the future. As firms only add capacity when the price of their output is high this means that firms always develop expecting prices to drop.

The skewness in the price of firms’ output is translated into stock prices, but only partially. Firm value is convex in the price of the firm’s output, and this tends to skew stock prices positively. When the price of firms’ output is low output prices are essentially unskewed, and the effect of convexity dominates. At these times stock prices are positively skewed. When the price of firms’ output is high, however, the effect of the strong negative skewness in the price of firms’ output dominates. At these times stock prices are also negatively skewed. That is, in recessions we expect to see positive skewness in stock returns, while during expansions we expect to see negative skewness.
The analysis also demonstrates that accounting for the price impact of development leads firms to develop later. That is, in equilibrium firms delay capital improvements longer than the standard, partial equilibrium analysis prescribes. Finally, we show that competition does not erode option premia. Even after properly accounting for the impact of competition, future development rights can contribute just as large a fraction of overall project value as they do in the standard analysis that ignores competition.

4.1 The Price of Firms’ Output

The negatively skewed equilibrium return to firms’ output is itself a major prediction of the model and provides testable implications. Because we characterize the equilibrium price process analytically we can make specific predictions regarding properties of the skewness. The model predicts the degree of skewness to be a function of both industry cost structure and industry history. Cross sectionally, we would expect more pronounced price skewness in industries where the supply elasticity of demand is high, i.e., prices should be more skewed in industries where the costs of adding capacity are low. We would expect more pronounced price skewness when the demand for firms’ output is elastic. In the time series, we expect to see less price skewness at short horizons. The model predicts supply to be more responsive when prices are near historic highs. At lower prices, when firms are less likely to add capacity, skewness should be less pronounced, and return skewness should tend to zero as the horizon becomes very short. At price maxima, on the other hand, skewness in prices should be especially pronounced.

Forward prices provide the most convenient method for studying the price process in greater detail. The forward price is an unbiased estimator of the risk adjusted future spot price. The closed form expression for the forward price of firms’ output is provided in the next proposition. It is somewhat complicated, but studying the behavior of some of its asymptotic properties proves illuminating.

**Proposition 4.1** The equilibrium instantaneous t-ahead forward price of the good (i.e., the
expected future spot price) is given by

$$F_{t_0+t} = P_{t_0}e^{\mu t} \left( N \left( \frac{\ln \left( \frac{P_{t_0}}{P_{t_0}} \right)}{\sigma \sqrt{t}} \right) \right)$$

$$+ \theta N \left( -\ln \left( \frac{T_{t_0}}{P_{t_0}} \right) - \left( \mu + \frac{\alpha^2}{2} \right) t \right) \left( \frac{\bar{P}_{t_0}}{P_{t_0}} \right)^{\left( \frac{1+2\mu}{\alpha^2} \right)}$$

$$+ (1 - \theta) e^{-\left( \frac{\alpha^2}{2(1+\alpha(\gamma-1))} \right) \left( \mu + \frac{\alpha^2}{2} \right) t} N \left( -\ln \left( \frac{T_{t_0}}{P_{t_0}} \right) + \left( \mu + \frac{\alpha^2}{2} - \frac{\alpha^2}{1+\alpha(\gamma-1)} \right) t \right) \left( \frac{\bar{P}_{t_0}}{P_{t_0}} \right)^{\left( \frac{1}{1+\alpha(\gamma-1)} \right)} \right).$$

(13)

where $\theta = \frac{1}{(1+\alpha(\gamma-1))(1+\frac{2\mu}{\alpha^2})-1}$.

Figure 3, which shows forward prices at horizons out to two years, demonstrates the importance of the price to price-high ratio on forward prices. The figure plots three possible relations between prices and the historical price high: 1) spot prices at the high; 2) spot prices one-sixth below the high; and 3) spot prices one-third below the high. In the figure the cost-to-scale of development is quadratic.

The basic shapes of the forward price curves in Figure 3 are similar to those observed in commodity markets. When the price of firms’ output is at or near historical highs, and increases in aggregate capacity are likely, the term structure of forward prices is downward sloping. When near term increases in capacity are unlikely the forward price curve will be upward sloping. In the parlance of commodity markets, forward prices may be either “backwardated,” with prices for future delivery of the good declining with time-to-delivery, or “in contango,” with prices increasing with time-to-delivery.

Two important special cases of equation (13) results from considering the extremes values of the possible relationships between current prices and the historical price maximum. At one extreme, when the previous maximum of the price process is very large relative to current prices, equation (13) reduces to $P_{t_0}e^{\mu t}$. This results because no building occurs below the maximum. When prices are low it is virtually certain that no development will occur for a significant interval, and prices are then essentially drifted geometric Brownian motion. At the other extreme, when the current spot is at the historical maximum, i.e.,
The equilibrium forward price is the unbiased estimator of the future spot price. In all three curves \( P_0 \), the spot price, is one. In the top curve, the spot price is one-third below the historical maximum, \( \overline{P}_0 = 1.5 \). In the middle curve, the spot price is one-sixth below the historical maximum, \( \overline{P}_0 = 1.2 \). In the bottom curve, the spot price is at the historical maximum, \( \overline{P}_0 = 1 \). The dashed line is the forward price in an economy in which total capacity to produce the good is fixed. Parameter values are \( \mu = .05, \sigma = .18, \alpha = 1 \) and \( \gamma = 2 \).

Figure 3: **Equilibrium Forward Prices**

The equilibrium forward price is the unbiased estimator of the future spot price. In all three curves \( P_0 \), the spot price, is one. In the top curve, the spot price is one-third below the historical maximum, \( \overline{P}_0 = 1.5 \). In the middle curve, the spot price is one-sixth below the historical maximum, \( \overline{P}_0 = 1.2 \). In the bottom curve, the spot price is at the historical maximum, \( \overline{P}_0 = 1 \). The dashed line is the forward price in an economy in which total capacity to produce the good is fixed. Parameter values are \( \mu = .05, \sigma = .18, \alpha = 1 \) and \( \gamma = 2 \).

When prices are at maxima, then in the short term are expected to decrease. These are also, however, precisely the times at which firms add new capacity, so firms optimally develop expecting prices to drop.

It is also useful to consider some of the other asymptotic properties of equilibrium forward price. As the costs-to-scale of building becomes high equation (13) again reduces to \( P_t = \overline{P}_t \). When it is too expensive to build the price process simply becomes drifted geometric Brownian motion. As the cost of building approaches linear forward prices go
to

\[
N \left( \frac{\ln \left( \frac{P_t}{P_{t0}} \right) - (\mu + \frac{\sigma^2}{2\mu}) t}{\sigma \sqrt{t}} \right) + \frac{\sigma^2}{2\mu} N \left( \frac{-\ln \left( \frac{P_t}{P_{t0}} \right) - (\mu + \frac{\sigma^2}{2\mu}) t}{\sigma \sqrt{t}} \right) \left( \frac{P_{t0}}{P_t} \right)^{(1 + \frac{2\mu}{\sigma^2})} P_{t0} e^{\mu t}
\]

\[+ \left( 1 - \frac{\sigma^2}{2\mu} \right) N \left( \frac{-\ln \left( \frac{P_t}{P_{t0}} \right) + (\mu - \frac{\sigma^2}{2\mu}) t}{\sigma \sqrt{t}} \right) \overline{P}_{t0}. \]

That is, the process begins at \( P_{t0} \). In the short run, prices evolve as a geometric Brownian motion, if \( P_{t0} < \overline{P}_{t0} \), but in the long term they reach the steady state expectation of \( (1 - \frac{\sigma^2}{2\mu}) \overline{P}_{t0} \) if \( \mu > \frac{\sigma^2}{2} \), or zero if \( \mu < \frac{\sigma^2}{2} \). When the cost to scale is linear, \( \overline{P}_{t0} \) is a reflecting barrier on prices. The factor \( (1 - \frac{\sigma^2}{2\mu}) \) captures the tension between the upward drift in the process and the variance. If the upward drift is large or the volatility is small, then the price process remains close to the barrier. On the other hand, when the drift is small or the volatility is large then prices, after bouncing off the barrier, may fall a long way prior to recovering.

Finally, we will consider more generally the long run average growth of the spot price, obtained by dividing the log of forward prices from equation (13) by \( t \) and letting \( t \) become large. Doing so in the interesting case, when \( \mu \) is not too small, we find that the average long term rate of expected price growth is

\[
\frac{\alpha(\gamma - 1)}{1 + \alpha(\gamma - 1)} \left( \frac{1}{1 + \alpha(\gamma - 1)} \frac{\sigma^2}{2} \right). \quad (14)
\]

4.2 Stock Returns

Our closed form expression for firm value as a function of the price of firms’ output and aggregate industry capacity allows us to study the distribution of stock returns. Stock returns are a combination of the returns to firms’ ongoing projects and their growth options. Away from historic highs in the price of firms’ output, aggregate capacity is unlikely to increase so the return to firms’ output exhibits little skew. This translates into positive skew in stock returns because the option component of firm value is convex in the price of firm output. When the price of firms’ output is high the effect of increasing aggregate capacity dominates and stock returns exhibit negative skew. This provides a hitherto untested empirical prediction: skewness in stock returns should vary over the business cycle. Stock returns should be negatively skewed during expansions, but positively skewed in recessions.
**Proposition 4.2** Away from maxima in the price of firms’ output stock returns are positively skewed at sufficiently short horizons. At maxima in the price of firms’ output stock returns are negatively skewed at short horizons.

### 4.3 The Investment Strategy

At this point we can compare the optimal behavior of a firm in the equilibrium economy, where firms’ development decisions affect aggregate capacity, to the optimal behavior of a firm in an economy in which firms’ investment decisions somehow do not affect aggregate capacity, which is implicitly the standard assumption in the literature.\(^\text{12}\)

The natural and instructive comparison is the case in which the expected long run average price growth and expected long run average price variances are the same in both economies.

An econometrician who calibrates a real options model under the assumption, standard in the literature, that the price of the firms’ output follows geometric Brownian motion, will be surprised to find firms delaying irreversible investment even longer than his model predicts is optimal. This occurs because, in addition to the delay related to option effects, the equilibrium return to firms’ output is actually negatively skewed.\(^\text{13}\) How this leads firms to delay investment even longer can be understood in the following way. Option value derives from the ability to avoid some investments that would have been mistakes \textit{ex post}. At the zero NPV threshold an investor may do two things: invest or delay. The choice to delay roughly reflects the fact that, at the current price level, the \textit{ex post} benefit to having not invested if the economic environment takes a turn for the worse exceeds the \textit{ex post} benefit to having invested earlier if the environment takes a turn for the better. Investment occurs when prices are just high enough that the loss-avoidance benefits to delaying balance the gains from not delaying. Asymmetric downside risk exceeding upside potential induces firms to delay option exercise longer. The oversized downside risk results in the potential \textit{ex post} benefits to delaying investment balancing the \textit{ex post} benefit to not delaying investment at a higher price level. The results of this comparison are summarized in the next proposition.

**Proposition 4.3** In equilibrium a firm redevelops a project to the same capacity as she

\(^{12}\)Alternatively, we are comparing optimal behavior in the equilibrium economy to the behavior of a small, price-taking firm in an economy in which no other firms have the ability to develop.

\(^{13}\)By return to firms’ output we simply mean the log change in prices. There is no sense in which this is a holding period return to a real asset.
would in a fixed capacity economy with the same long-run average price growth and variance. The development occurs later, however, in equilibrium.

4.4 The Value of the Option to Delay Investment

Our analysis also shows that competition leaves option premiums unmitigated. Competition erodes option values, but also erodes the value of the cash flows from assets in place. As a result, the premia, or percent of total option value attributable to future development opportunities, is undiminished relative to an economy in which prices follow geometric Brownian motion and have the same average long-run price growth and variance. That is, option premia are unmitigated relative to the partial equilibrium analysis that ignores competition.

Refer to the first term in the value-at-max function as the “intrinsic value” of the project, denoted by \( I(q, P) = \Pi qP \), and the second term as the “option value,” denoted by \( O(q, P) = Aq^\gamma (P / P^*_q)^\eta \).

**Proposition 4.4** The maximum ratio of option value to intrinsic value is given by

\[
\max_P \left( \frac{O(q, P)}{I(q, P)} \right) = \frac{(q^*_1 - 1)}{\eta \left( 1 - q^*_1^{(\gamma + \eta - \gamma \eta)} \right)}.
\]

This is strictly positive, implying that option values remain significant even when firms account for the impact of competition. The ratio exceeds one for some parameter choices, and the majority of a project’s value may reside in the option.

We would like to compare the equilibrium economy to a fixed capacity economy with the same long-run average price growth and long-run average price variance. Noting that the fixed capacity economy is simply the limit of the equilibrium economy as the cost to scale of building becomes very large provides an easy way to do so. Let overscore tildes denote parameters relating to the fixed capacity economy, and choose \( \mu \) and \( \sigma \) such that this economy has the same long-run average price growth and variance as the equilibrium economy. Then the maximal ratio of option value to intrinsic value in the fixed capacity economy is simply the right-hand side of equation (15) with the \( \eta \)’s replaced by \( \tilde{\beta} = \sqrt{\left( \frac{\mu}{\sigma^2} - \frac{1}{2} \right)^2 + \frac{2r}{\sigma^2} - \left( \frac{\mu}{\sigma^2} - \frac{1}{2} \right)} \). Finally, using the fact that \( \tilde{\beta} = \eta \), proved in the appendix as part of proposition 4.3, yields the following corollary.
Corollary 4.1 The maximal ratio of option value to intrinsic value in the equilibrium economy is the same as in a fixed capacity economy with the same long run average price growth and variance.

5 Conclusion

We characterize firms’ optimal investment strategy explicitly, and derive a closed form solution for firm value. We show that in the strategic equilibrium real option premia are significant. As a result firms delay investment, choosing optimally not to undertake some positive NPV projects. In fact, properly accounting for the effects of competition results in a negatively skewed equilibrium process for the price of firms’ output, which leads firms to delay investment longer than predicted by the standard options analysis. Finally, the equilibrium analysis in this paper has implications for equity markets. The model presented here predicts that firm returns should vary over the business cycle. Firm returns should be negatively skewed during expansions, but positively skewed in recessions.

These conclusions differ from those in previous studies of competition on option exercise, which have tended to conclude that competition reduces option premiums and leads to earlier investment. Earlier studies have considered the effects of oligopolistic competition and have quite naturally, therefore, compared valuation and optimal timing of investment to the case of a hypothetical strategic monopolist. While their conclusions are valid in this context, they should not be used to draw conclusions with respect to the standard analysis, which ignores competition. The standard analysis is not a fundamentally better yardstick than the strategic monopolist, but it is the theory that has informed the intuition which is now widely held, and the fact that our methodology allows for comparison to the standard analysis is, therefore, a great advantage.

The results in this paper differ from those in earlier papers because we consider heterogeneous firms facing opportunity costs to investing. While it has long been known that demand side heterogeneity can reduce competition, in this paper we demonstrate that supply side heterogeneity can have the same effect, though for different reasons. Demand side heterogeneity—variable consumer preferences—allows firms to segment a market, through horizontal product differentiation, and extract more of the consumer surplus. Supply side heterogeneity—variable opportunity costs—provides a natural ordering to firms’ investment decisions, allowing firms to act sometimes as local monopolists and extract more of the consumer surplus. The model predicts a “life-cycle” to firms investment decisions, because firms that have invested recently find it unprofitable to invest again for some
time. This prevents them from competing over new opportunities, and significantly limits the role of competition.

A Appendix

Proof of Proposition 3.1

Homogeneous firms will not choose to undertake the same investment at the same time, because if all firms were to follow such a strategy an individual firm would find it more profitable to deviate either by developing sooner than other firms, or by delaying investment.

Suppose firms do follow a symmetric strategy. That is, suppose firms are homogeneous, possessing the same initial capacities $q$, and follow the same development strategy. That is, at some point all firms will develop the same new capacity at the same time. Until this time aggregate capacity is fixed, so the price of firms’ output evolves like the multiplicative demand shock, as a geometric Brownian motion. The investment problem faced by firms is therefore Markovian up until the time of development, so the investment strategy may be characterized as a trigger strategy on the price of firms’ output. That is, there exist some $P^*$ such that the first time $P_t$, the price of firms’ output, reaches $P^*$ all firms add new capacity, developing to the new capacity $q^*$.

Because the cost of developing new capacity includes the opportunity cost of abandoning old capacity, individual firms do not undertake incremental investment, i.e., $q^*$ is bounded away from $q$. Aggregate capacity would also be lumpy (discontinuous) in a symmetric equilibrium, so the price of firms’ output immediately after development is strictly less than the trigger price, i.e., $P_{t^*} < P^*$, where $t^* = \min\{t > 0 | P_t = P^*\}$.

Because the opportunity cost for firms is higher after they have developed to the new, larger capacity $q^*$ no further development will happen until prices reach some price which is strictly higher than $P^*$. Letting $V_{s,t}$ denote the value of the firm at time $s$ when development to capacity $q^*$ will be undertaken at time $t$, we can write the value of a firm at the date of the initial development as the value of the expected, discounted cash flows from the new project up until prices return to $P^*$ plus the expected value of receiving the project on the date price first return to $P^*$,

$$V_{t^*, t^*} = q^* P_{t^*+\pi} - C(q^*) + E_{t^*} \left[ e^{-r(t^*+\pi-t^*)} \right] (V_{t^*+\pi} - q^* P^* \pi)$$

(16)

where $\pi = \frac{1}{r-\mu}$ is the standard geometric Brownian annuity factor, $t^* = \min\{t > \tau^* | P_t = P^*\}$ is the second passage time for the price process to $P^*$, and $V_{t^*+\pi}$ is the value of having capacity $q^*$ at time $t^*$. The value, at time $\tau^*$, of delaying development to capacity $q^*$ until $\tau^*$ is just

$$V_{t^*, \tau^*} = q^* P_{\tau^*+\pi} + E_{\tau^*} \left[ e^{-r(\tau^*+\pi-\tau^*)} \right] (V_{\tau^*+\pi} - C(q^*) - q^* P^* \pi),$$

(17)

so the value of delaying investment is given by

$$V_{t^*, \tau^*} - V_{t^*, t^*} = C(q^*) \left( 1 - E_{t^*} \left[ e^{-r(\tau^*+\pi-\tau^*)} \right] \right) - (q^* - q) P_{t^*+\pi} \left( 1 - E_{\tau^*} \left[ e^{-r(\tau^*+\pi-\tau^*)} \right] \right) \frac{P^*}{P_{t^*+\pi}}.$$
That is, delaying investment has the advantage of deferring the investment cost, but the disadvantage of
loosing some intermediate cash flows. Whether the net benefit is positive or negative depends on the relative
magnitudes of the two effects.

An identical argument shows that the value of investing early, to capacity \( q^* \) at time \( \tau_0 \equiv \min\{t > 0 | \tau_t = \tau_{t^*+} \} \), is given by

\[
V_{\tau_0, \tau_0} - V_{\tau_0, \tau^*} = (q^* - q) P_{\tau^*+} \left( 1 - E_{\tau_0} \left[ e^{-r(\tau^*-\tau_0)} \right] \frac{P^*}{P_{\tau^*+}} \right) - C(q^*) \left( 1 - E_{\tau_0} \left[ e^{-r(\tau^*\tau_0)} \right] \right)
\]

because \( E_{\tau_0} \left[ e^{-r(\tau^*\tau_0)} \right] = E_{\tau^*} \left[ e^{-r(\tau^*\tau_0)} \right] \).

So unless an investor is indifferent between developing to capacity \( q^* \) at times \( \tau_0, \tau^* \) and \( \tau^{**} \) she has a
strict preference for either deviating from the symmetric strategy and developing either early or late.

If she is indifferent between developing at \( \tau_0 \) and \( \tau^* \), however, she has a strict preference for developing
at \( \tau^{**} \equiv \min\{t > 0 | \tau_t = \sqrt{P^* P_{\tau^*+}} \} \), as shown by the following argument.

If the firm is indifferent between developing at \( \tau_0 \) and \( \tau^* \) then

\[
0 = V_{\tau_0, \tau_0} - V_{\tau_0, \tau^*}
\]

\[
= (q^* - q) P_{\tau^*+} \left( 1 - E_{\tau_0} \left[ e^{-r(\tau^*-\tau_0)} \right] \frac{P^*}{P_{\tau^*+}} \right) - C(q^*) \left( 1 - E_{\tau_0} \left[ e^{-r(\tau^*\tau_0)} \right] \right)
\]

\[
= (q^* - q) P_{\tau^*+} \left( 1 - \sqrt{E_{\tau_0} \left[ e^{-r(\tau^*-\tau_0)} \right]} \right) \frac{P^*}{P_{\tau^*+}} \left( 1 + \sqrt{E_{\tau_0} \left[ e^{-r(\tau^*\tau_0)} \right]} \right) - C(q^*) \left( 1 - \sqrt{E_{\tau_0} \left[ e^{-r(\tau^*\tau_0)} \right]} \right)
\]

\[
= (q^* - q) \pi \left( \frac{P^*}{P_{\tau^*+}} - 1 \right) \left( q^* - q \right) P_{\tau^*+} \left( 1 - \sqrt{E_{\tau_0} \left[ e^{-r(\tau^*\tau_0)} \right]} \right) \frac{P^*}{P_{\tau^*+}} \left( 1 + \sqrt{E_{\tau_0} \left[ e^{-r(\tau^*\tau_0)} \right]} \right)
\]

Now both \( \sqrt{E_{\tau_0} \left[ e^{-r(\tau^*\tau_0)} \right]} \left( \frac{P^*}{P_{\tau^*+}} - 1 \right) \left( q^* - q \right) P_{\tau^*+} \left( 1 - \sqrt{E_{\tau_0} \left[ e^{-r(\tau^*\tau_0)} \right]} \right) \frac{P^*}{P_{\tau^*+}} \left( 1 + \sqrt{E_{\tau_0} \left[ e^{-r(\tau^*\tau_0)} \right]} \right) \)

are strictly positive, so equation (20) implies

\[
(q^* - q)\pi \left( P_{\tau^*+} - \sqrt{E_{\tau_0} \left[ e^{-r(\tau^*\tau_0)} \right]} P^* P_{\tau^*+} \right) - C(q^*) \left( 1 - \sqrt{E_{\tau_0} \left[ e^{-r(\tau^*\tau_0)} \right]} \right) < 0. \quad (21)
\]

Now let \( P_{gm} = \sqrt{P^* P_{\tau^*+}} \) be the geometric mean of the two price levels. Then using the fact that
\( \sqrt{E_{\tau_0} \left[ e^{-r(\tau^*\tau_0)} \right]} = E_{\tau_0} \left[ e^{-r(\tau^*\tau_0)} \right] \) we have

\[
C(q^*) \left( 1 - E_{\tau_0} \left[ e^{-r(\tau^*\tau_0)} \right] \right) - (q^* - q)\pi P_{\tau^*+} \left( 1 - E_{\tau_0} \left[ e^{-r(\tau^*\tau_0)} \right] \frac{P_{gm}}{P_{\tau^*+}} \right) > 0. \quad (22)
\]
The left hand side of equation (22) is, however, precisely the value of delaying development to capacity \(q^*\) from time \(\tau_s\) to \(\tau_s^*\). That this is strictly positive implies that the firm strictly prefers investing at \(\tau_s^*\) to investing at \(\tau_s\), and as the firm is indifferent between developing at \(\tau_s\) to \(\tau_s^*\) equation (22) also implies that the firm also strictly prefers investing at \(\tau_s^*\) to investing at \(\tau_s\).

That is, if all agents develop to the same capacity at the same time an individual agent can behave more profitably by deviating either by developing earlier or later. Therefore no symmetric equilibrium exists. ■

Proof of Proposition 3.2

Proof of the proposition: Suppose firms’ log-capacities are initially distributed uniformly on \([\ln q_0^m, \ln q_0^M]\), and that each firm follows the strategy of developing to \(\kappa \equiv q_0^M / q_0^m\) times its existing capacity \(q\) whenever the price of firms’ output reaches \((q/q_0^m)^{(\gamma-1)} \bar{P}_0\), where \(\bar{P}_0\) is, by assumption, not less than \(P_0\).

The aggregate supply process is the integral of individual firms’ capacities,

\[
Q_t = \int q_t^M \, dq_v(q).
\]  

The development rule preserves the uniform distribution of log-capacities. That is, at all times \(\bar{P}_t\) is a sufficient statistic for aggregate capacity and the distribution of capacities is \((\kappa)q_t^M\), where \(\ln \kappa\) is distributed uniformly on \((0, 1]\) and

\[
q_t^M = q_0^M \kappa^{\ln \kappa (\gamma-1)} (\bar{P}_t / \bar{P}_0)
\]

\[
= q_0^M (\bar{P}_t / \bar{P}_0)^{\ln \kappa (\gamma-1)} \kappa
\]

\[
= q_0^M (\bar{P}_t / \bar{P}_0)^{\gamma-1},
\]  

so

\[
Q_t = q_0^M \left( \frac{\bar{P}_t}{\bar{P}_0} \right)^{\gamma-1} \int_0^\kappa \xi dq_v(\xi) = \left( \frac{\bar{P}_t}{\bar{P}_0} \right)^{1/(\gamma-1)} Q_0.
\]  

Now the inverse demand function, \(P_t = X_t Q_t^{-1/\alpha}\), is valid everywhere, so at historic price highs, which correspond to historic highs in the multiplicative demand shock, we have

\[
\bar{P}_t = X_t Q_t^{-1/\alpha}.
\]

Substituting for \(Q_t\) using equation (25), we then have that

\[
Q_t = \left( \frac{X_t Q_t^{-1/\alpha}}{\bar{P}_0} \right)^{1/(\gamma-1)} Q_0,
\]

or

\[
Q_t = c X_t^{(\alpha^{-1} + \gamma - 1)^{-1}}
\]  

where \(c = (Q_0 / P_0^{(\gamma-1)^{-1}})^{(1+\alpha(1-\gamma)^{-1})^{-1}}\). That is, the supply elasticity of the aggregate demand maximum (i.e., the multiplicative shock maximum) is constant, and equal to \((\alpha^{-1} + \gamma - 1)^{-1}\). Not surprisingly, supply
is more responsive to demand when the cost to scale of building is low and when demand is inelastic with respect to prices. If the cost to scale of building is high, or if demand is elastic with respect to prices, then supply is less responsive to demand.

Finally, substituting back into the inverse pricing function, we have

$$P_t = X_t \left( \frac{t}{r + \sigma^2 / 2} \right) X_t P_0,$$

(27)

which proves the proposition. ■

Proof of Proposition 3.3

Lemma A.1 Let $Y_t$ be a geometric Brownian motion with any drift, beginning at one. Then the value of cash flows proportional to the process and received until the first time the process reaches $\theta > 1$ is given by

$$E_0 \left[ \int_0^{\tau_\theta} e^{-rt} Y_t \, dt \right] = (1 - \theta^{1-\beta}) \pi. \quad (28)$$

Proof of lemma: We need to show that for all $\mu$

$$E_0 \left[ \int_0^{\tau_\theta} e^{-rt} e^{(\mu - \frac{\sigma^2}{2}) r + \sigma B_t} \, dt \right] = \frac{1 - \theta^{1-\beta}}{r - \mu}, \quad (29)$$

where $\tau_{\theta} \equiv \min\{t > 0 | e^{(\mu - \frac{\sigma^2}{2}) r + \sigma B_t} = \theta \}$. The left hand side of the previous equation can be rewritten in terms of a drifted Brownian motion with unit volatility,

$$E_0 \left[ \int_0^{\tau_\theta} e^{-rt} e^{\sigma (B_t + (\frac{\mu}{\sigma} - \frac{\sigma}{2}) r)} \, dt \right].$$

(30)

Changing measure, to demean the Brownian motion, and using the joint density for the value and the maximum of a standard Brownian then yields

$$\int_{t=0}^{\infty} \int_{m=0}^{\infty} \int_{b=-\infty}^{\infty} e^{-rt} e^{\frac{\sigma b}{\sqrt{2}} \left( \frac{2m - b}{\sqrt{t}} \right)^{\frac{1}{2}} \left( \frac{-2m + b / \sigma}{2} \right)^{\frac{1}{2}}} e^{\frac{(\frac{\mu}{\sigma} - \frac{\sigma}{2}) b}{\sqrt{2 t}} \left( \frac{2m - b}{\sqrt{t}} \right)^{\frac{1}{2}} \left( \frac{-2m + b / \sigma}{2} \right)^{\frac{1}{2}}} \, db \, dm \, dt. \quad (31)$$

Direct integration of (31) requires the fact, from the literature on Bessel functions, that

$$\int_{t=0}^{\infty} e^{-\alpha t} \left( \frac{2m - b}{\pi \sqrt{t}} \right)^{\frac{1}{2}} e^{-\frac{m^2 - m b / \sigma}{\pi \sqrt{t}}} \, dt = 2 e^{-\sqrt{\frac{2 m - b}{t}}} \quad (32)$$

Substituting into (31) with $\alpha = r + \frac{1}{2} \left( \frac{\mu}{\sigma} - \frac{\sigma}{2} \right)^2$ we have

$$2 \int_{m=0}^{\infty} e^{-2 \sqrt{2 r + \left( \frac{\mu}{\sigma} - \frac{\sigma}{2} \right)^2} m \left( \int_{b=-\infty}^{\infty} e^{\left( \frac{\mu}{\sigma} - \frac{\sigma}{2} \right) b + \sqrt{2 r + \left( \frac{\mu}{\sigma} - \frac{\sigma}{2} \right)^2} b} \, db \right) \, dm$$

That the joint density for the value and the maximum of a standard Brownian is given by $\sqrt{2/\pi} ((2m - b) / \sqrt{t}) e^{-((2m - b)^2 / 2t)}$ is a standard result in the probability literature that may be found in any good text on the subject; see, for example, Durrett (1996).
where

Suppose Lemma A.2
that the valuation is simple, and using analytic continuation to argue that the resulting solution is valid for all \( \mu \).

Lemma A.2 Suppose \( Y_t^\delta \) follows partially reflected geometric Brownian motion where \( 1 - \delta \) denotes the degree of reflection. That is, \( Y_t^\delta = \exp(X_t^\delta) \), where \( X_t^\delta = \delta \overline{X}_t - (\overline{X}_t - X_t) \). \( X_t \) is a drifted Brownian motion, \( \overline{X}_t \) is the maximum of the Brownian process up to time \( t \). Then the dollar price of a unit cash flow proportional to the process is given by

\[
\pi_\delta = E \left[ \int_0^\infty e^{-rt} Y_t^\delta \, dt \right] = \left( \frac{\beta - 1}{\beta - \delta} \right) \pi,
\]

subject to the parameter restriction \( \mu < \frac{r}{\delta} + (1 - \delta) \frac{\sigma^2}{2} \), which ensures \( \pi_\delta \) is finite.

Proof of lemma: The dollar price of a unit of cash flow at a price maximum is then given by

\[
\pi_\delta = E_0 \left[ \int_0^\infty e^{-rt} Y_t^\delta \, dt \right].
\]

That this value is finite if and only if \( \mu < \frac{r}{\delta} + (1 - \delta) \frac{\sigma^2}{2} \) will be proved later.

Letting \( \tau_\theta^\delta \) be the stopping time for the first time \( Y_t^\delta \) hits \( \theta > 1 \), we then have

\[
\pi_\delta = E_0 \left[ \int_0^{\tau_\theta^\delta} e^{-rt} Y_t^\delta \, dt \right] + E_0 \left[ \int_{\tau_\theta^\delta}^\infty e^{-rt} Y_t^\delta \, dt \right]
\]

\[
= E_0 \left[ \int_0^{\tau_\theta^\delta} e^{-rt} Y_t^\delta \, dt \right] + Y_{\tau_\theta^\delta} E_0 \left[ e^{-r(\theta - \tau_\theta^\delta)} Y_{\tau_\theta^\delta} \right] E_0 \left[ \int_{\tau_\theta^\delta}^\infty e^{-r(t - \tau_\theta^\delta)} Y_t^\delta \, dt \right]
\]

\[
= E_0 \left[ \int_0^{\tau_\theta^\delta} e^{-rt} Y_t^\delta \, dt \right] + \theta \rho_\delta(1, \theta) \pi_\delta
\]

where \( \rho_\delta(1, \theta) = \rho(1, \theta)^\delta = \theta^{-(\beta/\delta)} \). Rearranging gives

\[
\pi_\delta = E_0 \left[ \int_0^{\tau_\theta^\delta} e^{-rt} Y_t^\delta \, dt \right] \frac{1}{1 - \theta^{1-(\beta/\delta)}}.
\]
In the special case $\delta = 1$ we have, from the Lemma A.1, that

$$\mathbb{E}_0 \left[ \int_0^{\tau_0^1} e^{-rt} Y_t^1 \, dt \right] = \left( 1 - \theta^{1-\beta} \right) \pi. \quad (38)$$

We will now employ the fact that $\tau_0^1$ and $\tau_0^\delta$ have the same distribution. In particular, this implies $\tau_0^\delta$ is distributed the same as $\frac{\tau_0^1}{\sqrt{\sigma}}$. Now consider $\theta = 1 + \epsilon$, where $\epsilon \approx 0$ and positive. Then for all $t \leq \tau_0^\delta$, $Y_t^\delta \approx Y_t^1$, and then

$$\mathbb{E}_0 \left[ \int_0^{\tau_0^\delta} e^{-rt} Y_t^\delta \, dt \right] = (1 + o(\epsilon)) \mathbb{E}_0 \left[ \int_0^{\tau_0^1} e^{-rt} Y_t^1 \, dt \right]. \quad (39)$$

Substituting into the right hand side of this equation using the preceeding equation gives

$$\mathbb{E}_0 \left[ \int_0^{\tau_0^\delta} e^{-rt} Y_t^\delta \, dt \right] = (1 + o(\epsilon)) \left( 1 - \left( \theta^{(1/\delta)} \right)^{1-\beta} \right) \pi$$

$$= \left( \frac{\beta - 1}{\delta} \right) \epsilon \pi + o(\epsilon^2). \quad (40)$$

Substituting into equation (37) and taking the limit as $\epsilon$ goes to zero yields

$$\pi_\delta = \lim_{\epsilon \to 0} \left( \frac{\beta - 1}{\delta} \right) \epsilon \pi + o(\epsilon^2) = \left( \frac{\beta - 1}{\beta - \delta} \right) \pi. \quad (41)$$

The parameter restriction comes from requiring that $\mu$ is small enough that $\pi_\delta < \infty$, which is equivalent to requiring that $\beta > \delta$. We then have

$$\beta > \delta \iff \left( \frac{\mu}{\sigma^2} - \frac{1}{2} \right)^2 + \frac{2r}{\sigma^2} > \left( \delta + \left( \frac{\mu}{\sigma^2} - \frac{1}{2} \right) \right)^2$$

$$\iff \frac{2r}{\sigma^2} > \delta^2 + 2 \left( \frac{\mu}{\sigma^2} - \frac{1}{2} \right) \delta$$

$$\iff \mu < \frac{r}{\delta} + (1 - \delta) \frac{\sigma^2}{2}. \quad \blacksquare$$

Proof of the proposition: Rewriting the value of the intermediate cash flows as the difference between two perpetual cash flows, a long flow beginning immediately and a short flow which begins on the date of redevelopment, we have

$$\mathbb{E}_t \left[ \int_t^{\tau_q} e^{-r(s-t)} q \, ds \right] = \Pi_q P_t - \mathbb{E}_t \left[ e^{-r(\tau_q - t)} \right] \Pi q P^*_q. \quad (43)$$

where $\Pi = \mathbb{E}_t \left[ \int_t^{\infty} e^{-r(s-t)} \frac{P_s}{P_t} \, ds \right]$ is the dollar price of a unit cash flow, at today’s price, derived from a flow of the firm’s output.

The dollar price of a unit cash flow follows immediately from Lemma A.2. The assumed price process
follows partially reflected geometric Brownian motion with \( \delta = \frac{a(\gamma - 1)}{1 + \alpha(\gamma - 1)} \), so

\[
\Pi = \pi \frac{q(\gamma - 1)}{1 + \alpha(\gamma - 1)} = \frac{\pi}{1 + \frac{1}{(1 + \alpha(\gamma - 1))(\beta - 1)}}.
\]

(44)

The value of the state price for the first passage of prices to the development threshold, \( E_t \left[ e^{-r(t_{P,q}^*-t)} \right] \), can be calculated by using the standard results on the underlying process. We can determine the demand that will result in prices reaching the development price threshold \( P_q^* \) quite easily, because we know the evolution of the price of firms’ output as a function of demand. Inspection of the price–demand relation derived in Proposition 3.2 yields

\[
P_{t, P_q^*} = \frac{X_{q}^{r(\gamma - 1)}}{D_{q}^{r(\gamma - 1)}} \left( \frac{X_{t, P_q^*}}{X_{t, P_q^*}} \right) P_t.
\]

(45)

Using the fact that \( P_{t, P_q^*} = P_q^* \), and \( P_q^* \) is a price maximum so \( X_{t, P_q^*} = \bar{X}_{t, P_q^*} \), we have

\[
X_{t, P_q^*} = \left( \frac{P_q^*}{P_t} \right)^{(1 + \frac{1}{\alpha(\gamma - 1)})} X_t.
\]

(46)

Using this price-demand relation we then have that \( \tau_{P,q}^* \equiv \min \{ s \geq t | P_s = P_q^* \} = \min \{ s \geq t | X_s = \left( \frac{P_q^*}{P_t} \right)^{(1 + \frac{1}{\alpha(\gamma - 1)})} X_t \} \), so

\[
E_t \left[ e^{-r(\tau_{P,q}^*-t)} \right] = E_t \left[ e^{-r(\tau_{X_{P_q^*}}-t)} \right] = \left( \frac{X_t}{X_{t, P_q^*}} \right)^{\beta} = \left( \frac{P_t}{P_q^*} \right)^{(1 + \frac{1}{\alpha(\gamma - 1)})} \beta.
\]

(47)

Substituting for \( E_t \left[ e^{-r(\tau_{P,q}^*-t)} \right] \) and \( \Pi \) in equation (43) yields the proposition. ■

**Proof of Proposition 3.4**

**Lemma A.3** Suppose that the evolution of the log-price process only depends on the ratio of the price to the price maximum, i.e., that the distribution of instantaneous returns is a homogenous function of degree zero in \( P_t \) and \( \bar{P}_t \). Then the value-at-max function has the following scaling property:

\[
W(q, P) = q^{\gamma} W(1, q^{1-\gamma} P).
\]

(48)

*Proof of lemma:* Cash flow considerations imply the structure. An owner should be indifferent between holding 1) one project at a given capacity, price and price-maximum, and 2) more properties developed to lower capacities when the price and price-maximum are somewhat lower.

Suppose there are two firms, one with capacity \( q \) in an economy where the price of the good is \( P_t \) and the historically high price is \( \bar{P}_t \), and the other with unit capacity in an economy where the price of the good is \( \bar{P}_t = q^{1-\gamma} P_t \) and perfectly correlated with prices in the first economy. Then the instantaneous cash flows are \( qP_t \) in the first economy and \( q^{1-\gamma} P_t \) in the second. If either firm follows an optimal redevelopment
strategy the other firm can mirror the strategy by building developing at exactly the same time and keeping the new capacities in the ratio of \( q \) to one. The cost of following such a strategy is in the ratio of \( q^{\gamma} \) to one, and preserves the ratio of cash flows form production of the good in the same ratio. That is, the ratio of the cash flows from the project are always in the ratio \( q^{\gamma} \) to one, so firm values are as well,

\[
V(q, P_t, \overline{P}_t) = q^{\gamma} V(1, q^{1-\gamma} P_t, q^{1-\gamma} \overline{P}_t). \tag{49}
\]

Now we do not actually need that the prices are perfectly correlated, but only that the finite dimensional distributions of \( P_s \) and \( q^{\gamma-1} \overline{P}_s \) are the same for \( s > t \), which follows from the assumption that the evolution of the log-price process is a homogenous degree-zero function of \( P_t \) and \( \overline{P}_t \).

The scaling condition on the value-at-max function is inherited directly from equation (49).

**Proof of the proposition:** Lemma A.3 implies a relation between the investment behavior of an arbitrary firm and that of the firm with unit capacity, and consequently between the investment behaviors of two arbitrary firms. In particular, it implies a multiplicative investment rule, in which if the firm with unit capacity optimally redevelops to capacity \( q_1^* \) then a firm with capacity \( q \) optimally redevelops to capacity \( q_1^* \cdot q \). It is sufficient, consequently, to restrict attention to the strategy of a firm with unit capacity.

Substituting the functional form for the value-at-max function given by equation (7), \( W(q, P) = \Pi q P + a_q P^n \), into the value matching and smooth pasting conditions, equations (8) and (9), yields

\[
\Pi(q - 1) P_1^* = (a_1 - a_q) P_1^{*n} + q^{\gamma} \tag{50}
\]

\[
\Pi(q - 1) = \eta (a_1 - a_q) P_1^{*(n-1)}. \tag{51}
\]

Solving these immediately yields

\[
(a_1 - a_q) = \frac{q^{\gamma}}{(\eta - 1) P_1^{*n}} \tag{52}
\]

\[
P_1^* = \frac{\eta (a_1 - a_q)}{(\eta - 1) \Pi (q - 1)}. \tag{53}
\]

We need one additional constraint to pin down the three variables. The standard constraint, common assumption in the literature, is that \( a_q = 0 \). This constraint is generated by assuming a project may be developed one time only, so is unavailable here.

An appropriate constraint is, however, provided by the scaling condition on the value function, Lemma A.3. Because development rights are retained at exercise, the scaling condition holds across development boundaries. Using this condition we have

\[
q^{\gamma} W^1 \left( 1, q^{(1-\gamma)} P^* \right) = W^2 (q, P^*). \tag{54}
\]

Using the functional form for \( W^1 \) and \( W^2 \) given by equation (7), we then have

\[
q^{\gamma} \left( \Pi \left( q^{(1-\gamma)} P^* \right) + a_1 \left( q^{(1-\gamma)} P^* \right)^{\eta} \right) = \Pi q P^* + a_q P^{*n}, \tag{55}
\]

15 The investment behavior relationship implied by Lemma A.3 can actually be used to motivate the strategy hypothesis of proposition 3.2.
which, solving for \(a_q\), yields
\[
a_q = q^{r+(1-r)\eta} a_1. \tag{56}
\]

In conjunction with the equations for \((a_1 - a_q)\) and \(P_1^t\) this determines the value of a project in terms of the choice variable, \(q\). In particular, we have that
\[
a_1 = \frac{q^r}{(1 - q^{r+(1-r)\eta})(\eta - 1)P_1^{\eta}} = \frac{\Pi^\eta}{(\eta - 1) \left( \frac{\eta}{q^{r+(1-r)\eta}} \right)} \left( \frac{q - 1}{} \right). \tag{57}
\]
The optimal strategy is the one that maximizing firm value, so the redevelopment capacity must maximize \(a_1\).
\[
q_1^* = \arg\max_q \left\{ \frac{(q - 1)^\eta}{q^\eta(q^\eta q^{-\eta - 1})} \right\}. \tag{58}
\]
Let \(F(x) = \frac{(x-1)^\eta}{x(x^\eta q^{-\eta - 1})}\). Now \(q_1^*\) is greater than one and maximizes \(F(x)\), so \(F'(q_1^*) = 0\). Simple algebra shows that \(F'(q_1^*) = 0\) if and only if
\[
(y\eta - \eta - \eta)x^{\eta - \eta - 1} - (y\eta - \eta)\chi^{\eta - \eta - 1} + \eta \neq 0. \tag{59}
\]
Finally, to get the bounds on \(q_1^*\), let \(f(x) = (y\eta - \eta - \eta)x^{\eta - \eta - 1} - (y\eta - \eta)\chi^{\eta - \eta - 1} + \eta\), and note that the restriction on the drift implies that \(\eta > 1\). Because \(f'(x) = 0\) if and only if \(x = 0\) or \(x = \frac{y\eta - \eta}{y\eta - \eta - 1}\), which is strictly greater than one, \(f(x)\) can have at most two positive solutions. Since \(x = 1\) is a solution, it can have at most one other. Since \(f\left(\frac{y\eta - \eta}{y\eta - \eta - 1}\right) = \left(\frac{y\eta - \eta}{y\eta - \eta - 1}\right)^\eta - \eta - 1\), \(f(x)\) is clearly positive for sufficiently large \(x\), \(f(x)\) must have a root greater than \(\frac{\eta}{\eta - \eta - 1}\). That is, \(f(x)\) has a unique root greater than one. This is the only value greater than one for which \(F'(x) = 0\), so it must maximize \(F(x)\).

For the upper bound we will compare the behavior here to that of a firm that lacks redevelopment rights making the development decision in the same economy. Lacking redevelopment rights results, mathematically, in solving the free-boundary problem of proposition 3.4 using the constraint \(a_q = 0\) instead of the constraint implied by redevelopment rights, i.e. that implied by the scaling condition. Doing so, we get that the firm lacking redevelopment rights develops to a capacity \(x_1^\ast = \frac{\eta \eta - \eta}{y \eta - \eta - 1}\), where this comes from \(\arg\max_x \{G(x)\}\) for \(G(x) = \frac{1}{x^{\eta - \eta - 1}}\). Now letting \(H(x) = \frac{1}{1-x^{\eta - \eta - 1}}\), where we have \(F(x) = G(x)H(x)\), so \(F'(x) = G'(x)H(x) + G(x)H'(x)\). But if \(x > 1\) then \(G(x) > 1\), \(H(x) > 1\) and \(H'(x) < 1\), and if \(x > x_1^\ast\) then \(G'(x) < 1\). As a consequence, \(F'(x) < 0\) for all \(x > x_1^\ast\), so \(q_1^* < x_1^\ast\). □

**Proof of Proposition 3.5**

**Proof of the proposition:** Follows directly from the proof of Proposition 3.4. If \(d \ln P_t\) is a homogenous, degree-zero function of \(P_t\) and \(\overline{P}_t\) then \(V(q, P_t, \overline{P}_t) = q^r V(1, q^{1-r} P_t, q^{1-r} \overline{P}_t)\). This scaling condition on the value function implies a multiplicative investment rule of the type found in Proposition 3.4. For more details see the proof of the Proposition 3.4.
Proof of Proposition 3.6

The value of an arbitrary firm is given by

\[
V(q, P, \overline{P}_t) = E_t \left[ \int_t^\tau e^{-r(s-t)} qP_s ds \right] + E_t \left[ e^{-r(\tau-t)} \right] W(q, P^\ast_t)
\]

\[
= \pi q P_t - \left( \frac{P_t}{\overline{P}_t} \right)^\beta \pi q \overline{P}_t + \left( \frac{P_t}{\overline{P}_t} \right)^\beta W(q, \overline{P}_t).
\]

The functional form of the value-at-max is given by equation (7), \( W(q, P) = \Pi q P + a_q P^\eta \). After substituting for \( \Pi \) Propositions 3.4 and \( a_q \) from the proof of Propositions 3.4 we have

\[
W(q, P) = \Pi q P + (a_1 P^{\ast\eta}) q^\gamma \left( \frac{q^{1-\gamma} P}{P^{\ast}} \right)^\eta.
\]

Substituting this into the previous equation with \( P = \overline{P}_t \) yields the proposition, except for the parameter restriction.

The parameter restriction is needed to guarantee that firms have finite value. That is, \( \mu \) must be small enough such that \( A < \infty \), which is equivalent to demanding that \( \gamma \eta - \gamma - \eta > 0 \). Rearranging and using the definition of \( \eta \) results in \( \beta > \frac{\alpha \gamma}{1+\alpha(\gamma-1)} \). Substituting \( \delta = \frac{\alpha \gamma}{1+\alpha(\gamma-1)} \) yields the final parameter restriction:

\[
\mu < \left( 1 - \frac{\alpha - 1}{\alpha(\gamma - 1)} \right) r - \left( \frac{\alpha - 1}{1 + \alpha(\gamma - 1)} \right) \sigma^2.
\]

Proof of Proposition 4.1

The forward price is just the expected future price so, using the relationship between prices and the demand shock, is given by

\[
F_{t_0+t} = E_{t_0} \left[ P_{t_0} X_{t_0+t} \left( \frac{\overline{X}_{t_0}}{X_{t_0+t}} \right)^{\frac{1}{1+\alpha(\gamma-1)}} \right]
\]

(63)

where we have assumed with out loss of generality that \( X_{t_0} = 1 \), but allowed for the possibility that \( \overline{X}_{t_0} = \frac{\overline{P}_{t_0}}{P_{t_0}} > 1 \).

Now \( X_{t_0+t} \) is just a drifted Brownian motion, so distributed \( \text{Exp}[\mu t + \sigma \sqrt{t} \chi] \), where \( \chi \) is the standard normal. Using this in conjunction with the joint density for the value and the maximum of a standard Brownian up to time \( t \), and a change of measure, we have that the \( t \) ahead forward price is given by

\[
\int_{m=0}^\infty \int_{b=-\infty}^m P_{t_0} e^{\sigma b} \left( \frac{\overline{P}_{t_0}}{\max(\overline{P}_{t_0}, P_{t_0} e^{\sigma m})} \right)^{\frac{1}{1+\alpha(\gamma-1)}} e^{\left( \frac{\gamma-\alpha}{2} \right) b - \left( \frac{\gamma-\alpha}{2} \right)^2 \sigma^2} v(b, m) db dm
\]

(64)

where \( v(b, m) = \sqrt{\frac{2}{\pi}} \frac{2m-b}{\sqrt{1 + 4m^2}} e^{-b^2/(2m)} \) is the joint density for the value and the maximum of a standard
Brownian. Rearranging the previous equation yields

$$P_{t_0} e^{-\left( \frac{\sigma^2}{2} \right) t} \int_{m=0}^{\infty} \left( \frac{P_{t_0}}{\text{max}(P_{t_0}, P_{t_0}e^{\sigma m})} \right)^{1+a/\sigma - 11} \int_{b=-\infty}^{m} e^{\left( \frac{\mu}{\sigma} + \frac{\sigma}{2} \right) b} \gamma(b, m) db \, dm. \quad (65)$$

The interior integral, after completing the square in the exponential and simplifying, yields

$$e^{\left( \frac{\mu}{\sigma} + \frac{\sigma}{2} \right) \frac{b^2}{2}} \int_{b=-\infty}^{m} \left( \sqrt{\frac{2}{\pi}} \frac{2m - b}{t^{1/2}} e^{-\frac{(b-(2m+\left( \frac{\mu}{\sigma} + \frac{\sigma}{2} \right) t)^2}{2t}} \right) db. \quad (66)$$

The integration may be performed using

$$\frac{2m - b}{t^{1/2}} = -\frac{1}{\sqrt{t}} \left( b - (2m + \left( \frac{\mu}{\sigma} + \frac{\sigma}{2} \right) t \right) - \frac{\mu}{\sigma} + \frac{\sigma}{2}, \quad (67)$$

in which case we have

$$\int_{b=-\infty}^{m} \left( \sqrt{\frac{2}{\pi}} \frac{2m - b}{t^{1/2}} e^{-\frac{(b-(2m+\left( \frac{\mu}{\sigma} + \frac{\sigma}{2} \right) t)^2}{2t}} \right) db
\begin{equation}
= -\sqrt{\frac{2}{\pi t}} \int_{b=-\infty}^{m} \left( b - (2m + \left( \frac{\mu}{\sigma} + \frac{\sigma}{2} \right) t \right) e^{-\frac{(b-(2m+\left( \frac{\mu}{\sigma} + \frac{\sigma}{2} \right) t)^2}{2t}} db
\begin{equation}
= -2 \frac{\mu}{\sigma} + \frac{\sigma}{2} \int_{b=-\infty}^{m} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(b-(2m+\left( \frac{\mu}{\sigma} + \frac{\sigma}{2} \right) t)^2}{2t}} db
\begin{equation}
= \sqrt{\frac{2}{\pi}} e^{-\left( \frac{\mu}{\sigma} + \frac{\sigma}{2} \right)^2} - 2 \left( \frac{\mu}{\sigma} + \frac{\sigma}{2} \right) N \left( \frac{m - \left( \frac{\mu}{\sigma} + \frac{\sigma}{2} \right) t}{\sqrt{t}} \right). \quad (68)
\end{equation}
\end{equation}
\end{equation}
$$

Substituting back into the integral over the maximum, and looking separately at the regions $m < \frac{1}{\sigma} \ln \left( \frac{P_{t_0}}{P_{t_0}} \right)$ and $m > \frac{1}{\sigma} \ln \left( \frac{P_{t_0}}{P_{t_0}} \right)$, gives forward price as

$$P_{t_0} e^{\mu t} \left( \int_{0}^{\frac{1}{\sigma} \ln \left( \frac{P_{t_0}}{P_{t_0}} \right)} e^{2m\left( \frac{\mu}{\sigma} + \frac{\sigma}{2} \right)} \left( \sqrt{\frac{2}{\pi t}} e^{-\frac{(m(\mu + \frac{\sigma}{2}))}{2t}} \right) \, dm
\begin{equation}
- \int_{0}^{\frac{1}{\sigma} \ln \left( \frac{P_{t_0}}{P_{t_0}} \right)} e^{2m\left( \frac{\mu}{\sigma} + \frac{\sigma}{2} \right)} \left( \frac{\mu}{\sigma} + \frac{\sigma}{2} \right) N \left( \frac{m - \left( \frac{\mu}{\sigma} + \frac{\sigma}{2} \right) t}{\sqrt{t}} \right) \, dm
\begin{equation}
+ \left( \frac{P_{t_0}}{P_{t_0}} \right)^{1+a/\sigma - 11} \left( \int_{0}^{\infty} e^{2(m(\mu + \frac{\sigma}{2}) t^{-1+a/\sigma - 11})m} \left( \sqrt{\frac{2}{\pi t}} e^{-\frac{(m(\mu + \frac{\sigma}{2}))}{2t}} \right) \, dm
\begin{equation}
- \left( \frac{P_{t_0}}{P_{t_0}} \right)^{1+a/\sigma - 11} \left( \int_{0}^{\infty} e^{2(m(\mu + \frac{\sigma}{2}) t^{-1+a/\sigma - 11})m} \left( \frac{\mu}{\sigma} + \frac{\sigma}{2} \right) N \left( \frac{m - \left( \frac{\mu}{\sigma} + \frac{\sigma}{2} \right) t}{\sqrt{t}} \right) \, dm
\begin{equation}
\right) \right). \quad (69)
\end{equation}
\end{equation}
\end{equation}
$$
For the integral over the first region the first term is

\[ 2N \left( \ln \left( \frac{\mathcal{P}_{10}}{\mathcal{P}_{00}} \right) - \left( \mu + \frac{\sigma^2}{2} \right) t \right) \frac{\sigma}{\sqrt{T}} - \frac{\left( \mu + \frac{\sigma^2}{2} \right)}{\sqrt{T}}. \]  

(70)

The second term may be done by parts, and is

\[ e^{2\left( \frac{1}{2} - \frac{m}{\sigma^2} \right)t} N \left( -m - \left( \mu + \frac{\sigma^2}{2} \right) t \right) \frac{\sigma}{\sqrt{T}} + \int_{m=0}^{\frac{1}{\sigma}} \ln \left( \frac{\mathcal{P}_{10}}{\mathcal{P}_{00}} \right) \frac{1}{\sqrt{2\pi t}} e^{-\left( m - \left( \frac{\mu}{\sigma} + \frac{\sigma^2}{2} t \right)^2 \right)} dm. \]  

(71)

Substituting the previous two equations into the integral over the first region yields

\[ N \left( \ln \left( \frac{\mathcal{P}_{10}}{\mathcal{P}_{00}} \right) - \left( \mu + \frac{\sigma^2}{2} \right) t \right) \frac{\sigma}{\sqrt{T}} - N \left( -\ln \left( \frac{\mathcal{P}_{10}}{\mathcal{P}_{00}} \right) - \left( \mu + \frac{\sigma^2}{2} \right) t \right) \frac{\sigma}{\sqrt{T}} \left( \frac{1}{\sigma^2} \right). \]  

(72)

For the other region, when \( m > \frac{1}{\sigma} \ln \left( \frac{\mathcal{P}_{10}}{\mathcal{P}_{00}} \right) \), the integral is

\[ \left( \frac{\mathcal{P}_{10}}{\mathcal{P}_{00}} \right)^{-1 + \alpha (1 - \gamma)} \left( \int_{\frac{1}{\sigma} \ln \left( \frac{\mathcal{P}_{10}}{\mathcal{P}_{00}} \right)}^{\infty} e^{2\left( \frac{1}{2} - \frac{m}{\sigma^2} \right) t} \frac{2}{\sqrt{2\pi t}} e^{-\left( m - \left( \frac{\mu}{\sigma} + \frac{\sigma^2}{2} t \right)^2 \right)} dm \right) + \int_{\frac{1}{\sigma} \ln \left( \frac{\mathcal{P}_{10}}{\mathcal{P}_{00}} \right)}^{\infty} e^{2\left( \frac{1}{2} - \frac{m}{\sigma^2} \right) t} \frac{1}{\sqrt{2\pi t}} e^{-\left( m - \left( \frac{\mu}{\sigma} + \frac{\sigma^2}{2} t \right)^2 \right)} dm \right) \left( \frac{1}{\alpha} \right). \]  

(73)

The first integral, computed by completing the square in the exponent, is

\[ 2e^{-\left( \frac{1}{1 + \alpha(1 - \gamma)} \right) \left( \mu + \frac{\sigma^2}{\gamma - 1} - \frac{1}{1 + \alpha(1 - \gamma)} \right) t} \left( 1 - N \left( \frac{\ln \left( \frac{\mathcal{P}_{10}}{\mathcal{P}_{00}} \right) - \left( \mu + \frac{\sigma^2}{\gamma - 1} - \frac{\sigma^2}{\gamma - 1} \right) t}{\sigma \sqrt{T}} \right) \right) \]  

(74)

The second term in the integral, again done by parts, is

\[ \frac{2}{\left( \frac{1}{2} + \frac{\sigma^2}{\gamma - 1} \right) \frac{\sigma}{\gamma - 1} \frac{1}{\sqrt{T}}} \left( \int_{\frac{1}{\sigma} \ln \left( \frac{\mathcal{P}_{10}}{\mathcal{P}_{00}} \right)}^{\infty} e^{2\left( \frac{1}{2} - \frac{m}{\sigma^2} \right) t} \frac{1}{\sqrt{2\pi t}} e^{-\left( m - \left( \frac{\mu}{\sigma} + \frac{\sigma^2}{2} t \right)^2 \right)} dm \right) \]  

\[ + \int_{\frac{1}{\sigma} \ln \left( \frac{\mathcal{P}_{10}}{\mathcal{P}_{00}} \right)}^{\infty} e^{2\left( \frac{1}{2} - \frac{m}{\sigma^2} \right) t} \frac{1}{\sqrt{2\pi t}} e^{-\left( m - \left( \frac{\mu}{\sigma} + \frac{\sigma^2}{2} t \right)^2 \right)} dm \right) \]  

(75)

\[ - \int_{\frac{1}{\sigma} \ln \left( \frac{\mathcal{P}_{10}}{\mathcal{P}_{00}} \right)}^{\infty} e^{2\left( \frac{1}{2} - \frac{m}{\sigma^2} \right) t} \frac{1}{\sqrt{2\pi t}} e^{-\left( m - \left( \frac{\mu}{\sigma} + \frac{\sigma^2}{2} t \right)^2 \right)} dm \right). \]  

(76)
Substituting the previous two equations into the integral over the second region yields

\[
(1 + \theta)N \left( -\ln \left( \frac{\mathcal{F}_{t_0}}{P_{t_0}} \right) - \left( \mu + \frac{\sigma^2}{2} \right) t \right) \left( \frac{\mathcal{F}_{t_0}}{P_{t_0}} \right)^{1 + \frac{2\theta}{\sigma^2}} + (1 - \theta) e^{-\left( \frac{1}{1 + \alpha(y - 1)} \right) \left( \mu + \frac{\sigma^2}{2} \right) t} N \left( -\ln \left( \frac{\mathcal{F}_{t_0}}{P_{t_0}} \right) + \left( \mu + \frac{\sigma^2}{2} - \frac{\sigma^2}{1 + \alpha(y - 1)} \right) t \right) \left( \frac{\mathcal{F}_{t_0}}{P_{t_0}} \right)^{\frac{1}{1 + \alpha(y - 1)}} \right)
\]

where \( \theta = \frac{1}{(1 + \alpha(y - 1))(1 + \frac{\sigma^2}{\alpha}) - 1} \).

Finally, summing over both regions yields

\[
F_{t_0+t} = P_{t_0} e^{\mu t} \left( N \left( -\ln \left( \frac{\mathcal{F}_{t_0}}{P_{t_0}} \right) - \left( \mu + \frac{\sigma^2}{2} \right) t \right) \left( \frac{\mathcal{F}_{t_0}}{P_{t_0}} \right)^{1 + \frac{2\theta}{\sigma^2}} + \theta N \left( -\ln \left( \frac{\mathcal{F}_{t_0}}{P_{t_0}} \right) - \left( \mu + \frac{\sigma^2}{2} \right) t \right) \left( \frac{\mathcal{F}_{t_0}}{P_{t_0}} \right)^{\frac{1}{1 + \alpha(y - 1)}} \right) + (1 - \theta) e^{-\left( \frac{1}{1 + \alpha(y - 1)} \right) \left( \mu + \frac{\sigma^2}{2} - \frac{\sigma^2}{1 + \alpha(y - 1)} \right) t} N \left( -\ln \left( \frac{\mathcal{F}_{t_0}}{P_{t_0}} \right) + \left( \mu + \frac{\sigma^2}{2} - \frac{\sigma^2}{1 + \alpha(y - 1)} \right) t \right) \left( \frac{\mathcal{F}_{t_0}}{P_{t_0}} \right)^{\frac{1}{1 + \alpha(y - 1)}} \right) \right)
\]

**Proof of Proposition 4.2**

Heuristically, away from maxima in the price of firms’ output log-changes in the output price are normal at sufficiently short time intervals. As a result, returns to a firm, which has a price that is convex in the price of its output, is also convex. That is, if firm value is given by \( V(P) \), where \( P \) is the price of the firms’ output, then because \( V \) is convex so too is \( \ln \) composed with \( V \) composed with \( e^x \). Firm return is given by \( \ln(V(P)) = \ln(V(e^{lnP})) \), so because \( \ln P \) is normal large upward moves in \( \ln(V(e^{lnP}) \) are more likely than large downward moves.

More formally, we will show the result by perturbation analysis. We will consider the third center moment to returns as the time internal gets very small, and ignore dominated higher order terms., showing the skewness in returns up to second order terms in \( dt \) is positive. If \( \mathcal{F} > P_t \) then the probability that prices
of firms’ output returns to the maximum on the interval from \( t \) to \( t + dt \) is given by
\[
P(T_{t+dt}^f > P_{t+dt}) = P(X_{t+dt}^f > X_{t+dt})
\]
\[
= 2P(X_{t+dt}^f > X_{t+dt})
\]
\[
= o \left( N \left( \frac{X_t - \bar{X}_t}{\alpha \sqrt{dt}} \right) \right). \tag{78}
\]
and can be ignored in the second order expansion for sufficiently small \( dt \). Because the instantaneous return is essentially normal we will without loss of generality ignore the mean when computing higher order centered moments. That is, for convenience we will assume zero drift in demand (it is of course not zero, but the drift does not effect the higher order centered moments).

By inspection of equation (12), the decomposition of the value of a firm into cash flows until price of firms’ output returns to the historical maximum and the value of all cash flows after that, we have
\[
V(P) = c_1 P + c_2 P^\beta. \tag{79}
\]
where \( c_1, c_2 \) and \( \beta \) are all strictly positive.

Now firm returns are given by \( d \ln V(P_t) \), and \( d \ln V(P_t) = d \ln(k V(P_t)) \) for any constant \( k \). So let us consider \( f(x) = \frac{1}{1 + c} x + \frac{c}{1 + c} x^\beta \), for some \( c \) that makes \( f \) proportional to \( V \). To second order expansion around 1 we have
\[
f(x) \approx 1 + \left( \frac{1 + \beta c}{1 + c} \right) x + \left( \frac{\beta(\beta - 1)c}{1 + c} \right) \frac{(x - 1)^2}{2}. \tag{80}
\]
Using \( e^x \approx 1 + y + \frac{y^2}{2} \) we then have, to second order, that
\[
f(e^x) \approx 1 + \left( \frac{1 + \beta c}{1 + c} \right) \left( x + \frac{x^2}{2} \right) + \left( \frac{\beta(\beta - 1)c}{1 + c} \right) \frac{x^2}{2}
\]
\[
= 1 + \left( \frac{1 + \beta c}{1 + c} \right) x + \left( \frac{1 + \beta c + \beta(\beta - 1)c}{1 + c} \right) \frac{x^2}{2}. \tag{81}
\]
Finally, using \( \ln(1 + y) \approx y - \frac{y^2}{2} \) we have
\[
\ln f(e^x) \approx \left( \frac{1 + \beta c}{1 + c} \right) x + \left( \frac{1 + \beta c + \beta(\beta - 1)c}{1 + c} \right) \frac{x^2}{2} - \left( \frac{1 + \beta c}{1 + c} \right)^2 \frac{x^2}{2}
\]
\[
= \left( \frac{1 + \beta c}{1 + c} \right) x + \left( \frac{(1 + \beta^2 c)(1 + c) - (1 + \beta c)^2}{(1 + c)^2} \right) \frac{x^2}{2}. \tag{82}
\]
That is, for small \( x \) we have, ignoring terms higher than second order, that \( \ln f(e^x) = ax + bx^2 \) where \( a \) and \( b \) are both strictly positive. Using \( d \ln P_t \) for \( x \), and the fact that \( d \ln P_t \) is normally distributed with
variance $\sigma^2 dt$, we have that the third centered moment to stock returns is given by

$$
E \left[ (\ln f(e^x) - E[\ln f(e^x)])^3 \right] = E \left[ (ax + bx^2 - E[ax + bx^2])^3 \right] \\
= E \left[ (ax + b(x^2 - \sigma^2 dt))^3 \right] \\
= E \left[ b^3(x^2 - \sigma^2 dt)^3 + 3a^2x^2b(x^2 - \sigma^2 dt) \right] \\
= 8b^3\sigma^6 dt^3 + 18a^2bdt^2.
$$

In the limit as $dt$ gets very small stock returns’ third centered moment goes to $18a^2bdt^2$. This is strictly positive because $a$ and $b$ are both strictly positive. We have, therefore, that if $P_t < \bar{P}_t$ then at sufficiently short horizons stock prices are positively skewed.

When $P_t = \bar{P}_t$ the negative skewness in stock returns is inherited directly from the negative skewness in price of their output. The price of firms’ output is not Itô, and is negatively skewed even in the infinitesimal when $P_t = \bar{P}_t$. Because to order $dt$ stock prices are linear in the price of firms’s output, negative skewness in the returns to the price of firms’ output at short horizons (in the order $dt$ term) translates directly to negative skewness in stock returns at short horizons.

**Proof of Proposition 4.3**

For the redevelopment density, we need to compare the $q_1^*$ from the optimal strategy equation,

$$
q_1^* = \arg\max_q \left\{ \frac{(q - 1)^\eta}{q^{\eta(q^{\beta(\alpha - 1)} - 1)}} \right\},
$$

to the optimal strategy of a firm in the fixed capacity economy. Using the fact that the fixed capacity economy is just the limit of the equilibrium economy as the cost to scale of building gets very large, and letting overscore tildes denote parameters in the fixed capacity economy, we have that

$$
\tilde{q}_1^* = \arg\max_q \left\{ \frac{(q - 1)^{\tilde{\beta}}}{q^{\tilde{\beta}(q^{\tilde{\beta}(\alpha - 1)} - 1)}} \right\}. \quad (84)
$$

The long run average price growth was given in equation (14), $\tilde{\mu} = \frac{a(\gamma - 1)}{1 + a(\gamma - 1)} \left( \mu - \frac{1}{1 + a(\gamma - 1)} \frac{\sigma^2}{2} \right)$. The long run average price variance is implied directly from the equation for the equilibrium price process, and the fact that a Brownian process with positive drift stays “close” to its maximum, $\tilde{\sigma}^2 = \left( \frac{a(\gamma - 1)}{1 + a(\gamma - 1)} \right)^2 \sigma^2$. Using these in the formula for these in $\tilde{\beta} = \sqrt{\left( \frac{\mu}{\sigma^2} - \frac{1}{2} \right)^2 + \frac{2}{\sigma^2}} - \left( \frac{\mu}{\sigma^2} - \frac{1}{2} \right)$ yields $\tilde{\beta} = \eta$. The fact that $\tilde{q}_1^* = q_1^*$ then follows directly.

For the timing of the redevelopment, we will again use the fact that the fixed capacity economy is just the limit of the equilibrium economy as the cost to scale of building gets very large. Using $\tilde{\beta} = \eta$ and $\tilde{q}_1^* = q_1^*$ we have that

$$
\tilde{P}_1^* = \frac{\Pi}{\tilde{r}} P_1^*.
$$

That is, in equilibrium a firm redevelops later than it would in a fixed capacity economy with the same long
run average price growth and the long run average price variance if and only if \( \Pi < \tilde{\pi} \).

Using the formula for the price of a unit cash flow at a price maximum, and the formula for \( \tilde{\pi} \), we have \( \Pi < \tilde{\pi} \) if and only if

\[
(r - \mu) \left( 1 + \frac{1}{1 + \alpha(y - 1)(\beta - 1)} \right) > r - \left( \frac{\alpha(y - 1)}{1 + \alpha(y - 1)} \mu - \frac{\alpha(y - 1)}{1 + \alpha(y - 1)} \frac{\sigma^2}{2} \right). \tag{86}
\]

Simplifying and ignoring the strictly positive common denominator, this condition reduces to

\[
r + \frac{\alpha(y - 1)}{1 + \alpha(y - 1)} \frac{\sigma^2}{2} > \beta \left( \mu + \frac{\alpha(y - 1)}{1 + \alpha(y - 1)} \frac{\sigma^2}{2} \right), \tag{87}
\]

or

\[
\beta < \frac{r + \frac{\alpha(y - 1)}{1 + \alpha(y - 1)} \frac{\sigma^2}{2}}{\mu + \frac{\alpha(y - 1)}{1 + \alpha(y - 1)} \frac{\sigma^2}{2}}. \tag{88}
\]

Using the definition of \( \beta \) we then have

\[
\left( \frac{\mu}{\sigma^2} - \frac{1}{2} \right)^2 + \frac{2r}{\sigma^2} \leq \left( \frac{\mu}{\sigma^2} - \frac{1}{2} \right) + \frac{r + \frac{\alpha(y - 1)}{1 + \alpha(y - 1)} \frac{\sigma^2}{2}}{\mu + \frac{\alpha(y - 1)}{1 + \alpha(y - 1)} \frac{\sigma^2}{2}}. \tag{89}
\]

Simplifying and ignoring strictly positive common factors yields

\[
r < \left( \frac{\alpha(y - 1)}{1 + \alpha(y - 1)} \right) \left( \mu - \frac{1}{1 + \alpha(y - 1)} \frac{\sigma^2}{2} \right). \tag{90}
\]

This condition, that the discount rate exceeds the long run average price growth, is always satisfied by assumption. As a result, \( \Pi \) is always less than \( \tilde{\pi} \), and we then have that \( P_1^* > \tilde{P}_1^* \). ■

**Proof of Proposition 4.4**

The option value is convex in prices, whereas the intrinsic value is linear, and both value components go to zero with prices. As a result, it must be that the maximum ratio of option value to intrinsic value occurs at the highest possible price, i.e., at the moment before redevelopment occurs. We then have

\[
\max_P \left( \frac{O(q, P)}{I(q, P)} \right) = \frac{O(q, P_\bar{q}^*)}{I(q, P_\bar{q}^*)} = \frac{Aq^\gamma}{q \Pi P_\bar{q}^*}. \tag{91}
\]

Substituting for \( A \) and \( P_\bar{q}^* \) and simplifying results in

\[
\max_P \left( \frac{O(q, P)}{I(q, P)} \right) = \frac{(q_1^* - 1)}{\eta(1 - q_1^* \gamma - \gamma \eta)}. \tag{92}
\]
References


