

OPTIMAL STOCK TRADING WITH PERSONAL TAXES

Implications for Prices and the Abnormal January Returns*

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The tax law confers upon the investor a timing option – to realize capital losses and defer capital gains. With the tax rate on long term gains and losses being about half the short term rate, the law provides a second timing option – to realize losses short term and gains long term, if at all. Our theory and simulation over the 1962–1977 period establish that taxable investors should realize long term gains in high variance stocks and repurchase stock in order to realize potential future losses short term. Tax trading does not explain the small-firm anomaly but predicts a seasonal pattern in trading volume which maps into a seasonal pattern in stock prices, the January anomaly, only if investors are irrational or ignorant of the price seasonality.

1. Introduction

Capital gains and losses are taxed when the investor sells the stock – not when gains and losses actually occur. Suppressing the distinction between the short term tax rate on capital gains and losses and the long term rate, the optimal trading policy with zero transactions costs is to realize capital losses immediately and defer capital gains, thereby reducing the present value of the stream of tax payments on capital gains net of tax credits on capital losses. Constantinides (1983) derives the optimal trading policy, estimates the value of this timing option as a fraction of the stock price, and finds the effective tax rate on capital gains and dividends. He also explains how the capital gains tax influences the investor's optimal consumption and investment program and derives the equilibrium pricing implications – how the capital gains tax modifies the asset pricing models of Breeden (1979), Brennan (1973) and Merton (1973a).

With the tax rate on long term capital gains and losses being about half the rate on short term capital gains and losses, the tax law confers upon the

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investor a second timing option – to realize losses short term and realize gains long term, if at all. For those investors who can draw the distinction between the two tax rates, the second option is substantially more valuable than the first. Suppose that the investor bought the stock exactly one year ago. If the stock price has declined, he optimally sells the stock and repurchases it, realizing a short term capital loss immediately. If the stock price has increased instead, the investor optimally defers the realization of a short term capital gain and one day later faces two alternatives. First, he may defer the realization of the long term gain. Second, he may sell the stock and repurchase it, realizing a long term gain and reestablishing the favorable short term status, in order to realize future capital losses at the short term rate.

Our theory confirms that under broad conditions the investor ought to realize a long term gain in order to reestablish the short term status. The conditions apply to high variance stocks and even to medium variance stocks, if transactions costs are low and the interest rate is low (see tables 1 and 2). The results are intuitively appealing. The timing option to realize losses short term and gains long term is more valuable the higher the stock variance, for essentially the same reason that a call option is more valuable the higher the stock variance. Thus the higher the stock variance the stronger the incentive to reestablish the short term status even at the expense of the tax on long term gains. Also, the lower the interest rate the higher the present value of the future tax benefit of this option. Transactions costs inhibit trading but the order of magnitude of the potential tax benefit is large relative to even substantial transactions costs.

We simulate three trading policies and the buy-and-hold policy for a large sample of NYSE- and AMEX-listed stocks over the period 1962–1977. The performance under optimal trading far exceeds the performance under the buy-and-hold (see table 3). The tax trading benefit is large even when we allow for long term gains to offset short term losses dollar for dollar. The results readily translate into an annual tax subsidy which the government provides to taxable investors under various provisions of the tax law.

We relate our findings to the empirical evidence on positive abnormal returns by small firms, and positive abnormal returns in the month of January. We argue that tax trading not only fails to explain, but exacerbates the small firm anomaly, and cast doubts on a proposed explanation based on transactions costs. Regarding the January anomaly, we find that tax-loss selling predicts a seasonal pattern in trading volume, provided transactions costs are present. Tax-loss selling predicts a seasonal pattern in stock prices, only if we further assume irrationality or ignorance of the stock price seasonality on behalf of investors.

The paper is organized as follows: The tax environment is discussed in section 2. In section 3 we derive those properties of the optimal trading policy

which do not depend on detailed assumptions on the stock price distribution (see Proposition 1). In section 4 we assume that the stock price follows a binomial process and provide conditions under which it is optimal to incur the cost of realizing a long term gain in order to reestablish the tax-advantageous short term status (see Proposition 2 and tables 1 and 2). Section 5 reports the simulation of the trading policies. In section 6 we relate our findings to the empirical evidence on the small firm and January anomalies. Concluding remarks are offered in section 7.

2. The tax environment

Unrealized capital gains and losses are not taxed. Realized capital gains and losses are short term if the asset has been held for one year or less, and long term otherwise.¹ Net short term capital gains (or losses), X_S , are defined as the total short term capital gains net of total short term capital losses, including unused short term carryovers. Net long term capital gains (or losses), X_L , are defined analogously. If both X_S and X_L are non-negative, short term capital gains are taxed at the individual's marginal tax rate on ordinary income, and long term capital gains are taxed at 40% (until October 1979, 50%) of the individual's marginal tax rate on ordinary income.^{2,3} If both X_S and X_L are non-positive, then net short term capital losses and 50% of net long term capital losses are deductible from ordinary income. These deductions may jointly decrease the taxable ordinary income by a maximum of \$3,000 per tax year.⁴ Unused capital losses are carried forward indefinitely.

A complication of the tax code is that net short term losses offset net long term gains one to one. Thus, if $X_S < 0 < X_L$ and $X_S + X_L < 0$, the loss $X_S + X_L$ is taxed as short term; if $X_S + X_L > 0$, the gain $X_S + X_L$ is taxed as long term. Likewise, if $X_L < 0 < X_S$ and $X_S + X_L < 0$, the loss $X_S + X_L$ is taxed as long term; but if $X_S + X_L > 0$, the gain $X_S + X_L$ is taxed as short term.

If an asset is sold at a loss and repurchased within thirty days, the IRS terms the transaction a 'wash sale' and disallows the loss deduction. The investor has a high probability of circumventing this rule by waiting thirty days before repurchasing the same asset. A safer way to circumvent the rule is to repurchase a different asset with the same risk and return characteristics.

¹The holding period was six months in 1942–1976, nine months in 1978, and one year thereafter.

²Prior to 1969 the maximum rate on long term capital gains was 25%. In 1969–1976 (1976–1979) the marginal tax rate on capital gains above the first \$50,000 was as high as 42½% (49%).

³Upon the investor's death the assets' basis is adjusted to market and capital gains and losses remain untaxed.

⁴The deduction limit was \$1,000 until 1976, \$2,000 in 1977 and \$3,000 thereafter.

3. Properties of the optimal trading policy

An investor may sell stock to invest the proceeds in perceived underpriced assets, consume, or rebalance his portfolio. We consider those times at which the investor is not motivated by any of the above reasons to sell stock and investigate the conditions under which he would optimally sell stock and *repurchase it* for tax reasons.⁵ We defer the discussion of transactions costs until section 4.

There are at least three plausible tax scenarios. First, the investor is a tax-exempt institution or an individual who continually carries forward large capital losses and expects the deduction limit to remain binding for many years. The marginal tax rate on capital gains and losses is zero and the investor pays no attention to the realization of capital gains and losses in pursuing his optimal investment policy.

Second, the deduction limit is not binding and short term and long term capital gains and losses are taxed at the same rate. This scenario is plausible if (1) the individual investor is periodically forced to sell some assets by factors beyond his control and, on average, realizes large long term gains; and (2) he can defer the realization of short term capital gains until the holding period exceeds one year and then realize the capital gains long term. Then short term and long term capital losses simply offset some of the long term capital gains. The optimal trading policy in this case is derived in Constantinides (1983) and is restated here for completeness.

Proposition 0. Assume that transactions costs are zero and the tax rate on long term capital gains and losses equals the tax rate on short term capital gains and losses. Then at any time that the investor is not forced to sell the stock, he optimally realizes losses and repurchases the stock; and defers the realization of gains.

The third scenario distinguishes between the short and long term tax status and is the focus of the paper.⁶ We assume that short term capital gains and losses are taxed at the rate τ , and long term capital gains and losses are taxed at the lower rate τ_L .⁷ We examine the optimal trading policy under the

⁵Holt and Shelton (1962) discuss the investor's trade-off between the benefit of switching to relatively underpriced assets and the cost of the capital gains tax.

⁶Aspects of this problem are also discussed in Stiglitz (1983), Varian (1981), and Williams (1981).

⁷If the investor does not have short term gains, he offsets short term losses against long term gains one to one and, effectively, cannot use to his advantage the distinction between the short and long term tax status. This complication is suppressed here and discussed in section 5.4.

We suppress the distinction between the tax rate on net long term gains, τ_L (net gains) = 0.4τ , and the tax rate on net long term losses, τ_L (net losses) = 0.5τ because long term losses offset long term gains and the marginal tax rates are equal. In any case, Propositions 1 and 2 remain valid if the condition τ_L (marginal gain) = τ_L (marginal loss) < τ is replaced by the pair of weaker conditions, τ_L (marginal gain) < τ and τ_L (marginal loss) < τ .

simplifying assumption that trading occurs in discrete, one-year intervals. This assumption underestimates the benefits of tax trading. The length of the trading period conveniently coincides with the length of the holding period beyond which short term capital gains and losses become long term. If an asset is sold one year after purchase, the gain or loss is short term or long term at the investor's discretion. Obviously the investor delays the sale by at least one day if he has a short term gain, but not if he has a short term loss. Proposition 1 states some properties of the optimal trading policy which are free from distributional assumptions on the stock price.

Proposition 1. Assume that trading occurs in one-year intervals only, transactions costs are zero, and the tax rate on long term capital gains and losses is less than the tax rate on short term capital gains and losses, i.e., $\tau_L < \tau$. Then, at any time that the investor is not forced to sell the stock, he optimally:

- a. *Defers the realization of a short term gain.*
- b. *Sells the stock and repurchases it to realize a loss, short term if possible.*
- c. *Sells the stock and repurchases it to change the long term status to short term, whenever the stock price equals the basis and the status is long term.*

Proof. See the appendix.

The proposition is an incomplete characterization of the optimal policy because it leaves unspecified the optimal action when the stock price exceeds the basis and the capital gain is long term. Whether or not the investor should realize a long term gain and reestablish the short term status depends on (1) the stock price distribution, (2) the size of the gain, and (3) the tax rates on short term losses and long term gains. In the next section we state as Proposition 2 the conditions under which it is optimal to realize or defer a long term gain, in the special case that the stock price is generated by a stationary binomial process. Under the binomial assumption, Propositions 1 and 2 jointly provide a complete characterization of the optimal trading policy which is in agreement with the simulation results presented in section 5.

4. Optimal realization of long term gains in a binomial model of stock prices

Consider an investor holding a share of stock with price P , purchased t years ago at cost basis \hat{P} . It is optimal to defer the realization of the long term gain, $P - \hat{P} > 0$, if the value of the position in the stock with unrealized gain exceeds the after-tax proceeds of selling the stock, $(1 - \tau_L)P + \tau_L \hat{P}$.

Our first task is to define the value of a position. We define the value of a position, $V(P, \hat{P}, t)$, in one share of stock as the after-tax shadow price, such that the investor is indifferent (in the margin) between having one share with price P , basis \hat{P} and age t , or having $V(P, \hat{P}, t)$ after-tax dollars.

For example, at the time that the investor purchases the stock and is indifferent (in the margin) between owning the stock or not, the value of the position is $V(\hat{P}, \hat{P}, 0) = \hat{P}$. If the stock price falls one year later, the investor optimally realizes a short term loss and the value of the position is $V(P, \hat{P}, 1) = (1 - \tau)P + \tau\hat{P}$, $P \leq \hat{P}$.

We make the determination of the functional form of $V(P, \hat{P}, t)$ tractable, by assuming that the annual stock price, P_t, P_{t+1}, \dots , is generated by a binomial process as follows:

$$P_t \begin{cases} P_{t+1} = uP_t, & \text{with probability } 1 - q, \\ P_{t+1} = u^{-1}P_t, & \text{with probability } q, \end{cases}$$

where u is constant over time and is greater than one. No dividends are paid on the stock. Transactions costs are zero. The investor is infinitely lived and is never forced to realize a capital gain or loss. There exists a single-period riskless asset with after-tax interest rate $R - 1$. Irrespective of the tax rates τ, τ_L , the following condition is necessary, if the after tax cash flows of the risky and the riskless assets do not dominate one another:

$$u^{-1} < R < u. \quad (1)$$

As part of the argument, we first assume that the tax rates τ, τ_L and the parameters u, R are such that it is optimal to defer the realization of a long term gain one year after purchase, i.e., it is optimal to defer a long term gain when $P = u\hat{P}$. Under the above assumption we claim that it is optimal to defer a gain whenever $P \geq u\hat{P}$ (see Lemma 2 in the appendix). Second, we derive the functional form of V (see the appendix) by replicating a derivation in Constantinides (1983). Although that derivation assumes that the short and long term tax rates are equal, the same derivation applies in the present context because we have assumed that the tax rates τ, τ_L , although different, are such that it is optimal to defer all long term gains. The value of the position is given by

$$V(P, \hat{P}, t) = \left[1 - \frac{(1 - u^{-1})(u - R)\tau}{uR - 2 + u^{-1}R} \right] P + \frac{(1 - u^{-1})(u - R)\tau}{uR - 2 + u^{-1}R} P^m \hat{P}^{1-m}, \quad \begin{matrix} P \geq u^{-1}\hat{P}, \\ t \geq 1, \end{matrix} \quad (2)$$

where

$$m = [\ln(u - R) - \ln(uR - 1)] / \ln u < 0 \quad (3)$$

provided that the parameters τ, τ_L, u, R are such that it is optimal to defer a long term gain one year after purchase, i.e.,

$$V(u\hat{P}, \hat{P}, t) > (1 - \tau_L)u\hat{P} + \tau_L\hat{P}, \quad t \geq 1. \quad (4)$$

Third, we combine eqs. (2), (3) and (4) and obtain inequality (5) as the restriction on the parameters τ, τ_L, u, R . If this restriction is violated, it is optimal to realize a gain long term whenever $P = u\hat{P}$. If inequality (5) holds as an equality, the investor is indifferent between realizing or deferring a long term gain when $P = u\hat{P}$. These results are summarized below.

Proposition 2. Assume that trading occurs in one-year intervals only; there are no forced realizations; the stock price is generated by a binomial process with $u^{-1} < R < u$; there are no dividends; transactions costs are zero; and the tax rate on long term capital gains and losses is less than or equal to the tax rate on short term capital gains and losses, i.e., $\tau_L < \tau$.

a. *If*

$$(u - R)/(uR - 1) \leq \tau_L/\tau, \quad (5)$$

at any time that $P \geq u\hat{P}$, it is optimal to defer the gain.

b. *If*

$$\tau_L/\tau \leq (u - R)/(uR - 1), \quad (6)$$

at any time that $P = u\hat{P}$, it is optimal to realize the gain, and realize it long term.^{8,9}

⁸We may incorporate random forced realizations, provided that all forced realizations of gains are taxed as long term. Under condition (6) it is still optimal to realize a long term gain. Under condition (5) it may be optimal to defer or realize a long term gain, depending on the parameter values.

⁹In case (b) we leave unspecified the action when $P > u\hat{P}$ because this contingency never arises: Before $P > u\hat{P}$, the price will have equalled $u\hat{P}$ at some earlier period, a long term gain will have been realized and the basis will have been updated.

This argument hinges on our earlier assumption that the stock price can increase by only one step each year. If, instead, the stock price were generated by a trinomial process, $P_{t+1} = u^{-1}P_t$ or uP_t or u^2P_t , then we would have to consider separately three candidate policies:

- (a) Defer a gain if $P \geq u\hat{P}$.
- (b) Realize a gain if $P = u\hat{P}$, but defer a gain if $P > u^2\hat{P}$.
- (c) Realize a gain if $P \geq uP$.

The discussion following table 1 in this section suggests that the conclusions do not critically depend on the binomial process.

If the short term and long term tax rates are equal, then $\tau_L/\tau = 1 > (u - R)/(uR - 1)$ and it is optimal to defer a long term gain. For sufficiently low long term tax rate relative to the short term rate, it is optimal to realize long term gains. The function $(u - R)/(uR - 1)$ is decreasing in R . For sufficiently high interest rate condition (5) is satisfied and it is optimal to defer a long term gain because the future benefits associated with the short term tax status become less valuable. The function $(u - R)/(uR - 1)$ is increasing in u . For sufficiently high value of u condition (5) is violated and it is optimal to realize a long term gain. The higher u is, the higher the variance of the stock return. Numerical values are presented below and in table 1.

For the assumed binomial process, the conditional mean of the annual logarithmic stock return is

$$\mu = E[\ln(P_{t+1}/P_t)|P_t] = (1 - 2q)\ln u, \quad (7)$$

and the conditional variance is

$$\sigma^2 = \text{var}[\ln(P_{t+1}/P_t)|P_t] = 4q(1 - q)(\ln u)^2 \quad (8)$$

We eliminate q from eqs. (7) and (8) and obtain

$$u = e^{\sqrt{\mu^2 + \sigma^2}} \quad (9)$$

In table 1 we report u (in brackets) and the critical tax ratio, $(u - R)/(uR - 1)$, for a range of the parameters μ , σ , R . If the tax ratio, τ_L/τ , is below the critical ratio, the optimal policy is to realize long term gains; otherwise the optimal policy is to defer them. The range of the annual standard deviation is representative of the stocks listed on the NYSE and AMEX, with the median being about 0.35 in 1963. With $\tau_L/\tau = 0.4$ and the tax-exempt, annual interest rate 5%, the first panel states that the investor should sell the stock and repurchase it at the end of every year, realizing a short term loss or long term gain. The second panel, with the tax-exempt annual interest rate 10%, states that the investor should refrain from realizing long term gains only in a few cases, marked with an asterisk. These cases refer to stocks with very low variance (the bottom 25% of all NYSE and AMEX listed stocks) and expected rate of return lower than or equal to the expected rate of return on riskless tax-exempt bonds. We conclude that, at least in the absence of transactions costs, it is optimal to realize long term gains on most NYSE and AMEX listed stocks.

If condition (6) is satisfied, the investor realizes a long term gain whenever $P = u\hat{P}$. Suppose, however, that the investor has neglected to realize a long term gain until $P > u\hat{P}$. The investor realizes a long term gain, however large

Table 1

Critical ratio of the long term to the short term tax rate below which the optimal policy is to realize long term capital gains – the zero transactions costs case.^a

Expected annual stock return (μ)	Standard Deviation of annual stock return (σ)					
	0.10	0.20	0.40	0.60	0.80	1.00
Annual riskless return $R = 1.05$						
0.05	0.40 (1.12)	0.62 (1.23)	0.78 (1.50) [†]	0.85 (1.83) [†]	0.88 (2.23) [†]	0.90 (2.72) [†]
0.10	0.48 (1.15)	0.64 (1.25)	0.79 (1.51) [†]	0.85 (1.84) [†]	0.88 (2.24) [†]	0.90 (2.73) [†]
0.15	0.58 (1.20)	0.67 (1.28)	0.79 (1.53) [†]	0.85 (1.86) [†]	0.88 (2.26) [†]	0.90 (2.75) [†]
0.20	0.64 (1.25)	0.71 (1.33)	0.80 (1.56) [†]	0.85 (1.88) [†]	0.88 (2.28) [†]	0.90 (2.77) [†]
Annual riskless return $R = 1.10$						
0.05	0.90* (1.12)	0.37* (1.23)	0.62 (1.50) [†]	0.72 (1.83) [†]	0.78 (2.23) [†]	0.81 (2.72) [†]
0.10	0.19* (1.15)	0.40 (1.25)	0.62 (1.51) [†]	0.72 (1.84) [†]	0.78 (2.24) [†]	0.81 (2.73) [†]
0.15	0.31* (1.20)	0.44 (1.28)	0.63 (1.53) [†]	0.73 (1.86) [†]	0.78 (2.26) [†]	0.81 (2.75) [†]
0.20	0.40 (1.25)	0.50 (1.33)	0.64 (1.56) [†]	0.73 (1.88) [†]	0.78 (2.28) [†]	0.82 (2.77) [†]

^aIn parentheses the parameter $u \equiv \exp(\mu^2 + \sigma^2)$ of the binomial process generating annual prices. The asterisk marks stocks for which the optimal policy is to defer the realization of a long term gain when the price equals u times the basis and the ratio of the tax rates is $\tau_L/\tau = 0.40$. The dagger marks stocks for which the optimal policy is to realize a long term gain, however large the gain may be, where the ratio of the tax rates is $\tau_L/\tau = 0.40$.

the gain may be, provided

$$V(P, \hat{P}, t) \leq (1 - \tau_L)P + \tau_L \hat{P}, \quad \forall P/\hat{P} \geq 1. \tag{10}$$

Eqs. (2), (3) and (10) imply

$$\tau_L/\tau \leq [(1 - u^{-1})(u - R)] / (uR - 2 + u^{-1}R). \tag{11}$$

In table 1 a dagger marks the stocks for which the investor optimally realizes a long term gain, however large the gain may be, when the ratio of the tax rates is $\tau_L/\tau = .40$. We conclude that, in the absence of transactions costs, the investor optimally realizes a long term gain on medium and high variance stocks, however large the gain may be.

Transactions costs may dissipate the tax benefit of realizing a long term gain and even the benefit of realizing a short term loss. We introduce transactions costs, but assume that they are sufficiently small, so that Proposition 1 holds. We reexamine Proposition 2, i.e., investigate conditions on the parameters τ , τ_L , u , R and the transactions costs rate such that it is optimal to realize a long term gain one year after purchase.

We denote by γ the one-way proportional transactions costs rate, where $0 \leq \gamma \leq 1$. When the stock price is \hat{P} , the investor pays $(1 + \gamma)\hat{P}$ to buy one share, i.e.,

$$V(\hat{P}, (1 + \gamma)\hat{P}, 0) = (1 + \gamma)\hat{P}. \quad (12)$$

If the price falls to $u^{-1}\hat{P}$, the investor realizes a short term loss, and receives, net of tax and transactions costs,

$$V(u^{-1}\hat{P}, (1 + \gamma)\hat{P}, 1) = (1 - \tau)(1 - \gamma)u^{-1}\hat{P} + \tau(1 + \gamma)\hat{P}. \quad (13)$$

As part of the argument, we assume that the investor optimally defers a gain. If the price rises, the investor defers a gain. If and when the price falls to the level of the purchase price, the investor sells the stock and reestablishes the short term status, i.e.,

$$\begin{aligned} V(P, (1 + \gamma)\hat{P}, t) &= (1 - \tau_L)(1 - \gamma)P + \tau_L(1 + \gamma)\hat{P}, \quad P = \hat{P}, \quad t \geq 1, \\ &> (1 - \tau_L)(1 - \gamma)P + \tau_L(1 + \gamma)\hat{P}, \quad P > \hat{P}, \quad t \geq 1. \end{aligned} \quad (14)$$

As in the appendix, but now with proportional transactions costs, we compare portfolios with different bases and obtain the following:

$$\begin{aligned} V(P, (1 + \gamma)\hat{P}, t) &= R^{-1}[(1 - k)V(uP, (1 + \gamma)\hat{P}, t + 1) \\ &\quad + kV(u^{-1}P, (1 + \gamma)\hat{P}, t + 1)], \end{aligned} \quad (15)$$

where

$$k = (u - R)/(u - u^{-1}). \quad (16)$$

We combine eqs. (12), (13), (14) and (15) and obtain

$$\begin{aligned} (1 + \gamma)\hat{P} &> R^{-1}[(1 - k)\{(1 - \tau_L)(1 - \gamma)u\hat{P} + \tau_L(1 + \gamma)\hat{P}\} \\ &\quad + k\{(1 - \tau)(1 - \gamma)u^{-1}\hat{P} + \tau(1 + \gamma)\hat{P}\}]. \end{aligned} \quad (17)$$

Using eq. (16) we eliminate k from eq. (16) and obtain

$$\begin{aligned} \tau_L/\tau &> [(u - R)\{(1 + \gamma)u - (1 - \gamma)\} - 2(u - u^{-1})uR\gamma/\tau] \\ &\quad /[(uR - 1)\{(1 - \gamma)u - (1 + \gamma)\}]. \end{aligned} \quad (18)$$

Table 2

Critical ratio of the long term to the short term tax rate below which the optimal policy is to realize long term capital gains – the case with four percent round-trip transactions costs.^a

Expected annual stock return (μ)	Standard deviation of annual stock return (σ)					
	0.10	0.20	0.40	0.60	0.80	1.00
Annual riskless return $R = 1.05$						
0.05	Never* (1.12)	0.12* (1.23)	0.55 (1.50)	0.69 (1.83)	0.76 (2.23)	0.81 (2.72)
0.10	Never* (1.15)	0.18* (1.25)	0.56 (1.51)	0.70 (1.84)	0.76 (2.24)	0.81 (2.73)
0.15	Never* (1.20)	0.26* (1.28)	0.57 (1.53)	0.70 (1.86)	0.77 (2.26)	0.81 (2.75)
0.20	0.18* (1.25)	0.36* (1.33)	0.59 (1.56)	0.71 (1.88)	0.77 (2.28)	0.81 (2.77)
Annual riskless return $R = 1.10$						
0.05	Never* (1.12)	Never* (1.23)	0.38* (1.50)	0.56 (1.83)	0.66 (2.23)	0.72 (2.72)
0.10	Never* (1.15)	Never* (1.25)	0.39* (1.51)	0.57 (1.84)	0.66 (2.24)	0.72 (2.73)
0.15	Never* (1.20)	0.03* (1.28)	0.40 (1.53)	0.57 (1.86)	0.66 (2.26)	0.72 (2.75)
0.20	Never* (1.25)	0.14* (1.33)	0.43 (1.56)	0.58 (1.88)	0.67 (2.28)	0.72 (2.77)

^aThe short term tax rate is 50%. In parentheses the parameter $u \equiv \exp(\mu^2 + \sigma^2)$ of the binomial process generating annual prices. The asterisk marks stocks for which the optimal policy is to defer the realization of a long term gain when the price equals u times the basis and the ratio of tax rates is $\tau_L/\tau = 0.40$.

Note that, in the absence of transactions costs, eq. (17) becomes $\tau_L/\tau > (u - R)/(uR - 1)$, which is the earlier eq. (5).

In table 2 we assume 4% round-trip transactions costs and 50% short term tax rate.¹⁰ We report u (in brackets) and the critical tax ratio, i.e., the right-hand side of eq. (18). If τ_L/τ is below the critical ratio, it is optimal to realize long term gains when $P = u\hat{P}$; otherwise, it is optimal to defer long and short term gains. With $\tau_L/\tau = 0.40$ and with the tax-exempt annual interest

¹⁰Stoll and Whaley (1983) estimate the round-trip transactions costs to be 6.77% for the largest decile NYSE-listed stocks, and 2.71% for the smallest. These estimates exaggerate the costs applicable to medium size and large investors: First, negotiated commissions are small. Second, sophisticated investors, who can wait a few days before they execute a trade, need not have the full bid-asked spread work against them. Analyzing 215,000 individual tickets on over \$27 billion of equity trades, Beebower and Surz (1980) conclude that the average round-trip transactions costs rate is less than 1%.

rate either 5% or 10%, the table indicates that the investor should realize a short term loss or long term gain on high variance stocks ($\sigma \geq 0.60$), but do so on medium variance stocks ($\sigma = 0.40$) only if transactions costs are small. The conclusion is supported by the simulation reported next.

5. Post sample simulation of trading policies

5.1. Introduction

We simulate three trading policies over the fifteen year period, 1962–1977, and compare their performance against the buy-and-hold policy. The three trading policies are:

- I. Realize losses in every December, short term if possible; and defer gains.
- II. Realize losses in every December, short term if possible; and realize gains in every December, long term.
- III. In the Decembers of odd years, realize losses, short term if possible; and defer gains. In the Decembers of even years realize gains and losses, both long term.

Policy I illustrates the benefit of realizing losses, policy II illustrates the additional benefit of realizing gains, and policy III accommodates in a simple way the one to one offsetting of long term gains and short term losses.

We start in 1962, the first year covered by the University of Chicago's CRSP Daily Stock master file. The investment in each stock is made on December 3, 1962, and trading in future years occurs only once every year at the beginning or middle of December.¹¹ (The description of the trading policies explains how the exact trading day is chosen.)

We assume that the 1982 tax law was in force over the simulation period. The simulation then predicts the future effectiveness of the active trading policies under the current tax law, if the price fluctuations in the 1962–1977 period are representative of the future. The holding period beyond which short term capital gains and losses become long term is taken to be one year instead of six or nine months (which was the statutory time interval in the 1962–1977 period). The marginal tax rate on ordinary income and on short term capital gains and losses is taken to be 50%. The marginal tax rate on long term capital gains and losses is taken to be 20%. The limit on the capital loss deduction from ordinary income is assumed non-binding. This assumption overestimates the effectiveness of the active policies.

¹¹With individual investors' capital gains and losses taxed only once every year, investors have an incentive to realize their losses in December instead of the following January and defer their gains from December to the following January. The implied trading volume seasonality is discussed in section 6.2 in connection with the January stock return anomaly.

The sample includes all securities which were on the CRSP Daily Stock market file as of December 3, 1962, and remained on file until December 1977. Of the 2,027 securities on file on December 3, 1962, only 1,147 remained on file until December 1977, and qualify for inclusion in the sample. In a merger, reorganization, or exchange, if a security on file is replaced by another security on file until December 1977, then the security is included in the sample. If a security is delisted from the NYSE and immediately relisted on the AMEX (or vice versa), the security is included in the sample, provided it is on file until December 1977. Finally, suspension of trading by the SEC or halting of trading by the exchange for more than one year disqualifies a security for inclusion in the sample.

All securities on the CRSP file are classified by CRSP into ten portfolios with equal number of securities, based on the variance of the daily excess return in 1962. Portfolio one includes the securities with the highest variance and portfolio ten includes those with the lowest. We refer to the stocks with 1962 variance classification 1–4 as high variance stocks, those with classification 5–7 as medium variance stocks and those with classification 8–10 as low variance stocks. We classify the sample of 1,147 securities into three groups as follows: 367 high variance stocks with CRSP variance classification 1–4; 361 medium variance stocks with CRSP variance classification 5–7; and 419 low variance stocks with CRSP variance classification 8–10. We do not reclassify the stocks into groups in subsequent years although their CRSP variance classification may change annually.¹²

The highest variance stocks are the ones most likely to be delisted from the NYSE or AMEX. Our exclusion from the sample of all stocks not continuously listed over the 15-year simulation period eliminates from the sample many high variance stocks.¹³ Furthermore, the highest variance stocks are likely to be small firms traded over-the-counter and therefore excluded from the sample. The selection procedure eliminates many high variance stocks for which the trading policies work best.

By eliminating stocks which the investor would be forced to sell before the completion of the 15-year period, the selection procedure overestimates the effectiveness of the policy of deferring gains throughout the 15-year period (policy I) but not of the policy of realizing all gains and losses in every year

¹²Annual reclassification of stocks by variance would be meaningless in our simulation because we compare the performance of the active policies against a policy of buying the stock and holding it over 15 years. One of the conclusions derived from the simulation is that the active trading policies work best for high variance stocks. This conclusion would be reinforced if the simulation were done with stocks which not only had high variance in 1962 but which maintained high variance throughout the 15-year period.

¹³Before exclusion, 40% of the stocks are high variance, 30% medium variance, and 30% low variance. After eliminating the stocks not continuously listed over the 15-year period, 32% of the remaining stocks are high variance, 32% are medium variance, and 37% are low variance.

(policy II). The selection procedure only slightly overestimates the effectiveness of the policy of realizing gains in alternate years (policy III).

One hundred dollars are invested in each stock on December 3, 1962. The number of purchased shares is maintained constant throughout the 15-year period, adjusted only for stock splits, stock dividends, mergers, reorganizations and exchanges. Cash dividends, partial liquidations, and cash proceeds from selling rights are not used to repurchase stock. These cash distributions are taxed at the marginal tax rate of 50%, if appropriate, and are deposited each December in a cash fund that we associate with each stock and for each policy.¹⁴ The cash fund is invested each December in one-year Treasury bills, with the interest earned on the Treasury bills being taxed at 50% and reinvested until December 1977. If a loss is realized on the stock, the tax rebate is also deposited in this fund. If a gain is realized on the stock, the tax due is subtracted from the fund. The balance of the cash fund may be positive or negative. In December 1977 this balance is added to the after-tax proceeds from selling the stock.

5.2. Policy I: Realize losses in every December and defer gains

One hundred dollars are invested in the stock on December 3, 1962. One year later, on December 3, 1963, we observe the stock price.¹⁵ If the stock has a capital gain, we defer it. If the stock has a capital loss, we sell the stock, realize the loss short term and deposit the tax rebate in the cash fund associated with the stock. On the following trading day we repurchase the same number of shares.¹⁶ We repeat the procedure each December until 1977. After 1963 a capital loss is realized short term only if the stock was last sold and repurchased in the previous December; otherwise any loss realized is long term. In December 1977 the stock is sold and a loss or long term gain is realized. The cash fund, which has the after-tax cash distributions and tax rebates with interest, is added to the after-tax proceeds from selling the stock. The sum is the net proceeds in year 1977 under policy I.

Under the buy-and-hold, one hundred dollars are invested in the stock on December 3, 1962. On the same date in December 1977 that the stock is sold under policy I, the stock is sold under the buy-and-hold. Capital gains or losses

¹⁴For example, the cash distributions between December 3, 1962, and the next trading date in December 1963 are held, earning no interest, until the December 1963 trading date. At that time they are deposited in one-year Treasury bills.

¹⁵If this date is not a valid trading date, we observe the stock price on the last trading date on which capital losses qualify for short term status.

¹⁶If the number of shares in the active trading policy were not kept constant, the comparison of after tax cash values of the active and buy-and-hold policies would be meaningless because a different amount of risk would be associated with each policy.

Table 3
Post sample simulation of trading policies.^a

	High variance (367 stocks)		Medium variance (361 stocks)		Low variance (419 stocks)	
	$2\gamma = 0$	$2\gamma = 4\%b$	$2\gamma = 0$	$2\gamma = 4\%b$	$2\gamma = 0$	$2\gamma = 4\%b$
<i>Policy I: Realize losses in every December and defer gains – wealth relatives of policy I and buy-and-hold</i>						
Mean	1.120	1.084	1.058	1.028	1.039	1.000
25 percentile	1.002	1.000	1.000	0.990	1.000	0.982
50 percentile	1.043	1.016	1.007	1.000	1.019	1.000
75 percentile	1.168	1.110	1.059	1.022	1.060	1.008
<i>Policy II: Realize gains and losses in every December – wealth relatives of policy II and buy-and-hold</i>						
Mean	2.242	1.720	1.604	1.224	1.381	1.050
25 percentile	1.521	1.188	1.296	0.995	1.245	0.957
50 percentile	1.966	1.521	1.468	1.109	1.323	1.008
75 percentile	2.651	2.050	1.694	1.293	1.445	1.095
<i>Policy III: Realize gains in alternate Decembers and realize losses in every December – wealth relatives of policy III and buy-and-hold</i>						
Mean	1.747	1.359	1.383	1.094	1.275	1.011
25 percentile	1.308	1.040	1.173	0.964	1.176	0.930
50 percentile	1.547	1.214	1.300	1.034	1.238	0.982
75 percentile	1.922	1.501	1.444	1.162	1.324	1.064

^aThe marginal tax rate on ordinary income and on short term capital gains and losses is 50%. The marginal tax rate on long term capital gains and losses is 20%. The initial investment is made in December 1962 and the wealth relatives are compared in December 1977.

^b 2γ is the round-trip transactions costs rate.

are realized long term. The net liquidation proceeds in 1977 include the cash fund associated with this policy.

The ratio X_I/X_{BH} of the net proceeds of policy I and the buy-and-hold is a measure of their relative performance. Risk is controlled for in this comparison because the number of shares is held constant in both policies.¹⁷ In table 3 we report this ratio for the three variance groups of stocks with zero transactions costs and with 4% round-trip transactions costs. Policy I outperforms the buy-and-hold by a greater margin for high variance stocks than for medium or low variance stocks because the option to realize losses and defer gains is more valuable for high variance than for medium or low variance stocks.

The superiority of policy I over the buy-and-hold is modest. For the group of high variance stocks the median wealth ratio of policy I and the buy-and-hold is 1.043 with zero transactions costs, and 1.016 with 4% round-trip transactions

¹⁷This statement is correct only to a first approximation. By analogy, a portfolio consisting of a call option and a bond paying off the exercise price at the option's maturity, has the same risk as a share of stock only to a first approximation. There is no practical way to refine the adjustment for risk in our simulation or even estimate the direction of bias.

costs. In policy II the investor realizes even long term gains in order to reestablish the tax-advantageous short term status. As we shall see, this policy pays off handsomely.

5.3. Policy II: Realize gains and losses in every December

One hundred dollars are invested in the stock on December 3, 1962. One year later, on December 3, 1963, we observe the stock price. If the stock has a capital loss we realize it short term and repurchase the same number of shares on the following trading day. We deposit the tax rebate in the cash fund associated with the stock. If, instead, the stock has a capital gain, we wait until the following trading day. If the stock still has a capital gain, we realize it long term and repurchase the same number of shares on the following trading day. The capital gains tax is paid out of the cash fund associated with the stock. We repeat the procedure until 1977. In December 1977 the stock is sold. The after-tax proceeds plus the balance of the cash fund is the net proceeds under policy II.

The ratio of the net proceeds under policy II and the buy-and-hold is reported in table 3. Policy II does very well for high and medium variance stocks, even with 4% round-trip transactions costs. Updating the basis yearly and reestablishing the short term status is far more profitable than deferring gains. The results are in agreement with the theoretical prediction of section 4 and tables 1 and 2.

The optimal policy, as discussed in section 4, is to realize gains long term, provided the ratio of the stock price to the basis is below some critical number. In policy II gains are realized however large they may be. Therefore, the results underestimate the effectiveness of a more sophisticated policy of deferring large gains on low variance stocks.

Under the assumption that the holding period for long term capital gains and losses is six months (as was indeed the case in the years 1962–1976) instead of one year, policy II is dominated by the policy of realizing all capital gains long term and all capital losses short term every December and June. (This statement is true even though short term losses and long term gains incurred in the same year offset each other one to one.) Therefore policy II underestimates the effectiveness of a more sophisticated policy applicable in the years 1962–1976.

In a different sense policy II overestimates the benefits of tax trading. If the investor applies policy II to a portfolio of assets (instead of a single asset), each year he typically realizes short term losses on some stocks and long term gains on others. The offsetting of gains and losses may undermine the effectiveness of policy II. The next policy is designed to overcome this problem.

5.4. *Policy III: Realize gains in alternate Decembers and realize losses in every December*

One hundred dollars are invested in the stock on December 3, 1962. In Decembers of odd years gains are deferred and losses are realized, short term if possible. In odd years the problem of one to one offsetting between short term losses and long term gains does not arise because no gains are realized. In Decembers of even years both gains and losses are realized. Gains are realized long term. The investor may be able to realize the losses short term provided that the asset was last purchased one year ago and that there are no realized long term gains on other assets to offset the short term gain one to one. We cannot ascertain whether the investor has realized long term gains on other assets in the same year without having observed the realized returns on all of the portfolio assets. In policy III we simply assume that all losses realized in even years (but not in odd years) are realized long term. This assumption being conservative, policy III underestimates the effectiveness of the optimal policy.

On December 1977 the stock is sold. The after-tax proceeds plus the balance of the cash fund comprise the net proceeds under policy III. The ratio of the net proceeds under policy III and the buy-and-hold is reported in table 3. Even though policy III exaggerates the adverse effect of tax offsetting between short term losses and long term gains, it substantially outperforms the buy-and-hold for all categories of stocks. Even with 4% round-trip transactions costs, policy III outperforms the buy-and-hold for high variance and some medium variance stocks.¹⁸

5.5. *The timing option*

In buying stock, a taxable investor obtains a timing option on the realization of capital gains and losses. The option is utilized under the optimal trading

¹⁸A variant of policy III, which we may call policy IV, is to defer gains and realize losses, preferably short term, in odd years. So far the policy is the same as policy III. But in policy IV, in even years gains are realized long term and losses are deferred. This differs from policy III in that in policy III losses are realized long term in even years. A moment's reflection should convince the reader that policy III dominates policy IV in the absence of transactions costs. In the presence of transactions costs policy III does not dominate policy IV because it involves more frequent transactions. Simulations not reported here confirm that, in the absence of transactions costs, policy III dominates policy IV. With 4% round-trip transactions costs policy III still outperforms policy IV.

Yet another variant of policy III, which we may call policy V, is to realize losses, preferably short term, every December; and realize long term gains only on the Januaries of odd years (instead of realizing them on the Decembers of even years). An upper bound on the performance of this policy is obtained by replacing the tax rate on long term gains in policy III by $\tau_L/(1+r)$, where r is the interest rate; and leaving the tax rates on long term and short term losses unchanged. We conclude that policy V is only marginally superior to policy III.

policy but is wasted under the naive buy-and-hold policy. Suppose that one hundred dollars invested in stock become X_{BH} dollars after tax in fifteen years under the buy-and-hold and X_O under the optimal policy. If $X_O/X_{BH} = 2$ we say that the timing option represents fifty dollars of the initial investment. More generally, we say that the timing option represents fraction $1 - X_{BH}/X_O$ of the initial investment. The timing option provides an alternative interpretation of table 3. For example, for high variance stocks and trading policy III, the timing option represents, on average, fraction $1 - 1/1.747 = 0.43$ of the original investment without transactions costs, or fraction $1 - 1/1.359 = 0.26$ with 4% round-trip transactions costs.

6. Empirical implications

6.1. *The abnormal returns of small firms*

Banz (1981) and subsequently others document a negative association between average returns to stocks and the market value of the stocks after controlling for risk.¹⁹ Our contribution towards the search for an economic explanation of this anomaly is twofold. First, we argue that tax trading not only fails to explain, but exacerbates the anomaly: Small firms typically have higher variance than large firms. Then the tax timing option is more valuable for small than for large firms and the prediction is that small firms have lower before-tax mean returns than large firms.

Second, we argue that Stoll and Whaley's (1983) explanation of the anomaly in terms of differential transactions costs is incomplete.²⁰ Whereas they explicitly incorporate transactions costs, they omit from their calculation the perceived benefit which prompts investors to trade despite the presence of these costs. To illustrate the point, consider the subset of trades generated by taxable investors following trading policy III (see section 5.4). We capture the differential transactions costs between small and large firms by assuming that the round-trip transactions costs rate is 4% for small firms and zero for large firms. We also recognize the fact that the stock returns of small firms are more variable than the stock returns of large firms. For high variance stocks and 4% transactions costs, the mean wealth relative of policy III and the buy-and-hold is 1.359; for medium variance stocks and zero transactions costs, the mean wealth relative is very similar, being 1.383 (see table 3). For small firms, the high variance of stock return compensates for the high transactions costs. At least in this example, the differential transactions costs fail to explain the small-firm anomaly.

¹⁹Schwert (1983) critically reviews the empirical evidence on the small-firm anomaly, discusses the proposed economic explanations, and provides a comprehensive list of references.

²⁰See also Schultz (1983).

6.2. *The abnormal January returns*

Wachtel (1942) and subsequently others document abnormally high January stock returns, especially on stocks of small firms and on stocks that had losses during the previous year.²¹ The tax-loss-selling hypothesis is put forth, most recently, in Roll (1983) as an explanation of the January anomaly, and is critically discussed in Brown, Keim, Kleidon and Marsh (1983). Our contribution to this debate is to identify the critical assumptions in a tax-related explanation of the seasonality in trading volume and stock prices. We consider four scenarios with different assumptions on tax rates and transactions costs.

In the first scenario there is no distinction between the short term and long term tax rates, and transactions costs are zero. The investor optimally realizes a loss whenever it occurs, and defers gains. Even though taxes are paid only at year-end, the investor realizes a loss immediately, lest the stock price rises and the opportunity of taking the loss vanishes. This scenario does not predict an increase in tax-loss selling at year-end.

In the second scenario realized long term gains and losses are taxed at the lower long term rate; long term gains do not offset short term losses dollar for dollar, even if incurred in the same tax year; and transactions costs are zero. The investor realizes short term losses immediately, lest they become long term or vanish. He also realizes long term losses immediately, lest they vanish. This scenario does not predict an increase in tax-loss selling at year-end. The investor has an incentive to realize long term gains on medium and high variance stocks and reinstate the short term status. He also has an incentive to defer the realization of gains from the end of one year to the beginning of the next and save the interest on the tax. This scenario predicts decreased tax-gain selling at the end of a year and increased tax-gain selling at the beginning of a year. If the selling pressure were to depress the price, this scenario would predict a negative abnormal January return for stocks that have had gains during the previous year, but would not predict the observed January anomaly.

In the third scenario realized long term gains and losses are taxed at the lower long term rate; long term gains offset short term losses dollar for dollar in the same tax year; and transactions costs are zero. The optimal policy is complex but we identify the relevant factors: The investor wishes to realize short term losses immediately, lest they become long term or vanish. However, if he has already realized large long term gains earlier in the year, he wishes to defer the short term losses to the following year, if possible. He wishes to realize long term losses immediately, lest they vanish. He also wishes to realize long term gains on medium and high variance stocks, preferably at the beginning of the following year. The important message is that none of these incentives predicts increased tax-loss selling at year-end.

²¹Schwert (1983) critically reviews the empirical evidence on the January anomaly. See the references provided in Schwert (1983) and also Dyl (1973, 1977) and Branch (1977).

In the fourth scenario we assume that there is no distinction between the short term and long term tax rates, but there are transactions costs. If there is no price seasonality, the investor realizes his losses by following a control-limit policy, as in fig. 1. At the beginning of the year he is reluctant to incur the transactions costs and realize a small capital loss. Towards the end of the year his reluctance is overcome by his preference for a tax rebate this year rather than the next. This policy predicts that tax-loss selling gradually increases from January to December and suddenly ceases in the first few days of January; realizing the loss at the end of December dominates the realization of the same loss at the beginning of January.

An objection to the above argument is that, without the distinction between the short term and long term tax rates, transactions costs may dissipate the benefit of tax trading and the optimal policy may be to refrain from tax-loss selling. However, when we draw the distinction between the short term and long term tax rates, our simulation demonstrates that transactions costs do not significantly decrease the benefit from tax trading (see table 3). The optimal trading policy becomes complex, but the essential point remains that, with transactions costs, tax-loss selling gradually increases from January to December and suddenly ceases in the first few days of January.

The selling pressure may or may not affect the stock price. The crucial assumption is that, after selling the stock to realize a loss, investors do not repurchase the same stock or the stock sold by other investors to realize a loss.

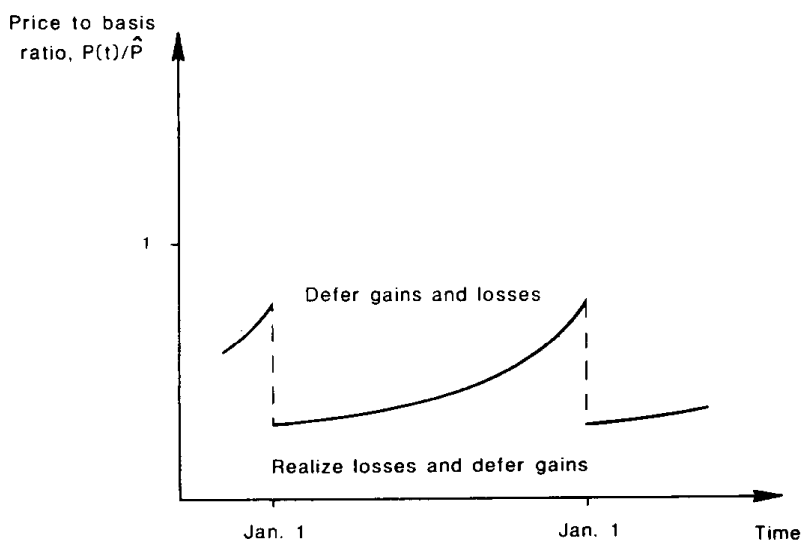


Fig. 1. The optimal trading policy with annual taxation of capital gains and losses, transactions costs, and without the distinction between the short and long term tax status. The figure illustrates the point that at year-end the investor is prone to realize a capital loss, despite the transactions costs, in order to take the loss deduction in the current tax year.

Then tax-loss selling depresses the price of stocks traded in illiquid markets, such as the stocks of small firms. With the tax-loss selling drying up at the beginning of January, the stocks experience positive abnormal returns in January.

This explanation is subject to the criticism that otherwise sophisticated tax-loss sellers are irrational or, at least, ignorant of the stock price seasonality. Rational tax-loss sellers should repurchase different stock sold by other tax-loss sellers at about the same time. Effectively, pairs of tax-loss sellers swap stocks and this activity does not depress the price of either one of the stocks.

If a tax-loss seller has a special reason (e.g., inside information) to repurchase the same stock, he can wait at least one month before the repurchase, and bypass the wash sale provision. Being rational, he can modify the trading policy illustrated in fig. 1, accelerate tax-loss sales from December to the previous October or November, and thereby repurchase the stock in time to receive the abnormal January returns. Alternatively, the investor can change his tax year to end in a month other than December.

7. Concluding remarks

The basic message of our theoretical calculations and simulation is that for those investors who can draw the distinction between the short term and long term tax rates, the benefit of optimal tax trading on medium and high variance stocks outweighs even large transactions costs. Tax trading does not explain the small-firm anomaly, but predicts a seasonal pattern in trading volume which maps into a seasonal pattern in stock prices, the January anomaly, only if we assume irrationality or ignorance on behalf of investors.

Appendix: Proof of Proposition 1

a. Realizing the gain long term one year and one day after purchase dominates the policy of realizing the gain short term one year after purchase. This proves (a).

b. Suppose that the stock was purchased longer than one year ago with cost basis P_0 , and the stock price now is P_t , $P_t < P_0$. If the investor does nothing now and sells the stock at time T , $T > t$, the after-tax proceeds are $(1 - \tau_L)P_T + \tau_L P_0$.

We show that the active policy of realizing the loss at time t and repurchasing the stock dominates the above policy. In selling the stock at time t the investor receives a tax rebate $\tau_L(P_0 - P_t)$ which he invests in a riskless bond with after-tax annual return R , $R > 1$. The basis of the stock is P_t and the status is short term. At time T he sells the stock and the bond. The after-tax proceeds are at least $(1 - \tau_L)P_T + \tau_L P_t + R^{T-t}\tau_L(P_0 - P_t)$ and exceed the after-tax proceeds of the passive policy by at least $(R^{T-t} - 1)\tau_L(P_0 - P_t) > 0$. We

say at least, because if $T = t + 1$ and $P_T < P_t$, the loss realized at time T is short term and $(1 - \tau)P_T + \tau P_t > (1 - \tau_L)P_T + \tau_L P_t$. If the investor dies at time T and the effective tax rate becomes zero, the proceeds of the optimal policy exceed the proceeds of the passive policy by at least $R^{T-t}\tau_L(P_0 - P_t) > 0$.

Next suppose that the stock was purchased just one year ago. By the above argument, realizing a long term loss (by waiting one day before selling the stock) dominates the passive policy. Also realizing the loss short term dominates the policy of realizing it long term. This proves (b).

c. The short term status dominates the long term status. With the short term status, if capital gains are realized in the future, they are realized long term irrespective of the current status; if capital losses are realized one period hence, they are realized short term. Therefore the investor switches to the short term status whenever he can do so costlessly. This proves (c).

Lemma 1. Assume that the distribution of the stock return is price independent. Then $V(P, \hat{P}, t)$ is convex in P .

The proof, available from the author, is similar to Merton's (1973b) proof of his Theorems 4, 9 and 10.

Lemma 2. Assume that the distribution of the stock return is price independent. If it is optimal to defer a long term gain when $P = h\hat{P}$, $h > 1$, then it is optimal to defer a long term gain for all P , $P \geq h\hat{P}$.

The proof, available from the author, is a direct implication of $V(\hat{P}, \hat{P}, t) = \hat{P}$ and Lemma 1.

Determination of the function $V(P, \hat{P}, t)$ when the short term and long term tax rates are equal

We suppress the argument t . We determine the functional form of $V(P, \hat{P})$ in the case $P \geq \hat{P}$ by comparing two positions in the same stock but with different bases. Consider two portfolios with the following composition:

First portfolio:

- (i) $V(uP, \hat{P}) - V(u^{-1}P, \hat{P})$ shares with basis \hat{P} , $\hat{P} \leq P$.
- (ii) A riskless, one-period bond with price $R^{-1}[V(uP, \hat{P}) - V(u^{-1}P, \hat{P})]V(uP, \hat{P})$.

Second portfolio:

- (i) $V(uP, \hat{P}) - V(u^{-1}P, \hat{P})$ shares with basis \hat{P} , $\hat{P} \leq P$.
- (ii) A riskless, one-period bond with price $R^{-1}[V(uP, \hat{P}) - V(u^{-1}P, \hat{P})]V(uP, \hat{P})$.

One period later, the stock price is either uP or $u^{-1}P$. If the stock price is uP , the first portfolio's value is

$$\begin{aligned} & [V(uP, \hat{P}) - V(u^{-1}P, \hat{P})]V(uP, \hat{P}) \\ & + [V(uP, \hat{P}) - V(u^{-1}P, \hat{P})]V(uP, \hat{P}), \end{aligned}$$

and equals the second portfolio's value. If the stock price is $u^{-1}P$, the first portfolio's value is

$$\begin{aligned} & [V(uP, \hat{P}) - V(u^{-1}P, \hat{P})]V(u^{-1}P, \hat{P}) \\ & + [V(uP, \hat{P}) - V(u^{-1}P, \hat{P})]V(uP, \hat{P}), \end{aligned}$$

and again equals the second portfolio's value. Therefore, the investor is indifferent between the two portfolios at the beginning of the period and we obtain

$$\begin{aligned} & [V(uP, \hat{P}) - V(u^{-1}P, \hat{P})]V(P, \hat{P}) \\ & + R^{-1}[V(uP, \hat{P}) - V(u^{-1}P, \hat{P})]V(uP, \hat{P}) \\ & = [V(uP, \hat{P}) - V(u^{-1}P, \hat{P})]V(P, \hat{P}) \\ & + R^{-1}[V(uP, \hat{P}) - V(u^{-1}P, \hat{P})]V(uP, \hat{P}). \end{aligned}$$

Rearranging, we obtain

$$\begin{aligned} \frac{V(P, \hat{P}) - R^{-1}V(uP, \hat{P})}{V(uP, \hat{P}) - V(u^{-1}P, \hat{P})} &= \frac{V(P, \hat{P}) - R^{-1}V(uP, \hat{P})}{V(uP, \hat{P}) - V(u^{-1}P, \hat{P})} \\ &\equiv -R^{-1}k, \end{aligned} \tag{A.1}$$

where k is a constant to be determined. Since the left-hand side is independent of \hat{P} and the right-hand side is independent of \hat{P} , k is independent of \hat{P} . Since $V(P, \hat{P})$ is homogeneous of degree one in (P, \hat{P}) , it follows that $V(P, 0)$ is homogeneous of degree one in P . Setting $\hat{P} = 0$ in (A.1) we obtain

$$k = (u - R)/(u - u^{-1}). \tag{A.2}$$

Eqs. (1) and (A.2) imply $0 < k < 1$.

We rewrite eq. (A.1) as

$$V(P, \hat{P}) = R^{-1}[(1 - k)V(uP, \hat{P}) + kV(u^{-1}P, \hat{P})], \quad P \geq \hat{P}. \tag{A.3}$$

We interpret $1 - k$, k as the pseudoprobabilities of the two states which occur in the binomial process, as in Cox, Ross and Rubinstein (1979). Eq. (A.3) has the following interpretation: $V(P, \hat{P})$ is the expected value of positions (uP, \hat{P}) and $(u^{-1}P, \hat{P})$ discounted by R , where the expectation is with respect to the pseudoprobabilities $1 - k$ and k . The actual probabilities $1 - q$, q do not enter eq. (A.3). The corresponding result in the option pricing theory of Black and Scholes (1973) is that the option price is independent of the expected rate of return of the underlying security.

We eliminate k from eqs. (A.2) and (A.3) and obtain the difference equation in P ,

$$V(P, \hat{P}) = R^{-1} \left[\left(\frac{R - u^{-1}}{u - u^{-1}} \right) V(uP, \hat{P}) + \left(\frac{u - R}{u - u^{-1}} \right) V(u^{-1}P, \hat{P}) \right], \quad P \geq \hat{P}, \quad (\text{A.4})$$

with general solution

$$V(P, \hat{P}) = AP + BP^m \hat{P}^{1-m}, \quad P \geq u^{-1}\hat{P}, \quad (\text{A.5})$$

where

$$m \equiv [\ln(u - R) - \ln(uR - 1)] / \ln u < 0. \quad (\text{A.6})$$

It remains to determine the constants A and B . Continuity of the function V at $P = \hat{P}$ requires

$$V(\hat{P}, \hat{P}) = \hat{P}, \quad (\text{A.7})$$

and continuity at $P = u^{-1}\hat{P}$ requires

$$V(u^{-1}\hat{P}, \hat{P}) = (1 - \tau)u^{-1}\hat{P} + \tau\hat{P}. \quad (\text{A.8})$$

The boundary conditions (A.7) and (A.8) uniquely determine the constants A and B and we obtain eq. (2).

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