

WARRANT EXERCISE AND BOND CONVERSION IN COMPETITIVE MARKETS

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We develop a theory of warrants held by competitive warrant holders not constrained to exercise their warrants as one block; the theory also applies to convertible bonds held by competitive bondholders not constrained to convert their bonds as one block. We prove that the warrant (bond) price in each of the competitive equilibria is less than or equal to the price in an economy with the block constraint; and for at least one competitive equilibrium the warrant (bond) price equals the warrant (bond) price in the block-constrained economy. We illustrate the paths of competitive warrant exercise and bond conversion and conclude that under realistic assumptions they can be long.

1. Introduction

Ingersoll (1977) and Brennan and Schwartz (1977) price American convertible bonds, imposing the artificial constraint that bondholders must convert their bonds as one block. They apply the option pricing theory of Black and Scholes (1973) and Merton (1973), modified to account for the increase in the number of shares upon conversion. The same theory applies to the pricing of warrants under the constraint that warrant holders must exercise their warrants as one block.¹

In relaxing the block constraint, there are at least two interesting cases. In the first case, the entire warrant issue is held by one individual. At each time that the warrants are exercisable, this individual may exercise any fraction of the remaining outstanding warrants. We refer to this individual as a monopolist warrant holder. In the second case, the warrants are held in small amounts by a large number of individuals. At each time that the warrants are exercisable, each warrant holder may exercise any fraction of his or her remaining

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¹See Smith (1977), Galai and Schneller (1978) and Cox and Rubinstein (1983).

outstanding warrants. The important assumption is that the individuals may not collude in their exercise policy. We refer to these individuals as competitive warrant holders. Other cases are briefly discussed in section 7.

On the maturity date of a European or American warrant (convertible bond) issue, but not necessarily on dates prior to maturity, either all outstanding warrants or none are exercised, depending on the state of the economy and of the firm. This result is easily demonstrated, irrespective of whether the warrants are held by a monopolist or by competitive warrant holders. For European warrants the block constraint is not binding and the price of a block warrant equals the price of divisible warrants, whether held by a monopolist or by competitive individuals.

Emanuel (1979, 1983) demonstrates by example that a monopolist warrant holder exercises American warrants sequentially so that the imposition of the block constraint would lower the warrant price. Emanuel casts doubts on the validity of the extant theories of American warrants and convertible bonds which impose the block constraint.

In this paper we eliminate the block constraint and develop a theory of divisible American warrants and convertible bonds held by competitive investors. We assume that markets are complete and that the firm chooses only zero-net-present-value projects. In general there exist multiple competitive equilibria. For each equilibrium the price of a divisible warrant (bond) is less than or equal to the price of a block warrant (bond). For at least one, the price of a divisible warrant (bond) equals the price of a block warrant (bond). We obtain this surprising result despite the fact that the path of warrant exercise (bond conversion) for the divisible warrant in the competitive economy is generally different from the exercise path of the block warrant.

Curiously, our results lend support to the conclusions, if not the assumptions, of the earlier theories of warrant and convertible bond pricing: if we are strictly interested in the price of warrants (bonds) in a competitive market, but not in the path of warrant exercise (bond conversion), the earlier theories with their inherent simplicity provide exact pricing in at least one competitive equilibrium and provide an upper bound to the price in all competitive equilibria, if there exist multiple equilibria.

It is obvious that the warrant price in a monopolist's economy is greater than or equal to the price of a block warrant: the monopolist can always replicate the exercise path which the constrained warrant holder follows; and, in general, the monopolist can do better. The result of our paper is not obvious, however. Competitive warrant holders have the advantage over the constrained warrant holder that they can exercise their warrants sequentially. On the other hand, the competitive warrant holders may not collude in their exercise policy. It turns out that the absence of collusion gives rise to certain rational expectations competitive equilibria in which the warrant holders, expecting the other warrant holders to exercise 'too early', panic and follow suit,

thereby fulfilling their expectations. If we rule out rational expectations equilibria which involve panic, we may claim a stronger result: the price of a divisible warrant (bond) in the competitive equilibrium equals the price of a block warrant (bond); this price is less than or equal to the price in the monopolist's equilibrium.

The price of a block warrant is independent of the manner in which the firm uses the proceeds of warrant exercise, if the firm chooses only zero-net-present-value projects. Therefore the divisible warrant price in the no-panic equilibrium is independent of the manner in which the firm uses the proceeds of warrant exercise. This is an important finding because covenants which limit the firm's policy on the use of warrant exercise proceeds rarely exist or are binding (see also footnote 8). As we illustrate in several examples, the path of warrant exercise does depend on the firm's policy regarding the use of these proceeds. This observation provides the clue to the puzzle: the rate at which warrants are exercised in the no-panic equilibrium depends on the firm's policy regarding the use of warrant exercise proceeds and thereby offsets the effect of this policy on the warrant price.

We illustrate the path of divisible warrant exercise in several examples. If the firm uses the proceeds of warrant exercise to declare an extraordinary dividend, all warrants are exercised simultaneously. If the firm uses the proceeds to increase the scale of production or to buy back stock, the length of time over which warrants are exercised is long: under reasonable assumptions, if there are as many warrants as shares outstanding the exercise time is about 7 years; if the number of warrants is 20% of the number of shares outstanding, the exercise time is 1.8 years (see section 6.5). Likewise the path of bond conversion may be long: under reasonable assumptions, if the bonds are convertible to 20% of the shares outstanding, the conversion time is 4 years. As the dividend yield or the number of bonds increases, the conversion time tends to infinity (see section 6.4).

The paper is organized as follows. In section 2 we describe the model and assumptions. General properties of all competitive equilibria are presented in section 3 as Propositions 1–3. We illustrate these properties in the example of section 4. The existence of a well-behaved competitive equilibrium is stated in section 5 as Proposition 4, and proved in the appendix. In section 6 we illustrate the path of warrant exercise and bond conversion. Finally, in section 7 we pose some questions which arise in non-competitive markets.

2. The model and assumptions

At the beginning of period t , $t = 1, 2, \dots, T$, the firm has $N_t - n_t$ shares of common stock with *cum dividend* price S_t per share, and one issue of n_t warrants with *cum coupon* price W_t per warrant. In sections 2, 3 and 5 the term 'warrant' generically refers to a warrant, a callable convertible bond, or a

callable convertible preferred stock. At the beginning of period t , the value of the firm is $x_t = (N_t - n_t)S_t + n_t W_t$.

The state of the economy is summarized by a vector s_t . We assume the following:

Assumption 1. The market is complete, so that at time t there exists a competitive market for state contingent claims for all realizations s_{t+1} .

The role of this assumption is simply to make the span of the market independent of the existence of the warrants, so that value additivity holds. We define the prices of state-contingent claims as follows: $\pi(s_{t+1}|s_t)$ is the price at time t and state s_t of a claim which pays one unit of wealth at time $t+1$, if state s_{t+1} occurs, and pays zero otherwise.

The sequence of events in period t is described below. First, the firm decides whether to call the warrant issue or not. If the firm calls, it must call all n_t outstanding warrants.² The call price is $K_t = K_t(s_t)$ per warrant.³ The firm's policy is assumed to be known to the stockholders and warrant holders: the firm calls the warrants at time t , if and only if $W_t \geq K_t$.⁴

Second, each warrant holder decides whether to exercise his warrant or not. In exercising his warrant, the investor pays $\beta_t = \beta_t(s_t)$ to the firm and receives one *cum dividend* share of stock. If the warrants have been called in period t , the investor's choice is between exercising his warrant at price β_t or surrendering it to the firm at the call price K_t . If the warrants have not been called in period t , the investor's choice is between exercising his warrants at price β_t or doing nothing until period $t+1$. We denote by y_t the number of warrants exercised in period t . Then the number of warrants at the beginning of period $t+1$ is $n_{t+1} = n_t - y_t$ if the warrants are not called in period t , or $n_{t+1} = 0$ if the remaining warrants are called in period t .

²The calling of a bond should not be confused with the retirement of a bond in accordance with sinking fund provisions. Some sinking fund provisions give the firm the choice between depositing funds in an escrow account, buying in the market a fraction of the outstanding bonds, or calling a fraction of the outstanding bonds. Whereas this feature of sinking fund provisions can be incorporated in our model, it would unnecessarily complicate the discussion and is therefore omitted.

³Some bond covenants give the firm the choice of exchanging the called bond with another bond of par value K_t in lieu of the cash payment K_t . Since we assume that the firm has no bonds (other than the warrant issue) our model does not capture this feature directly.

We may introduce debt in the present model without altering the main results. (See, however, the caveat in footnote 10.) Let $B(s_t)$ be the market value of a \$1 par value bond. At call, the firm optimally pays out $K_t \max[1, B(s_t)]$. Effectively the call price is state dependent. Our assumption that $K_t(s_t)$ is state dependent captures this feature indirectly.

⁴If $W_t \geq K_t$ the firm may be indifferent between calling the warrants or not. We break this indeterminacy by specifying that the firm calls the warrants whenever $W_t \geq K_t$. This policy is consistent with the firm's objective of maximizing the current stockholders' wealth, but may not be unique. See Constantinides and Grundy (1984).

Third, the firm pays coupon $c_t = c_t(x_t, n_t, n_{t+1}, N_t, s_t)$ per (live) warrant;⁵ it pays dividend $d_t = d_t(x_t, n_t, n_{t+1}, N_t, s_t)$ per share;⁶ and it issues (or, repurchases in the market) *ex dividend* shares of common stock so that the sum of the number of shares and warrants outstanding becomes $N_{t+1} = N_{t+1}(x_t, n_t, n_{t+1}, N_t, s_t)$.⁷

We assume that the stockholders and warrant holders are rational and know the firm's policy functions $c_t(\cdot)$, $d_t(\cdot)$, and $N_{t+1}(\cdot)$.⁸ For the general results proved in sections 3 and 5 it is unnecessary to assume anything more specific about these policy functions.

Finally the firm chooses a linear technology at time t from a set of technologies. Each technology in the set specifies the output at time $t + 1$ as a function of the input, x_t , and of the state realization next period, s_{t+1} . We allow the firm's choice of a technology to depend on the capital structure, to reflect the fact that the firm may use the proceeds of warrant exercise to invest in a riskless project, invest in a risky project, or pay an extraordinary dividend. Therefore the policy function, which describes the output of the chosen technology, depends not only on x_t and s_{t+1} but also on n_t , n_{t+1} , and N_t . We assume that this policy function is known to the stockholders and warrant holders. We also assume the following:

Assumption 2. All the projects chosen by the firm have zero net present value of after-tax cash flows.

If the firm's investment policy is independent of the capital structure, Assumption 1 on market completeness implies the Modigliani–Miller (1958) theorem,

⁵At maturity T , c_T is the last coupon plus the par value of the convertible bond. In the event of bankruptcy or reorganization at time t , c_t stands for whatever the warrant holders or bondholders receive in this event.

⁶In the event of bankruptcy or reorganization at time t , d_t stands for whatever the stockholders receive in this event.

⁷The assumption, that the warrant holder receives one *cum dividend* share of stock upon exercise in period t , is innocuous. We may model the situation in which the warrant holder receives one *ex dividend* share of stock upon exercise in period t , by having the dividend declared and paid in period $t - 1$.

Likewise, the assumption, that only the warrant holders who do not exercise in period t receive a coupon in period t , is innocuous. We may model the situation in which the warrant holders receive a coupon and then exercise their warrant in period t , by having the coupon paid in period $t - 1$.

⁸The firm's policy functions are rarely determined by bond covenants. Bond covenants which limit the firm's policies on distributions to its shareholders may or may not exist. Even if they exist they are rarely binding. See Kalay (1982).

The restriction applies to the sum total of the distribution to shareholders. Therefore the firm is typically free to decide whether to pay a cash dividend to the stockholders within the allowable limit or disburse the same amount of funds to the stockholders in the form of stock repurchase. Furthermore the firm is typically unrestricted by bond covenants in the use of the proceeds from the issue of stock or the exercise of warrants after the peg date. See American Bar Foundation (1971, sec. 10) and Smith and Warner (1979).

that the firm value is independent of the capital structure. With Assumption 2, even though the firm's investment policy generally depends on its capital structure, the firm value is independent of the capital structure.

We consider the game played by the warrant holders and stockholders, where the firm's policy functions are assumed to be known and given exogenously. Each warrant holder's action is his choice of the time to exercise his warrant. The stockholders are passive players in this game because they take no action. The significance of Assumption 2 is that the game between the warrant holders and stockholders is zero-sum. Propositions 1–4 hinge on the zero-sum property.

The equilibrium concept is Nash: each warrant holder takes the other warrant holders' action as given and acts to maximize his payoff (i.e., the present value of his stream of cash flows). We assume that each warrant holder is small. Formally, there is a continuum of warrant holders distributed on the interval $[0, 1]$ and each warrant holder holds a number of warrants of measure zero. Therefore, each warrant holder not only takes the other warrant holders' action as given, but also takes the total number of warrants exercised in each period and in each state as given; being small, his action does not change the total number of exercised warrants. By assumption, the firm's policy is a function not of the individual warrant holders' action but of the total number of exercised warrants. Therefore each warrant holder perceives that his action leaves the firm's action and the price of stock and warrants unchanged. We informally refer to this equilibrium as a competitive equilibrium. A competitive equilibrium exists, although it may not be unique.⁹ Indeed, as we demonstrate by example in section 4, there is a multiplicity of equilibria. Our discussion explicitly take into account this multiplicity.

3. Properties of all competitive equilibria

The definition of the firm's value is

$$x_t = (N_t - n_t)S_t(x_t, n_t, N_t, s_t) + n_tW_t(x_t, n_t, N_t, s_t). \quad (1)$$

This implies the following useful lemma:

Lemma 1. The statements (2), (3), and (4) are equivalent:

$$W_t(x_t, n_t, N_t, s_t) \geq S_t(x_t, n_t, N_t, s_t) - \beta_t, \quad (2)$$

$$S_t(x_t, n_t, N_t, s_t) \geq (x_t + n_t\beta_t)/N_t, \quad (3)$$

$$W_t(x_t, n_t, N_t, s_t) \geq (x_t + n_t\beta_t)/N_t - \beta_t = (x_t - (N_t - n_t)\beta_t)/N_t. \quad (4)$$

⁹Existence is proved in Constantinides and Rosenthal (1984). The warrant holders' problem is formally modeled as a non-cooperative game in which the set of players is a non-atomic continuum. Schmeidler's (1973) results about existence of equilibria are shown to apply in the context of the warrant holders' game.

Eq. (1) together with any one of inequalities (2), (3), and (4) implies the other two inequalities, and this completes the proof of the lemma. Note that the proof does not rely on Assumptions 1 and 2.

Proposition 1. *If Assumptions 1 and 2 hold, and if the competitive warrant holders are indifferent between exercising their warrants or not at time t , then the stock and warrants are priced at time t as if all warrants are exercised immediately.*

Proof. Since the warrant holders are indifferent between exercising their warrants or not, inequality (2) holds as an equality and therefore inequalities (3) and (4) also hold as equalities. By Assumptions 1 and 2, the right-hand sides of eqs. (3) and (4) are the prices per share and warrant, respectively, if all warrants are exercised immediately. This completes the proof.

Proposition 1 also holds if the firm has senior debt, provided that the value of the senior debt is independent of the warrant exercise.¹⁰ The proof of Proposition 1 (and Propositions 2–4) hinges on the property that the game between the warrant holders and stockholders is zero-sum. This property is preserved if the value of the senior debt is independent of the warrant exercise.

Proposition 1 motivates the definition of a useful theoretical construct, the block warrant.

Definition. *A block warrant is an indivisible warrant issue which may be traded and exercised only as one block.*

We consider a firm identical to the one with competitive warrant holders, except that the warrant is a block. We define the following:

\hat{W}_t : *Ex-coupon warrant price, if the warrant is a block.*

\hat{S}_t : *Cum-dividend stock price, if the warrant is a block.*

In the next proposition we argue that the holder of a block warrant can do as well as the competitive warrant holders by following the (generally suboptimal) policy of exercising the warrant block at the first time that some warrants are exercised in the competitive equilibrium. We conclude that the price of the divisible warrant is less than or equal to the block warrant price.

¹⁰The senior debt may be subject to default risk. The assumption is that the exercise of warrants does not redistribute wealth between the senior bondholders and the other claimants to the firm, although it may redistribute wealth between the stockholders and the warrant holders.

The senior debt may also be subject to term structure risk. Its price may change with changes in the term structure of interest rates, as summarized by the state variable s_t .

Proposition 2. If Assumptions 1 and 2 hold, then

$$W_t(x_t, n_t, N_t, s_t) \leq \hat{W}_t(x_t, n_t, N_t, s_t), \quad \text{for all } t, x_t, n_t, N_t \text{ and } s_t. \quad (5)$$

Proof. Given x_t, n_t, N_t and s_t in period t , let $\tau, \tau \geq t$, be the first time at which the competitive warrant holders exercise some of their warrants. Then

$$W_\tau(x_\tau, n_\tau, N_\tau, s_\tau) = S_\tau(x_\tau, n_\tau, N_\tau, s_\tau) - \beta_\tau. \quad (6)$$

By Lemma 1,

$$W_\tau(x_\tau, n_\tau, N_\tau, s_\tau) = (x_\tau + n_\tau \beta_\tau) / N_\tau - \beta_\tau.$$

If the holder of the block warrant follows the generally suboptimal policy of doing nothing over $[t, \tau)$ and exercising his warrant at time τ , then

$$\hat{W}_\tau(x_\tau, n_\tau, N_\tau, s_\tau) = (x_\tau + n_\tau \beta_\tau) / N_\tau - \beta_t = W_\tau(x_\tau, n_\tau, N_\tau, s_\tau),$$

and therefore

$$\hat{W}_t(x_t, n_t, N_t, s_t) = W_t(x_t, n_t, N_t, s_t). \quad (8)$$

If the holder of the block warrant follows his optimal policy instead, his warrant is at least as valuable as when he follows the suboptimal policy. This completes the proof.

The next proposition provides further insight as to why the price of the divisible warrant is less than or equal to the price of a block warrant: the competitive warrant holders may start exercising their warrants too early, i.e., before the holder of the block warrant optimally exercises his block.

Proposition 3. Given x_t, n_t, N_t and s_t , let $\tau, \tau \geq t$, be the first time after t at which the competitive warrant holders are indifferent between exercising or not some warrants; and let $\hat{\tau}, \hat{\tau} \geq t$, be the first time after t at which the block warrant holder is indifferent between exercising or not his warrant block. If Assumptions 1 and 2 hold, then

$$\tau \leq \hat{\tau}. \quad (9)$$

Proof. By the definition of the stopping time $\hat{\tau}$,

$$\hat{W}_{\hat{\tau}}(x_{\hat{\tau}}, n_{\hat{\tau}}, N_{\hat{\tau}}, s_{\hat{\tau}}) = (x_{\hat{\tau}} + n_{\hat{\tau}} \beta_{\hat{\tau}}) / N_{\hat{\tau}} - \beta_{\hat{\tau}}. \quad (10)$$

If $\tau > \hat{\tau}$, then in the competitive equilibrium

$$\begin{aligned} W_{\hat{\tau}}(x_{\hat{\tau}}, n_{\hat{\tau}}, N_{\hat{\tau}}, s_{\hat{\tau}}) &> S_{\hat{\tau}}(x_{\hat{\tau}}, n_{\hat{\tau}}, N_{\hat{\tau}}, s_{\hat{\tau}}) - \beta_{\hat{\tau}} \\ &> (x_{\hat{\tau}} + n_{\hat{\tau}}\beta_{\hat{\tau}})/N_{\hat{\tau}} - \beta_{\hat{\tau}} \quad [\text{by Lemma 1}] \\ &> \hat{W}(\hat{x}_{\tau}, n_{\hat{\tau}}, N_{\hat{\tau}}, N_{\hat{\tau}}, s_{\hat{\tau}}) \quad [\text{by eq. (10)}], \end{aligned}$$

and contradicts Proposition 2. Therefore $\tau \leq \hat{\tau}$, as claimed.

In the next section an example illustrates the results developed so far.

4. An example

4.1. Introduction

A firm has 500 shares of stock with *cum dividend* price S per share, and 500 warrants with price W per warrant. Each warrant may be exercised only at one of two times, now and at maturity, which is one year from now. Each warrant's exercise price is \$9, and the conversion ratio is unity. Dividends are declared and paid one day from now and one day after the warrants' maturity; if a warrant is exercised now, the warrant holder receives one *cum dividend* share of stock.

The firm value now, before any warrants are exercised and before any dividends are paid, is given to be \$10,000, i.e., $500S + 500W = 10,000$. If y warrants are exercised now, the firm value becomes $10,000 + 9y$. One day from now, the firm declares and pays out 5% of its assets as dividends, i.e., pays $((10,000 + 9y) \times 0.05)/(500 + y)$ per share. The firm's remaining capital, $(10,000 + 9y) \times 0.95$, is invested in a riskless production process over the next year, and earns the riskless rate of interest, taken to be 10%. One year from now, the firm value becomes $(10,000 + 9y) \times 0.95 \times 1.10$. If y' warrants are exercised at maturity, the firm value becomes $(10,000 + 9y) \times 0.95 \times 1.10 + 9y'$ and the price per share becomes $[(10,000 + 9y) \times 0.95 \times 1.10 + 9y']/(500 + y + y')$.

4.2. The block warrant holder's problem

One investor, the block warrant holder, holds the entire issue of 500 warrants and is (artificially) constrained to exercise them as one block, whenever he decides to exercise them. He chooses $(y = y' = 0)$ or $(y = 0, y' = 500)$ or $(y = 500, y' = 0)$ to maximize the present value of his cash flows,

$$\begin{aligned} &-9y + (10,000 + 9y) \times 0.05 \times \frac{y}{500 + y} - \frac{9y'}{1.1} \\ &+ \frac{(10,000 + 9y) \times 0.95 \times 1.10 + 9y'}{1.1} \times \frac{y + y'}{500 + y + y'}. \end{aligned}$$

The optimal decision is to exercise all his warrants now. The present value of his cash flow is \$2,750 and the warrant price is $\hat{W} = 2,750/500 = \$5.50$. This price is compared to the warrant price in the competitive warrant holders' equilibrium.

4.3. *The competitive warrant holders' problem*

Assume that the warrants are infinitely divisible and are held by a large number of non-colluding, rational investors, none of which holds a sufficiently large fraction of the warrants to be able to affect prices through warrant exercise. Suppose that the competitive warrant holders conjecture that y warrants will be exercised now. Then each warrant holder makes the following calculation: taking as given the aggregate number, y , of exercised warrants, the present value of the proceeds of a live warrant exercised one year from now, is

$$W = \frac{(10,000 + 9y) \times 0.95 \times 1.10 + (500 - y) \times 9}{1,000 \times 1.1} - \frac{9}{1.1}. \quad (11)$$

This function is a straight line and is illustrated in fig. 1. If $y = 500$, then $W = \$5.59 > \5.50 . This demonstrates that the warrant exercise path in the block warrant holder's problem, to exercise all 500 warrants now, cannot be sustained in a competitive equilibrium. For, taking as given that everybody else exercises his warrant now, each rational warrant holder has the incentive of \$0.09 per warrant to defer the exercise of his warrant by one year.

Taking as given the aggregate number, y , of exercised warrants, the present value of the proceeds of exercising one warrant now is

$$S - 9 = -9 + \frac{(10,000 + 9y) \times 0.05}{500 + y} + \frac{(10,000 + 9y) \times 0.95 \times 1.10 + (500 - y) \times 9}{1.1 \times 1,000}. \quad (12)$$

The function $S - 9$ is illustrated in fig. 1. It demonstrates that it cannot be a competitive equilibrium to have none of the warrants exercised now: if $y = 0$, $S - 9 = \$5.59$, $W = \$5.41$, and each warrant holder has an incentive to exercise his warrant now. The competitive equilibrium occurs at $W = S - 9 = \$5.50$ and $y = 247$, $y' = 253$. At equilibrium the dividend received as a result of early exercise is the same as the interest on the exercise price advanced.

The example illustrates three points. First, it illustrates that the divisible warrants held by competitive warrant holders are not necessarily exercised in a block. Second, it illustrates Lemma 1. In the competitive equilibrium $W_0 = 5.50$ and $S_0 = 14.50$. The firm's value is $x_0 = 500 \times 5.50 + 500 \times 14.50 = 10,000$.

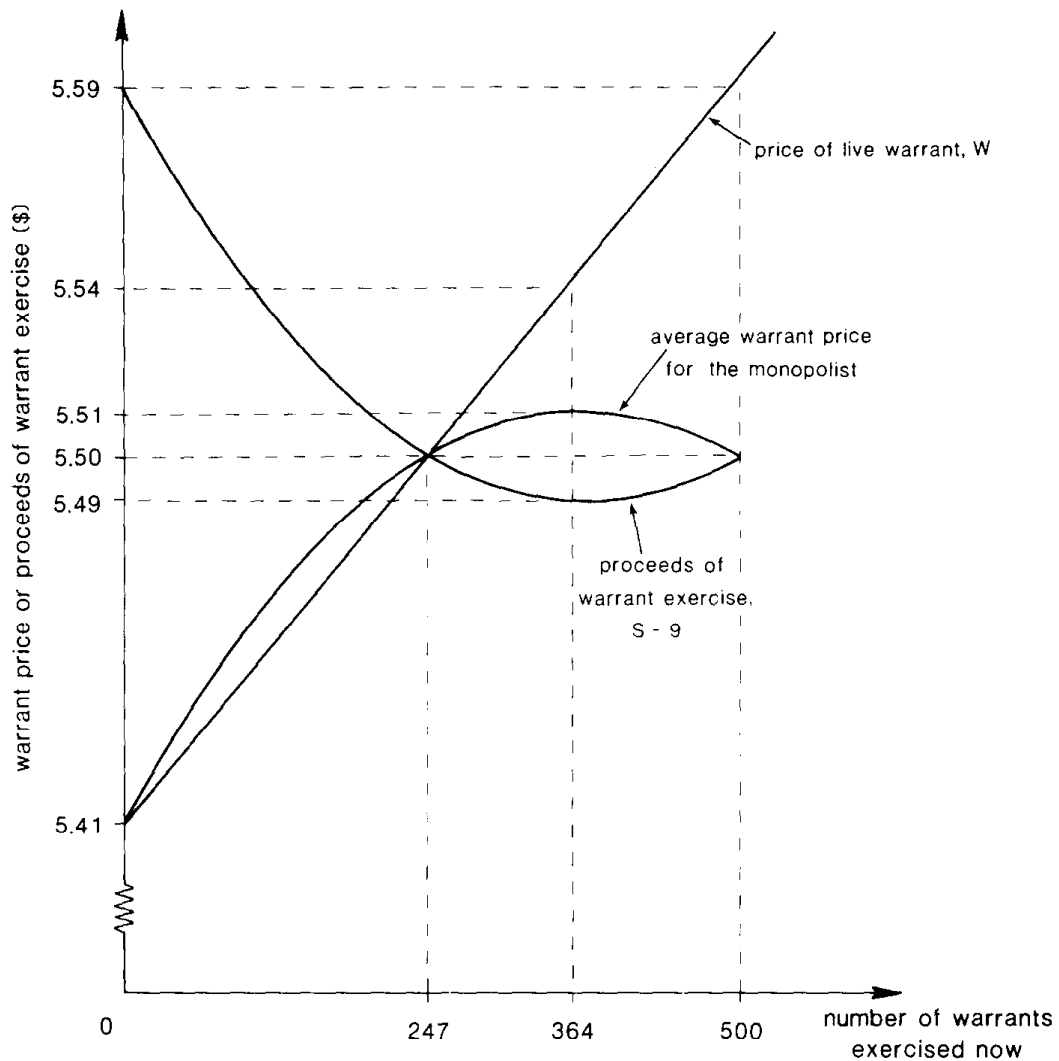


Fig. 1. The warrant price (W) given by eq. (11) and the warrant exercise proceeds ($S - 9$) given by eq. (12) as functions of the number of warrants exercised now (y), for the example of sections 4.1–4.3. The competitive equilibrium occurs at the intersection of the curves W and $S - 9$ with $W = S - 9 = \$5.50$ and $y = 247$ warrants. But $\$5.50$ is also the warrant price in the block warrant holder's problem. The example illustrates Proposition 1. It also illustrates that the average warrant price for the monopolist warrant holder is $\$5.51$, maximized with the exercise of 364 warrants, and exceeds the warrant price in the competitive equilibrium.

In the example, the firm has 500 shares and 500 warrants with total market value $\$10,000$. Each warrant may be exercised at a cost of $\$9$ into one *cum dividend* share of stock now or at maturity which is one year from now. If y warrants are exercised, the firm value becomes $10,000 + 9y$. The firm distributes 5% of $10,000 + 9y$ as a dividend one day from now and invests the rest of its capital at the riskless rate of interest, taken to be 10%.

Each competitive warrant holder is indifferent between exercising his warrant or not, since $W_0 = 14.50 - 9$, where the exercise price is 9. Since (2) is satisfied as an equality, Lemma 1 states that $S_0 = (10,000 + 500 \times 9)/1,000 = 14.50$, as is indeed the case. Third, it illustrates Proposition 1. The stock and warrants are priced now as if all warrants are exercised immediately.

4.4. Illustration of Propositions 2 and 3

We illustrate Propositions 2 and 3 by an example in which there exist two competitive equilibria. In the first, the competitive warrant holders exercise their warrants at the same time that the holder of the block warrant exercises his warrant, i.e., $\tau = \hat{\tau}$. The price of divisible warrants equals the block warrant price. In the second, the competitive holders of the divisible warrants begin exercising their warrants before the holder of the block warrant exercises his block, i.e., $\tau < \hat{\tau}$. The competitive divisible warrant price is less than the price of the block warrant.

We modify the previous example in two respects: (1) We set the exercise price to be $K = 12$ (instead of $K = 9$). (2) We set the firm's dividend policy now to be

$$\begin{aligned} (\text{Total dividend now}) &= 10,000 \times 0.05 & \text{if } 0 \leq y \leq 1, \\ &= 10,000 + Ky & \text{if } 1 < y \leq 500. \end{aligned}$$

That is, the firm pays out 5% of its assets as a dividend, if no more than one warrant is exercised now; and pays out a *liquidating* dividend if more than one warrant is exercised. Whereas this dividend policy is artificial, it illustrates the point in a forceful way.

The reader may easily verify that the holder of the block warrant waits until maturity to exercise his warrant. The warrant price now is $\hat{W} = 4.05$, the *cum dividend* stock price is $\hat{S}_{\text{cum}} = 15.95$ and the *ex dividend* stock price is $\hat{S}_{\text{ex}} = 14.95$.

In the first competitive equilibrium, none of the warrants are exercised now, and all the warrants are exercised at maturity. As the reader may verify, this is a rational expectations equilibrium. Now the warrant price is $W = 4.05$, the *cum dividend* stock price is $S_{\text{cum}} = 15.95$ and the *ex dividend* stock price is $S_{\text{ex}} = 14.95$. In this equilibrium $\tau = \hat{\tau}$ and $W = \hat{W}$.

In the second competitive equilibrium, all of the warrants are exercised now. This also is a rational expectations equilibrium. The warrant price is $W = 4.00$, the *cum dividend* stock price is $S_{\text{cum}} = 16.00$ and the *ex dividend* stock price is $S_{\text{ex}} = 0$. In this equilibrium, $\tau < \hat{\tau}$ and $W < \hat{W}$. Whereas the warrant holders are better off if they all wait until maturity to exercise their warrants, the fear that some warrant holder will exercise his warrant now leads all the other warrant holders to follow suit and the expectation is fulfilled.

The above example suggests the conjecture that, whereas there may exist several competitive equilibria, for at least one equilibrium the divisible warrant price equals the price of the block warrant. This conjecture is indeed true and is formalized in the next section as Proposition 4.

5. Existence of a well-behaved competitive equilibrium

It is shown in Constantinides and Rosenthal (1984) that there exists at least one competitive equilibrium. We now prove that, of these equilibria, at least one is well-behaved: the warrant exercise begins at the time that the holder of a block warrant would exercise the block, and the competitive divisible warrant price equals the price of the warrant block. (In this section the term 'warrant' generically refers to both warrants and convertible bonds.)

Proposition 4. If Assumptions 1 and 2 hold, then for at least one competitive equilibrium,

- (a) $W_t(x_t, n_t, N_t, s_t) = \hat{W}_t(x_t, n_t, N_t, s_t), \quad t = 1, 2, \dots, T. \quad (13)$
- (b) *The first time at which the block warrant is optimally exercised coincides with the first time at which some divisible warrants are exercised by the competitive warrant holders.*
- (c) *Whenever the block warrant is called, the competitive divisible warrants are also called.*
- (d) *Given x_t, n_t, N_t and s_t , the price of competitive warrants is independent of the firm's policy on the use of warrant exercise proceeds.*

We outline the proof and refer the reader to the appendix for the details. To prove (a), we first recognize that eq. (13) holds at $t = T + 1$. We then proceed by induction: we assume that eq. (13) holds at time $t + 1$ and prove that it also holds at time t . We consider separately two cases. Suppose first that the block warrant is optimally exercised at time t . We prove that it is a rational expectations equilibrium to have some of the divisible warrants exercised at time t . Invoking Lemma 1, we conclude that eq. (13) is satisfied at time t . Suppose next that the block warrant is optimally left unexercised at time t . We cannot prove that in all rational expectations equilibria none of the divisible warrants are exercised. Hence the possibility of multiple equilibria. We do prove, however, that it is a rational expectations equilibrium to have none of the divisible warrants exercised at time t . If a warrant holder believes that none of the other warrant holders exercise at time t , he has no incentive to diverge from this strategy, for under it the warrant is worth the block warrant price which exceeds the proceeds of immediate exercise. We conclude that eq. (13) is satisfied at time t and this completes the proof of part (a).

Parts (b) and (c) follow from part (a) and Proposition 3. Part (d) follows from part (a) and the fact that the price of the block warrant is independent of the firm's policy on the use of warrant exercise proceeds.

Proposition 4 states that the price of competitive warrants is independent of the manner in which the firm uses the proceeds of warrant exercise. This statement is puzzling because, as we shall see in the examples of section 6, the competitive warrant holders may exercise their warrants over a period of time and one would expect that the price of the unexercised warrants depends on the manner in which the firm uses these proceeds. The examples of section 6 illustrate that the path of warrant exercise depends on the firm's policy regarding the use of these proceeds. This observation provides the clue to the puzzle. *The equilibrium path of competitive warrant exercise depends on the firm's policy regarding the use of warrant exercise proceeds and thereby offsets the effect of this policy on the warrant price.*

6. Path of warrant exercise and bond conversion

6.1. Introduction

We consider a firm with $n(t)$ non-callable warrants or convertible bonds with infinite maturity, constant exercise price β , conversion ratio unity, and constant coupon c per warrant. For actual warrants $c = 0$ and for convertible bonds $\beta = 0$. The firm also has $N(t) - n(t)$ shares of stock. Unlike the earlier sections, trading and warrant exercise may occur continuously. The firm's capital, $x(t)$, grows at the riskless rate of interest, r , gross of dividends, coupons, and warrant exercise proceeds. The firm pays out a dividend at the rate $\alpha x(t)$, where α is a constant, $0 < \alpha < r$. As long as no warrants are exercised, the firm does not issue new financial claims nor does it repurchase any existing ones. Net of dividends and coupons, but gross of warrant exercise proceeds, the firm's capital grows as

$$\dot{x}(t) = (r - \alpha)x(t) - n(t)c. \quad (14)$$

To simplify the notation we set the initial sum of the number of warrants and shares equal to one, i.e., $N_0 = 1$. We assume

$$x_0 > n_0 c / (r - \alpha), \quad (15)$$

so that the firm is never bankrupt, even if none of the warrants are ever exercised.

6.2. *The block warrant holder's and convertible bondholder's problem*

We need not specify the firm's policy on the use of warrant exercise proceeds beyond Assumption 2.¹¹ The block warrant holder's problem is stated below, where subscripts denote partial derivatives and the arguments x, t are suppressed,

$$\{(r - \alpha)x - n_0c\} \hat{W}_x - r\hat{W} + c = 0, \quad x < \bar{x}, \tag{16}$$

$$\hat{W} = x - (1 - n_0)\beta, \quad x \geq \bar{x}, \tag{17}$$

$$\hat{W}_x = 1, \quad x \geq \bar{x}. \tag{18}$$

\bar{x} is the firm value at which the warrant is exercised and eq. (16) states that, so long as the warrant remains unexercised, its rate of return equals r . Eq. (17) states that, when the firm value equals or exceeds \bar{x} , the warrant is exercised, the firm value becomes $x + n_0\beta$, and the price per share becomes $x + n_0\beta$. Since each warrant is exchanged for one share at cost β , the warrant price equals $\hat{W}(x, n_0) = x + n_0\beta - \beta$, which is eq. (17). Eq. (18), known as the 'smooth-pasting condition', follows from the requirement that the barrier, \bar{x} , is chosen optimally by the block warrant holder,

$$(\partial/\partial\bar{x})[\hat{W}(\bar{x}, n_0) - \{\bar{x} - (1 - n_0)\beta\}] = 0. \tag{19}$$

The latter equation simplifies to eq. (18).

The solution to eq. (16) subject to the boundary conditions (17) and (18) is

$$\begin{aligned} \hat{W}(x, n_0) &= \frac{c}{r} + \left(\bar{x} - (1 - n_0)\beta - \frac{c}{r} \right) \left[\frac{x - n_0c/(r - \alpha)}{\bar{x} - n_0c/(r - \alpha)} \right]^\nu, \\ x \leq \bar{x} &\equiv (1 - n_0)\beta\nu/(\nu - 1) + (\nu/r - n_0/(r - \alpha))c/(\nu - 1) \\ &= x - (1 - n_0)\beta, \quad x > \bar{x}, \end{aligned} \tag{20}$$

where

$$\nu \equiv r/(r - \alpha). \tag{21}$$

Using eq. (21), the critical value of the firm at which the warrant is exercised is $\bar{x} = (1 - n_0)(c + r\beta)/\alpha$. The critical value is increasing in the coupon yield c ,

¹¹Recall that Assumption 2 guarantees that the firm either redistributes these proceeds in the form of dividends or in the form of stock or warrant repurchase; or invests them in a project with zero net present value.

the interest rate r , and the exercise price β . It is decreasing in the dividend yield α , and in the number of warrants, n_0 .

The warrants are exercised at time τ , where τ is the solution to the equation

$$x(\tau) = \frac{n_0 c}{r - \alpha} + \left(x(0) - \frac{n_0 c}{r - \alpha} \right) e^{(r - \alpha)\tau} = (1 - n_0)(c + r\beta)/\alpha. \quad (22)$$

We shift the time origin to τ . In the new notation,

$$x(t) = \frac{n_0 c}{r - \alpha} + \left(\frac{(1 - n_0)(c + r\beta)}{\alpha} - \frac{n_0 c}{r - \alpha} \right) e^{(r - \alpha)t}, \quad t \leq 0. \quad (23)$$

When $t < 0$, the rate of return on the warrant equals the interest rate and we obtain

$$\hat{W}(t) = c/r + (\bar{x} - (1 - n_0)\beta - c/r)e^{rt}, \quad t \leq 0. \quad (24)$$

Finally, using the equation $x(t) = n_0 \hat{W}(t) + (1 - n_0)\hat{S}(t)$ we obtain the path of the stock price as

$$\hat{S}(t) = (x(t) - n_0 \hat{W}(t))/(1 - n_0). \quad (25)$$

6.3. A competitive equilibrium

We describe a competitive equilibrium of divisible warrants where none of the warrants are exercised prior to time zero, the time that the block warrant would be exercised. Prior to time zero, the divisible warrant price in the competitive equilibrium equals the price of the warrant block and is given by eq. (24). Also the stock price in the competitive equilibrium is given by eq. (25). The reader may verify that $W(t) > S(t) - \beta$ for $t < 0$, so that the expectation that none of the warrant holders exercises a warrant prior to time zero is rational.

For all firm policies along the exercise path, a competitive warrant holder is indifferent between exercising his warrant or not. The rate of return on a warrant equals the interest rate and we obtain

$$W(t) = c/r + (\bar{x} - (1 - n_0) - c/r)e^{rt}, \quad t \leq T_0, \quad (26)$$

where T_0 denotes the time that the last warrant is exercised, i.e., T_0 is defined by

$$n(T_0) = 0. \quad (27)$$

Since each competitive warrant holder is indifferent between exercising his warrant or not, we obtain

$$S(t) = W(t) + \beta, \quad 0 \leq t \leq T_0. \quad (28)$$

For future reference we also write

$$\begin{aligned} x(t) &= \{N(t) - n(t)\}S(t) + n(t)W(t) \\ &= \{N(t) - n(t)\}\beta \\ &\quad + N(t)[c/r + (\bar{x} - (1 - n_0) - c/r)e^{rt}], \quad 0 \leq t \leq T_0. \end{aligned} \quad (29)$$

In the remainder of this section we draw the distinction between convertible bonds and warrants and examine the path of bond conversion and warrant exercise separately. For warrant exercise we consider three representative firm policies on the use of warrant exercise proceeds: reinvestment, stock repurchase, and payment of an extraordinary dividend.

6.4. Path of bond conversion

For a convertible bond the conversion is costless, $\beta = 0$, and the sum of the number of shares and bonds remains constant, $N(t) = 1$. The dynamics of the firm's value is

$$\dot{x}(t) = (r - \alpha)x(t) - n(t)c. \quad (30)$$

Combining eqs. (29) and (30) we obtain the path of bond conversion,

$$\begin{aligned} n(t) &= n_0 - (1 - n_0 - \alpha/r)(e^{rt} - 1), \\ 0 \leq t \leq T_0 &\equiv \frac{1}{r} \ln \left(\frac{1 - \alpha/r}{1 - n_0 - \alpha/r} \right), \\ &= 0, \quad T_0 \leq t. \end{aligned} \quad (31)$$

The length of time over which bonds are converted is long. If $r = 0.10$ per year, $\alpha = 0.05$ per year and the bonds are convertible to 20% of the shares outstanding, the conversion time is 4 years. As the dividend yield relative to the interest rate increases, or as the number of bonds increases, the conversion time tends to infinity.

6.5. *Path of warrant exercise with reinvestment of warrant exercise proceeds (policy I)*

When $-\dot{n}(t)dt$ warrants are exercised, the production input increases by $-\beta\dot{n}(t)dt$. The dynamics of the firm value is

$$\dot{x}_I(t) = (r - \alpha)x_I(t) - \beta\dot{n}_I(t), \quad (32)$$

where the subscript reminds us that this dynamics is specific to policy I. The total number of shares and warrants remains constant, $N_I(t) = 1$. Combining eqs. (29) and (32), we obtain the path of warrant exercise,

$$\begin{aligned} n_I(t) &= 1 - (1 - n_0)e^{rt}, & 0 \leq t \leq T_I \equiv -r^{-1}\ln(1 - n_0), \\ &= 0, & T_I \leq t. \end{aligned} \quad (33)$$

The path is illustrated in fig. 2. The warrants are exercised at an increasing rate because the proceeds of warrant exercise are retained; this increases the stock price and induces more warrant holders to exercise their warrant. Formally, the function $n(t)$ is decreasing and concave in t .

The length of time over which warrants are exercised is long. For example, if there are as many warrants as shares outstanding, i.e., $n_0 = \frac{1}{2}$, and the interest rate is $r = 0.10$ per year, the exercise time is $-r^{-1}\ln(1 - n_0) = 6.9$ years. If the number of warrants is 20% of the number of shares outstanding, the exercise time is 1.8 years. As the number of warrants increases, or as the interest rate decreases, the exercise time increases.

We may easily modify the discussion to allow for finite warrant maturity T . If $T < 0$, the warrants mature out-of-the-money and neither the block warrant holder nor the competitive warrant holders exercise their warrants; the warrants are worthless at all times. If $T > T_I$ under firm policy I, the exercise of warrants in the competitive equilibrium is completed before the warrants' maturity and our discussion of the warrant exercise path and the warrant price remains intact. The finite maturity changes the path of warrant exercise in the case $0 < T < T_I$ as follows: the warrants are exercised as in eq. (33) for $0 < t < T$; at time T all the remaining outstanding warrants, $n_I(T)$, are immediately exercised. Nonetheless, the warrant price is independent of T for $t \leq T$. This is an implication of Proposition 1.

6.6. *Path of warrant exercise with the proceeds of warrant exercise used for stock repurchase (policy II)*

The dynamics of the firm value is

$$\dot{x}_{II}(t) = (r - \alpha)x_{II}(t). \quad (34)$$

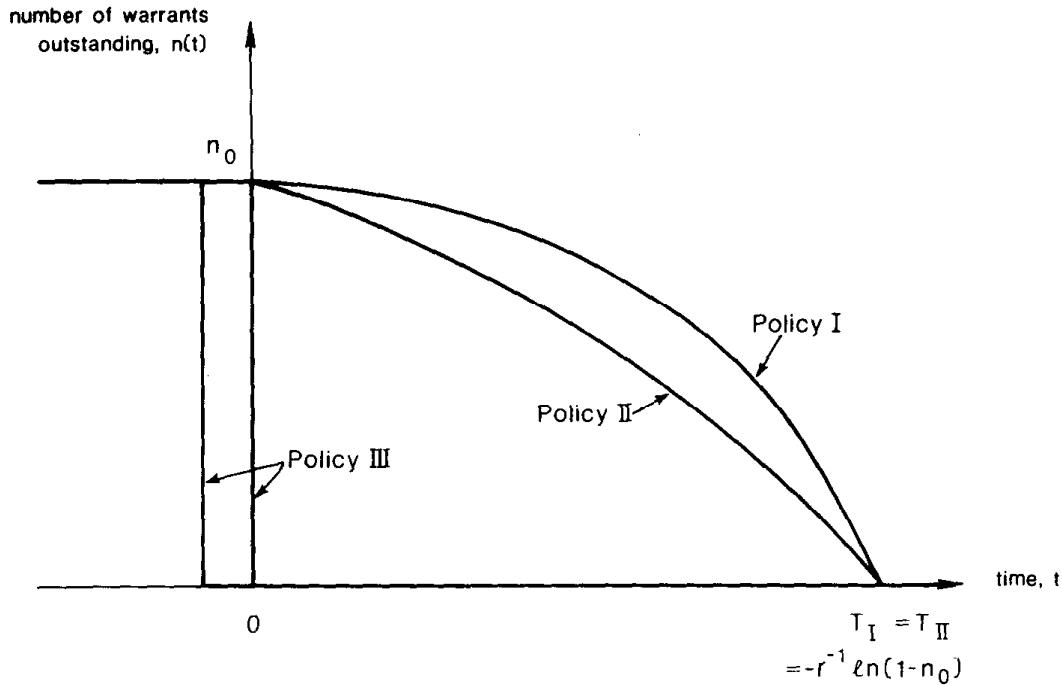


Fig. 2. The number of outstanding warrants, $n(t)$, in a competitive equilibrium of divisible warrants, as a function of time under three firm policies on the use of warrant exercise proceeds: (I) reinvestment [eq. (33)], (II) stock repurchase [eq. (36)], (III) extraordinary dividend to stockholders on record after the exercise. The time origin is the time at which a block warrant holder would exercise the warrant. Under policies I and II, the warrants are exercised over a length of time. Under policy III, the warrants are exercised simultaneously at time zero in a no-panic equilibrium and before time zero in a panic equilibrium.

In the example, the firm's value grows at the interest rate, r , gross of dividend payouts. The firm pays an ordinary dividend rate equal to fraction α of its assets, $\alpha < r$. The total number of shares and warrants is one and the exercise price is β per warrant. For further details see section 6.

The firm repurchases $-\beta \dot{n}(t)/S(t)$ shares and the total number of shares and warrants decreases accordingly, i.e.,

$$\dot{N}_{II}(t) = \beta \dot{n}_{II}(t)/S_{II}(t). \tag{35}$$

Combining eqs. (29), (34) and (35) we obtain

$$\begin{aligned} n_{II}(t) &= [1 - (1 - n_0)e^{rt}]e^{-\alpha t}, & 0 \leq t \leq T_{II} \equiv -r^{-1} \ln(1 - n_0), \\ &= 0, & T_{II} \leq t. \end{aligned} \tag{36}$$

The path of warrant exercise is illustrated in fig. 2. The total time of warrant exercise, T_{II} , equals the total time of warrant exercise under policy I. At every

point in time $0 < t < T_{II}$, however, there are fewer warrants outstanding under policy II than under policy I. The warrants are exercised at an increasing rate, i.e. $n(t)$ is concave in t , only when the dividend yield is sufficiently small, $\alpha^2/(r - \alpha)^2 < 1 - n(t)$, as the reader may verify.

6.7. *Path of warrant exercise with the proceeds used to pay an extraordinary dividend to stockholders on record after the exercise (policy III)*

Suppose that $-\Delta n(t)$ warrants are exercised over time $[t, t + dt]$. We write $-\Delta n(t)$ and not $-\dot{n}(t)dt$ to allow for the possibility that a finite number of warrants are exercised over an infinitesimal time interval. The proceeds of warrant exercise, $-\Delta n(t) \cdot \beta$, are distributed as an extraordinary dividend to the $1 - n(t) - \Delta n(t)$ stockholders on record at time $t + dt$.

We show that it is a competitive equilibrium to have all the warrants exercised at time zero. It suffices to show that an (infinitesimal) warrant holder optimally exercises his warrant, if he expects that all other warrant holders exercise their warrants. Then the equilibrium is rational. If all the warrant holders exercise their warrants at time zero, the proceeds of warrant exercise is

$$n_0 + n_0\beta - \beta = \beta(r/\alpha - 1)(1 - n_0) \tag{37}$$

per warrant. If one (infinitesimal) warrant holder does not exercise his warrant, the price of the live warrant, after the extraordinary dividend $n_0\beta$ is paid, is given by eq. (20),

$$\begin{aligned} \hat{W}((r\beta/\alpha)(1 - n_0), 0) &= (r\beta/\alpha - \beta)(1 - n_0)^v \\ &< \beta(r/\alpha - 1)(1 - n_0). \end{aligned} \tag{38}$$

Therefore the price of the live warrant is less than the proceeds of warrant exercise and it is a competitive equilibrium to have all warrants exercised at time zero.

The path of warrant exercise is discontinuous and is illustrated in fig. 2: all the warrants are exercised at time zero. This path is sharply different from the paths of warrant exercise under policies I and II.

Under policy III there exist several competitive equilibria. If for some $t < 0$, $x(t)$ satisfies the inequality,

$$\begin{aligned} \hat{W}(x(t), 0) &= \left(\frac{r\beta}{\alpha} - \beta\right) \left(\frac{x(t)}{r\beta/\alpha}\right)^v \\ &< x(t) - (1 - n_0)\beta, \end{aligned} \tag{39}$$

then it is a competitive, rational equilibrium to have all the warrants exercised at time $t, t < 0$. The proceeds of warrant exercise are less than in the equilibrium where warrant holders wait until time zero to exercise their warrants. If, however, warrant holders believe that all the warrants will be exercised at time $t < 0$ where inequality (39) is satisfied, the warrant holders exercise their warrants in order to receive the extraordinary dividend as stockholders, and the expectation is fulfilled. The path of warrant exercise is illustrated in fig. 2.

6.8. *Path of warrant exercise with reinvestment of warrant exercise proceeds and stochastic firm growth*

We set the coupon rate equal to zero and generalize the firm's rate of return to be stochastic, i.e., net of dividend payouts, but gross of warrant exercise proceeds, the firm value grows as

$$dx(t) = (\mu - \alpha)x(t) dt + \sigma x(t) dz(t), \tag{40}$$

where μ, σ are constants and $dz(t)$ is the increment of a Wiener process. We retain all the other assumptions of section 6.1 and show that the expected time of warrant exercise under policy I increases and tends to infinity as the variance σ^2 increases and tends to $\mu - \alpha$.

Eq. (16) is now derived by the Black-Scholes (1973) no-arbitrage argument and has one more term on the left-hand side, the Ito term $(\sigma^2/2)x^2\dot{W}_{xx}$. The solution is given by eq. (20) with $c = 0$ and ν defined not by eq. (21) but by the following:

$$\nu \equiv - (r - \alpha - \sigma^2/2)/\sigma^2 + \left[(r - \alpha - \sigma^2/2)^2 + 2r\sigma^2 \right]^{1/2} / \sigma^2 > 1. \tag{41}$$

The critical value of the firm which triggers warrant exercise is $(1 - n_0)\beta\nu / (\nu - 1)$. It is increasing in the exercise price β , and is decreasing in the number of warrants, n_0 . From eq. (41) we obtain $\partial\nu/\partial\alpha > 0$, $\partial\nu/\partial r < 0$ and $\partial\nu/\partial(\sigma^2/2) < 0$. Since $\nu > 1$, $\nu/(\nu - 1)$ is decreasing in ν . Therefore the critical value of the firm, \bar{x} , is decreasing in the dividend rate α , and increasing in the interest rate r and in the variance of the firm's rate of return σ^2 . The higher the variance, the higher the value of the live warrants and the higher the firm value required to trigger warrant exercise.

We now describe the path of warrant exercise in the competitive warrant holders' equilibrium. As long as $x(t) < (1 - n_0)\beta\nu / (1 - \nu)$, none of the warrants are exercised. At $x(t) = (1 - n_0)\beta\nu / (1 - \nu)$ the warrants begin to be exercised, but the path of warrant exercise is discontinuous and depends on the stochastic changes of the firm's value. By some future time t , with $n(t)$ outstanding warrants, the firm's value may fall below the critical value, i.e.,

$$x(t) < (1 - n(t))\beta\nu / (1 - \nu), \tag{42}$$

whereupon the warrant exercise stops. The warrant exercise starts again when the firm value grows sufficiently to violate inequality (42).

The length of time from the exercise of the first warrant until the exercise of the last warrant is stochastic. When the firm follows policy I on the use of warrant exercise proceeds, the expected exercise time is given below:¹²

$$E[T_I] = - \frac{\ln(1 - n_0)}{(\mu - \alpha - \sigma^2/2)v}. \quad (43)$$

In the special case that the firm's value, gross of dividend payments, grows at the constant rate of interest, $\mu = r$, $\sigma^2 = 0$ and the exercise time is as given in eq. (33). In the stochastic growth case, the expected exercise time is increasing in the variance σ^2 , in the interest rate r , and in the number of warrants n_0 . It is decreasing in the expected growth rate μ . As $\mu - \alpha - \sigma^2/2$ tends to zero the expected exercise time tends to infinity. To understand the intuition behind this result, note from eq. (40) that $E[d \ln x(t)] = (\mu - \alpha - \sigma^2/2) dt$. At $\mu - \alpha - \sigma^2/2 = 0$ there is a finite probability that the $\ln x(t)$ never reaches or exceeds any given level $\ln x_1$, $x_1 > x(0)$; hence $E[T_I]$ is infinite.

6.9. Testable implications

The path of warrant exercise depends on the firm's policy on the use of warrant exercise proceeds, as illustrated above for policies I, II, and III. This prediction of the theory is not directly testable because neither covenants nor the firm's charter specify the firm's policy on the use of these proceeds. Also it is difficult, if not impossible, to isolate the firm's actual use of warrant exercise proceeds from the firm's other financing and investment decisions.

Under the reasonable maintained assumption that the firm reinvests in production the proceeds of warrant exercise (policy I) and that the firm's growth rate is stochastic, testable implications are derived in section 6.8 on the expected duration of warrant exercise: it is increasing in the variance of the growth rate, in the interest rate and in the number of warrants outstanding, and is decreasing in the expected growth rate.

Testable implications on the path of voluntary bond conversion are derived in section 6.4. However, bond conversion is oftentimes forced by the firm's calling the bond. In work in progress, Constantinides and Grundy (1984) analyze the observed pattern of bond conversion and calling as well as the bond prices relative to their call price and conversion value in order to better understand the objective of the firm and of the bondholders.

¹²The proof is available from the author upon request.

7. Extensions

A market with competitive warrant holders is the one extreme of a market setting. The other extreme is a market with a monopolist holding the entire issue of warrants and exercising them without the block constraint. This situation naturally arises in the private placement of a bond issue accompanied with a warrant issue. In the context of the example presented in section 4, fig. 1 illustrates the average warrant price for the monopolist as a function of the number of warrants exercised now. In maximizing the average warrant price, the monopolist exercises 364 warrants now and the remaining 136 warrants next period. The average warrant price is \$5.51 and exceeds the warrant price in the competitive or in the block warrant holder's equilibrium. The monopolist warrant holder uses the monopoly power by exercising 'too many' warrants (by the competitive warrant holders' standards), and thereby driving the warrant price of his 136 remaining live warrants to \$5.54. This path of warrant exercise cannot be sustained as a competitive equilibrium because each warrant holder prefers to exercise a warrant in the second than in the first period. The monopolist maximizes the difference between the aggregate share of dividends he gets by exercising now and the interest he must pay. The competitive warrant holders ignore the effect of their exercise on the dividend received by other warrant holders exercising their warrants. Other examples of warrant exercise by a monopolist or oligopolists are discussed in Emanuel (1983), Cox and Rubinstein (1984), and Constantinides and Rosenthal (1984).

Another interesting market setting consists of one monopolist holding a fraction of the outstanding warrants and a fringe of competitive investors holding the remaining fraction of outstanding warrants. In exercising some of the warrants early, the monopolist benefits by increasing the price of his remaining warrants. The competitive fringe behave as free riders, taking full advantage of the price increase due to the monopolist's sacrifice. If the monopolist holds a sufficiently small fraction of the warrants, the free rider problem may become sufficiently severe and it may be optimal for the monopolist to act as a competitive warrant holder. It is interesting to investigate the critical fraction of warrant ownership, below which the monopolist acts as a competitive warrant holder.

Finally consider a market with competitive warrant holders and a potential monopolist who holds zero warrants and offers to buy a certain fraction of the outstanding warrants at the uniform price p . Obviously, the potential monopolist will not set p higher than the average price of the warrants to the monopolist. Also none of the competitive warrant holders will tender a warrant if the price p is less than the price of warrants held by the competitive fringe. One may prove quite generally that the average warrant price to the monopolist is less than the price of a warrant held by the competitive fringe, because of the free rider problem. Therefore the potential monopolist is never successful

in acquiring monopoly power. Suppose, however, that the potential monopolist secretly acquires a certain fraction of the warrants at the competitive price before announcing his intentions. It is interesting to determine the critical fraction of warrants which the potential monopolist needs to acquire at the competitive price before announcing his or her intentions, if the bid is to be successful.

Appendix: Proof of Proposition 4

(a) Eq. (13) is satisfied at $t = T + 1$ because

$$W_{T+1}(x_{T+1}, n_{T+1}, N_{T+1}, s_{T+1}) = \hat{W}_{T+1}(x_{T+1}, n_{T+1}, N_{T+1}, s_{T+1}) = 0.$$

The proof proceeds by induction: We assume

$$W_{t+1}(x_{t+1}, n_{t+1}, N_{t+1}, s_{t+1}) = \hat{W}_{t+1}(x_{t+1}, n_{t+1}, N_{t+1}, s_{t+1}), \quad (\text{A.1})$$

and prove that eq. (13) holds. As an intermediate step, we prove the following:

$$\begin{aligned} & W_t(x_t, n_t, N_t, s_t) \\ &= \max \left[\{x_t - (N_t - n_t)\beta_t\} / N_t, \right. \\ & \quad \left. \min \left\{ \sum_{s_{t+1}} \pi(s_{t+1}|s_t) (W_{t+1}(x_{t+1}, n_{t+1}, N_{t+1}, s_{t+1}) + c_{t+1}), K_t \right\} \right]. \end{aligned} \quad (\text{A.2})$$

But eq. (A.2) also holds for the block warrant holder, where W_t and W_{t+1} are replaced by \hat{W}_t and \hat{W}_{t+1} , respectively. Then combining eqs. (A.1) and (A.2) we obtain eq. (13). In proving eq. (A.2) we consider separately two cases:

First case: Assume the following:

$$\begin{aligned} & \{x_t - (N_t - n_t)\beta_t\} / N_t \\ & > \min \left[\sum_{s_{t+1}} \pi(s_{t+1}|s_t) \{W_{t+1}(x_{t+1}, n_t, N_{t+1}, s_{t+1}) + c_{t+1}\}, K_t \right]. \end{aligned} \quad (\text{A.3})$$

Since

$$\begin{aligned}
 &W_t(x_t, n_t, N_t, s_t) \\
 &\geq S_t(x_t, n_t, N_t, s_t) - \beta_t \\
 &\geq \{x_t - (N_t - n_t)\beta_t\} / N_t \quad [\text{by Lemma 1}] \\
 &> \min_{s_{t+1}} \left[\sum \pi(s_{t+1}|s_t) \{W_{t+1}(x_{t+1}, n_t, N_{t+1}, s_{t+1}) + c_{t+1}\}, K_t \right] \\
 &\hspace{15em} [\text{by inequality (A.3)}], \quad (\text{A.4})
 \end{aligned}$$

at least some warrants are exercised at time t . Then

$$\begin{aligned}
 W_t(x_t, n_t, N_t, s_t) &= S_t(x_t, n_t, N_t, s_t) - \beta_t \\
 &= \{x_t - (N_t - n_t)\beta_t\} / N_t \quad [\text{by Lemma 1}]. \quad (\text{A.5})
 \end{aligned}$$

Combining eq. (A.5) and inequality (A.3) we obtain eq. (A.2).

Second case: Assume the following:

$$\begin{aligned}
 &\{x_t - (N_t - n_t)\beta_t\} / N_t \\
 &\leq \min_{s_{t+1}} \left[\sum \pi(s_{t+1}|s_t) \{W_{t+1}(x_{t+1}, n_t, N_{t+1}, s_{t+1}) + c_{t+1}\}, K_t \right]. \quad (\text{A.6})
 \end{aligned}$$

Whereas there may exist more than one competitive equilibrium, we prove that it is a competitive equilibrium to have none of the warrants exercised at time t , i.e., $n_{t+1} = n_t$. If the investors expect $n_{t+1} = n_t$, then the warrants are priced accordingly, and

$$\begin{aligned}
 &W_t(x_t, n_t, N_t, s_t) \\
 &= \min_{s_{t+1}} \left[\sum \pi(s_{t+1}|s_t) \{W_{t+1}(x_{t+1}, n_t, N_{t+1}, s_{t+1}) + c_{t+1}\}, K_t \right] \\
 &\geq \{x_t - (N_t - n_t)\beta_t\} / N_t \quad [\text{by inequality (A.6)}] \\
 &\geq S_t(x_t, n_t, N_t, s_t) - \beta_t \quad [\text{by Lemma 1}]. \quad (\text{A.7})
 \end{aligned}$$

Thus a warrant is worth more live than exercised and the investors' expectation that $n_{t+1} = n_t$ is rational. Eq. (A.7) implies eq. (A.2). This proves (a).

(b) Let τ be the first time at which the competitive warrant holders exercise some warrants. Then

$$W_\tau(x_\tau, n_\tau, N_\tau, s_\tau) = \{x_\tau - (N_\tau - n_\tau)\beta_\tau\}/N_\tau. \quad (\text{A.8})$$

Eqs. (13) and (A.8) imply

$$\hat{W}_\tau(x_\tau, n_\tau, N_\tau, s_\tau) = \{x_\tau - (N_\tau - n_\tau)\beta_\tau\}/N_\tau. \quad (\text{A.9})$$

Therefore τ is also a stopping time at which the block warrant holder exercises the warrant block. Defining $\hat{\tau}$ to be the first stopping time at which the block warrant holder exercises the warrant block, we obtain

$$\hat{\tau} \leq \tau. \quad (\text{A.10})$$

But we proved in Proposition 3 that

$$\tau \leq \hat{\tau}. \quad (\text{A.11})$$

Combining eqs. (A.10) and (A.11) we obtain

$$\tau = \hat{\tau}. \quad (\text{A.12})$$

This proves (b).

(c) Suppose that the block warrant is called at time τ . This implies that the block warrant has not been exercised prior to time τ , i.e., $n_\tau = n_0$ for the block warrant. It also implies (see Proposition 3) that none of the competitive warrants have been exercised prior to time τ , i.e., $n_\tau = n_0$ for the competitive warrants also.

Since the block warrant is called at time τ , we obtain

$$\sum_{s_{\tau+1}} \pi(s_{\tau+1}|s_\tau)(\hat{W}_{\tau+1}(x_{\tau+1}, n_0, N_{\tau+1}, s_{\tau+1}) + c_{\tau+1}) \geq K_\tau. \quad (\text{A.13})$$

Combining eqs. (13) and (A.13) we obtain

$$\sum_{s_{\tau+1}} \pi(s_{\tau+1}|s_\tau)(W_{\tau+1}(x_{\tau+1}, n_0, N_{\tau+1}, s_{\tau+1}) + c_{\tau+1}) \geq K_\tau. \quad (\text{A.14})$$

Thus the competitive warrants are called at time τ . This proves (c).

(d) The price of the block warrant is $W_{\bar{t}} = \{x_{\bar{t}} - (N_{\bar{t}} - n_{\bar{t}})\beta_{\bar{t}}\} / N_{\bar{t}}$ at the stopping time at which it is exercised. Therefore the price of the block warrant is independent of the firm's policy on the use of warrant exercise proceeds, at all times. From eq. (A.4) we conclude that the price of competitive warrants also is independent of the firm's policy on the use of warrant exercise proceeds. This proves (d), and completes the proof.

References

- American Bar Foundation, 1971, Commentaries (Chicago, IL).
- Black, F. and M.S. Scholes, 1973, The pricing of options and corporate liabilities, *Journal of Political Economy* 81, 637–654.
- Brennan, M.J. and E.S. Schwartz, 1977, Convertible bonds: Valuation and optimal strategies for call and conversion, *Journal of Finance* 32, 1699–1715.
- Constantinides, G.M. and B.D. Grundy, 1984, Are convertibles called late?, Mimeo. (Graduate School of Business, University of Chicago, Chicago, IL).
- Constantinides, G.M. and R.W. Rosenthal, 1984, Strategic analysis of the competitive exercise of certain financial options, *Journal of Economic Theory* 32, 128–138.
- Cox, J.C. and M. Rubinstein, 1984, *Options markets* (Prentice-Hall, Englewood Cliffs, NJ) forthcoming.
- Emanuel, D.C., 1980, Extensions and developments of option pricing, Unpublished doctoral dissertation (Stanford University, Stanford, CA).
- Emanuel, D.C., 1983, Warrant valuation and exercise strategy, *Journal of Financial Economics* 12, 211–235.
- Galai, D. and M.I. Schneller, 1978, Pricing of warrants and the value of the firm, *Journal of Finance* 33, 1333–1342.
- Ingersoll, J.E., Jr., 1977, A contingent-claims valuation of convertible securities, *Journal of Financial Economics* 4, 289–322.
- Kalay, A., 1982, Stockholder–bondholder conflict and dividend constraints, *Journal of Financial Economics* 10, 211–233.
- Merton, R.C., 1973, Theory of rational option pricing, *Bell Journal of Economics and Management Science* 4, 141–183.
- Modigliani, F. and M.H. Miller, 1958, The cost of capital, corporation finance, and the theory of investment, *American Economic Review* 48, 261–297.
- Schmeidler, D., 1973, Equilibrium points of nonatomic games, *Journal of Statistical Physics* 7, 295–300.
- Smith, C.W., Jr., 1977, Alternative methods for raising capital: Rights versus underwritten offerings, *Journal of Financial Economics* 5, 273–307.
- Smith, C.W., Jr. and J.B. Warner, 1979, On financial contracting: An analysis of bond covenants, *Journal of Financial Economics* 7, 117–161.