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# Notes\*

## VI

### OPTIMAL PORTFOLIO REVISION WITH PROPORTIONAL TRANSACTION COSTS: EXTENSION TO HARA UTILITY FUNCTIONS AND EXOGENOUS DETERMINISTIC INCOME†

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Kamin's theorem [1] on portfolio revision with proportional transaction costs is extended to allow for hyperbolic absolute risk averse investors and for the possibility of exogenous deterministic income, assuming that one of the two investment opportunities is riskless.

Kamin [1] considered the problem of portfolio revision with proportional transaction costs. The investor manages a two-asset portfolio and revises his portfolio in every time period with the objective of maximizing the expected utility of his end-of-the-horizon wealth.<sup>1</sup> Kamin assumed that (a) the utility function exhibits constant relative risk aversion, that is to say, the utility of wealth  $W$  is  $U(W) = (1 - \gamma)W^\gamma/\gamma$  or  $U(W) = \log W$  or a linear transformation thereof, and (b) there is no exogenous income. The purpose of this note is to show that, provided one of the two assets in the portfolio is riskless, we may extend Kamin's theorem to the case of (a)' hyperbolic absolute risk aversion utility (the HARA family) or a linear transformation thereof, and (b)' exogenous deterministic income. Unless otherwise stated we follow Kamin's assumptions.

Specifically we assume (a)' The investor's utility of wealth  $W$  at the end of the horizon is:  $V(W) = (1 - \gamma)(W - \hat{W})^\gamma/\gamma$  or  $= \log(W - \hat{W})$  where  $\gamma < 1$ ,  $\gamma \neq 0$ ,  $W > \hat{W}$  or  $\gamma = 2$ ,  $W < \hat{W}$ ;  $\gamma$  and  $\hat{W}$  are known constants.<sup>2</sup>

(b)' The investor receives exogenous, deterministic income  $I(s)$  at the end of stage  $s$ . This income increases the wealth in asset 1 by  $I(s)$ .

(c) Asset 1 is riskless; that is to say, the one period wealth relative  $t_a(s)$  for asset 1

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<sup>1</sup> If  $A_0$  and  $B_0$  represent the end-of-the-horizon values of the two assets then Kamin defines the end-of-the-horizon wealth as  $W = A_0 + B_0$ . If we further assume that  $A_0$  is the liquid asset which may be consumed without incurring additional transaction costs, while  $B_0$  has to be converted into the liquid asset before consumption, then it is more appropriate to maximize the utility of the liquidated wealth  $A_0 + B_0(1 - C)/(1 + C)$ , where the transaction cost rate  $C$  is defined in Kamin. If we define  $W = A_0 + B_0(1 - C)/(1 + C)$  rather than Kamin's  $W = A_0 + B_0$ , Kamin's theorem still holds true: it will suffice to note that Kamin's corollary to Lemma 4 holds true under the alternative definition of  $W$ .

<sup>2</sup> The constant  $\hat{W}$  may be considered as the minimum required wealth at the end of the horizon.

Note that this utility function implies hyperbolic absolute risk aversion (the HARA property):  $-U''(W)/U'(W) = (1 - \gamma)/(W - \hat{W}) > 0$ . By the proper choice of the parameters  $\hat{W}$ ,  $\gamma$  we obtain some interesting special cases: (i) power utility function for  $\gamma < 1$ ,  $\gamma \neq 0$ ,  $\hat{W} = 0$ ,  $W > 0$ ; (ii) quadratic utility function for  $\gamma = 2$ ,  $W < \hat{W}$ ; (iii) unit relative risk aversion for  $\hat{W} = 0$ ,  $W > 0$  and  $\gamma \rightarrow 0$ , which is a property of logarithmic utility; and (iv) constant absolute risk aversion for  $W$  bounded,  $\hat{W} \rightarrow -\infty$  and  $(\gamma/\hat{W}) \rightarrow h$ , a positive constant; then  $-U''(W)/U'(W) = (1/\hat{W} - (\gamma/\hat{W}))/((W/\hat{W}) - 1) \rightarrow h$ , that is the absolute risk aversion tends to a positive constant, which is a property of exponential utility.

from stage  $s$  to  $s - 1$  is deterministic. The one period wealth relative  $t_b(s)$  for asset 2 from stage  $s$  to  $s - 1$  is stochastic.

Our approach consists of formulating Kamin's problem under assumptions (a), (b) and (c) (Problem 1), then formulating a more general problem under assumptions (a)', (b)' and (c) (Problem 2) and through a transformation of variables showing that Problem 2 may be reduced to Problem 1.

Denote the wealth invested in asset 1 (the riskless asset) at the beginning of stage  $s$  and before any transactions by the symbol  $X_s$ , and the wealth invested in asset 2 (the risky asset) at the beginning of stage  $s$  and before any transactions by the symbol  $Y_s$ . Transaction costs are defined as follows: one unit of wealth in asset 2 is convertible into  $1 - k_2$  units of asset 1; and  $1 + k_1$  units of wealth in asset 1 are convertible into one unit of asset 2.<sup>3</sup> Define  $k_u$  by

$$\begin{aligned} k_u &= k_1 & u &\geq 0 \\ &= -k_2 & u &< 0 \end{aligned}$$

where  $k_1, k_2 \geq 0$ . Then Kamin's problem under assumptions (a), (b) and (c) may be formulated as<sup>4</sup>

Problem 1: Given  $X_n, Y_n$

$$\text{maximize}_{u_1, u_2, \dots, u_n} E_{t_b(1), t_b(2), \dots, t_b(n)} U_0(X_0, Y_0) \quad \text{subject to} \quad (1.1)$$

$$X_{s-1} = (X_s - u_s - k_u u_s) t_a(s) \quad s = 1, 2, \dots, n \quad (1.2)$$

$$Y_{s-1} = (Y_s + u_s) t_b(s) \quad s = 1, 2, \dots, n \quad \text{and} \quad (1.3)$$

$$X_s, Y_s \geq 0 \quad s = 1, 2, \dots, n \quad \text{where} \quad (1.4)$$

$$U_0(X_0, Y_0) = (1 - \gamma)(X_0 + Y_0)^\gamma / \gamma \quad \text{or} = \log(X_0 + Y_0). \quad (1.5)$$

Under the more general assumptions (a)', (b)' and (c) the problem is stated as Problem 2: Given  $X_n, Y_n$

$$\text{maximize}_{u_1, u_2, \dots, u_n} E_{t_b(1), t_b(2), \dots, t_b(n)} V_0(X_0, Y_0) \quad \text{subject to} \quad (2.1)$$

$$X_{s-1} = (X_s - u_s - k_u u_s) t_a(s) + I(s) \quad s = 1, 2, \dots, n \quad (2.2)$$

$$Y_{s-1} = (Y_s + u_s) t_b(s) \quad s = 1, 2, \dots, n \quad \text{and} \quad (2.3)$$

$$X'_s, Y_s \geq 0^5 \quad s = 1, 2, \dots, n \quad \text{where} \quad (2.4)$$

$$V_0(X_0, Y_0) = (1 - \gamma)(X_0 + Y_0 - \hat{W})^\gamma / \gamma \quad \text{or} = \log(X_0 + Y_0 - \hat{W}). \quad (2.5)$$

Define a new variable  $X'_s$  as  $X'_0 = X_0 - \hat{W}$ ,  $X'_s = X_s + \sum_{i=1}^s R_s(i)I(i) - R_s(1)\hat{W}$ ,  $s = 1, \dots, n$ , where  $R_s(i) = 1/\prod_{j=i}^s t_a(j)$ .

<sup>3</sup> Kamin defined transaction costs in terms of a transaction rate  $C$ , pre-adjustment wealth in assets 1 and 2,  $A$  and  $B$  respectively and post-adjustment wealth in assets 1 and 2,  $X$  and  $Y$  respectively by  $X + Y = A + B - C|A - X + Y - B|$ . In terms of our notation, Kamin essentially assumed  $k_1 = 2C/(1 - C)$  and  $k_2 = 2C/(1 + C)$  which implies  $-k_1^{-1} + k_2^{-1} = 1$ . The reader may retrace Kamin's proof and verify that his proof holds even when the transaction rate  $C$  for transferring wealth from asset 1 to asset 2 is not the same as the rate of transferring wealth from asset 2 to asset 1. Then  $k_1$  and  $k_2$  will be taken to be independent of each other.

<sup>4</sup> The reader may easily verify that a linear transformation of the utility of the end-of-the-horizon wealth leaves the problem formulations 1 and 2 unchanged.

<sup>5</sup> Note that we impose  $X'_s \geq 0$  and not  $X_s \geq 0$ . The meaning of this constraint is discussed in the following paragraphs.

We call  $X'_s$  the *effective* riskless asset at stage  $s$ . It is the actual riskless asset at stage  $s$ , plus the capitalized future income, less the capitalized minimum required wealth at the end of the horizon, both capitalized at the riskless rate. Then condition (2.4) states that the effective riskless asset and not the actual riskless asset is constrained to nonnegative values.

Problem 2 is defined in terms of the state variables  $(X_s, Y_s, s)$ ,  $s = 1, 2, \dots, n$ . Rewrite Problem 2 in terms of the state variables  $(X'_s, Y_s, s)$ ,  $s = 1, 2, \dots, n$ . Then Problem 2 is equivalent to Problem 2': Given  $X'_n, Y_n$

$$\underset{u_1, u_2, \dots, u_n}{\text{maximize}} E_{t_b(1), t_b(2), \dots, t_b(n)} U_0(X'_0, Y_0) \quad \text{subject to} \quad (2.1)$$

$$X'_{s-1} = (X'_s - u_s - k_u u_s) t_a(s) \quad s = 1, 2, \dots, n \quad (2.2)$$

$$Y_{s-1} = (Y_s + u_s) t_b(s) \quad s = 1, 2, \dots, n \quad \text{and} \quad (2.3)$$

$$X'_s, Y_s \geq 0 \quad s = 1, 2, \dots, n \quad \text{where} \quad (2.4)$$

$$U_0(X'_0, Y_0) = (1 - \gamma)(X'_0 + Y_0)^\gamma / \gamma \quad \text{or} = \log(X'_0 + Y_0). \quad (2.5)$$

Problem 2' is identical to Problem 1 with  $X'_s$  replacing  $X_s$ . Thus any statement which is true for the actual riskless asset and risky asset of Problem 1 is also true for the effective riskless asset and risky asset of Problem 2. Then Kamin's theorem which refers to the actual riskless and risky assets of Problem 1 is also true for the effective riskless asset and the risky asset of Problem 2. This result is summarized below:

**THEOREM.** *Under assumptions (a)', (b)' and (c) in every time period there exist two limits defining a range of no transactions. The investor refrains from transacting if the ratio of his risky asset to his effective riskless asset lies in the range of no transactions; he transacts to the closer boundary of the no transactions range if the ratio of his risky asset to his effective riskless asset lies outside this range.*

The theorem is easily translated to a transactions policy regarding the actual riskless asset and the risky asset. In particular, if the range of no transactions at stage  $s$  for Problem 1 is determined by parameters  $\lambda_1$  and  $\lambda_2$  such that at stage  $s$ ,  $s > 0$ :  $\lambda_1 \leq X_s / Y_s \leq \lambda_2$ , then the range of no transactions at stage  $s$  for Problem 2 in terms of the actual riskless asset  $X_s$  and the risky asset  $Y_s$  is such that  $\lambda_1 \leq \{X_s + \sum_{i=1}^s R_s(i) I(i) - R_s(1) \hat{W}\} / Y_s \leq \lambda_2$ . In a  $(X_s, Y_s)$  plane, the area that defines the range of no transactions is a cone with vertex which does not, in general, coincide with the origin.

**References**

1. KAMIN, JULES H., "Optimal Portfolio Revision with a Proportional Transaction Cost," *Management Science*, Vol. 21, No. 11 (July 1975).