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George M. Constantinides

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MULTIPERIOD CONSUMPTION AND INVESTMENT BEHAVIOR WITH CONVEX TRANSACTIONS COSTS*

GEORGE M. CONSTANTINIDES†

The effect of convex transactions costs on consumers' derived utility functions and on optimal consumption and investment decisions is examined in a general multiperiod framework. The extent to which multiperiod consumption-investment behavior and capital market equilibrium may be studied in a single period framework is discussed. Optimal investment policy, in terms of a region of no transactions, is shown to be of a particularly simple form.

(FINANCE–INVESTMENT CRITERIA; DECISION ANALYSIS–SEQUENTIAL; UTILITY/PREFERENCE–THEORY)

1. Introduction

The paper examines the effect of convex transactions costs on consumers' derived utility functions and on optimal consumption-investment decisions. In the absence of transactions costs and under otherwise weak assumptions, Fama [7], [8] proved that the observable behavior in any period of a consumer who has a multiperiod horizon is indistinguishable from the behavior of some other consumer who has a single period horizon and maximizes his expectation of a concave, state dependent (derived) utility function. Furthermore, under plausible assumptions the derived utility function is state independent. These observations by Fama make possible the study of consumption-investment behavior and capital market equilibrium in a single period framework rather than in a cumbersome intertemporal framework.

The first issue addressed in this paper is the extent to which multiperiod consumption-investment behavior and capital market equilibrium may be studied in a single period framework in the presence of convex transactions costs. We prove that derived utility is monotone increasing and concave in each of the assets (Proposition 1). Derived utility is concave in total wealth also, provided that asset proportions remain unchanged (Proposition 2). These properties suggest that consumption-investment behavior may be studied in a single period framework, provided that assets are treated as distinct goods (which may be transformed to one another at some transactions cost). Under plausible assumptions the derived utility function is also shown to be state independent.

The second issue discussed in this paper is the properties of optimal investment decisions. Locally optimal investment decisions are shown to be globally optimal (Proposition 3). With proportional transactions costs optimal investment policy is conveniently described in terms of a region of no transactions (Proposition 4). Under some homogeneity assumptions, the region of no transactions is a cone and optimal investment is homogeneous of degree one in the asset holdings (Proposition 5). If there exist only two assets (or mutual funds) optimal investment policy is simply described in terms of two parameters, $\underline{a} \leqslant \bar{a}$: The consumer refrains from transacting so long as portfolio proportions lie in the interval $[\underline{a}, \bar{a}]$; and transacts to the closer boundary $\underline{a}$ or $\bar{a}$ whenever the portfolio proportions lie outside this interval (Proposition 7). These properties not only increase our understanding of optimal investment behavior but also simplify numerical estimation of these decisions.

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† University of Chicago.

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The third issue concerns the properties of optimal consumption behavior. Locally optimal consumption decisions are shown to be globally optimal (Proposition 3). Under some homogeneity assumptions, optimal consumption is homogeneous of degree one in the asset holdings (Proposition 5), and marginal propensity to consume lies between zero and one (Proposition 6).

2. The Model

A consumer makes sequential consumption and investment decisions at dates $t = 0, 1, 2, \ldots, T$. There are $n + 1$ goods, or "assets" at each date. At date t the consumer owns x_t^i units of account (e.g., dollars) of the ith asset, before he makes investment decisions and before he consumes at date t. We define vectors $x_t \equiv [x_t^1, x_t^2, \ldots, x_t^n]$ and $X_t \equiv [x_t^0, x_t]$. The zeroth asset plays a special role, which will be shortly explained. The consumer makes investment decisions at date t denoted by $u_t \equiv [u_t^1, u_t^2, \ldots, u_t^n]$. u_t^i denotes the number of units of account by which the consumer increases (decreases) his holding of the ith asset; after the transaction the consumer's holdings of the n assets $i = 1, 2, \ldots, n$ are $x_t + u_t$. The total amount transferred to the n assets is $u_t I$ where I is a unit column vector. This amount is subtracted from the zeroth asset. The consumer incurs transactions costs, $T(u_t)$, which are charged to the zeroth asset. After the transaction the consumer's holding of the zeroth asset is $x_t^0 - u_t I - T(u_t)$. We assume that $T(0) = 0$, $T(u)$ is convex in u, and $|T(u) - T(v)| < \|u - v\|$, $\forall u, v$, $u \neq v$.

After the transactions are completed, the consumer consumes $c_t^i \geqslant 0$ units of account of the ith asset, $i = 0, 1, \ldots, n$. We define vectors $c_t \equiv [c_t^1, c_t^2, \ldots, c_t^n]$ and $C_t \equiv [c_0^0, c_0, c_1^0, c_1, \ldots, c_t^0, c_t]$. Consumption depletes the $n + 1$ assets. After consumption the consumer holds $y_t^0 = x_t^0 - u_t I - T(u_t) - c_t^0$ units of account of the zeroth asset and $y_t = x_t + u_t - c_t$ units of account of the remaining n assets.

The consumer's holdings $[y_t^0, y_t]$ are transformed at date $t + 1$ to $X_{t+1} = f_t(y_t^0, y_t, \phi_{t+1})$. f_t is a monotone increasing and concave function of y_t^0, y_t and $f_t(0, 0, \phi_{t+1}) \geqslant 0$. ϕ_{t+1} is a state variable which is realized at date $t + 1$. This state variable will be further discussed shortly.

The simplest kind of transformation occurs when (y_t^0, y_t) represent investments in the stock or bond market. Let r_{t+1}^i be the capital gain rate and d_{t+1}^i be the dividend rate for the ith asset. The transformation on the ith asset is $x_{t+1}^i = (1 + r_{t+1}^i)y_t^i$; also, the cash asset grows by $d_{t+1}^i y_t^i$. There is no need, however, to limit the interpretation of the transformation function to simple investments. Consider, for example, investment in a European call option which matures at date $t + 1$; if the option is exercised, the option is transformed to the underlying stock and the cash asset is decreased by the exercise price of the option plus commission fee. The transformation function also allows for the possibility that the consumer receives stochastic endowments of the $n + 1$ assets at date $t + 1$, i.e., receives exogenous income. More generally, the transformation may be interpreted as a stochastic production function where y_t^0, y_t are the inputs and X_{t+1} is the vector of outputs. For the general results discussed in this and the next section it is unnecessary to specify the function f_t in further detail.

At date t the state of the consumer is summarized by his asset holdings X_t, by his stream of consumption C_{t-1} at all dates prior to t, and by a vector ϕ_t. The vector ϕ_t is of minimal dimension and is observable by the consumer. ϕ_t summarizes the consumer's beliefs at date t regarding the state variable ϕ_{t+1}, i.e., $F_t(\phi_{t+1} \mid \phi_t, X_t, C_{t-1}) = F_t(\phi_{t+1} \mid \phi_t)$; F_t is a probability distribution function at date t. Since the transformation f_t of asset holdings over $(t, t + 1)$ is a function of ϕ_{t+1}, the state variable ϕ_t summarizes the consumer's beliefs regarding the transformation f_t. For example, ϕ_t summarizes the consumer's probability distribution of the rates of return on assets over $(t, t+1)$. As we shall shortly explain, ϕ_t also summarizes the consumer's tastes at date t.

The consumer's utility of lifetime consumption is $U(C_T, \phi_T)$. U is monotone increasing and concave in C_T. Note that the consumer's tastes are state dependent. The consumer makes sequential consumption-investment decisions with the objective to maximize the expectation of his utility of lifetime consumption.

We define the function $V_t(C_{t-1}, X_t, \phi_t)$ as the derived utility of consumption C_{t-1} and assets X_t at time t and state ϕ_t, before consumption-investment decisions have been made at date t and assuming that optimal consumption-investment policies are followed from date t to T. The dynamic program is stated as:

$$\begin{aligned} &V_t(C_{t-1}, X_t, \phi_t) \\ &\quad \equiv \max_{\{c_t^0, c_t, u_t\} \in \Omega_t} \int_{\phi_{t+1}} V_{t+1}\big(C_t, f_t(x_t^0 - u_t I - T(u_t) - c_t^0, x_t + u_t - c_t, \\ &\qquad\qquad\qquad \phi_{t+1}), \phi_{t+1}\big)\, dF_t(\phi_{t+1} \mid \phi_t), \qquad (1) \\ &\qquad\qquad\qquad\qquad t = 0, 1, \dots, T, \end{aligned}$$

with boundary condition

$$V_{T+1}(C_T, X_{T+1}, \phi_{T+1}) = U(C_T, \phi_T). \qquad (2)$$

At date t the feasible set Ω_t of consumption-investment decisions c_t^0, c_t, u_t is a function of the state. The set Ω_t ensures that consumption at date t is nonnegative and that there exist feasible future consumption paths. We may define Ω_t as:

$$\begin{aligned} &c_t^0 \geqslant 0, \quad c_t \geqslant 0 && \text{(nonnegative consumption)}, \\ &x_t^0 - u_t I - T(u_t) - c_t^0 \geqslant 0, && \\ &x_t + u_t - c_t \geqslant 0 && \text{(nonnegative investment)}. \end{aligned} \qquad (3)$$

If the transformation function f_t is such that nonnegative inputs at date t ensure nonnegative outputs at $t+1$ (limited liability), then the set Ω_t defined by (3) ensures feasible consumption paths.

A more general definition of Ω_t which allows negative investment, i.e., borrowing and selling assets short, is given below:

$$\begin{aligned} &c_t^0 \geqslant 0, \quad c_t \geqslant 0 && \text{(nonnegative consumption)}, \\ &x_{t+1}^0 + x_{t+1} I - T(-x_{t+1}) \geqslant 0, \text{ for all } \phi_{t+1} && \text{(nonnegative net worth)}. \end{aligned} \qquad (3')$$

The latter definition ensures that the consumer is solvent at date $t+1$ onwards and that there exists at least one feasible consumption plan, namely $c_\tau^0 = c_\tau = 0$ for $\tau = t+1, t+2, \dots, T$.

The set Ω_t is convex whether it is defined by (3) or by (3′). This property is stated and proved as a lemma for future reference.

LEMMA 1 *Ω_t is a convex set.*

PROOF. Consider first the case where Ω_t is defined by (3). $x_t^0 - u_t I - T(u_t) - c_t^0$ is a concave function of u_t, c_t^0 and therefore $x_t^0 - u_t I - T(u_t) - c_t^0 \geqslant 0$ defines a convex set. Also $c_t^0 \geqslant 0$, $c_t \geqslant 0$ and $x_t + u_t - c_t \geqslant 0$ define convex sets. Ω_t is the intersection of these sets and is therefore convex.

Consider next the case where Ω_t is defined by (3′). Since $y_t^0 = x_t^0 - u_t I - T(u_t) - c_t^0$ and $y_t = x_t + u_t - c_t$ are concave functions of c_t^0, c_t, u_t, and since $f_t(y_t^0, y_t, \phi_{t+1})$ is a monotone increasing and concave function of y_t^0, y_t, it follows that $X_{t+1} = f_t$ is a concave function of c_t^0, c_t, u_t, for given ϕ_{t+1}. By assumption, $|T(u) - T(v)| < \|u - v\|$, $\forall u, v$, $u \neq v$. Therefore $x_{t+1}^0 + x_{t+1}I - T(-x_{t+1})$ is a monotone increasing and concave function of X_{t+1} and $x_{t+1}^0 + x_{t+1}I - T(-x_{t+1})$ is a concave function of c_t^0, c_t, u_t. For each $\phi_{t+1}, x_{t+1}^0 + x_{t+1} - T(-x_{t+1}) \geqslant 0$ defines a convex set. The intersection of these sets defined by all ϕ_{t+1}, and of the sets $c_t^0 \geqslant 0$ and $c_t \geqslant 0$ is the convex set Ω_t. The proof is complete.

The consumer's initial state X_0, ϕ_0, the set of feasible consumption-investment decisions Ω_t defined by (3) or (3′), the sequential optimization defined by (1) and the boundary condition (2) completely specify the consumer's decision problem. It is assumed that the initial state and the model specification are such that the set Ω_0 is nonempty. In the next section we discuss properties of the derived utility function and of the optimal consumption-investment decisions.

3. Properties of Derived Utility, Consumption and Investment

Proposition 1. *For all* ϕ_t *and* $t = 0, 1, \ldots, T$, *derived utility* $V_t(C_{t-1}, X_t, \phi_t)$ *is monotone increasing and concave in* C_{t-1}, X_t.

Proof. By the boundary condition (2), $V_{T+1}(C_T, X_{T+1}, \phi_{T+1}) = U(C_T, \phi_T)$. Therefore V_{T+1} is monotone increasing and concave in C_T, X_{T+1}. The proof proceeds by induction. We assume that $V_{t+1}(C_t, X_{t+1}, \phi_{t+1})$ is monotone increasing and concave in C_t, X_{t+1}.

We first prove monotonicity of $V_t(C_{t-1}, X_t, \phi_t)$ in C_{t-1}, X_t. Let $\bar{c}_t^0$, $\bar{c}_t$ and $\bar{u}_t$ be the optimal decisions corresponding to state C_{t-1}, X_t, ϕ_t. For any vectors $\delta C_{t-1} \geqslant 0, \delta X_t \geqslant 0$, the decisions $\bar{c}_t^0$, $\bar{c}_t$ and $\bar{u}_t$ are feasible given state $C_{t-1} + \delta C_{t-1}, X_t + \delta X_t, \phi_t$. Therefore:

$$\begin{aligned} & V_t(C_{t-1} + \delta C_{t-1}, X_t + \delta X_t, \phi_t) \\ & \quad \geqslant \int_{\phi_{t+1}} V_{t+1}\big(C_{t-1} + \delta C_{t-1}, \bar{c}_t^0, \bar{c}_t, f_t\big(x_t^0 - \bar{u}_t I - T(\bar{u}_t) \\ & \qquad - \bar{c}_t^0 + \delta x_t^0, x_t + \bar{u}_t - \bar{c}_t + \delta x_t, \phi_{t+1}\big), \phi_{t+1}\big)\, dF_t(\phi_{t+1} \mid \phi_t) \\ & \quad \geqslant \int_{\phi_{t+1}} V_{t+1}\big(C_{t-1}, \bar{c}_t^0, \bar{c}_t, f_t\big(x_t^0 - \bar{u}_t I - T(\bar{u}_t) \\ & \qquad - \bar{c}_t^0, x_t + \bar{u}_t - \bar{c}_t, \phi_{t+1}\big), \phi_{t+1}\big)\, dF_t(\phi_{t+1} \mid \phi_t) \\ & \quad \geqslant V_t(C_{t-1}, X_t, \phi_t). \end{aligned}$$

We next prove concavity of $V_t(C_{t-1}, X_t, \phi_t)$ in C_{t-1}, X_t. Consider the recursive equation (1). $y_t^0 = x_t^0 - u_t I - T(u_t) - c_t^0$ and $y_t = x_t + u_t - c_t$ are concave in x_t^0, x_t, c_t^0, c_t and u_t. Also $f_t(y_t^0, y_t, \phi_{t+1})$ is monotone increasing and concave in y_t^0, y_t. Therefore f_t is concave in x_t^0, x_t, c_t^0, c_t and u_t. $V_{t+1}(C_t, f_t, \phi_{t+1})$ is monotone increasing and concave in C_t, f_t, given ϕ_{t+1}. Therefore V_{t+1} is concave in x_t^0, x_t, C_{t-1}, c_t and u_t. The concavity is preserved under integration (addition) and therefore the integral is concave in x_t^0, x_t, C_{t-1}, c_t and u_t. The concavity is also preserved under the operation of maximization since Ω_t is a convex set, by Lemma 1. (See, for example, Rockafellar [16, Theorem 5.3].) Therefore $V_t(C_{t-1}, X_t, \phi_t)$ is concave in C_{t-1}, X_t. The proof is complete.

Some variations of Proposition 1 are stated without proof. If $U(C_T \mid \phi_T)$ is strictly monotone increasing in C_T then $V_t(C_{t-1}, X_t, \phi_t)$ is strictly monotone increasing in

C_{t-1}, X_t for all ϕ_t, t. Alternatively, if $U(C_T | \phi_T)$ is monotone increasing in C_T and strictly monotone increasing in $c_0^0, c_1^0, \ldots, c_T^0$ then $V_t(C_{t-1}, X_t, \phi_t)$ is strictly monotone increasing in $c_0^0, c_1^0, \ldots, c_{t-1}^0$ and X_t, for all ϕ_t, t.

In interpreting Proposition 1, we first assume that the $n+1$ assets are different consumption or production goods. Then the recursive equation (1) and Proposition 1 state that the consumer's intertemporal optimization problem reduces to a single period problem: At each date t, and given the state C_{t-1}, X_t, ϕ_t, the consumer maximizes his expectation of a monotone increasing and concave (derived) utility function of consumption c_t^0, c_t in the $n+1$ goods at date t and of wealth X_{t+1} in the $n+1$ goods at date $t+1$. Furthermore, if tastes and relative asset prices are state independent, i.e., $U(C_T, \phi_T) = U(C_T)$ for all ϕ_T, and if investment opportunities are state independent, i.e., $f_t(y_t^0, y_t, \phi_{t+1}) = f_t(y_t^0, y_t)$ for all ϕ_{t+1}, then the consumer's derived utility function is state independent, i.e., $V_{t+1}(C_t, X_{t+1}, \phi_{t+1}) = V_{t+1}(C_t, X_{t+1})$ for all ϕ_{t+1}. The state variable ϕ_{t+1} becomes superfluous in the sense that the consumer's optimal consumption-investment decisions are independent of this variable. This discussion essentially generalizes earlier results by Fama [7] which were derived under the assumption of zero transactions costs.

It is useful to express the consumer's holdings X_t in terms of total wealth, $W_t \equiv x_t^0 + x_t I$, and fractional holdings in the n assets, $\xi_t \equiv x_t / W_t$. Clearly $x_t = W_t \xi_t$ and $x_t^0 = (1 - \xi_t I) W_t$. Derived utility $\bar{V}_t(C_{t-1}, W_t, \xi_t, \phi_t)$ in terms of state variables $C_{t-1}, W_t, \xi_t, \phi_t$ is defined by

$$\bar{V}_t(C_{t-1}, W_t, \xi_t, \phi_t) \equiv V_t(C_{t-1}, (1 - \xi_t I) W_t, W_t \xi_t, \phi_t). \tag{4}$$

We now prove:

PROPOSITION 2. *For all* ξ_t, ϕ_t, $t = 0, 1, \ldots, T$, *derived utility* $\bar{V}_t(C_{t-1}, W_t, \xi_t, \phi_t)$ *is monotone increasing and concave in* C_{t-1}, W_t; *for all* W_t, ϕ_t, $t = 0, 1, \ldots, T$, *derived utility* $\bar{V}_t(C_{t-1}, W_t, \xi_t, \phi_t)$ *is concave in* C_{t-1}, ξ_t.

PROOF. Given ξ_t, ϕ_t, t, then X_t is monotone increasing and concave in W_t. By Proposition 1, V_t is monotone increasing and concave in C_{t-1}, X_t. Therefore $\bar{V}_t$ is monotone increasing and concave in C_{t-1}, W_t. The second part is similarly proved.

Before we further study the effect of transactions costs on consumption-investment decisions, we briefly review the implications of the model in the absence of transactions costs. We set $T(u) = 0$ and obtain Fama's [7] model:[1] State (C_{t-1}, X_t, ϕ_t) is concisely represented by (C_{t-1}, W_t, ϕ_t), where ξ_t becomes a superfluous state variable. At date t the consumer makes optimal consumption-investment decisions with the objective to maximize his expectation of a monotone increasing and concave (derived) utility function of consumption at date t and of wealth W_{t+1} at date $t+1$.[2] These observations by Fama provide a general theoretical justification for the study of consumption-investment behavior and capital market equilibrium in a single period model rather than in a cumbersome intertemporal model.

In the presence of transactions costs ξ_t becomes a relevant state variable at date t. At date t the consumer maximizes the expectation of $\bar{V}_{t+1}(C_t, W_{t+1}, \xi_{t+1}, \phi_{t+1})$. This function is not in general concave in W_{t+1} unless ξ_{t+1} is taken as fixed at date t. These remarks qualify the justification for the study of consumption-investment behavior in a single period model. Additional qualifications appear in [4].

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[1] Our transformation function f_t is slightly more general than the transformation function in Fama's model.

[2] Strict concavity of the derived utility function is proven in Fama [8]. See also the discussion in Ziemba [22].

In characterizing the consumer's optimal consumption-investment decisions, we state:

PROPOSITION 3. *Locally optimal consumption-investment decisions are globally optimal.*

PROOF. In the proof of Proposition 1, we showed that the expectation at t of V_{t+1} is concave in $x_t^0, x_t, C_{t-1}, c_t^0, c_t, u_t$. The consumer's objective at date t is the maximization of the expectation of V_{t+1}, given $\phi_t, x_t^0, x_t, C_{t-1}$. The objective is concave in c_t^0, c_t, u_t and the result follows.

Thus numerical procedures which search for local optima yield globally optimal consumption-investment decisions.

4. Proportional Transactions Costs

We sharpen the characterization of optimal consumption-investment behavior under the additional assumption that the transactions costs function is positively homogeneous of degree one, i.e., $T(\lambda u) = \lambda T(u)$, $\lambda > 0$. The joint assumption of convexity and homogeneity of the transactions costs function implies $T(u + v) \leqslant T(u) + T(v)$ (see, for example, Rockafellar [16, Theorem 4.7]): The cost of transaction $u + v$ is less than the sum of the costs of transactions u and v. This property is crucial in the subsequent discussion.

A transactions costs function of considerable practical importance is the proportional transactions costs function defined by $T(u) \equiv \sum_{i=1}^n \max[k_{i1}u_i, -k_{i2}u_i]$, where $0 \leqslant k_{i1} < 1$, $0 \leqslant k_{i2} < 1$, and k_{i1}, k_{i2} are given constants. Note that this function has the desirable properties of convexity and positive homogeneity of degree one; also $T(u) = 0$ and $|T(u) - T(v)| < \|u - v\|$, $\forall u, v$, $u \neq v$.

We define the function g_t as

$$g_t(u_t, X_t; C_{t-1}, \phi_t) \equiv \max_{\{c_t^0, c_t\}} \int_{\phi_{t+1}} V_{t+1}\big(C_t, f_t\big(x_t^0 - u_t I - T(u_t) - c_t^0, x_t + u_t - c_t, \phi_{t+1}\big), \phi_{t+1}\big)\, dF_t(\phi_{t+1} \mid \phi_t) \tag{5}$$

where $\{c_t^0, c_t\}$ is such that $\{c_t^0, c_t, u_t\} \in \Omega_t$. We shall abbreviate $g_t(u_t, X_t; C_{t-1}, \phi_t)$ by $g(u, X)$, when t, C_{t-1}, ϕ_t are easily understood. We also define the subset ω_t as $\omega_t = \{u_t \mid (c_t^0, c_t, u_t) \in \Omega_t,\ (c_t^0, c_t) \text{ are optimal}\}$. Clearly, the consumer makes investment decisions $u_t \in \omega_t$ at date t with the objective to maximize $g_t(u_t, X_t; C_{t-1}, \phi_t)$.

Optimal investment policy is conveniently described in terms of a *region of no transactions*, $\Psi_t = \{X_t \mid g(u_t, X_t) \leqslant g(0, X_t), u_t \in \omega_t\}$. If the consumer enters date t with assets $X_t \in \Psi_t$, an optimal investment policy is to carry no transactions, i.e., $u_t = 0$. We note that the region of no transactions Ψ_t, in general depends upon past history, C_{t-1}, ϕ_t. When X_t lies outside the region of no transactions, optimal investment is described as follows:

PROPOSITION 4. *Assume homogeneous of degree one transactions costs. The consumer enters date t with assets X_t. An optimal investment decision u_t is such that:*

(a) *After the transaction, the resulting asset holdings lie in the region of no transactions, i.e.,* $(x_t^0 - u_t I - T(u_t), x_t + u_t) \in \Psi_t$.

(b) *If there exists a* $0 \leqslant \lambda < 1$ *such that* $(x_t^0 - \lambda u_t I - T(\lambda u_t), x_t + \lambda u_t) \in \Psi_t$, *then* λu_t *is an optimal investment decision also.*

PROOF. (a) The proof is by contradiction. Assume that $(x_t^0 - u_t I - T(u_t), x_t + u_t)$

is not in Ψ_t. Then there exists some $v \neq 0$ such that:

$$\begin{aligned} & g\big(0, x_t^0 - u_t I - T(u_t), x_t + u_t\big) \\ & \quad < g\big(v, x_t^0 - u_t I - T(u_t), x_t + u_t\big) \\ & \quad < g\big(0, x_t^0 - u_t I - T(u_t) - vI - T(v), x_t + u_t + v\big) \\ & \quad < g\big(0, x_t^0 - (u_t + v)I - T(u_t + v), x_t + (u_t + v)\big) \end{aligned}$$

where we used the property $T(u_t) + T(v) \leqslant T(u_t + v)$. Therefore the investment decision u_t is inferior to the investment decision $u_t + v$, given state X_t.

It remains to show that investment $u_t + v$ is feasible, given state X_t. Consider first the case where the set Ω_t is defined by equation (3). By assumption, the investment v is feasible, given state $(x_t^0 - u_t I - T(u_t), x_t + u_t)$; i.e., there exists a consumption vector $c_t^0 \geqslant 0, c_t \geqslant 0$ such that $x_t^0 - u_t I - T(u_t) - c_t^0 - vI - T(v) \geqslant 0$ and $x_t + u_t - c_t + v_t \geqslant 0$. Since $x_t^0 - (u_t + v)I - T(u_t + v) - c_t^0 \geqslant x_t^0 - u_t I - T(u_t) - c_t^0 - vI - T(v)$, investment $u_t + v$ is feasible, given state X_t. A similar argument applies when Ω_t is defined by (3′).

Since the investment u_t is inferior to the feasible investment $u_t + v$, given state X_t, the assumption is contradicted.

(b) We break up the investment u_t into two parts λu_t and $(1 - \lambda)u_t$, $0 \leqslant \lambda < 1$. The total transactions cost remains unchanged because $T(\lambda u_t) + T((1 - \lambda)u_t) = T(u_t)$. Also transaction $(1 - \lambda)u_t$ is feasible given state $(x_t^0 - \lambda u_t I - T(\lambda u_t), x_t + \lambda u_t)$. Therefore

$$\begin{aligned} g_t(u_t, X_t) &= g_t\big((1 - \lambda)u_t, x_t^0 - \lambda u_t I - T(\lambda u_t), x_t + \lambda u_t\big) \\ &\leqslant g_t\big(0, x_t^0 - \lambda u_t I - T(\lambda u_t), x_t + \lambda u_t\big). \end{aligned}$$

The second step follows from the fact that $(x_t^0 - \lambda u_t I - T(\lambda u_t), x_t + \lambda u_t) \in \Psi_t$. Also

$$g_t\big(0, x_t^0 - \lambda u_t I - T(\lambda u_t), \lambda u_t\big) = g_t(\lambda u_t, X_t)$$

since the resulting asset holdings are equal. Combining the above two equations we obtain $g_t(u_t, X_t) \leqslant g_t(\lambda u_t, X_t)$; i.e., the investment decision λu_t is at least as good as the investment decision u_t. Since u_t is optimal, λu_t is optimal also.

The proof is complete.

The region of no transactions, Ψ_t, plays a double role. First (by definition) it is the set of asset holdings X_t such that an optimal investment policy is to carry no transactions. Second (by Proposition 4) it is the set of asset holdings $(x_t^0 - \bar{u}_t I - T(\bar{u}_t), x_t + \bar{u}_t)$ after optimal transaction $\bar{u}_t$, given any initial holdings X_t (X_t may or may not lie in the region of no transactions). Without the assumption of homogeneous transactions costs the region of no transactions will not, in general, play the second role: An optimal policy might be to transact to a point which lies outside the region of no transactions.

Part (b) of Proposition 4 may be explained as follows. Let $\bar{u}_t$ be an optimal investment decision. The points $(x_t^0 - \lambda \bar{u}_t I - T(\lambda \bar{u}_t), x_t + \lambda \bar{u}_t)$, $0 \leqslant \lambda \leqslant 1$ describe the path of the asset holdings from the initial position ($\lambda = 0$) to the final position ($\lambda = 1$). If some point for which $0 < \lambda < 1$, lies in the region of no transactions, then $\lambda \bar{u}_t$ is an optimal investment also. Informally, whenever there exists an optimal investment decision such that the resulting asset holdings lie within the region of no transactions, there exists another optimal investment decision such that the resulting asset holdings lie on the boundary of the region of no transactions. Thus, *whenever the initial asset* *holdings lie outside the region of no transactions, an optimal investment decision is to transact to the boundary of this region.* In contrast, with concave transactions costs we would expect all optimal (nonzero) investment decisions to be such that the resulting asset holdings lie within the region of no transactions.

If the utility function is additively separable, i.e., $U(C_T, \phi_T) = \sum_{t=0}^{T} \hat{U}_t(c_t^0, c_t, \phi_t)$, then the consumer's objective at date t is to maximize his expectation of $\sum_{\tau=t}^{T} \hat{U}_\tau(c_\tau^0, c_\tau, \phi_\tau)$ which is independent of past consumption. Also Ω_t is independent of C_{t-1}. C_{t-1} is a superfluous state variable and the state at date t is simply (X_t, ϕ_t). The recursive equation (1) is simplified to

$$J_t(X_t, \phi_t) = \max_{\{c_t^0, c_t, u_t\} \in \Omega_t} \left[\hat{U}_t(c_t^0, c_t, \phi_t) + \int_{\phi_{t+1}} J_{t+1}\Big(f_t\big(x_t^0 - u_t I - T(u_t) - c_t^0, x_t + u_t - c_t, \phi_{t+1}\big), \phi_{t+1}\Big) \, dF_t(\phi_{t+1} \mid \phi_t) \right], \tag{1'}$$

$$t = 0, 1, \ldots, T,$$

with boundary condition

$$J_{T+1}(X_{T+1}, \phi_{T+1}) \equiv 0. \tag{2'}$$

If the utility function is multiplicatively separable, i.e., $U(C_T \mid \phi_T) = \prod_{t=0}^{T} \hat{U}_t(c_t^0, c_t, \phi_t)$, the state at date t is simply (X_t, ϕ_t). Also the recursive equation (1) simplifies accordingly. In the subsequent discussion we shall assume additively separable utility and leave the corresponding development in the case of multiplicatively separable utility as an exercise to the reader.

We now state:

PROPOSITION 5. *Assume that the transactions costs function $T(u)$ is positively homogeneous of degree one in u; the utility function $U(C_T, \phi_T)$ is additively or multiplicatively separable; $\hat{U}_t(c_t^0, c_t, \phi_t)$ is positively homogeneous of degree α in c_t^0, c_t;*[3] *and the transformation function $f_t(Y_t, \phi_t)$ is positively homogeneous of degree one in Y_t. Then*

(a) *The optimal policy functions $\bar{c}_t^0(X_t, \phi_t), \bar{c}_t(X_t, \phi_t), \bar{u}_t(X_t, \phi_t)$ are homogeneous of degree one in X_t.*

(b) *The region of no transactions is a cone, i.e., $X_t \in \Psi_t$ implies $\lambda X_t \in \Psi_t$, for all $\lambda > 0$.*

PROOF. (a) Inspection of (3) or (3′) indicates that $(c_t^0, c_t, u_t) \in \Omega_t(X_t, \phi_t)$ implies $(\lambda c_t^0, \lambda c_t, \lambda u_t) \in \Omega_t(\lambda X_t, \phi_t)$, $\lambda > 0$. If $J_{t+1}(\lambda X_{t+1}, \phi_{t+1}) = \lambda^\alpha J_t(X_{t+1}, \phi_{t+1})$, inspection of (1′) and the above result indicates that $J_t(\lambda X_t, \phi_t) = \lambda^\alpha J_t(X_t, \phi_t)$ and optimal c_t^0, c_t, u_t are homogeneous of degree one in X_t. To complete the induction proof we note that (2′) implies that $V_{T+1}(\lambda X_{T+1}, \phi_{T+1}) = \lambda^\alpha V_{T+1}(X_{T+1}, \phi_{T+1})$.

(b) The above results imply that, if $u_t \in \omega_t(X_t, \phi_t)$ then $\lambda u_t \in \omega_t(\lambda X_t, \phi_t)$. They also imply that $g_t(\lambda u_t, \lambda X_t, \phi_t) = \lambda^\alpha g_t(u_t, X_t, \phi_t)$. If $g_t(u_t, X_t) \leqslant g_t(0, X_t)$ and $u_t \in \omega_t(X_t, \phi_t)$, then $g_t(\lambda u_t, \lambda X_t) \leqslant g_t(0, \lambda X_t)$. But $\lambda u_t \in \omega_t(\lambda X_t, \phi_t)$. Therefore, if $X_t \in \Psi_t$ then $\lambda X_t \in \Psi_t$, i.e., Ψ_t is a cone.

The proof is complete.

We express state (X_t, ϕ_t) as (W_t, ξ_t, ϕ_t) where $W_t \equiv x_t^0 + x_t I$ is total wealth and $\xi_t \equiv x_t / W_t$ are portfolio proportions. We define the optimal consumption function of the ith asset in terms of state variables (W_t, ξ_t, ϕ_t) as $\hat{c}_t^i(W_t, \xi_t, \phi_t) \equiv \bar{c}_t^i(X_t, \phi_t)$. The marginal propensity to consume out of wealth is described as follows:

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[3] If the utility function is not homogeneous in consumption, it may sometimes be converted to a homogeneous function by a linear transformation of the asset variables. See [3] for details.

PROPOSITION 6. *Under the homogeneity assumptions of Proposition* 5, *the marginal propensity to consume each asset and the sum of the assets out of wealth lies between zero and one, i.e.*,

(a) $0 \leqslant \partial \hat{c}_t^i(W_t, \xi_t, \phi_t)/\partial W_t \leqslant 1$, $i = 0, 1, \ldots, n$;
and
(b) $0 \leqslant \sum_{i=0}^n [\partial \hat{c}_t^i(W_t, \xi_t, \phi_t)/\partial W_t] \leqslant 1$.

PROOF. (a) By the homogeneity of optimal policy functions, $\partial \hat{c}_t^i(W_t, \xi_t, \phi_t)/\partial W_t = \hat{c}_t^i / W_t$. Also $0 \leqslant \hat{c}_t^i / W_t \leqslant 1$, by the conditions (3) or (3′). The result follows.

(b) By homogeneity,

$$\sum_{i=0}^{n} \frac{\partial \hat{c}_t^i(W_t, \xi_t, \phi_t)}{\partial W_t} = \frac{\sum_{i=0}^n \hat{c}_t^i}{W_t} .$$

Also $0 \leqslant \sum_{i=0}^n \hat{c}_t^i / W_t \leqslant 1$. The result follows.

This Proposition was earlier proved by Zabel [21] and Magill and Constantinides [12] under stronger assumptions.

5. The Two Asset Case with Proportional Transactions Costs

The region of no transactions attains a particularly simple form under the assumption that there exist only two assets. One may conveniently think of the two assets as money (or a bank account) and a mutual fund of all other assets; or, as a mutual fund of bonds and a mutual fund of stocks. Clearly the two-asset case is of considerable practical importance.

The investment decision u is a scalar. The assumptions $T(\lambda u) = \lambda T(u)$ for $\lambda > 0$, $T(0) = 0$ and $|T(u) - T(v)| < \|u - v\|$ for $u \neq v$, imply that $T(u) = \max[ku, -k'u]$, where $0 \leqslant k < 1$, $0 \leqslant k' < 1$ and k, k' are given constants; i.e., transactions costs are proportional.

We now prove:

PROPOSITION 7. *Under the homogeneity assumptions of Proposition* 5 *and the assumption that there exist only two assets, i.e.*, $i = 0, 1$, *the region of no transactions is convex.*

PROOF. It will suffice to prove that, if two rays A, B are in Ψ_t, so is any point C which lies in the cone defined by rays A, B.

Consider some point C which lies in the cone defined by rays A, B. Point C stands for (x_c^0, x_c^1), where we have suppressed the time subscript for convenience. Reference to Figure 1 is helpful. From point (x_c^0, x_c^1) we draw a line with slope $-(1 + k)^{-1}$. This

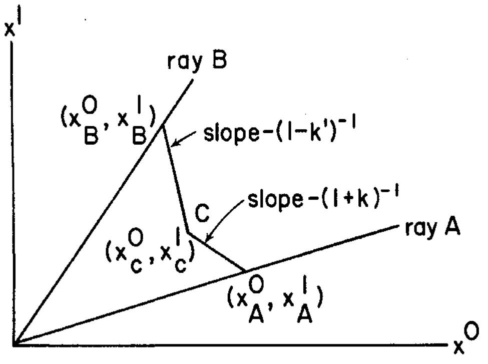

FIGURE 1

line meets ray A at point (x_A^0, x_A^1). Define $u \equiv x_c^1 - x_A^1 > 0$. Then $x_c^0 = x_A^0 - u - T(u)$, $x_c^1 = x_A^1 + u$ and

$$g(u, x_A^0, x_A^1) = g(0, x_c^0, x_c^1). \tag{6}$$

We need to prove that the line through (x_c^0, x_c^1) always intersects ray A. If Ω_t is defined by (3), rays A, B lie in the positive orthant and the slope of A is nonnegative. If Ω_t is defined by Equation (3′), the slope of ray A equals or exceeds $-(1+k)^{-1}$. So long as the slope of ray A exceeds $-(1+k)^{-1}$, the line through C intersects ray A at some finite point, i.e., u is finite. (If the slope of ray A is $-(1+k)^{-1}$, the point of intersection is at infinity.)

Since $(x_A^0, x_A^1) \in \Psi_t$,

$$g(u, x_A^0, x_A^1) \leqslant g(0, x_A^0, x_A^1).$$

By the concavity of g in u and the above, for any $\lambda > 0$,

$$g((1+\lambda)u, x_A^0, x_A^1) \leqslant g(u, x_A^0, x_A^1). \tag{7}$$

Thus

$$\begin{aligned} g(\lambda u, x_c^0, x_c^1) &= g((1+\lambda)u, x_A^0, x_A^1) \\ &\leqslant g(u, x_A^0, x_A^1) \quad \text{(by (7))} \\ &\leqslant g(0, x_c^0, x_c^1) \quad \text{(by (6))}. \end{aligned}$$

By a similar argument, for some $u < 0$ and for any $\lambda > 0$,

$$g(\lambda u, x_c^0, z_c^1) \leqslant g(0, x_c^0, x_c^1).$$

Therefore $(x_c^0, x_c^1) \in \Psi_t$. The proof is complete.

Zabel [21] proved Proposition 7 under the following additional assumptions, none of which were necessary in our proof: The decision horizon is limited to two periods, i.e., $T = 2$; the zeroth asset is riskless; the returns on the risky asset are uncorrelated in the two periods; borrowing and selling short are prohibited; single period utility is of power form, is state independent and is identical in the two periods; no dividends are paid. Kamin [10] also proved Proposition 7 under the following assumptions, none of which were necessary in our proof: The consumer consumes only at the end of the horizon; borrowing and selling short are prohibited; rates of return are independent of earlier realizations; utility is state independent and is of power or logarithmic form; no dividends are paid.

Propositions 4, 5 and 7 imply that at any point in time optimal investment policy is determined in terms of two control limits $\underline{a} \leqslant \bar{a}$. These limits are in general functions of the state variable ϕ_t. If $x_t^1/x_t^0 \in [\underline{a}, \bar{a}]$, an optimal investment policy is to refrain from transacting. If $x_t^1/x_t^0 < \underline{a}$ ($> \bar{a}$) an optimal investment policy is to transact to level $\underline{a}$ ($\bar{a}$). That the optimal policy is of the control limit type is not surprising. What is far from obvious is that the region of no transactions is convex and that there is only one connected interval $[\underline{a}, \bar{a}]$ of portfolio proportions in which it is optimal to refrain from transacting.

6. Concluding Remarks

A comprehensive discussion of the properties of derived utility and optimal consumption-investment decisions has been presented under fairly weak assumptions. Recently, building upon the results of this paper, Abrams and Karmarkar [1] have presented conditions under which the derived utility function is differentiable.

A promising direction of future research is the derivation of closed-form expressions for the optimal decisions. Magill and Constantinides [12] made a first step in this direction under the assumption that asset returns are lognormally distributed. Another promising direction is the development of efficient algorithms for the numerical estimation of optimal decisions. The general properties of derived utility, optimal decisions and the region of no transactions discussed in this paper should prove useful in the development of such algorithms.[4]

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[4] This paper is a revised version of [5]. I would like to thank two anonymous referees for several helpful comments. As usual, I remain responsible for errors.

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