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## Existence of Optimal Simple Policies for Discounted-Cost Inventory and Cash Management in Continuous Time

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We formulate a continuous-time, infinite-horizon, discounted-cost cash management model with both fixed and proportional transactions costs and with linear holding and penalty costs. We model the cumulative demand for cash by a Wiener process with drift and use the optimal control technique of "impulse control" to find sufficient conditions under which an optimal policy exists. We show that these conditions are always met. Therefore, we prove that there always exists an optimal policy for the cash management problem and that this policy is of a simple form. When the proportional transactions cost of decreasing the cash balance is sufficiently high, it is never optimal to decrease the cash balance. Then the cash management model degenerates to the inventory model. We prove that there always exists an optimal policy for the inventory model and that this policy is of a simple form. Under special cases of the cash management model we obtain analytic expressions for the parameters of the optimal policy.

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SCARF [17] has shown that an  $(S, s)$  policy is always optimal for the discrete-time inventory problem. A natural and intuitively appealing generalization of  $(S, s)$  inventory policies is what we call simple policies for the cash management problem. Simple policies are defined by parameters  $(d, D, U, u)$ , where  $d \leq D \leq U \leq u$ , such that we do not transact as long as the cash level lies within the open set  $(d, u)$  and we adjust the cash level to  $D(U)$  whenever the cash level falls to  $d$  (rises to  $u$ ). From a practical viewpoint, these policies are easy to implement.

The qualitative form of optimal policies of the cash management problem has been addressed by Whistler [19], Eppen and Fama [8, 9], Girgis [11] and Neave [14]. Neave provides the most general sufficient conditions to date for a simple policy to be optimal: no proportional transactions costs, symmetric fixed transactions costs, equal holding and penalty costs, and symmetric about zero and quasiconcave cash-demand density func-

tions. Without the symmetry assumptions and in the presence of proportional transactions costs, Neave shows that the optimal policy, in terms of parameters  $(d, D, U, u)$ , is such that one adjusts the cash level to  $D(U)$  whenever the cash level falls below  $d$  (exceeds  $u$ ); but the optimal action is quite complex if the cash level lies within the range  $(d, u)$ .

In this paper we show that many of the difficulties that Neave and others encountered in a discrete-time framework largely disappear when we assume that decisions are made continuously in time and that demand is generated by a Wiener process with drift. Indeed, we prove that under these assumptions there always exists an optimal policy for the cash management system and that this policy is of simple form. We do not assume symmetry in any of the costs or in the demand distribution. Instead we assume first that decisions are made continuously in time. We find this a natural assumption (especially in light of real-time computer monitoring) and at least as acceptable as the alternative assumption that decisions can be made only at arbitrarily determined discrete points in time. Our second assumption is that demand is generated by a Wiener process with drift. This means that the observed demand over any length of time is normally distributed with both mean and variance proportional to the length of time.

The inventory problem is a special case of the cash management problem where the proportional transaction cost of decreasing the cash balance is sufficiently high that it is never optimal to decrease the cash balance. This translates in inventory theory to the condition that an order to replenish the inventory must be non-negative. Assuming that decisions are made continuously in time and that demand is generated by a Wiener process with drift, we prove that there always exists an optimal policy for the inventory system and that this policy is of simple form.

There have been earlier formulations of the inventory and cash management problems in a continuous-time framework with demand generated by a Wiener process with drift. However, these models address a different set of issues, and they all assume the existence of an optimal policy. Bather [2] and Antelman and Savage [1] formulated an inventory model and Constantinides [6] formulated a cash management model; these authors assume that a policy exists and is of simple form, and explicitly derive the parameters of the optimal policy. Puterman [15] shows that this type of problem can be handled as an application of Mandl's [13] theory of controlled diffusion processes. Puterman applies this theory to a model of a storage system. Vial [18] formulates the cash management problem in continuous time and examines the form of the optimal policy. Vial assumes that an optimal policy exists and is one from a set of admissible policies, and that there exists a value function that satisfies some consistency re-

quirements.<sup>1</sup> Under these assumptions he proves that an optimal policy is of simple form. It will become apparent from our discussion that the proof of his assumptions is a nontrivial task. It must be stressed that our approach is free from such assumptions.

We address the questions of existence and form of an optimal policy through the optimal control technique of "impulse control," originated by Bensoussan and Lions [3, 4] and extended by Richard [16]. The technique rigorously handles continuous-time optimal control problems where there are fixed costs of control and hence the optimal control is discontinuous and consists of impulses, applied at stochastic stopping times. The strength of our results testifies to the power of impulse control.

The paper is organized as follows: In Section 1 we formulate the cash management model. In Section 2 we prove that an optimal policy exists and is of simple form. The special case of inventory control is discussed in Section 3, where similar results are obtained. In Section 4, for special cases of the model, we derive explicit solutions of the parameters of the optimal cash management policy.

## 1. MODEL FORMULATION

In this section we formulate a cash management model. The inventory model is treated as a special case of the cash management model and is discussed in Section 3. We define  $x = x(t)$  to be the cash balance at time  $t$ . The holding-penalty cost rate of keeping cash balance  $x$  is  $C(x) = \max [hx, -px]$ , where  $h, p > 0$ . The cost of transferring funds and changing the cash level from  $x_0$  to  $x_1$  is  $B(x_1 - x_0)$

$$B(x_1 - x_0) = \begin{cases} K^+ + (x_1 - x_0)k^+, & x_0 \leq x_1, \\ K^- + (x_0 - x_1)k^-, & x_1 < x_0, \end{cases}$$

where  $k^+, k^-, K^+, K^- > 0$ . Note that  $B(0) = K^+ > 0$  so that a zero control incurs a fixed cost. We distinguish between no control, which costs nothing, and a zero control, which does nothing but nevertheless has a cost associated with it. This distinction is important in the proof of Lemma 1.

We denote the cumulative demand for cash in the time interval  $[t, s]$  by  $D(t, s)$ . The demand for cash is defined to be such that, for any partition  $t_0 \leq t_1 \leq t_2 \leq \dots$  of the time interval  $[t_0, \infty)$ , the random variables  $D(t_0, t_1), D(t_1, t_2), \dots, D(t_i, t_{i+1}), \dots$  are independent and normally distributed. The mean and variance of the demand  $D(t_i, t_{i+1})$  are

<sup>1</sup> Vial [18] defined an admissible policy as "... a stationary policy with the properties that: (i) its no-transfer set is a finite union of open intervals, (ii) every order point belongs to the no-transfer set" (p. 254). Vial also assumed that there exists a value function that satisfies his conditions (23) and (24) (p. 273), which correspond to our equations (6) and (7). Establishing that (6) and (7) can be met is far from obvious, as we show below.

$ED(t_i, t_{i+z}) = (t_{i+1} - t_i)\mu$  and  $\text{var } D(t_i, t_{i+1}) = (t_{i+1} - t_i)\sigma^2$ , where  $\mu$  and  $\sigma^2$  are constants independent of time. Thus the cumulative demand from  $t$  to  $s$  is given by  $D(t, s) = (s - t)\mu + (w(s) - w(t))\sigma$ ,  $s \geq t$  where  $w$  is a Wiener process in  $R^1$  with zero drift and diffusion coefficient one. Further details on Wiener processes can be found in Kushner [12, ch. 10], Breiman [5] or Gihman and Skorohod [10].

The decision maker continuously observes the cash level and intervenes when necessary to make adjustments. Since his decision to intervene at time  $t$  depends in general on the random state of the system just prior to time  $t$ , the times of intervention will be random variables, called stopping times, that depend only on events up to time  $t$  and are independent of future events. Any given policy will generate an increasing sequence of stopping times  $\tau_1 \leq \tau_2 \leq \dots \leq \tau_i \leq \dots$  at which the cash level will be controlled. To ensure that policies have finite cost we require that, with probability one, only a finite number of stopping times will occur in a bounded time interval. Furthermore, for the optimal policy it should be clear that, with probability one,  $\tau_1 < \tau_2 < \dots < \tau_i < \dots$  because if  $\tau_i = \tau_{i+1}$ , then the fixed transactions cost will be incurred twice.

At stopping time  $\tau_i$  the control applied, denoted  $\phi_i$ , may in general be any random variable that is independent of the future state of the system. Such a random variable is called non-anticipating. Included in this class of permissible controls are those that are deterministic functions of the state of the system,  $x(\tau_i^-)$ , immediately prior to the control. Hence  $\phi_i = \eta(x(\tau_i^-))$ , where  $\eta(\cdot)$  is a real-valued measurable function, is only a special case of the permissible controls. An impulse control policy  $v$  can then be represented as a sequence of stopping times and corresponding controls:  $v = \{\tau_1, \phi_1; \tau_2, \phi_2; \dots\}$ . We define  $\Omega$  to be the class of all impulse control policies  $v$  such that  $\tau_1, \tau_2, \dots$  are stopping times and  $\phi_1, \phi_2, \dots$  are non-anticipating random variables. Additionally, we require that  $\Omega$  contain only those policies with a finite number of stopping times in any bounded interval.

If policy  $v$  is adopted, then the cash level dynamics are given as follows. Denote by  $x(\tau_i^-)$  the cash level at stopping time  $\tau_i$  before the control  $\phi_i$  is applied, and denote by  $x(\tau_i)$  the cash level at stopping time  $\tau_i$  and after the control  $\phi_i$  is applied. Then the state equations of the cash level are

$$\begin{aligned} dx(t) &= -\mu dt - \sigma dw(t), \quad 0 \leq t < \tau_1; \quad x(0^-) = x_0; \quad \text{and} \\ x(\tau_i) &= x(\tau_i^-) + \phi_i, \quad dx(t) = -\mu dt - \sigma dw(t), \quad \tau_i \leq t < \tau_{i+1}^-, \quad i \geq 1. \end{aligned}$$

Implicit in the state equations is the assumption that the holding-penalty costs and transactions costs are not charged to the cash balance. This has been a standard assumption in the cash management literature.

Given a transactions policy, its equivalent impulse control,  $v$ , and the

initial cash level,  $x(0^-) = x_0$  at time  $t=0$ , the joint probability distribution of the stochastic variables  $x(t)$  for all  $t>0$  and of  $\tau_i, \phi_i, \forall i \geq 1$  is defined. Also, the expectation operator  $E_{x_0}^v$  with respect to the stochastic variables  $x(t), \tau_i$  and  $\phi_i$ , conditional on the cash level  $x(0^-) = x_0$  and transaction policy  $v$ , is defined.

The total cost of the cash management system is the sum of the transactions costs, holding costs, and penalty costs. Given policy  $v$ , the expected total cost from time zero to infinity, discounted to time zero and conditional on the cash level  $x(0^-) = x_0$ , is

$$J_{x_0}(v) \equiv E_{x_0}^v \left\{ \sum_{i=1}^{\infty} \exp(-\beta \tau_i) B(\phi_i) + \int_0^{\infty} \exp(-\beta s) C(x(s)) ds \right\},$$

where  $\beta$  is the discount rate. The objective is to choose a policy  $\hat{v}$  such that  $J_{x_0}(\hat{v}) \leq J_{x_0}(v), \forall v \in \Omega$ .

## 2. EXISTENCE AND FORM OF OPTIMAL CASH MANAGEMENT POLICY

If an optimal policy  $\hat{v}$  exists, we define the value function  $V(x)$  as  $V(x) = J_x(\hat{v})$ . Essentially  $V(x)$  is the minimized expected total cost from time zero to infinity, discounted to time zero and conditional on the cash level  $x(0^-) = x$ . Since the problem is invariant to time translations,  $V(x)$  may be more generally interpreted as the minimized expected total cost from time  $t$  to infinity, discounted to time  $t$  and conditional on the cash level  $x(t^-) = x$ .

Before we formally discuss the set of sufficient conditions that characterize the function  $V(x)$  and ensure the existence of an optimal policy, we gain further insight into the problem by heuristically deriving necessary conditions on the function  $V(x)$ . First, we note that  $V(x) \geq 0$  since all costs are non-negative. Let us express the value function at time  $t$  in terms of the value function at time  $t+dt$  through dynamic programming. We have to explicitly allow for two cases, depending on whether or not  $t$  is a stopping time. We make the simplifying assumption that, whether  $t$  is or is not a stopping time,  $dt$  is sufficiently small so that we need not be concerned with the possibility of stopping times occurring in the time interval  $(t, t+dt]$ . We then obtain

$$V(x(t^-)) = \min \left\{ \inf_{\xi} [B(\xi) + E_{x(t)}^v(C(x(t))) dt + \exp(-\beta dt) V(x(t)+dx)] \right. \\ \left. E_{x(t^-)}^v(C(x(t^-))) dt + \exp(-\beta dt) V(x(t^-)+dx) \right\},$$

which is equivalent to the following two inequalities:

$$V(x(t^-)) \leq \inf_{\xi} [B(\xi) + E_{x(t)}^v(C(x(t))) dt \\ + \exp(-\beta dt) V(x(t)+dx)] \quad (1)'$$

and

$$V(x(t^-)) \leq E_{x(t^-)}^v(C(x(t^-))) dt + \exp(-\beta dt) V(x(t^-)+dx), \quad (2)'$$

where at least one of them holds as an equality. Taking the limit in (1)' as  $dt \rightarrow 0$ , we obtain

$$V(x) \leq \inf_{\xi} [B(\xi) + V(x + \xi)] \equiv QV(x), \quad (1)$$

where  $x(t^-) = x$  and  $x(t) = x + \xi$ ; the operator  $Q$  is defined by (1) and is given here for future reference. Expanding (2)' in a Taylor series and taking the limit<sup>2</sup> as  $dt \rightarrow 0$ , we obtain

$$\beta V(x) + \mu V'(x) - \frac{1}{2}\sigma^2 V''(x) - C(x) \leq 0, \quad (2)$$

where as before we have replaced the symbol  $x(t^-)$  by  $x$ . Finally, we state the requirement that at least one of inequalities (1) or (2) must hold as an equality by

$$[\beta V(x) + \mu V'(x) - \frac{1}{2}\sigma^2 V''(x) - C(x)][V(x) - QV(x)] = 0. \quad (3)$$

Thus (1), (2), and (3) are necessary conditions on the function  $V(x)$ . The first lemma states that these conditions, together with some regularity conditions on the function  $V(x)$ , are sufficient to determine the value function and also determine an optimal transactions policy  $\hat{v}$ . Furthermore, if these conditions are met, then an optimal policy exists.

We suppose there exists a function  $V(x)$  that satisfies (1)–(3) and the regularity condition

$$V(x) \geq 0, \quad V'(x) \text{ is absolutely continuous and bounded} \quad (4)$$

and  $V''(x)$  is square integrable on  $R^1$ , i.e.,  $V''(x) \in L^2(R^1)$ .

We then claim that an optimal policy  $\hat{v}$  does exist and is such that we do not transact as long as the cash level lies within the continuation region  $H$  given by  $H = \{x: V(x) < QV(x)\}$ ; we transact by the amount  $\hat{\xi}(x)$  for  $x \notin H$ , where  $\hat{\xi}(x)$  is the minimizing value of  $\xi$  in (1).

**LEMMA 1.** *If there exists a function  $V(x)$  satisfying (1)–(4), then there exists an optimal impulse control  $\hat{v}$  such that  $V(x) = J_x(\hat{v}) \leq J_x(v) \forall v \in \Omega$ .*

*Proof.* See [16].

In Lemma 1 we provided conditions that guarantee the existence of an optimal policy. These same conditions reduce the problem of finding an optimal control policy to that of finding a solution to a set of differential inequalities, subject to some regularity conditions. The remaining part of this section is organized as follows. First, we heuristically show that the value function must satisfy the boundary conditions (6)–(11) and the differential equation (12). Second, we prove in Lemma 2 that there always exists a solution to the differential equation (12) that satisfies the boundary conditions (6)–(11) and the regularity conditions (4). Third, we prove in

<sup>2</sup> For the details of this heuristic derivation see Dreyfus [7, p. 215–219].

Theorem 1 that the above solution to the differential equation also satisfies all the conditions of Lemma 1. Thus we establish that there always exists an optimal solution to the cash management problem. More to the point, however, we also prove in Theorem 1 that an optimal solution to the cash management problem is of simple form.

For the purpose of our preliminary heuristic discussion we assume that the optimal policy is determined by parameters  $d, D, U, u$  where  $d \leq D \leq U \leq u$ , such that whenever the cash level is  $x$ , we immediately transact to level  $y(x)$  where

$$y(x) = \begin{cases} D, & x \leq d \\ x, & d < x < u \\ U, & u \leq x. \end{cases} \quad (5)$$

A stopping time is signified by  $x=d$  or  $x=u$ , and at these times (1) holds as an equality. Setting  $x=d$ ,  $\xi=D-d$  in (1) we obtain

$$V(d) = V(D) + K^+ + k^+(D-d), \quad (6)$$

and setting  $x=u$ ,  $\xi=U-u$  in (1) we obtain

$$V(u) = V(U) + K^- + k^-(u-U). \quad (7)$$

Since  $V(d+\xi) + B(\xi)$  is minimized at  $\xi=D-d$ , assuming differentiability of the function  $V$  and setting the first derivative w.r.t.  $\xi$  equal to zero, we obtain

$$V'(D) + k^+ = 0; \quad (8)$$

i.e., the marginal saving on future costs in increasing the cash balance by one unit equals the marginal transactions cost of increasing the cash balance by one unit. A similar argument at  $x=u$  yields

$$V'(U) - k^- = 0 \quad (9)$$

with a similar marginal cost interpretation.

The assumed policy is such that, once the cash level is within the closed region  $[d, u]$  it remains in that region thereafter. But assume that initially  $x=d-\delta$ , where  $0 < \delta \ll 1$ . Then the optimal policy is  $\xi=D-d+\delta$  and (1) becomes  $V(d-\delta) = V(D) + K^+ + k^+(D-d+\delta)$  or  $V(d) - V(d-\delta) = -k^+\delta$ . Assuming that the first derivative of  $V$  exists at  $x=d$  and taking the limit as  $\delta \rightarrow 0$ , we obtain

$$V'(d) + k^+ = 0. \quad (10)$$

Similarly, we obtain

$$V'(u) - k^- = 0. \quad (11)$$

For  $x \in (d, u)$ , the optimal control is  $\xi=0$ . From (1) and (3) we deduce that  $V$  must satisfy (3) as an equality for  $x \in (d, u)$ , i.e.,

$$\beta V(x) + \mu V'(x) - \frac{1}{2}\sigma^2 V''(x) = C(x). \quad (12)$$

So far we have heuristically shown that the value function must satisfy (12) subject to (6)-(11).

We argue that the cost parameters must satisfy

$$h - \beta k^- > 0 \quad \text{and} \quad p - \beta k^+ > 0. \tag{13}$$

$h/\beta$  is the present value of the holding cost of keeping one unit of cash from now to infinity. If  $h/\beta \leq k^-$ , it will never be optimal to reduce the cash level as long as  $K^- > 0$ . Similarly, if  $p/\beta \leq k^+$ , it will never be optimal to increase the cash level as long as  $K^+ > 0$ . Thus (13) provides necessary conditions for the cash balance to be controlled when it becomes sufficiently high or low. If  $p - \beta k^+ > 0$  but  $h - \beta k^- < 0$ , it is optimal to increase the cash level when it becomes sufficiently low, but it is never optimal to decrease the cash level, however high it may be. This special case is, of course, the inventory model and is studied in Section 3.

In order to prove Theorem 1 we need

LEMMA 2. *Suppose (13) holds. There always exist parameters  $d \leq D \leq U \leq u$  and a twice continuously differentiable solution  $V(x)$  of the differential equation (12) satisfying the regularity conditions (4) and the boundary conditions (6)–(11).*

*Proof.* See appendix.

We now prove our main result.

THEOREM 1. *Suppose (13) holds. Then there exists an optimal policy to the cash management problem. This policy is simple and is given by (5).*

*Proof.* We define the function  $V(x)$  as

$$V(x) = \begin{cases} V(u) + (x-u)k^-, & u \leq x \\ hx/\beta - h\mu/\beta^2 + c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}, & 0 \leq x \leq u \\ -px/\beta + p\mu/\beta^2 + \{c_1 - (h+p)\lambda_2/(\lambda_1\beta(\lambda_1 - \lambda_2))\} e^{\lambda_1 x} \\ \quad + \{c_2 + (h+p)\lambda_1/(\lambda_2\beta(\lambda_1 - \lambda_2))\} e^{\lambda_2 x}, & d \leq x \leq 0 \\ V(d) + (d-x)k^+, & x \leq d \end{cases} \tag{14}$$

where

$$\lambda_1 = \mu/\sigma^2 + (\mu^2 + 2\beta\sigma^2)^{1/2}/\sigma^2, \quad \lambda_2 = \mu/\sigma^2 - (\mu^2 + 2\beta\sigma^2)^{1/2}/\sigma^2 \tag{15}$$

and  $c_1, c_2$  are parameters yet to be determined. It is easily shown that  $V(x)$  satisfies (12) for  $d \leq x \leq u$ . We choose the parameters  $d \leq D \leq U \leq u$  and  $c_1, c_2$  so that  $V(x)$  satisfies (4) and (6)–(11). Lemma 2 specifies that such parameters can always be found. In Figures 1 and 2 we sketch the functions  $V'(x)$  and  $V(x)$ .

We proceed to show that the function  $V(x)$  satisfies all conditions of Lemma 1. We have already shown that  $V(x)$  satisfies the regularity conditions (4). Consider (1) under three distinct cases  $x \leq d, d < x < u$ , and

$x \geq u$ , and refer to Figure 2. If  $x \leq d$ ,

$$\begin{aligned} \inf_{\xi} [B(\xi) + V(x+\xi)] &= [B(\xi) + V(x+\xi)]_{\xi=D-x} \\ &= K^+ + k^+(D-x) + V(D) \\ &= K^+ + k^+(D-x) + V(d) - K^+ - k^+(D-d) \\ &= k^+(d-x) + V(d), = V(x). \end{aligned}$$

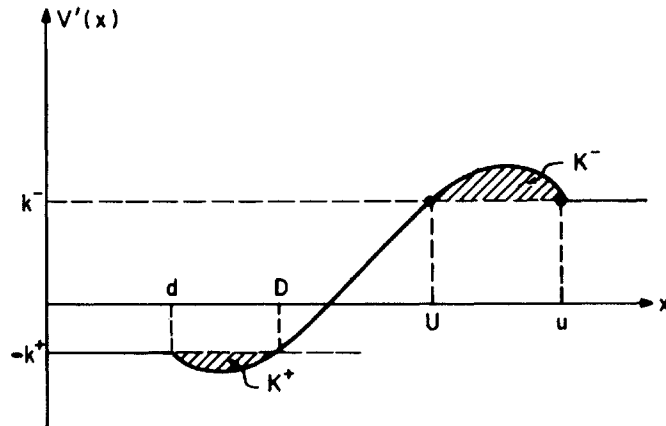


Figure 1. The function of  $V'(x)$ .

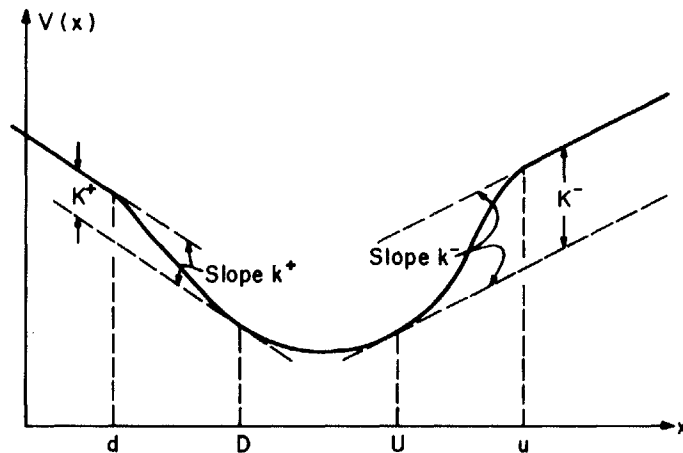


Figure 2. The function of  $V(x)$ .

If  $d < x < u$ ,  $\inf_{\xi} [B(\xi) + V(x+\xi)] = [B(\xi) + V(x+\xi)]_{\xi=0} = B(0) + V(x)$ ,  
 $> V(x)$ . If  $x \geq u$ ,

$$\begin{aligned} \inf_{\xi} [B(\xi) + V(x+\xi)] &= [B(\xi) + V(x+\xi)]_{\xi=U-x} \\ &= K^- + k^-(U-x) + V(U) \\ &= K^- + k^-(x-U) + V(u) - K^- - k^-(u-U) \\ &= k^-(x-u) + V(u) = V(x). \end{aligned}$$

Thus  $\inf_{\xi} [B(\xi) + V(x + \xi)] \begin{cases} > V(x), & x \in (d, u) \\ = V(x), & x \leq d \text{ or } x \geq u, \end{cases}$

and (1) is satisfied.

Next consider (2). If  $x \in (d, u)$ ,  $\beta V(x) + \mu V'(x) - \frac{1}{2}\sigma^2 V''(x) - C(x) = 0$ . If  $x \leq d$ ,

$$\begin{aligned} \beta V(x) + \mu V'(x) - \frac{1}{2}\sigma^2 V''(x) - C(x) & \\ &= \beta k^+(d-x) + \beta V(d) - \mu k^+ - C(x) \\ &= \beta V(d) + \mu V'(d) - \frac{1}{2}\sigma^2 V''(d^+) + pd \\ &\quad - \mu(k^+ + V'(d)) + \frac{1}{2}\sigma^2 V''(d^+) + (p - \beta k^+)(x-d), \end{aligned}$$

where  $V''(d^+)$  stands for the right-hand second derivative of  $V(x)$ . From the definition of  $V(x)$  in the range  $(d, u)$  it easily follows that  $\beta V(d) + \mu V'(d) - \frac{1}{2}\sigma^2 V''(d^+) + pd = 0$ . From (10) we obtain  $k^+ + V'(d) = 0$ . From Figure 2 we obtain  $V''(d^+) \leq 0$ . We have  $(p - \beta k^+) > 0$  by (13) and  $x - d < 0$  by assumption. Therefore,  $\beta V(x) + \mu V'(x) - \frac{1}{2}\sigma^2 V''(x) - C(x) < 0$ . Similarly, we prove that if  $x \geq u$ ,  $\beta V(x) + \mu V'(x) - \frac{1}{2}\sigma^2 V''(x) - C(x) < 0$ . Thus

$$\beta V(x) + \mu V'(x) - \frac{1}{2}\sigma^2 V''(x) - C(x) \begin{cases} = 0, & x \in (d, u) \\ < 0, & x \leq d \text{ or } x \geq u, \end{cases}$$

and (2) is satisfied. Note also that (3) is satisfied. Therefore,  $V(x)$  is the value function. The continuation region is  $(d, u)$  and from (1) we obtain the optimal control  $\xi(x)$

$$\xi(x) = \begin{cases} d-x, & x \leq d \\ 0, & d < x < u \\ x-u, & u \leq x, \end{cases}$$

which is equivalent to (5).

### 3. EXISTENCE AND FORM OF OPTIMAL INVENTORY POLICY

Although the existence and optimality of  $(S, s)$  policies have been proved by Scarf [17] for discrete-time models, the corresponding result has not been proved in continuous time. We briefly indicate how the methodology of the last section may be specialized to examine the problem of inventory control.

Our cash management model reduces to an inventory model under appropriate reinterpretation of the parameters. We interpret  $C(x) = \max [hx, -px]$  as the holding and backlogging cost rates. We assume the time lag between order and delivery to be zero. The order cost for an order of size  $\xi$  is  $B(\xi) = K + k\xi$ , where  $\xi \geq 0$ ,  $K > 0$ ,  $k \geq 0$ . Note that, unlike the cash management model,  $\xi$  is constrained to be non-negative.

The development of the proof is exactly analogous to that of Section 2. Lemma 1 remains valid under the restriction  $\xi \geq 0$ . We consider a policy

$y(x)$  where

$$y(x) = \begin{cases} S, & x \leq s \\ x, & s < x. \end{cases} \quad (16)$$

A modified version of Lemma 2 states that, if  $p - \beta k > 0$ , then there always exist parameters  $s \leq S$  such that a solution of (12) for  $x \geq s$  satisfies the regularity conditions (4) and the boundary conditions

$$V'(s) = V'(S) = -k \quad (17)$$

and

$$V(s) - V(S) = K + (S - s)k. \quad (18)$$

We now state the main theorem on inventory control.

**THEOREM 2.** *Suppose  $p - \beta k > 0$ . Then there exists an optimal policy to the inventory problem. This policy is simple and is given by (16).*

*Outline of proof.* Define the value function by

$$V(x) = \begin{cases} hx/\beta + h\mu/\beta^2 + c_2 e^{\lambda_2 x}, & 0 \leq x \\ -px/\beta - p\mu/\beta^2 + c_3 e^{\lambda_1 x} + c_4 e^{\lambda_2 x}, & s \leq x \leq 0 \\ k(s - x) + V(s), & x < s \end{cases} \quad (19)$$

where  $\lambda_1, \lambda_2$  are defined by (15). Note that since  $x$  is unbounded from above, we have omitted the term  $c_1 e^{\lambda_1 x}$  in the definition of  $V(x)$  for  $x \geq 0$  in order to satisfy  $V''(x) \in L^2(R^1)$ . The proof of the theorem follows the same steps as that of Theorem 1 and is omitted.

We now examine the cash management system when one or both of conditions  $h/\beta - k^- > 0$  and  $p/\beta - k^+ > 0$  do not hold. If  $h/\beta - k^- < 0$ , we argued earlier that it is never optimal to decrease the cash balance, that is to say,  $\xi \geq 0$ . As long as  $p/\beta - k^+ > 0$ , the cash management system reduces to the special case of an inventory system and the optimal policy is described by Theorem 2. If  $h/\beta - k^- > 0$  and  $p/\beta - k^+ < 0$ , we obtain an inverse inventory system. Finally, if  $h/\beta - k^- < 0$  and  $p/\beta - k^+ < 0$ , we never control the system.

#### 4. EXPLICIT SOLUTIONS OF THE CASH MANAGEMENT POLICY

In this section and under additional assumptions we explicitly solve for the policy parameters and perform a comparative statics analysis. We first consider the symmetric case with zero fixed transactions costs, i.e.,  $\mu = 0$ ,  $K^+ = K^- = 0$ ,  $k^+ = k^-$ ,  $h = p$ . We obtain

$$\begin{aligned} u = U = -d = -D \\ &= \ln [(h/\beta - \{k(2h/\beta - k)\}^{1/2}) / (h/\beta + \{k(2h/\beta - k)\}^{1/2})] \\ &= (k\sigma^2/h) \{1 - k\beta/4h + \theta(\beta)\}. \end{aligned}$$

The policy is described by reflecting boundaries  $d$ ,  $u$ , which confine the cash balance within  $(d, u)$ . As expected, the boundaries lie farther away from the origin the higher the transaction cost  $k$ , the greater the variation  $\sigma^2$  of the demand for cash, the lower the holding cost, and the lower the discount rate. If there are no transaction costs,  $k=0$ , the cash balance is constantly maintained at zero.

We next consider the case where costs are not symmetric and there are fixed as well as proportional transactions costs. We retain our earlier assumption that there is no drift in the demand for cash,  $\mu=0$ . In general, we cannot solve the equations explicitly, except in the case where the discount rate is small,  $\beta \rightarrow 0$ . We further assume that  $D < 0$ ,  $U > 0$ . Taking the limit  $\beta \rightarrow 0$ , we obtain

$$\begin{aligned} h(u+U) &= -p(d+D), & huU + pdD &= \sigma^2(k^+ + k^-), \\ (h/2\sigma^2)(u-U)^3 &= K^-, & (p/2\sigma^2)(D-d)^3 &= K^+ \end{aligned}$$

with solution

$$\begin{aligned} U &= \gamma h^{-1} - h^{-1}(3h^2\sigma^2K^-/4)^{1/3}, & u &= \gamma h^{-1} + h^{-1}(3h^2\sigma^2K^-/4)^{1/3} \\ D &= -\gamma p^{-1} + p^{-1}(3p^2\sigma^2K^+/4)^{1/3}, & d &= -\gamma p^{-1} - p^{-1}(3p^2\sigma^2K^+/4)^{1/3} \end{aligned}$$

where  $\gamma^2 = [hp\sigma^2(k^+ + k^-) + p(3h^2\sigma^2K^-/4)^{2/3} + h(3p^2\sigma^2K^+/4)^{2/3}]/(h+p)$ . Theoretically, the procedure of taking the limit  $\beta \rightarrow 0$  may be questioned because the value function tends to infinity. Therefore, we consider this solution as an approximation when  $\beta$  is sufficiently small. A rigorous treatment of the undiscounted case,  $\beta=0$ , is discussed in [6], where these equations were first derived. There  $\gamma$  is interpreted as the average cost rate of the cash management system. These equations lend themselves to straightforward comparative statics analysis. For example, the average cost rate is an increasing function of all cost parameters and the variance of the demand for cash; the policy parameter  $u$  is an increasing function of  $p$ ,  $K^+$ ,  $K^-$ ,  $k^+$ ,  $k^-$ ,  $\sigma^2$ ;  $U$  is an increasing function of  $K^+$ ,  $k^+$ ,  $k^-$ ,  $p$ ; and  $u-U$  is an increasing function of  $\sigma^2$ ,  $K^-$  and a decreasing function of  $h$  and is independent of other parameters. Since we assumed  $\mu=0$ , we may not draw useful implications for a deterministic system as  $\sigma \rightarrow 0$  because the joint assumption  $\mu=0$  and  $\sigma \rightarrow 0$  implies that there is no demand for cash and the problem is degenerate.

## APPENDIX

**LEMMA 2.** *Suppose (13) holds. There always exist parameters  $d \leq D \leq U \leq u$  and a twice continuously differentiable solution  $V(x)$  of (12) that satisfies (4) and (6)–(11).*

*Proof.* Consider  $V(x)$  defined by

$$V(x) = \begin{cases} hx/\beta - h\mu/\beta^2 + c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}, & x \geq 0 \quad (\text{A1}) \\ -px/\beta + p\mu/\beta^2 + \{c_1 - (h+p)\lambda_2/(\beta(\lambda_1 - \lambda_2)\lambda_1)\} e^{\lambda_1 x} \\ \quad + \{c_2 + (h+p)\lambda_1/(\beta(\lambda_1 - \lambda_2)\lambda_2)\} e^{\lambda_2 x}, & x < 0 \quad (\text{A2}) \end{cases}$$

where  $\lambda_1$  and  $\lambda_2$  are given by (15). By substitution we verify that  $V(x)$  satisfies (12) and the continuity requirements  $V(0^-) = V(0^+)$ ,  $V'(0^-) = V'(0^+)$  and  $V''(0^-) = V''(0^+)$ . In carrying out these substitutions it is helpful to remember the identity  $\mu/\beta = -(\lambda_1 + \lambda_2)/\lambda_1\lambda_2$ .

We consider next restrictions on the free parameters  $c_1$ ,  $c_2$ , which guarantee the existence of parameters  $d$ ,  $D$ ,  $U$ ,  $u$  to satisfy conditions (8)–(11). For  $\hat{x} \geq 0$ ,  $V''(\hat{x}) = 0$  implies  $\hat{x} = (\lambda_1 - \lambda_2)^{-1} \ln(-c_2\lambda_2^2/c_1\lambda_1^2)$ , which is real and positive provided

$$-c_2\lambda_2^2/c_1\lambda_1^2 \geq 1. \quad (\text{A3})$$

Thus  $V''(x)$  has either one or no roots for  $x \geq 0$ , depending on whether or not (A3) is satisfied. By a similar argument  $V''(x)$  has either one or no roots for  $x \leq 0$ , depending on whether the condition

$$0 < -\{c_2 + (h+p)\lambda_1/(\beta(\lambda_1 - \lambda_2)\lambda_2)\}\lambda_2^2 / \{c_1 - (h+p)\lambda_2/(\beta(\lambda_1 - \lambda_2)\lambda_1)\}\lambda_1^2 \leq 1 \quad (\text{A4})$$

is or is not satisfied. If (A4) is satisfied, then for some  $\hat{x} \leq 0$ ,  $V''(\hat{x}) = 0$ . Thus the function  $V''(x)$  has no more than two roots in the entire range of  $x$ . We conclude that  $V'(x)$  has no more than two turning points. As long as  $d < D$  and  $U < u$ , the only feasible configuration of  $V'(x)$  that is continuous, has no more than two turning points, and satisfies (8)–(11) is shown in Figure 1.

Note from Figure 1 that  $V'(x)$  attains a maximum at  $\hat{x}$ , i.e.,  $V'''(\hat{x}) < 0$ , which implies

$$c_1\lambda_1^3 \exp(\lambda_1\hat{x}) + c_2\lambda_2^3 \exp(\lambda_2\hat{x}) < 0. \quad (\text{A5})$$

But (A3) implies that  $c_1$ ,  $c_2$  have opposite signs and since  $\lambda_1 > 0$ ,  $\lambda_2 < 0$ , (A5) requires

$$c_1 < 0, \quad c_2 > 0. \quad (\text{A6})$$

Note that (A6) implies (A5). Therefore, (A5) will be ignored hereafter.

Similar arguments apply for  $x < 0$ . Since  $V'(x)$  attains a minimum at  $\hat{x} < 0$ , then  $V'''(\hat{x}) > 0$ , i.e.,

$$\{c_1 - (h+p)\lambda_2/(\beta(\lambda_1 - \lambda_2)\lambda_1)\}\lambda_1^3 \exp(\lambda_1\hat{x}) + \{c_2 + (h+p)\lambda_1/(\beta(\lambda_1 - \lambda_2)\lambda_2)\}\lambda_2^3 \exp(\lambda_2\hat{x}) > 0. \quad (\text{A7})$$

But (A4) and (A7) imply

$$c_1 - (h+p)\lambda_2/(\beta(\lambda_1 - \lambda_2)\lambda_1) > 0, \quad c_2 + (h+p)\lambda_1/(\beta(\lambda_1 - \lambda_2)\lambda_2) < 0. \quad (\text{A8})$$

Note that (A8) implies (A7). Therefore, (A7) will be ignored hereafter.

With the aid of (A6) we rewrite (A3) as

$$c_1\lambda_1^2 + c_2\lambda_2^2 \geq 0. \tag{A9}$$

We now show that (A8) and (A9) imply (A4). From (A9)

$$\{c_1 - (h+p)\lambda_2/(\beta(\lambda_1 - \lambda_2)\lambda_1)\}\lambda_1^2 + \{c_2 + (h+p)\lambda_1/\beta(\lambda_1 - \lambda_2)\lambda_2\}\lambda_2^2 \geq 0$$

and with the aid of (A8) we obtain (A4).

To summarize, so far we have shown that (A6), (A8), and (A9) are necessary conditions for Figure 1 to obtain. The space from which the parameter pair  $(c_1, c_2)$  may be chosen, consistent with these constraints, is the shaded area in Figure 3.

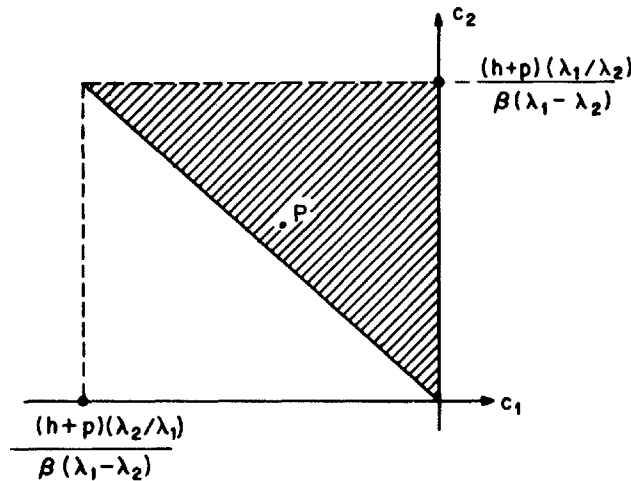


Figure 3. Permissible choices of  $(c_1, c_2)$ .

We now return to Figure 1. There exist parameters  $U, u$ , where  $U < u$  satisfying  $V'(U) = V'(u) = k^-$  if and only if  $V'(\hat{x}) > k^-$ , where  $\hat{x} > 0$  is the value at which the function  $V'(x)$  attains its maximum. We may write condition  $V'(\hat{x}) > k^-$  as  $c_1 \geq -Bc_2^{\lambda_1/\lambda_2}$ , where

$$B \equiv \{[(\lambda_1 - \lambda_2)/(h/\beta - k^-)]^{(\lambda_1 - \lambda_2)/\lambda_2} (-\lambda_2/\lambda_1)^{(\lambda_1 + \lambda_2)/\lambda_2}\} > 0.$$

Consider the function  $\hat{c}_1(c_2) = -Bc_2^{\lambda_1/\lambda_2}$ . Note that  $d\hat{c}_1/dc_2 > 0$ ,  $d^2\hat{c}_1/dc_2^2 < 0$ ; as  $c_2 \rightarrow 0^+$ ,  $\hat{c}_1 \rightarrow -\infty$ ; as  $c_1 \rightarrow \infty$ ,  $\hat{c}_1 \rightarrow 0^-$ . The function  $\hat{c}_1$  is plotted in Figure 4. To summarize, if  $c_1 \geq \hat{c}_1(c_2)$  then  $V'(\hat{x}) > k^-$  and there exist parameters  $U < u$  such that conditions  $V'(U) = V'(u) = k$  are satisfied.

We consider next the requirement that there exist parameters  $d, D$  where  $d < D$ , such that  $V'(d) = V'(D) = -k^+$ . This requirement is met if and only if  $V'(\hat{x}) < -k^+$ , where  $\hat{x} < 0$  is the value at which the function  $V'(x)$  attains its minimum. We may simplify condition  $V'(\hat{x}) < -k^+$  as  $c_1 < \hat{\hat{c}}_1(c_2)$

where

$$\hat{c}_1(c_2) \equiv (h+p)\lambda_2/\lambda_1\beta(\lambda_1-\lambda_2) + [(\lambda_1-\lambda_2)/(p/\beta-k^+)]^{(\lambda_1-\lambda_2)/\lambda_2} \cdot (-\lambda_2/\lambda_1)^{(\lambda_1+\lambda_2)/\lambda_2} \{-c_2 - (h+p)\lambda_1/\lambda_2\beta(\lambda_1-\lambda_2)\}^{\lambda_1/\lambda_2}.$$

Note that as  $c_2 \rightarrow -(h+p)\lambda_1/\lambda_2\beta(\lambda_1-\lambda_2)$ ,  $\hat{c}_1 \rightarrow \infty$ ; as  $c_2 \rightarrow -\infty$ ,  $\hat{c}_1 \rightarrow (h+p)\lambda_2/\lambda_1\beta(\lambda_1-\lambda_2)$ . The function  $\hat{c}_1(c_2)$  is plotted in Figure 4. To summarize, if  $c_1 < \hat{c}_1(c_2)$ , then  $V'(\hat{c}) < -k^+$  and there exist parameters  $d < D$  such that conditions  $V'(d) = V'(D) = -k^+$  are satisfied.

We now establish that the curves  $\hat{c}_1(c_2)$  and  $\hat{c}_1(c_2)$  must intersect at least once in the shaded region of Figure 3. Consider the point

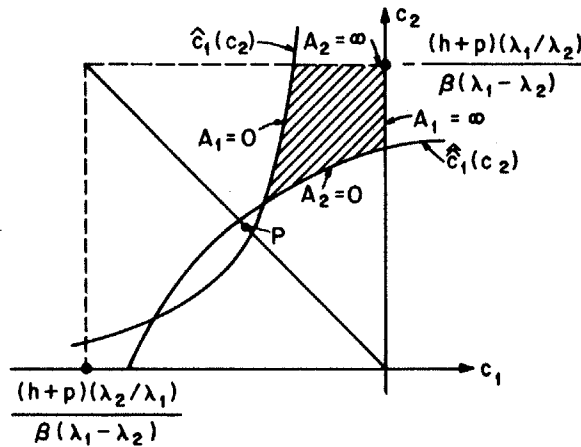


Figure 4. The functions of  $\hat{c}_1(c_2)$  and  $\hat{c}_1(c_2)$ .

$P = (h\lambda_2/(\lambda_1\beta(\lambda_1-\lambda_2)), -h\lambda_1/(\lambda_2\beta(\lambda_1-\lambda_2)))$ . Since  $\lambda_1^2 h\lambda_2/(\lambda_1\beta(\lambda_1-\lambda_2)) - \lambda_2^2 h\lambda_1/(\lambda_2\beta(\lambda_1-\lambda_2)) = 0$ , point  $P$  satisfies (A9) as an equality. Also,

$$\hat{c}_1(-h\lambda_1/(\lambda_2\beta(\lambda_1-\lambda_2)))/(h\lambda_2/(\lambda_1\beta(\lambda_1-\lambda_2))) = (h/(h-k^-\beta))^{(\lambda_1-\lambda_2)/\lambda_2} < 1$$

since  $h/(h-k^-\beta) > 1$  and  $(\lambda_1-\lambda_2)/\lambda_2 < 0$ . Therefore, the point  $P$  lies to the left and above curve  $\hat{c}_1(c_2)$  (see Figure 4). Consider next at point  $P$  the difference

$$\begin{aligned} \hat{c}_1(c_2) - c_1 &= (h+p)\lambda_2/(\lambda_1\beta(\lambda_1-\lambda_2)) \\ &+ [(\lambda_1-\lambda_2)/(p/\beta-k^+)]^{(\lambda_1-\lambda_2)/\lambda_2} (-\lambda_2/\lambda_1)^{(\lambda_1+\lambda_2)/\lambda_2} \\ &\cdot \{h\lambda_1/(\lambda_2\beta(\lambda_1-\lambda_2)) - (h+p)\lambda_1/(\lambda_2\beta(\lambda_1-\lambda_2))\}^{\lambda_1/\lambda_2} \\ &- h\lambda_2/(\lambda_1\beta(\lambda_1-\lambda_2)) \\ &< p\lambda_2/(\lambda_1\beta(\lambda_1+\lambda_2)) \\ &+ [(\lambda_1-\lambda_2)/(p/\beta)]^{(\lambda_1-\lambda_2)/\lambda_2} (-\lambda_2/\lambda_1)^{(\lambda_1+\lambda_2)/\lambda_2} \\ &\cdot [(p/\beta)/(\lambda_1-\lambda_2)]^{\lambda_1/\lambda_2} (-\lambda_1/\lambda_2)^{\lambda_1/\lambda_2} \end{aligned}$$

since  $[(\lambda_1 - \lambda_2)/(p/\beta - k^+)]^{(\lambda_1 - \lambda_2)/\lambda_2} < [(\lambda_1 - \lambda_2)/(p/\beta)]^{(\lambda_1 - \lambda_2)/\lambda_2}$ . Finally,  $c_1(c_2) - c_1 < p\lambda_2/(\lambda_1\beta(\lambda_1 - \lambda_2)) + [(p/\beta)/(\lambda_1 - \lambda_2)](-\lambda_2/\lambda_1) = 0$ . Therefore, point  $P$  lies to the right and below the curve  $c_1(c_2)$  (see Figure 4). We have established that point  $P$ , which lies on the boundary of the shaded region of Figure 3, lies to the left and above curve  $\hat{c}_1(c_2)$  and lies to the right and below curve  $c_1(c_2)$ .

The important feature of Figure 4 is that, given the relative positions of the three curves to point  $P$ , the curves  $\hat{c}_1(c_2)$  and  $\hat{\hat{c}}_1(c_2)$  must intersect at least once in the shaded region of Figure 3. Therefore, there exists a region (the shaded region of Figure 4) in which parameters  $d, D, U, u$  can always be found to satisfy constraints (8)–(11). Furthermore, this region is bounded by the curves  $c_1 = \hat{c}_1(c_2)$ ,  $c_1 = \hat{\hat{c}}_1(c_2)$ ,  $c_1 = 0$  and  $c_2 = -(h+p)\lambda_1/(\lambda_2\beta(\lambda_1 - \lambda_2))$ . These results are crucial in the next step of our proof, which is to show that there always exists a pair  $(c_1, c_2)$  in the shaded region of Figure 4, which satisfies the remaining two conditions (6) and (7).

We write  $V(x; c_1, c_2) = V(x)$  to emphasize that  $V(x)$ , as defined in equation (A1), is a function of the parameters  $c_1, c_2$ . Of course,  $c_1, c_2$  will be chosen so that  $V$  satisfies the boundary conditions. Note from Figure 1 that (7) implies that the area underneath the curve  $V'(x)$  and the  $x$ -axis between  $x = U$  and  $x = u$ , less the area  $(u - U)k^-$  of the rectangle, equals  $K^-$  (shaded area). A similar interpretation applies to the area above the curve and below the  $x$ -axis between  $x = d$  and  $x = D$ . For any pair  $(c_1, c_2)$ , define  $A_1(c_1, c_2) = V(u; c_1, c_2) - V(U; c_1, c_2) - (u - U)k^-$  and  $A_2(c_1, c_2) = V(d; c_1, c_2) - V(D; c_1, c_2) - (D - d)k^+$ .

We shall demonstrate that we can always find a feasible pair  $(c_1, c_2)$  to satisfy  $A_1(c_1, c_2) = K^-$ ,  $A_2(c_1, c_2) = K^+$ . When  $c_1 = \hat{c}_1(c_2)$ ,  $\hat{x} = k^-$  and  $A_1(\hat{c}_1(c_2), c_2) = 0$ . Similarly, when  $c_1 = \hat{\hat{c}}_1(c_2)$ ,  $\hat{x} = k^+$  and  $A_2(\hat{\hat{c}}_1(c_2), c_2) = 0$ . As  $c_1 \rightarrow 0^-$ ,  $U$  remains finite,  $u \rightarrow \infty$ ,  $V'(\hat{x}) \rightarrow \infty$  and therefore  $A_1(0, c_2) \rightarrow \infty$ . As  $c_2 + (h+p)\lambda_1/(\lambda_2\beta(\lambda_1 - \lambda_2)) \rightarrow 0^-$ ,  $D$  remains finite,  $d \rightarrow -\infty$ ,  $V'(\hat{x}) \rightarrow -\infty$  and therefore  $A_2(c_1, -(h+p)\lambda_1/(\lambda_2\beta(\lambda_1 - \lambda_2))) \rightarrow \infty$ . Since  $V(x; c_1, c_2)$  is continuous in  $x, c_1, c_2$ , it follows that  $A_1(c_1, c_2)$  and  $A_2(c_1, c_2)$  are continuous in  $(c_1, c_2)$ . By the intermediate value theorem, there exists a feasible pair  $(c_1, c_2)$  (i.e., a pair that lies in the feasible region of Figure 4) that satisfies  $A_1 = K^-$ ,  $A_2 = K^+$ . Therefore, we have established that there always exist parameters  $c_1, c_2, d, D, U, u$  to satisfy (6)–(11).

We finally prove that the remaining parts of (4), namely  $V(x) > 0$  and  $V''(x) \in L^2(R^1)$ , always hold. It will suffice to show that  $V(x) > 0$  at the minimum value of  $V(x)$ . Rearranging (12) and bearing in mind that  $V'(x) = 0$  at the minimum, we obtain  $\beta V(x) = \frac{1}{2}\sigma^2 V''(x) + C(x) > 0$  since  $V''(x) > 0$ ,  $C(x) \geq 0$ . Lastly, note that  $V''(x)$  is square integrable since it is everywhere finite and vanishes off a bounded interval.

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