

# The Predictability of Returns with Regime Shifts in Consumption and Dividend Growth\*

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## Abstract

We model consumption and dividend growth as different processes across two latent regimes. We estimate the equilibrium model over 1930-2009 and show that the second regime is associated with recessions, market downturns, higher risk premia, lower consumption and dividend growth, higher volatility of returns and growth rates, and lower market-wide price-dividend ratio. The model performs better at in-sample forecasting and significantly better at out-of-sample prediction of the equity, size, and value premia and consumption and dividend growth rates and their variances than the price-dividend ratio and risk free rate do. The calibrated model replicates several features of the data.

*Keywords:* Regime Shifts, Return Predictability, Consumption Growth Predictability, Dividend Growth Predictability, Cross-Section of Returns, Equity Premium, Size Premium, Value Premium.

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# 1 Introduction

The predictability of the aggregate stock market return, dividend growth, and consumption growth have for long been the subject of both theoretical and empirical research in economics and finance. In this paper, we present evidence that the aggregate stock market return, dividend growth, and consumption growth, and their variances are predictable both in sample and out of sample. The key insight is that the aggregate consumption and dividend growth processes differ across (at least) two latent economic regimes. We estimate the equilibrium model and identify the probability that the economy is in one of two economic regimes as a non linear function of two financial variables, the short term risk free rate and the market-wide price-dividend ratio. The regimes are related to the business cycle: the probability of a recession in a year is 44.4% (8.2%), if the probability of being in the second (first) regime at the beginning of the year exceeds 50%. The regimes are also related to the major stock market downturns, as identified in Barro and Ursua (2009): the probability of a stock market downturn in a year is 44.4% (8.2%), if the probability of being in the second (first) regime at the beginning of the year exceeds 50%.

Attempts to predict the aggregate stock market return have a long history in economics going back to as early as 1920 when Dow (1920) explored the role of dividend ratios in predicting the market return. Over the last three decades, the academic literature has explored numerous macroeconomic and financial variables as potential predictors of the market return and equity premium. The price-dividend ratio has received extensive scrutiny as a predictive variable because, as a mathematical identity, all variation in the price-dividend ratio must be accounted for by changing expectations on future returns and/or future dividend growth (Campbell and Shiller (1988)).<sup>1</sup> Welch and Goyal (2008) review this literature and undertake a comprehensive study of the in-sample and out-of-sample performance of the price-dividend ratio and other variables in predicting the equity premium. They conclude that “by and large these models have predicted poorly both in sample and out of sample for 30 years now; these models seem unstable, as diagnosed by their out-of-sample predictions and other statistics; and these models would not have helped an investor with access only to available information to profitably time the market.”

The benchmark in the consumption growth literature is Hall’s (1978) demonstration that, under rational expectations and time- and state-separable preferences, the marginal utility of consumption is unpredictable. The statistical difficulty in distinguishing between an i.i.d. process from one with a small but persistent predictable component has led to considerable controversy regarding the presence of a predictable component in the aggregate consumption growth rate. The debate on consumption growth predictability remains open.

In this paper, we shed light on this debate by arguing that there exist (at least) two latent economic regimes. The aggregate consumption and dividend growth rates

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<sup>1</sup>See also Ang and Bekaert (2007), van Binsbergen and Koijen (2010), Boudoukh Richardson, and Whitelaw (2008), Campbell and Thompson (2008), Cochrane (2008), Fama and French (1988), Kelly and Pruitt (2010), and Lettau and Van Nieuwerburgh (2008).

have a predictable component that has persistence and volatility that are different across regimes. We identify the regimes in the context of a dynamic equilibrium asset pricing model with two regimes. The probability that the economy is in the first regime is obtained as a *non linear* function of the market-wide price-dividend ratio and short term risk free rate, with parameters estimated from the Euler equations of the market return, risk free rate, and the cross-section of size- and book-to-market-equity-sorted portfolio returns, plus unconditional moments of the consumption and dividend processes. Furthermore, this non-linearity cannot be captured by simple nonlinear functions like a quadratic function of the market-wide price-dividend ratio and risk free rate.

In the model, a state variable  $x_t$  that simultaneously drives the conditional means of the aggregate consumption and dividend growth rates reverts to its unconditional mean with a process that differs across two regimes. Based on his information set, the consumer observes  $x_t$  and also calculates the probability,  $p_t$ , that the economy is in the first regime. The conditional means of the aggregate consumption and dividend growth rates are affine functions of the two state variables  $(x_t, p_t)$ . The market-wide log price-dividend ratio and risk free rate are approximately affine functions of  $(x_t, p_t)$  and their product, thereby rendering the (potentially latent) state variables and the expected return of each asset class known nonlinear functions of the price-dividend ratio and risk free rate. The model parameters are estimated from the Euler equations of the market return, risk free rate, and the cross-section of size and book-to-market-equity sorted portfolio returns plus unconditional moments of the consumption and dividend processes.

A calibrated version of the model replicates several features of observed macroeconomic and financial data over the sample period 1930 – 2009. The model matches the unconditional means and volatilities of the aggregate consumption and dividend growth rates, the market-wide price-dividend ratio, market return, and risk free rate. The model understates the volatility of the risk free rate which we attribute to the lack of a good proxy for the real risk free rate. The model also implies size and value premia comparable to those observed in the data. Finally, it implies higher mean and lower volatility of the consumption and dividend growth rates in the first regime compared to the second one, higher mean and volatility of the market return and the equity, size, and value premia in the second regime, and higher mean of the market-wide price-dividend ratio in the first regime. These features are consistent with the data.

We also show that the model has superior in-sample forecasting performance for the equity premium and its variance relative to a linear forecasting model with the market-wide price-dividend ratio and risk free rate as predictive variables. Moreover, unlike linear forecasting regressions with the price-dividend ratio and risk free rate as predictive variables, the model-implied state variables have robust forecasting performance across subperiods.

While most of the predictability literature focuses on predicting the aggregate US stock market return and equity premium, the literature on the time series forecastability of the cross-section of size and book-to-market-equity sorted portfolio returns is scant.<sup>2</sup> Forecastability of the cross-section of returns is important for at least two

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<sup>2</sup>There are two notable exceptions. First, Baker and Wurgler (2006) find that the cross-section of

reasons. First, the historical size premium (9.4%) and value premium (7.3%) are of the same order of magnitude as the equity premium (7.9%), based on arithmetic annual returns. Therefore, the predictability of these premia is important in active portfolio management.<sup>3</sup> Second, it is also important in providing an alternative channel to examine the empirical plausibility of a given set of state variables that purport to explain the cross-section of returns. We show that the model has superior forecasting performance for the size and value premia relative to the linear forecasting model; furthermore this performance is robust across subperiods.

We demonstrate that our model retains its predictive power out of sample. The model-implied state variables give an out-of-sample  $R^2$  of 5.2%, 22.6%, and 0.0%, respectively, for the equity, size, and value premia over the period 1976 – 2009. When used as predictive variables in a linear predictive model, the price-dividend ratio and risk free rate have poor predictive performance with out-of-sample  $R^2$  of  $-2.6\%$ ,  $-6.8\%$ , and  $-11.9\%$ , respectively, for the equity, size, and value premia.

Finally, we show that the model-implied state variables have strong forecasting power for the aggregate consumption and dividend growth rates and their variances. In-sample forecasting regressions for the consumption and dividend growth rates give statistically significant coefficients on the state variables and  $\bar{R}^2$  8.0% and 11.7%, respectively.<sup>4</sup> The corresponding  $\bar{R}^2$  are 12.7% and 15.5%, respectively, for the conditional variances of the growth rates. The model-implied state variables give an out-of-sample  $R^2$  of 0.7% and 4.4%, respectively, for the consumption and dividend growth rates over the period 1976 – 2009. The price-dividend ratio and risk free rate perform poorly at predicting the consumption and dividend growth rates out of sample, giving large negative  $R^2$  of  $-123.2\%$  and  $-60.3\%$ , respectively. These results provide strong support for the risk channels highlighted in the model and the precise mechanism by which they drive the dynamics of the consumption and dividend growth processes.

Our paper is related to equilibrium models of exchange economies by Bansal and Shaliastovich (2011), Bansal and Yaron (2004), Drechsler (2009), Hansen, Heaton and Li (2008), Lettau and Ludvigson (2001), and Menzly, Santos, and Veronesi (2004) with implications on forecasting the market return and dividend growth. It is closely related to Johannes, Lochstoer, and Mou (2011) who establish that there are at least two or three economic regimes of the consumption and dividend growth processes driving asset prices. Whereas our model only allows for learning about the probability of being in a regime at any point in time, their model also allows for learning about the number of regimes and the model parameters in each regime. In a different context, Constantinides, Jackwerth, and Savov (2011) highlight the importance of regime shifts by finding that a pricing factor that tracks jumps in the volatility of the market return partly explains the cross-section of index option returns and equity returns.

Our paper is also related to Lettau and Van Nieuwerburgh (2008), Pastor and Stambaugh (2001), and Paye and Timmermann (2006) who find evidence of structural

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future stock returns is conditional on beginning-of-period proxies for sentiment. Second, the January dummy has strong predictive power for size and book-to-market-equity sorted portfolio returns.

<sup>3</sup>Whereas the average size premium mostly vanished in the latter part of the twentieth century, the forecastability of the conditional one-year-ahead size premium is of independent interest.

<sup>4</sup>Throughout the paper,  $\bar{R}^2$  denotes adjusted- $R^2$ .

breaks and argue that allowance for these breaks has important implications for return predictability. Finally, our paper is related to Brandt and Kang (2004), van Binsbergen and Koijen (2010), Kelly and Pruitt (2010), and Pastor and Stambaugh (2009) who focus on predictability using reduced form models. Whereas these papers rely on filtering techniques, we avoid using filtering techniques by arguing that, under the model assumptions, the (potentially latent) state variables and the expected return of each asset class are known functions of observable financial variables like the price-dividend ratio and risk free rate. Constantinides and Ghosh (2011) earlier applied a similar inversion methodology to extract latent state variables in the context of the Bansal and Yaron (2004) long run risks model and its cointegrated extension.

The paper is organized as follows. In Section 2, we present the regime shifts model. We express the price-dividend ratio, risk free rate, expected equity premium, and expected consumption and dividend growth rates as functions of the state variables ( $x_t$ ,  $p_t$ ). In Section 3, we discuss the data. In Section 4, we estimate the model parameters with *GMM* from the set of the Euler equations for the market return, risk free rate, and portfolios of "Small", "Large", "Growth" and "Value" stocks, and the unconditional moments of the consumption and dividend growth processes. We also calibrate the model and compare the model-generated moments of returns, consumption and dividend growth, and market-wide price-dividend ratio with their sample counterparts.

Using the point estimates of the model parameters, we invert the expressions for the price-dividend ratio and risk free rate as functions of the state variables and express the state variables as functions of the price-dividend ratio and risk free rate. Armed with the time series of the state variables, we address the questions raised in this paper. Section 5 provides an economic interpretation of the two regimes. In Section 6, we present evidence on the in-sample and out-of-sample predictability of the equity, size, and value premia. In Section 7, we present evidence on the in-sample and out-of-sample predictability of the aggregate consumption and dividend growth rates. In Section 8, we present evidence on the predictability of the variance of the market return and the growth rates of consumption and dividends. Section 9 concludes. The Appendix contains the derivation of the main results.

## 2 The Model and Implications for Predictability

We present the regime shifts model and its implications for the predictability of the equity, size, and value premia and consumption and dividend growth.

### 2.1 Model

The model stipulates that the state variable,  $x_t$ , that simultaneously drives the conditional means of the aggregate consumption and dividend growth rates reverts to its

unconditional mean with a process that differs across two regimes:

$$x_{t+1} = \rho_{s_{t+1}}x_t + \varphi_e\sigma_{s_{t+1}}e_{t+1}, \quad (1)$$

$$\Delta c_{t+1} = \mu + x_t + \sigma_{s_{t+1}}\eta_{t+1}, \quad (2)$$

$$\Delta d_{t+1} = \mu_d + \phi x_t + \varphi_d\sigma_{s_{t+1}}u_{t+1}, \quad (3)$$

where  $c_{t+1}$  is the logarithm of the aggregate consumption level;  $d_{t+1}$  is the logarithm of the aggregate stock market dividends; and  $s_t = 1, 2$  is a variable that denotes the economic regime. The persistence parameter,  $\rho_{s_t}$ , of the state variable  $x_t$  and the level of its volatility,  $\sigma_{s_t}$ , are generally different in the two regimes. The shocks  $e_{t+1}$ ,  $\eta_{t+1}$ , and  $u_{t+1}$  are assumed to be distributed with mean 0 and variance 1 and independent of the past.

Given his information set,  $F(t)$ , the representative consumer observes  $x_t$  and calculates his probability,  $p_t$ , at time  $t$  of being in regime  $s_t = 1$ :

$$p_t \equiv Prob(s_t = 1|F(t)) \quad (4)$$

We do not take a stand on the content of the information set,  $F(t)$ . In one extreme case, it may be limited to the history of consumption, dividends, and past realizations of  $x$ . In the other extreme case, it may include all publicly available information. Furthermore, we do not take a stand on the optimality of the filter that the consumer applies to form his belief,  $p_t$ . The econometrician does not directly observe the state variables,  $p_t$  and  $x_t$ , and, hence, they are latent.

We assume that  $s_t$  follows a Markov process with the following transition probability matrix:

$$\Pi = \begin{pmatrix} \pi_1 & 1 - \pi_2 \\ 1 - \pi_1 & \pi_2 \end{pmatrix}, \quad (5)$$

where  $0 < \pi_i < 1$  for  $i = 1, 2$ . Thus, the consumer's probability of being in regime  $s_{t+1} = 1$  at time  $t + 1$ , given his information set,  $F(t)$ , is

$$Prob(s_{t+1} = 1|F(t)) = \pi_1 p_t + (1 - \pi_2)(1 - p_t) \equiv f(p_t). \quad (6)$$

Note that  $0 < f(p_t) < 1$  for all  $p_t$ ,  $0 \leq p_t \leq 1$ .

Once the consumer updates his information set at time  $t + 1$ , his probability of being in regime  $s_{t+1} = 1$  at time  $t + 1$  is  $p_{t+1} \equiv Prob(s_{t+1} = 1|F(t + 1))$ . We assume that the consumer's expectations are unbiased in that

$$p_{t+1} = f(p_t) + \varepsilon_{t+1}, \quad (7)$$

where  $E[\varepsilon_{t+1}|F(t)] = 0$ . We also assume that the domain of  $\varepsilon_{t+1}$  is such that  $0 \leq p_{t+1} \leq 1$ .

We make the following assumptions regarding the shocks  $\eta_{t+1}$ ,  $u_{t+1}$ ,  $e_{t+1}$ , and  $\varepsilon_{t+1}$ :

$$E[y_{t+1}|F(t), s_{t+1} = 1] = E[y_{t+1}|s_{t+1} = 1] \equiv y(1), \text{ a constant}, \quad (8)$$

where  $y = \eta, u, e$ , and  $\varepsilon$ ;

$$E [y_{t+1}w_{t+1}|F(t), s_{t+1} = 1] = E [y_{t+1}w_{t+1}|F(t)] \equiv \sigma_{y,w}, \text{ a constant}, \quad (9)$$

where  $y, w = \eta, u, e, \varepsilon$ , and  $y \neq w$ ; and

$$E [y_{t+1}^2|F(t), s_{t+1} = 1] = E [y_{t+1}^2] = 1, \quad (10)$$

where  $y = \eta, u$ , and  $e$ .

Equation (8) recognizes that the means of the residuals  $\eta_{t+1}, u_{t+1}, e_{t+1}$ , and  $\varepsilon_{t+1}$ , conditional on the regime at time  $t+1$ , may differ from their unconditional value of zero. In the estimation, we impose the following sufficient (but not necessary) conditions that ensure that  $p_t$  lies in the permissible interval  $[0, 1]$ :

$$\varepsilon(1) \in \begin{cases} \left[ \max \left( -(1 - \pi_2), -\frac{(1 - \pi_1)^2}{1 - \pi_2} \right), 1 - \pi_1 \right] & , \text{ if } \pi_1 + \pi_2 - 1 > 0, \\ \left[ \max \left( -\pi_1, -\frac{(1 - \pi_1)\pi_2}{1 - \pi_2} \right), \min \left( \pi_2, \frac{(1 - \pi_1)\pi_1}{1 - \pi_2} \right) \right] & , \text{ if } \pi_1 + \pi_2 - 1 < 0, \end{cases} \quad (11)$$

(see Appendix A.1 for derivation of this result). Equation (9) recognizes that the residuals  $\eta_{t+1}, u_{t+1}, e_{t+1}$ , and  $\varepsilon_{t+1}$  may be correlated. Finally, equation (10) limits the number of parameters to be estimated by setting the second moments of the residuals  $\eta_{t+1}, u_{t+1}$ , and  $e_{t+1}$ , conditional on the regime at time  $t+1$ , equal to their unconditional value of one.

We assume that the consumer has the version of Kreps and Porteus (1978) preferences adopted by Epstein and Zin (1989) and Weil (1989). These preferences allow for a separation between the coefficient of risk aversion and the elasticity of intertemporal substitution. The utility function is defined recursively as

$$V_t = \left[ (1 - \delta)C_t^{\frac{1-\gamma}{\theta}} + \delta (E [V_{t+1}^{1-\gamma}|F(t)])^{\frac{1}{\theta}} \right]^{\frac{\theta}{1-\gamma}}, \quad (12)$$

where  $\delta$  denotes the subjective discount factor,  $\gamma > 0$  is the coefficient of risk aversion,  $\theta \equiv \frac{1-\gamma}{1-\frac{1}{\psi}}$ , and  $\psi > 0$  is the elasticity of intertemporal substitution. Note that the sign of  $\theta$  depends on the relative magnitudes of  $\gamma$  and  $\psi$ . The standard time-separable power utility is obtained as a special case when  $\theta = 1$ , i.e.  $\gamma = \frac{1}{\psi}$ .

For this specification of preferences, Epstein and Zin (1989) and Weil (1989) show that, for any asset  $j$ , the first-order conditions of the consumer's utility maximization yield the following Euler equations,

$$E [\exp(m_{t+1} + r_{j,t+1})|F(t)] = 1, \quad (13)$$

$$m_{t+1} = \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1)r_{c,t+1}, \quad (14)$$

where  $m_{t+1}$  is the natural logarithm of the intertemporal marginal rate of substitution,  $r_{j,t+1}$  is the continuously compounded return on asset  $j$ , and  $r_{c,t+1}$  is the unobservable continuously compounded return on an asset that delivers aggregate consumption as its dividend each period.

We rely on log-linear approximations for the return on the consumption claim,  $r_{c,t+1}$ ,

and that on the market portfolio (the observable return on the aggregate dividend claim),  $r_{m,t+1}$ , as in Campbell and Shiller (1988),

$$r_{c,t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + \Delta c_{t+1}, \quad (15)$$

$$r_{m,t+1} = \kappa_{0,m} + \kappa_{1,m} z_{m,t+1} - z_{m,t} + \Delta d_{t+1}, \quad (16)$$

where  $z_t$  is the log price-consumption ratio and  $z_{m,t}$  the log price-dividend ratio. In equation (15),  $\kappa_1 = \frac{e^{\bar{z}}}{1+e^{\bar{z}}}$  and  $\kappa_0 = \log(1 + e^{\bar{z}}) - \kappa_1 \bar{z}$  are log-linearization constants, where  $\bar{z}$  denotes the long run mean of the log price-consumption ratio. Similarly, in equation (16),  $\kappa_{1,m} = \frac{e^{\bar{z}_m}}{1+e^{\bar{z}_m}}$  and  $\kappa_{0,m} = \log(1 + e^{\bar{z}_m}) - \kappa_{1,m} \bar{z}_m$ , where  $\bar{z}_m$  denotes the long run mean of the log price-dividend ratio.

Note that the current model specification involves two state variables,  $x_t$  and  $p_t$ . We conjecture the following approximate expressions for the log price-consumption ratio, log price-dividend ratio and log risk free rate and derive expressions for their parameters in Appendices A.2.1, A.2.2, and A.2.3, respectively:

$$z_t = p_t [A_0(1) + A_1(1)x_t] + (1 - p_t) [A_0(2) + A_1(2)x_t], \quad (17)$$

$$z_{m,t} = p_t [A_{0,m}(1) + A_{1,m}(1)x_t] + (1 - p_t) [A_{0,m}(2) + A_{1,m}(2)x_t], \quad (18)$$

$$r_{f,t} = A_{0,f} + A_{1,f}x_t + A_{2,f}p_t + A_{3,f}p_t x_t. \quad (19)$$

The pricing kernel is a function of the two latent state variables,  $x_t$  and  $p_t$ , and their lags, in addition to consumption growth (see Appendix A.2.4 for derivation). We invert the two non-linear equations (18) and (19) to express the latent state variables,  $x_t$  and  $p_t$ , as functions of the observables,  $z_{m,t}$  and  $r_{f,t}$ .<sup>5</sup> This gives a quadratic equation for  $p_t$ , with coefficients that depend on  $z_{m,t}$  and  $r_{f,t}$ , and the time-series and preference parameters. We set  $p_t$  equal to the bigger root of the quadratic equation.<sup>6</sup> Finally, we obtain  $x_t$  which is given as a function of  $p_t$ . This procedure gives a pricing kernel entirely

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<sup>5</sup>The model relates the two latent state variables not only to the market-wide price-dividend ratio and risk free rate but also to the conditional volatility of the market return, the price-dividend ratios of other portfolios, and the term premia of real default-free bonds. In Constantinides and Ghosh (2011) we experiment with extracting the state variables from the simultaneously observed market-wide price-dividend ratio, risk free rate, and the conditional volatility of the market return. We find that the results are similar to extracting the state variables from the price-dividend ratio and risk free rate alone. Therefore, we do not pursue this approach here. We do not use the price-dividend ratio of some other portfolio, for example the value portfolio, as an observable variable. In doing so, we would be introducing a maintained hypothesis on the dividend growth process of the value portfolio and also would need to estimate the parameters of this process. Finally, we do not use the term premium on, say, a 5-year Treasury bond as an observable variable as the conversion of this premium to the term premium on a hypothetical 5-year real bond would introduce a maintained hypothesis on the inflation process and also require estimation of the parameters of this process.

<sup>6</sup>We justify this choice via simulation. Specifically, we calibrate the model, generate a time series of  $x_t$  and  $p_t$  of length 1 million years, and then generate the time series of  $z_{m,t}$  and  $r_{f,t}$ . Each year, we obtain the quadratic equation for  $p_t$ , with coefficients that depend on the generated values of  $z_{m,t}$  and  $r_{f,t}$ . We invariably find that the known value of  $p_t$  that generated the time series equals the bigger root of the quadratic equation. In the data, due to parameter estimation error, we infrequently find that the bigger root is slightly greater than one and, in this case, set it equal to 0.99; we also infrequently find that the bigger root is slightly smaller than zero and set it equal to 0.01.

in terms of observables. We use this expression for the pricing kernel to estimate the parameters of the model using a cross-section of asset returns in Section 4. Figures 1 and 2 display  $p$  and  $x$ , respectively, as highly non-linear functions of  $z_m$  and  $r_f$  using the point estimates of the model parameters in Section 4.<sup>7</sup>

— Figures 1 and 2 about here —

## 2.2 Predictive Implications for Returns and Growth Rates

Equations (3), (16), and (18) imply that the expected market return is given by:

$$E[r_{m,t+1}|F(t)] = B_0 + B_1x_t + B_2p_t + B_3p_tx_t. \quad (20)$$

Hence, from Equations (20) and (19), the expected equity premium is given by:

$$\begin{aligned} E[(r_{m,t+1} - r_{f,t})|F(t)] &= E_0 + E_1x_t + E_2p_t + E_3p_tx_t, \\ E_i &= B_i - A_{i,f}, \quad i = 0, 1, \dots, 3. \end{aligned} \quad (21)$$

Therefore, the model generates time-varying expected market return and equity premium. The coefficients  $\{B_i, E_i\}_{i=0}^3$  are known functions of the underlying time-series and preference parameters (see Appendix A.2.5 for derivation). Under the assumption that the dividend growth processes of the "Small", "Large", "Growth" and "Value" portfolios are similar to that for the market, the expected returns on these portfolios can also be shown to be affine functions of the state variables,  $x$  and  $p$ , and their product.

The regime shifts model also has implications for the predictability of the aggregate consumption and dividend growth rates (see Appendix A.2.6 for derivation). The time series specification of the model implies that the expected consumption growth rate is given by

$$E(\Delta c_{t+1}|F(t)) = \mu + (\sigma_1 - \sigma_2)\eta(1)(1 - \pi_2) + x_t + (\sigma_1 - \sigma_2)\eta(1)(\pi_1 + \pi_2 - 1)p_t, \quad (22)$$

and the expected dividend growth rate is given by

$$E(\Delta d_{t+1}|F(t)) = \mu_d + \varphi_d(\sigma_1 - \sigma_2)u(1)(1 - \pi_2) + \phi x_t + \varphi_d(\sigma_1 - \sigma_2)u(1)(\pi_1 + \pi_2 - 1)p_t, \quad (23)$$

both linear functions of the state variables,  $x_t$  and  $p_t$ .

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<sup>7</sup>At the point estimates of the parameters, we obtain the following quadratic equation for  $p_t$  at time  $t$ :  $ap_t^2 + b_t p_t + h_t = 0$ , where  $a = -2.3 \times 10^{-14}$ ,  $b_t = 8.94r_{f,t} + 0.24$ , and  $h_t = 5.63r_{f,t} - 1.11z_{m,t} + 2.67$ , and  $x_t = \frac{r_{f,t} - 0.001 - (2.6 \times 10^{-15})p_t}{1.11}$ .

Finally, the model implies that the conditional variances of the aggregate consumption and dividend growth rates are functions of the probability,  $p_t$ , alone:

$$\text{Var}(\Delta c_{t+1}|F(t)) = a_c + b_c p_t, \quad (24)$$

and

$$\text{Var}(\Delta d_{t+1}|F(t)) = a_d + b_d p_t. \quad (25)$$

### 3 Data

Throughout our investigation, we use annual data over the sample period 1930 – 2009. The asset menu consists of the market return, risk free rate, and portfolios of "Value", "Growth", "Small" capitalization, and "Large" capitalization stocks. Our market proxy is the Centre for Research in Security Prices (CRSP) value-weighted index of all stocks on the NYSE, AMEX, and NASDAQ. The proxy for the annual real risk free rate is the inflation-adjusted rolled-over return of one-month Treasury Bills from Ibbotson Associates. The equity premium is the difference in average log returns on the market and the risk free rate.

The construction of the size and book-to-market portfolios is as in Fama and French (1993). In particular, for the size sort, all NYSE, AMEX, and NASDAQ stocks are allocated across 10 portfolios according to their market capitalization at the end of June of each year. Value-weighted returns on these portfolios are then computed over the following twelve months. NYSE breakpoints are used in the sort. "Small" and "Large" denote the bottom and top market capitalization deciles, respectively. The size premium is the difference in average returns between the "Small" and "Large" portfolios. Similarly, value-weighted returns are computed for portfolios formed on the basis of BE/ME at the end of June of each year using NYSE breakpoints. The BE used in June of year  $t$  is the book equity for the last fiscal year end in  $t - 1$  and ME is the price times shares outstanding at the end of December of  $t - 1$ . "Growth" and "Value" denote the bottom and top BE/ME deciles, respectively. The value premium is the difference in average returns between the "Value" and "Growth" portfolios. Annual returns for the "Small", "Large", "Growth", and "Value" portfolios are computed by compounding monthly returns within each year. The premia are computed as the difference in the average annual log returns.

Also used in the empirical analysis are the price-dividend ratio and dividend growth rate of the market portfolio. These two time series are computed using the monthly returns with and without dividends on the market portfolio obtained from the CRSP files. The monthly dividend payments within a year are added to obtain the annual aggregate dividend, i.e. we do not reinvest dividends either in T-Bills or in the aggregate stock market. The annual price-dividend ratio is computed as the ratio of the price at the end of each calendar year to the annual aggregate dividends paid out during that year. Finally, the consumption data consists of the per capita personal consumption expenditure on nondurable goods obtained from the Bureau of Economic Analysis. All

nominal quantities are converted to real, using an  $ARMA(1, 1)$  forecast of the annual inflation.

## 4 Parameter Estimation and Interpretation

We estimate the model parameters over the period 1930 – 2009 with GMM on the following set of 23 moment conditions, weighted by the identity matrix: the 18 Euler equations for the risk free rate, market return, and "Small", "Large", "Growth", and "Value" portfolio returns, with the risk free rate and the lagged log price-dividend ratio of the market as instruments; and the five moment restrictions implied by the unconditional means and variances of the aggregate consumption and dividend growth rates and the covariance between consumption and dividend growth rates (see Appendix A.3 for derivation of these moments). The total number of parameters to be estimated is 21: 3 preference parameters ( $\gamma, \psi, \delta$ ); 16 time-series parameters ( $\mu, \mu_d, \phi, \varphi_d, \rho_1, \rho_2, \sigma_1, \sigma_2, \pi_1, \pi_2, \varphi_e, e(1), \eta(1), u(1), \varepsilon(1), \sigma_{\varepsilon,e}$ ); and 2 combinations of all the parameters that appear in the Euler equations.

The estimation of all the parameters and state variables is done simultaneously as follows: (a) take a set of time series and preference parameters; (b) calculate the state variables through the inversion; (c) calculate the stochastic discount factor; (d) calculate the J-stat in the GMM system which consists of the time series restrictions and the Euler equations on the risk free rate, market return, and the return on the Small, Large, Growth, and Value portfolios; (e) take another set of time series and preference parameters and repeat steps (b)-(d); (e) keep repeating step (e) until the J-stat is globally minimized.<sup>8</sup>

The estimation results are reported in Table 1. The first and second rows report the point estimates of the parameters along with the associated standard errors in parentheses.<sup>9</sup> The point estimates of the subjective discount factor (0.976) and risk aversion coefficient (12) are economically sensible. The point estimate of the IES is 0.9 and is slightly smaller than one.

— Table 1 about here —

The parameter estimates of the time-series processes illustrate the presence of (at least) two regimes, with more persistent and less volatile consumption and dividend growth rates in the first regime than in the second one. The persistence parameter of the state variable  $x$  is 0.94 in the first regime (half-life longer than 11 years) and

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<sup>8</sup>This is equivalent to imposing that the model-implied market return (based on the market's dividend growth process that is specified) matches the observed stock market return. Based on the current values of the state variables (and a set of parameter values), the model delivers the conditional expectation of the market return but not the future realization. What is needed in the Euler equation is the future realization and that is what we plug in the Euler equation. If we wish, instead, to calculate the model-implied realization on the market return, we will calculate next period's values of the state variables by observing next period's risk free rate and price-dividend ratio. The approaches are identical given the identity that links market return with its price-dividend ratio and dividend growth.

<sup>9</sup>Standard errors are Newey-West (1987) corrected using 2 lags.

0.60 in the second one (half life of just over one year); and the volatility of  $x$  is 0.5% in the first regime and 3.5% in the second one. The point estimates of the transition probabilities imply that the first regime has a mean duration of 20 years while the second regime has a much shorter duration of just over 6 years.

We gain further insights into the workings of the model by simulating data from a calibrated version of the model. First, we generate a time series of the regime by drawing from the Markov process in equation (5) with transition probabilities  $\pi_1=0.95$  and  $\pi_2=0.85$ , the point estimates in Table 1. Recall that the model does not take a stand on the content of the information set that the consumer uses to form his belief,  $p_t$ . We choose to calibrate the model in the case that the consumer's information set at date  $t$  fully reveals the regime:  $p_t=1$  if  $s_t=1$  and  $p_t=0$  if  $s_t=2$ . This implies that  $\varepsilon(1)=0.068$ , well within one standard error of the point estimate of  $\varepsilon(1)$  in Table 1.<sup>10</sup>

The calibrated values of the parameters are displayed in Table 2. Almost all the parameter values are well within one standard error of their point estimates in Table 1. One exception is the elasticity of intertemporal substitution which we set equal to 1.5 to emphasize the preference for early resolution of uncertainty. Another one is the parameter  $\phi$  which we set equal to 7.5 in order to obtain a better fit for the volatility of the dividend process and returns.

— Table 2 about here —

We generate the time series of the state variable  $x_t$ , consumption growth, and dividend growth by assuming that the distribution of each of the error terms  $\{e_t, \eta_t, u_t\}$  is independent of each other and normal. Specifically, conditional on the economy being in the first regime at date  $t$ ,  $y_t \sim N(y(1), 1)$ ,  $y_t=e_t, \eta_t, u_t$ ; and conditional on the economy being in the second regime at date  $t$ ,  $y_t \sim N\left(-y(1)\frac{1-\pi_2}{1-\pi_1}, 1\right)$ ,  $y_t=e_t, \eta_t, u_t$ .

Table 1 displays the model-implied values along with the 95% confidence intervals (in square brackets) of the moments of consumption and dividend growth rates and returns generated by the calibrated model. The model-implied values of the moments are generated from a single long simulation of 1 million observations. The 95% confidence intervals are obtained as the 2.5% and 97.5% quantiles from 10000 simulations of the same length (80) as the historical time series. The model matches almost perfectly the unconditional mean of the consumption and dividend growth rates and the volatility of dividend growth. It matches reasonably well the first and second moments of the market-wide price-dividend ratio, market return, and risk free rate. An exception is the volatility of the risk free rate which we attribute to the lack of a good proxy for the real risk free rate.

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<sup>10</sup>The probability that  $s_t=1$  and  $s_{t+1}=1$  is  $\frac{1-\pi_2}{2-\pi_1-\pi_2} \times \pi_1$  and  $\varepsilon_{t+1}=1-\pi_1$ . The probability that  $s_t=0$  and  $s_{t+1}=1$  is  $\frac{1-\pi_1}{2-\pi_1-\pi_2} \times (1-\pi_2)$  and  $\varepsilon_{t+1}=\pi_2$ . Therefore,

$$\begin{aligned} \varepsilon(1) &= \frac{1-\pi_2}{2-\pi_1-\pi_2} \times \pi_1 \times (1-\pi_1) + \frac{1-\pi_1}{2-\pi_1-\pi_2} \times (1-\pi_2) \times \pi_2 \\ &= \frac{(1-\pi_1)(1-\pi_2)(\pi_1+\pi_2)}{2-\pi_1-\pi_2} = 0.068 \end{aligned}$$

Finally, the model generates a size premium of 8.3%, close to the 9.4% value in the data, and a value premium of 3.7%, that is within the 95% confidence interval of the value 7.3% in the data.

## 5 Economic Interpretation of the Two Regimes

The regimes are correlated with the business cycle. In Figure 3, we plot the time-series of the probability,  $p_t$ , that the economy is in the first regime over 1931 – 2009. The shaded areas mark recession years, defined here as years with two or more quarters in NBER-designated recession. The correlation between the probability series and a dummy variable that takes the value one in a recession year and zero otherwise is  $-0.42$ . Conditional on lower than 50% probability that the economy is in the first regime ( $p_t < 0.5$ ), the probability of a recession in that year is 44.4%; conditional on higher than 50% probability that the economy is in the first regime ( $p_t > 0.5$ ), the probability of a recession in that year is 8.2%. The association of the second regime with recessions is consistent with our earlier finding that the second regime is associated with lower persistence and higher volatility of the predictable component of consumption growth and has a shorter duration compared to the first regime.

— Figure 3 about here —

The regimes are also correlated with major stock market downturns. In Figure 3, the vertical dashed lines mark major stock market downturns, as defined in Barro and Ursua (2009). The correlation between the probability series and a dummy variable that takes the value one in years with a stock market downturn and zero otherwise is  $-0.40$ . Conditional on lower than 50% probability that the economy is in the first regime, the probability of a stock market downturn in that year is 44.4%; conditional on higher than 50% probability that the economy is in the first regime, the probability of a stock market downturn in that year is 8.2%. Note that the second regime does double duty by capturing both economic recessions and periods of stock market downturns. This rendition is necessarily imperfect because economic recessions and stock market downturns are related but distinct economic phenomena.<sup>11</sup>

In Table 3, we report the annual sample mean and volatility, along with the associated asymptotic standard errors in parentheses, of the consumption, dividend, and GDP growth rates, the rate of inflation, the market-wide price-dividend ratio, risk free rate, market return, and equity, size, and value premia. In Panel *A*, we present these summary statistics for the 61 years over the period 1931 – 2009 in which the probability that the economy is in the first regime exceeds 50%. In Panel *B*, we present these summary statistics for the 18 years in which the probability that the economy is in the first regime is below 50%. Given the small size of these subsamples, particularly the second one, the standard errors are large and differences in the point estimates across the two regimes are often statistically insignificant. However, consistent with

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<sup>11</sup>The correlation between a dummy variable that takes the value one in a recession year and zero otherwise and a dummy variable that takes the value one in years with a stock market downturn and zero otherwise over the period 1930 – 2008 is 0.45.

the interpretation of the second regime as the regime associated with recessions, we find that the sample mean of consumption and dividend growth are higher in the first regime and their volatilities are higher in the second regime. The same pattern across regimes is reflected in the simulated moments.

The sample mean and volatility of the equity, size, and value premia are higher in the second regime than the first one, consistent with the interpretation of the second regime as being the regime associated with recessions. The same pattern across regimes is reflected in the simulated moments. The sample mean and volatility of the price-dividend ratio are higher in the first regime than the second one; the simulated mean of the price-dividend ratio is higher in the first regime than the second one while the simulated volatility is higher in the second regime than the first one.

The sample mean of the risk free rate is lower in the second regime than the first one while the reverse pattern is observed in the simulated mean, possibly because the Federal Reserve’s policy in recessions is not captured by the model. Finally, both the sample and simulated volatility of the risk free rate are higher in the second regime.

— Table 3 about here —

In Figure 4, we plot the time-series of the second state variable,  $x_t$ , over 1931–2009. Whereas Figure 4 does not provide a direct economic interpretation of the regimes, we show in Section 7 that  $x_t$  strongly forecasts the aggregate consumption and dividend growth rates which are obviously related to the business cycle.

— Figure 4 about here —

## 6 Forecasting the Equity, Size, and Value Premia

We perform forecasting regressions of the equity, size, and value premia on the model state variables and compare the results with those obtained from corresponding regressions of the premia on the market-wide price-dividend ratio and risk free rate. In Section 6.1, we estimate the model parameters over the period 1930–2009, extract the time series of the state variables, and perform in-sample forecasting regressions over the same period. In Section 6.2, we estimate the model parameters over the subperiod 1930–1975, extract the time series of the state variables, and perform in-sample forecasting regressions and out-of-sample predictive regressions over the non-overlapping subperiod 1976–2009. The results provide strong evidence in favor of the model, with the out-of-sample predictive regressions providing the strongest evidence.

### 6.1 In-Sample Forecasting: 1930-2009

The expected equity premium implied by the model is an affine function of the two state variables and their product (equation (21)). We estimate the model parameters over the period 1930–2009 and extract the time series of the state variables, as described in Section 4. We then perform an in-sample forecasting regression of the realized equity premium on the lagged state variables and their product. The results are displayed in

the first row of Table 4, Panel *A*. The coefficient on  $p$  is statistically significant and the  $\bar{R}^2$  is 7.4%. The performance of the model is superior to the forecasting performance of the price-dividend ratio with  $\bar{R}^2$  2.8% (second row); the joint forecasting performance of the price-dividend ratio and risk free rate with  $\bar{R}^2$  3.8% (third row); and the joint forecasting performance of the price-dividend ratio, risk free rate, and their product, with  $\bar{R}^2$  3.4% (fourth row). The regression in the third row is performed to facilitate comparison with the single-regime Bansal and Yaron (2004) model that implies that the expected equity premium is an affine function of the lagged price-dividend ratio and risk free rate. The regression in the fourth row is motivated by the model implication that the two state variables are non-linear functions of the aggregate log price-dividend ratio and risk free rate ((see equations (18) and (19))).

— Table 4 about here —

The sample size premium (9.4%) and value premium (7.3%) provide an alternative channel to examine the empirical plausibility of the model. We perform an in-sample forecasting regression of the realized size premium on the lagged state variables and their product. The results are displayed in the first row of Table 4, Panel *B*. The coefficients on  $p$  and the product,  $xp$ , are strongly statistically significant and the  $\bar{R}^2$  is 14.3%. The performance of the model is superior to the forecasting performance of the price-dividend ratio with  $\bar{R}^2$  0.6% (second row); the joint forecasting performance of the price-dividend ratio and risk free rate with  $\bar{R}^2$  2.8% (third row); and the joint forecasting performance of the price-dividend ratio, risk free rate, and their product, with  $\bar{R}^2$  8.9% (fourth row).<sup>12</sup> Finally, we perform an in-sample forecasting regression of the realized value premium on the lagged state variables and their product. The results are displayed in the first row of Table 4, Panel *C*. The coefficients on  $x$  and  $p$  are strongly statistically significant and the  $\bar{R}^2$  is 4.8%. The performance of the model is superior to the forecasting performance of the price-dividend ratio with  $\bar{R}^2$  -0.7% (second row); the joint forecasting performance of the price-dividend ratio and risk free rate with  $\bar{R}^2$  -2.0% (third row); and the joint forecasting performance of the price-dividend ratio, risk free rate, and their product, with  $\bar{R}^2$  -1.6% (fourth row).

The results are illustrated in Figure 5. Panels *A*, *C*, and *E* display the realized equity, size, and value premia (black solid line), respectively, along with their predicted values from the forecasting regressions implied by the model (green dashed line), and linear forecasting regressions using the log price-dividend ratio as a predictor variable (red dashed line). The time series of the premia predicted by the model line up more closely with the actual realized time series compared to the time series predicted by the price-dividend ratio. Panels *B*, *D*, and *F* display the cumulative squared demeaned

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<sup>12</sup>Since the model implies that the stochastic discount factor is a function of the two state variables  $x$  and  $p$  and their lags, it is reasonable to expect that these state variables should predict all asset returns, including the returns of the Small, Large, Growth, and Value portfolios. If we were to model the fundamental cash flow processes of these portfolios, we would generally be introducing additional state variables that might well improve the predictability. We have not modeled the fundamental cash flow processes of these portfolios because doing so would considerably increase the number of parameters to be estimated.

equity, size, and value premia, respectively, minus the cumulative squared regression residuals from the alternative forecasting regression specifications: the forecasting regression implied by the model (black solid line) and a linear forecasting regression with the log price-dividend ratio as a predictor variable (green dashed line). The figure reveals the superior forecasting performance of the regime shifts model relative to the price-dividend ratio for the equity, size, and value premia.

— Figure 5 about here —

## 6.2 In-Sample Forecasting and out-of-Sample Prediction, 1976-2009

We re-examine the ability of the regime shifts model to forecast in sample over the subperiod 1976 – 2009 for two reasons. First, it facilitates comparison with the extant literature that documents poor in-sample performance of forecasting models over this particular subperiod (see Welch and Goyal (2008)). Second, it allows us to estimate the model parameters over the first subperiod 1930 – 1975 and examine the forecasting performance of the model over the non-overlapping second subperiod 1976 – 2009, thereby eliminating the potential look-ahead bias introduced by estimating the model parameters over the same period over which we forecast the premia.

We also examine the ability of the regime shifts model to predict out of sample over the subperiod 1976 – 2009 and compare our results to the extant literature over the same subperiod (see Welch and Goyal (2008)). At each year  $t$ , starting from 1975, we forecast the premia in the year  $t+1$  as follows. First, we estimate the model parameters over the period 1930 – 1975 and extract the time series of the state variables. This approach is conservative because we do not use all the information in the history from 1930 to time  $t$  in estimating the model parameters. Second, we estimate the coefficients of the lagged values of  $x$ ,  $p$ , and  $xp$  from a regression over the period 1930 to time  $t$  and use these coefficients to forecast the premia at time  $t + 1$ .<sup>13</sup> The out-of-sample performance of these forecasts is evaluated using an out-of-sample  $R^2$  statistic as in Campbell and Thompson (2008) and Welch and Goyal (2008):

$$R_{OOS}^2 = 1 - \frac{MSE_A}{MSE_N}, \quad (26)$$

where  $MSE_A$  denotes the mean-squared prediction error from the predictive regression implied by the model and  $MSE_N$  denotes the mean-squared prediction error of the historical average return. If  $R_{OOS}^2$  is positive, then the predictive regression has lower mean-squared prediction error than the historical average return.

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<sup>13</sup>Campbell and Thompson (2008) point out that the rolling out-of-sample predictive regressions are estimated over short sample periods, particularly at the beginning of the forecast evaluation period, and can, therefore, easily generate perverse results, such as a negative coefficient when theory suggests that the coefficient should be positive. In all of our out-of-sample predictive regressions for the equity premium, we impose two restrictions suggested in Campbell and Thompson (2008): *a*) we set the regression coefficients to zero whenever they have the “wrong” sign (different from the theoretically expected sign obtained from the model), and *b*) we assume that investors rule out a negative equity premium, and set the forecast to zero whenever it is negative.

The in-sample and out-of-sample results on the equity premium are reported in Table 5, Panel A. The first row displays results of a forecasting regression with the state variables and their product as predictive variables. The  $\bar{R}^2$  of the regression is 8.5%. The performance of the model is superior to the forecasting performance of the price-dividend ratio with  $\bar{R}^2$  2.2% (second row); superior to the joint forecasting performance of the price-dividend ratio and risk free rate with  $\bar{R}^2$  -1.0% (third row); and comparable to the joint forecasting performance of the price-dividend ratio, risk free rate, and their product, with  $\bar{R}^2$  9.5% (fourth row). However, the regime shifts model retains its predictive performance out of sample with  $R_{OOS}^2$  5.2%, while the price-dividend ratio, the combined price-dividend ratio and risk free rate, and the combined price-dividend ratio, risk free rate, and their product all have negative  $R_{OOS}^2$ .

— Table 5 about here —

Panel B displays results for the size premium. The first row shows that the in-sample forecasting regression with  $x$ ,  $p$ , and their product as predictive variables yields an  $\bar{R}^2$  of 25.5%. The model retains its predictive performance out of sample with  $R_{OOS}^2$  22.6%. By contrast, the price-dividend ratio and the combined price-dividend ratio and risk free rate have negative  $R_{OOS}^2$  while the combined price-dividend ratio, risk free rate, and their product yields an  $R_{OOS}^2$  of 5.3%.

Panel C, displays results for the value premium. The first row shows that the in-sample forecasting regression with  $x$ ,  $p$ , and their product as predictive variables yields an  $\bar{R}^2$  of 5.9%. The model gives an  $R_{OOS}^2$  of 0%. The price-dividend ratio, the combined price-dividend ratio and risk free rate, and the combined price-dividend ratio, risk free rate, and their product yield negligible or negative  $R^2$  both in sample and out of sample.

The in-sample and out-of-sample results are illustrated in Figures 6 and 7, respectively. The description of the figures is similar to that of Figure 5. The overall conclusion is that, over the subperiod 1976 – 2009, the model forecasts in sample and predicts out of sample the equity, size, and value premia far better than the price-dividend ratio, the combined price-dividend ratio and risk free rate, and the combined price-dividend ratio, risk free rate, and their product.

— Figures 6 and 7 about here —

## 7 In-Sample Forecasting and out-of-Sample Prediction of Consumption and Dividend Growth

The model implies that the expected consumption growth rate is linear in  $x_t$  with coefficient one (equation (22)); and linear in  $p_t$  with coefficient close to zero calculated at the point estimates of the parameters. It also implies that the expected dividend growth rate is linear in  $x_t$  with coefficient  $\phi$  where  $\phi$  is estimated to be greater than one (equation (23)); and linear in  $p_t$  with coefficient close to zero calculated at the point estimates of the parameters. We show that the state variables forecast the consumption

and dividend growth rates, consistent with the presence of a predictable component in the consumption and dividend processes and the mechanism by which the state variables drive the dynamics.

We estimate the model parameters over the period 1930 – 2009, extract the time series of the state variables, and perform an in-sample linear forecasting regression of consumption growth on the two state variables over the same period. The results are reported in the first row of Table 6, Panel A. The  $\bar{R}^2$  is 8.0% but the regression coefficient on  $x_t$  has the wrong sign. We attribute this to the anomalous behavior of consumption and dividend growth during the thirties. The coefficient is positive and statistically significant when we remove the thirties from the sample. Furthermore, we show below in Panels B and C, respectively, that the coefficient of  $x_t$  has the right sign in the post-war and post 1976 subperiods. The regression on the price-dividend ratio yields  $\bar{R}^2$  6.8% (second row) and on the price-dividend ratio and risk free rate yields  $\bar{R}^2$  8.4% (third row).

— Table 6 about here —

We repeat the above forecasting regressions over the post-war subperiod 1947–2009. The results are reported in Panel B. The forecasting regression of consumption growth rate on the two state variables yields statistically significant coefficient on  $x_t$  of the right sign and  $\bar{R}^2$  21.4%. The regression on the price-dividend ratio yields  $\bar{R}^2$  6.7% (second row) and on the price-dividend ratio and risk free rate yields  $\bar{R}^2$  23.9% (third row).

In Panel C, we report the results of in-sample forecasting regressions and out-of-sample predictive regressions over the subperiod 1976–2009. The forecasting regression of consumption growth rate on the two state variables yields statistically significant coefficient on  $x_t$  of the right sign. The  $\bar{R}^2$  is 15.6% in sample and remains positive albeit small (0.7%) out of sample. The regression on the price-dividend ratio yields zero  $\bar{R}^2$  in sample and large negative  $R_{OOS}^2$  out of sample (second row); and the regression on the price-dividend ratio and risk free rate yields  $\bar{R}^2$  23.6% in sample and large negative  $R_{OOS}^2$  out of sample (third row).

In Table 7, we report corresponding results for in-sample forecasting and out-of-sample prediction of the aggregate dividend growth rate. Over the period 1930 – 2009, an in-sample linear forecasting regression of dividend growth on the two state variables yields  $\bar{R}^2$  11.7% but the coefficient on  $x_t$  is not statistically significant (first row, Panel A). The regression on the price-dividend ratio yields  $\bar{R}^2$  8.0% (second row) and a regression on the price-dividend ratio and risk free rate yields  $\bar{R}^2$  7.0% (third row).

— Table 7 about here —

Over the subperiod 1947 – 2009, an in-sample linear forecasting regression of dividend growth on the two state variables yields  $\bar{R}^2$  8.7% and positive and statistically significant regression coefficient on  $x_t$  (first row, Panel B). The regression on the price-dividend ratio yields negative  $\bar{R}^2$  (second row) and on the price-dividend ratio and risk free rate yields  $\bar{R}^2$  8.4% (third row).

In Panel *C*, we report the results of in-sample forecasting regressions and out-of sample predictive regressions over the subperiod 1976–2009. The forecasting regression of the dividend growth rate on the two state variables yields a coefficient on  $x_t$  of the right sign. The  $\bar{R}^2$  is 2.9% in sample and 4.4% out of sample. The regression on the price-dividend ratio yields zero  $\bar{R}^2$  in sample and large negative  $R_{OOS}^2$  out of sample (second row); and the regression on the price-dividend ratio and risk free rate yields  $\bar{R}^2$  7.6% in sample and large negative  $R_{OOS}^2$  out of sample (third row).

Overall, the results of this section show that the model-implied state variables forecast the aggregate consumption and dividend growth rates consistent with the implications of the model. This lends empirical support for the risk channels highlighted in the model. Whereas the market-wide price-dividend ratio and risk free rate have comparable in-sample forecasting performance, they lose their predictive power out of sample, delivering large negative  $R_{OOS}^2$ . This is consistent with the findings of the earlier literature on the price-dividend ratio as an unreliable predictor of consumption and dividend growth. By contrast, the model-implied state variables retain their predictive power out of sample.

## 8 Forecasting the Variance of the Market Return and Consumption and Dividend Growth

We estimate the conditional variance of the annual market return as the sum of squares of the twelve monthly log returns. In the first row of Table 8, Panel *A*, we report the results of the in-sample forecasting regression of this conditional variance on the state variables and their product over 1931 – 2009. The regression coefficient on  $p$  is statistically significant and the  $\bar{R}^2$  is 2.1%. We also report results of in-sample forecasting regressions on the price-dividend ratio (Row 2), the price-dividend ratio and risk free rate (Row 3), and the price-dividend ratio, risk free rate, and their product (Row 4). None of the regression coefficients is statistically significant and the  $\bar{R}^2$  varies from  $-0.9\%$  to  $1.0\%$ .

— Table 8 about here —

The superior performance of the model in forecasting the conditional variance of the market return is illustrated in Figure 8 that plots the realized variance (black solid line) along with its predicted value from the forecasting regression implied by the regime shift model (green dashed line) and a linear forecasting regression using the market-wide price-dividend ratio as a predictor variable (red dashed line). Note that the time series of the variance predicted by the model lines up much more closely with the actual realized time series compared to the time series predicted by the price-dividend ratio.

— Figure 8 about here —

The model also implies that the conditional variance of consumption growth is linear in the state variable  $p_t$  (equation (24)). The conditional variance is computed

as the squared residual from a regression of consumption growth on the two state variables. In the first row of Table 8, Panel *B*, we report the results of the in-sample forecasting regression of this conditional variance over 1931 – 2009 on the state variable  $p_t$ . The regression coefficient is strongly statistically significant with the right sign and the  $\bar{R}^2$  is 12.7%. A low value of  $p_t$  signifies high probability of being in a recession and, therefore, a high variance of the consumption growth rate. In the second row, we add as an additional predictor variable the state variable  $x_t$  even though the model states that the conditional variance is a function of  $p_t$  alone. Consistent with the theory, we find that the coefficient of  $x_t$  is not statistically significant and the  $\bar{R}^2$  decreases. The in-sample forecasting regression on the price-dividend ratio yields a statistically insignificant coefficient and  $\bar{R}^2$  2.7% (Row 3). The in-sample forecasting regression on the price-dividend ratio and risk free rate yields a marginally significant coefficient for the price-dividend ratio and  $\bar{R}^2$  of only 0.6% (Row 4).

Finally, the model implies that the conditional variance of dividend growth is linear in the state variable  $p_t$  (equation (25)) with negative coefficient and same economic interpretation as for the consumption growth rate. In the first row of Table 8, Panel *C*, we report the results of the in-sample forecasting regression of this conditional variance over 1931 – 2009 on the state variable  $p_t$ . The regression coefficient is statistically significant with the right sign and the  $\bar{R}^2$  is 15.5%. As for the variance of the consumption growth rate, the state variable  $x_t$  has no incremental forecasting power for the variance of the dividend growth rate. In the second row,  $x_t$  is included as a second predictor variable in the forecasting regression. Its regression coefficient is not statistically significant and the  $\bar{R}^2$  decreases. The in-sample forecasting regression on the price-dividend ratio yields a statistically significant coefficient and  $\bar{R}^2$  3.4% (Row 3). The in-sample forecasting regression on the price-dividend ratio and risk free rate yields a significant coefficient only for the price-dividend ratio and  $\bar{R}^2$  2.4% (Row 4).

Overall, the results show that the state variable  $p_t$  has strong forecasting power for the conditional variance of the consumption and dividend growth rates, consistent with the implications of the model. This state variable gives an  $\bar{R}^2$  that is an order of magnitude higher than that obtained from linear forecasting regressions for the consumption and dividend growth rates with the market-wide price-dividend ratio and risk free rate as predictor variables. This offers support for the regime-switching model presented in this paper.

## 9 Concluding Remarks

We present an exchange economy with consumption and dividend growth processes that differ across two regimes. We identify the second regime as the regime associated with recessions and market downturns along several criteria. The second regime is shorter in duration than the first one. In the second regime, expected consumption and dividend growth rates are less persistent and more volatile; the realized consumption and dividend growth rates have lower mean and higher volatility; the market return and the equity, size, and value premia have higher mean and volatility; and the mean market-wide price-dividend ratio is lower. Finally, we show that the model-implied

state variables perform better at in-sample forecasting and significantly better at out-of-sample prediction of the equity, size, and value premia, and the aggregate consumption and dividend growth rates than linear regressions with the market-wide price-dividend ratio and risk free rate as predictive variables.

High on our agenda is the application of the model to explain of the cross-section of equity, bond, and derivative returns. Also high on our agenda is the investigation of the number of regimes that are needed to adequately describe the economy. At present, our second regime does double duty by capturing both economic recessions and market downturns. This rendition is necessarily imperfect because economic recessions and market downturns are related but distinct economic phenomena. The challenge is the judicious increase of the number of regimes in a model that retains computational and empirical tractability.

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# A Appendix

Here, we derive the time series and pricing implications of the regime shifts model.

## A.1 Restriction on $\varepsilon(1)$

The law of motion of the probability,  $p_t$ , is

$$p_{t+1} = f(p_t) + \varepsilon_{t+1}, \quad (27)$$

where

$$\begin{aligned} f(p_t) &\equiv \pi_1 p_t + (1 - \pi_2)(1 - p_t) \\ &= (1 - \pi_2) + (\pi_1 + \pi_2 - 1)p_t. \end{aligned}$$

**Case 1:**  $\pi_1 + \pi_2 - 1 > 0$

In this case,  $f(p_t)$  is a monotonically increasing function of  $p_t$ . Since  $p_t \in [0, 1]$ , we have

$$f(p_t) \in [1 - \pi_2, \pi_1]. \quad (28)$$

Given this range for  $f(p_t)$ , equation (27) implies the following restriction on  $\varepsilon_{t+1}$  so as to keep  $p_{t+1}$  in its permissible range, i.e. in the unit interval:

$$\varepsilon_{t+1} \in [-(1 - \pi_2), 1 - \pi_1]. \quad (29)$$

In Section 2.1, we recognize that the mean of the residual  $\varepsilon_{t+1}$  conditional on the regime at time  $t + 1$ , may differ from its unconditional value of zero:

$$E(\varepsilon_{t+1} | s_{t+1} = i) = \varepsilon(i), \quad i = 1, 2.$$

Since the unconditional expectation of  $\varepsilon_{t+1}$  is zero, the law of iterated expectations implies

$$\frac{1 - \pi_2}{2 - \pi_1 - \pi_2} \varepsilon(1) + \left[ 1 - \frac{1 - \pi_2}{2 - \pi_1 - \pi_2} \right] \varepsilon(2) = 0,$$

which implies

$$\varepsilon(2) = -\varepsilon(1) \frac{1 - \pi_2}{1 - \pi_1}. \quad (30)$$

Now, from equation (28),  $f(p_t) + \varepsilon(1) \leq 1$  provided  $\varepsilon(1) \leq (1 - \pi_1)$ . This condition is satisfied by equation (29).

We also require  $f(p_t) + \varepsilon(2) \leq 1$ , i.e.

$$f(p_t) - \varepsilon(1) \frac{1 - \pi_2}{1 - \pi_1} \leq 1.$$

Since the above restriction must hold for all values of  $p_t$  and since the maximum possible

value of  $f(p_t)$  is  $\pi_1$  (equation (28)), we have

$$\pi_1 - \varepsilon(1) \frac{1 - \pi_2}{1 - \pi_1} \leq 1,$$

which implies

$$\varepsilon(1) \geq -\frac{(1 - \pi_1)^2}{1 - \pi_2}. \quad (31)$$

Now, from equation (28),  $f(p_t) + \varepsilon(1) \geq 0$  provided  $\varepsilon(1) \geq -(1 - \pi_2)$ . This condition is satisfied by equation (29).

Finally, we require  $f(p_t) + \varepsilon(2) \geq 0$ , i.e.

$$f(p_t) - \varepsilon(1) \frac{1 - \pi_2}{1 - \pi_1} \geq 0.$$

Since the above restriction must hold for all values of  $p_t$  and since the minimum possible value of  $f(p_t)$  is  $(1 - \pi_1)$  (equation (28)), we have

$$(1 - \pi_1) - \varepsilon(1) \frac{1 - \pi_2}{1 - \pi_1} \geq 0,$$

which implies  $\varepsilon(1) \leq 1 - \pi_1$ . This condition is satisfied by equation (29).

Therefore, from equations (29) and (31), the permissible range for  $\varepsilon(1)$  is

$$\varepsilon(1) \in \left[ \max \left( -(1 - \pi_2), -\frac{(1 - \pi_1)^2}{1 - \pi_2} \right), 1 - \pi_1 \right].$$

**Case 2:**  $\pi_1 + \pi_2 - 1 < 0$

In this case,  $f(p_t)$  is a monotonically decreasing function of  $p_t$ . Since  $p_t \in [0, 1]$ , we have

$$f(p_t) \in [\pi_1, 1 - \pi_2], \quad (32)$$

$$\varepsilon_{t+1} \in [-\pi_1, \pi_2]. \quad (33)$$

Now, from equation (32),  $f(p_t) + \varepsilon(1) \leq 1$  provided  $\varepsilon(1) \leq \pi_2$ . This condition is satisfied by equation (33).

We also require  $f(p_t) + \varepsilon(2) \leq 1$ , i.e.

$$f(p_t) - \varepsilon(1) \frac{1 - \pi_2}{1 - \pi_1} \leq 1.$$

Since the above restriction must hold for all values of  $p_t$  and since the maximum possible value of  $f(p_t)$  is  $1 - \pi_2$  (equation (32)), we have

$$1 - \pi_2 - \varepsilon(1) \frac{1 - \pi_2}{1 - \pi_1} \leq 1,$$

which implies

$$\varepsilon(1) \geq -\frac{(1-\pi_1)\pi_2}{1-\pi_2}. \quad (34)$$

Now, from equation (32),  $f(p_t) + \varepsilon(1) \geq 0$  provided  $\varepsilon(1) \geq -\pi_1$ . This condition is satisfied by equation (33).

Finally, we require  $f(p_t) + \varepsilon(2) \geq 0$ , i.e.

$$f(p_t) - \varepsilon(1) \frac{1-\pi_2}{1-\pi_1} \geq 0.$$

Since the above restriction must hold for all values of  $p_t$  and since the minimum possible value of  $f(p_t)$  is  $\pi_1$  (equation (32)), we have

$$\pi_1 - \varepsilon(1) \frac{1-\pi_2}{1-\pi_1} \geq 0,$$

which implies

$$\varepsilon(1) \leq \frac{(1-\pi_1)\pi_1}{1-\pi_2}. \quad (35)$$

Therefore, from equations (33), (34) and (35), the permissible range for  $\varepsilon(1)$  is

$$\varepsilon(1) \in \left[ \max \left( -\pi_1, -\frac{(1-\pi_1)\pi_2}{1-\pi_2} \right), \min \left( \pi_2, \frac{(1-\pi_1)\pi_1}{1-\pi_2} \right) \right].$$

## A.2 Derivation of Pricing Restrictions

Note that Assumptions (8) to (9) imply the following results:

i)

$$\begin{aligned} & E[\sigma_{s_{t+1}} e_{t+1} | F(t)] \\ &= f(p_t) \sigma_1 E[e_{t+1} | F(t), s_{t+1} = 1] + (1 - f(p_t)) \sigma_2 E[e_{t+1} | F(t), s_{t+1} = 2], \\ &= (\sigma_1 - \sigma_2) e(1) f(p_t), \end{aligned} \quad (36)$$

where the first equality follows from the law of iterated expectations, and the second equality follows since

$$\begin{aligned} & f(p_t) E[e_{t+1} | F(t), s_{t+1} = 1] + (1 - f(p_t)) E[e_{t+1} | F(t), s_{t+1} = 2] \\ &= E[e_{t+1} | F(t)], \\ &= 0. \end{aligned}$$

ii)

$$\begin{aligned}
& E [\sigma_{s_{t+1}} \eta_{t+1} | F(t)] \\
&= f(p_t) \sigma_1 E [\eta_{t+1} | F(t), s_{t+1} = 1] + (1 - f(p_t)) \sigma_2 E [\eta_{t+1} | F(t), s_{t+1} = 2], \\
&= (\sigma_1 - \sigma_2) \eta(1) f(p_t), \tag{37}
\end{aligned}$$

where the first equality follows from the law of iterated expectations, and the second equality follows since

$$\begin{aligned}
& f(p_t) E [\eta_{t+1} | F(t), s_{t+1} = 1] + (1 - f(p_t)) E [\eta_{t+1} | F(t), s_{t+1} = 2] \\
&= E [\eta_{t+1} | F(t)], \\
&= 0.
\end{aligned}$$

iii)

$$\begin{aligned}
& E [\rho_{s_{t+1}} \varepsilon_{t+1} | F(t)] \\
&= f(p_t) \rho_1 E [\varepsilon_{t+1} | F(t), s_{t+1} = 1] + (1 - f(p_t)) \rho_2 E [\varepsilon_{t+1} | F(t), s_{t+1} = 2], \\
&= (\rho_1 - \rho_2) \varepsilon(1) f(p_t), \tag{38}
\end{aligned}$$

where the first equality follows from the law of iterated expectations, and the second equality follows since

$$\begin{aligned}
& f(p_t) E [\varepsilon_{t+1} | F(t), s_{t+1} = 1] + (1 - f(p_t)) E [\varepsilon_{t+1} | F(t), s_{t+1} = 2] \\
&= E [\varepsilon_{t+1} | F(t)], \\
&= 0.
\end{aligned}$$

iv)

$$\begin{aligned}
& E [\varepsilon_{t+1} \sigma_{s_{t+1}} e_{t+1} | F(t)] \\
&= f(p_t) \sigma_1 E [\varepsilon_{t+1} e_{t+1} | F(t), s_{t+1} = 1] + (1 - f(p_t)) \sigma_2 E [\varepsilon_{t+1} e_{t+1} | F(t), s_{t+1} = 2], \\
&= \sigma_{\varepsilon, e} \{ \sigma_1 f(p_t) + \sigma_2 (1 - f(p_t)) \}, \tag{39}
\end{aligned}$$

where the first equality follows from the law of iterated expectations, and the second equality follows from equation (9).

### A.2.1 Consumption Claim

We rely on the log-linear approximation for the continuous return on the consumption claim,  $r_{c,t+1}$ ,

$$r_{c,t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + \Delta c_{t+1},$$

where  $z_t$  is the log price-consumption ratio. Note that the current model specification involves two latent state variables,  $x_t$  and  $p_t$ . We conjecture that the log price-

consumption ratio at date  $t$  takes the form,

$$z_t = p_t [A_0(1) + A_1(1)x_t] + (1 - p_t) [A_0(2) + A_1(2)x_t].$$

The Euler equation for the consumption claim is,

$$E [\exp (m_{t+1} + r_{c,t+1}) | F(t)] = 1, \quad (40)$$

$$m_{t+1} = \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1)r_{c,t+1}.$$

Substituting the above expression for  $m_{t+1}$  into (40), we have,

$$E \left[ \exp \left( \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + \theta r_{c,t+1} \right) | F(t) \right] = 1,$$

which implies

$$E \left[ \exp \left( \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + \theta (\kappa_0 + \kappa_1 z_{t+1} - z_t + \Delta c_{t+1}) \right) | F(t) \right] = 1$$

By Taylor series expansion up to quadratic terms, we obtain the following:

$$\begin{aligned} & \theta \log \delta + \theta (\kappa_0 - z_t) + \left( \theta - \frac{\theta}{\psi} \right) E [\Delta c_{t+1} | F(t)] + \theta \kappa_1 E [z_{t+1} | F(t)] \\ & + \frac{1}{2} var \left( \left( \theta - \frac{\theta}{\psi} \right) \Delta c_{t+1} + \theta \kappa_1 z_{t+1} | F(t) \right) \\ = & 0, \end{aligned}$$

which implies,

$$\begin{aligned} & \theta \log \delta + \theta \kappa_0 - \theta \{ p_t [A_0(1) + A_1(1)x_t] + (1 - p_t) [A_0(2) + A_1(2)x_t] \} \\ & + \left( \theta - \frac{\theta}{\psi} \right) E [\Delta c_{t+1} | F(t)] + \theta \kappa_1 E [z_{t+1} | F(t)] + \frac{1}{2} var \left( \left( \theta - \frac{\theta}{\psi} \right) \Delta c_{t+1} + \theta \kappa_1 z_{t+1} | F(t) \right) \\ = & 0 \end{aligned}$$

We approximate the conditional variance,  $var \left( \left( \theta - \frac{\theta}{\psi} \right) \Delta c_{t+1} + \theta \kappa_1 z_{t+1} | F(t) \right)$ , with the constant,  $\Sigma$ , and write the above equation as

$$\begin{aligned} & \theta \log \delta + \theta \kappa_0 - \theta \{ p_t [A_0(1) + A_1(1)x_t] + (1 - p_t) [A_0(2) + A_1(2)x_t] \} \\ & + \left( \theta - \frac{\theta}{\psi} \right) E [\Delta c_{t+1} | F(t)] + \theta \kappa_1 E [z_{t+1} | F(t)] + \frac{1}{2} \Sigma \\ = & 0 \end{aligned}$$

The parameter  $\Sigma$  is a function of the deeper parameters of the joint distribu-

tion of the error terms  $e_{t+1}$ ,  $\eta_{t+1}$ ,  $u_{t+1}$ ,  $\varepsilon_{t+1}$ , and  $s_{t+1}$  (e.g.  $E [e_{t+1}\eta_{t+1}|F(t), s_{t+1}]$ ,  $E [\varepsilon_{t+1}\eta_{t+1}|F(t), s_{t+1}]$ ,  $E [\varepsilon_{t+1}e_{t+1}\eta_{t+1}|F(t), s_{t+1}]$ ). In our empirical work, we treat  $\Sigma$  as a free parameter.

We calculate  $E [\Delta c_{t+1}|F(t)]$  as follows:

$$\begin{aligned} & E [\Delta c_{t+1}|F(t)] \\ &= \mu + x_t + E [\sigma_{s_{t+1}}\eta_{t+1}|F(t)] \\ &= \mu + x_t + (\sigma_1 - \sigma_2) \eta(1) f(p_t), \end{aligned}$$

where the second equality follows from equation (37).

We calculate  $E [z_{t+1}|F(t)]$  as follows:

$$\begin{aligned} & E [z_{t+1}|F(t)] \\ &= E \left[ \left( \begin{array}{l} (f(p_t) + \varepsilon_{t+1}) \{A_0(1) + A_1(1) (\rho_{s_{t+1}}x_t + \varphi_e\sigma_{s_{t+1}}e_{t+1})\} + \\ (1 - f(p_t) - \varepsilon_{t+1}) \{A_0(2) + A_1(2) (\rho_{s_{t+1}}x_t + \varphi_e\sigma_{s_{t+1}}e_{t+1})\} \end{array} \right) |F(t) \right] \\ &= f(p_t) [A_0(1) + A_1(1) \{x_t (f(p_t)\rho_1 + (1 - f(p_t)) \rho_2) + \varphi_e E [\sigma_{s_{t+1}}e_{t+1}|F(t)]\}] \\ &\quad + A_0(1) E [\varepsilon_{t+1}|F(t)] \\ &\quad + A_1(1)x_t E [\rho_{s_{t+1}}\varepsilon_{t+1}|F(t)] \\ &\quad + A_1(1)\varphi_e E [\varepsilon_{t+1}\sigma_{s_{t+1}}e_{t+1}|F(t)] \\ &\quad + (1 - f(p_t)) [A_0(2) + A_1(2) \{x_t (f(p_t)\rho_1 + (1 - f(p_t)) \rho_2) + \varphi_e E [\sigma_{s_{t+1}}e_{t+1}|F(t)]\}] \\ &\quad - A_0(2) E [\varepsilon_{t+1}|F(t)] \\ &\quad - A_1(2)x_t E [\rho_{s_{t+1}}\varepsilon_{t+1}|F(t)] \\ &\quad - A_1(2)\varphi_e E [\varepsilon_{t+1}\sigma_{s_{t+1}}e_{t+1}|F(t)] \end{aligned}$$

We use equations (36), (38), and (39) to simplify the above expression as follows:

$$\begin{aligned} & E [z_{t+1}|F(t)] \\ &= f(p_t) [A_0(1) + A_1(1) \{x_t (f(p_t)\rho_1 + (1 - f(p_t)) \rho_2)\}] \\ &\quad + (1 - f(p_t)) [A_0(2) + A_1(2) \{x_t (f(p_t)\rho_1 + (1 - f(p_t)) \rho_2)\}] \\ &\quad + \varphi_e (\sigma_1 - \sigma_2) e(1) \{A_1(1)f(p_t) + A_1(2) (1 - f(p_t))\} f(p_t) \\ &\quad + (A_1(1) - A_1(2)) (\rho_1 - \rho_2) \varepsilon(1)x_t f(p_t) \\ &\quad + (A_1(1) - A_1(2)) \varphi_e \sigma_{\varepsilon,e} \{\sigma_1 f(p_t) + \sigma_2 (1 - f(p_t))\} \\ &= [A_0(1) - A_0(2) + (A_1(1) - A_1(2)) \rho_2 x_t] f(p_t) \\ &\quad + A_0(2) + A_1(2)x_t (f(p_t)\rho_1 + (1 - f(p_t)) \rho_2) \\ &\quad + (A_1(1) - A_1(2)) (\rho_1 - \rho_2) \varepsilon(1)x_t f(p_t) \\ &\quad + (A_1(1) - A_1(2)) \varphi_e \sigma_{\varepsilon,e} \{\sigma_1 f(p_t) + \sigma_2 (1 - f(p_t))\} \\ &\quad + (A_1(1) - A_1(2)) [(\rho_1 - \rho_2) x_t + \varphi_e (\sigma_1 - \sigma_2) e(1)] \{f(p_t)\}^2 \\ &\quad + \varphi_e (\sigma_1 - \sigma_2) e(1) A_1(2) f(p_t) \end{aligned}$$

Finally, we write the Euler equation as

$$\begin{aligned}
& \theta \log \delta + \theta \kappa_0 - \theta \{p_t [A_0(1) + A_1(2)x_t] + (1 - p_t) [A_0(2) + A_1(2)x_t]\} \\
& + \left( \theta - \frac{\theta}{\psi} \right) (\mu + x_t + (\sigma_1 - \sigma_2) \eta(1) f(p_t)) + \frac{1}{2} \Sigma \\
& + \theta \kappa_1 \left( \begin{array}{l} [A_0(1) - A_0(2) + (A_1(1) - A_1(2)) \rho_2 x_t] f(p_t) \\ + A_0(2) + A_1(2)x_t (f(p_t) \rho_1 + (1 - f(p_t)) \rho_2) \\ + (A_1(1) - A_1(2)) (\rho_1 - \rho_2) \varepsilon(1) x_t f(p_t) \\ + (A_1(1) - A_1(2)) \varphi_e \sigma_{\varepsilon, e} \{ \sigma_1 f(p_t) + \sigma_2 (1 - f(p_t)) \} \\ + (A_1(1) - A_1(2)) [(\rho_1 - \rho_2) x_t + \varphi_e (\sigma_1 - \sigma_2) e(1)] \{ f(p_t) \}^2 \\ + \varphi_e (\sigma_1 - \sigma_2) e(1) A_1(2) f(p_t) \end{array} \right) \\
& = 0
\end{aligned}$$

Collecting terms, we obtain

$$\begin{aligned}
& \left( \begin{array}{l} \theta \log \delta + \theta \kappa_0 - \theta A_0(2) + \left( \theta - \frac{\theta}{\psi} \right) (\mu + (\sigma_1 - \sigma_2) \eta(1) (1 - \pi_2)) + \theta \kappa_1 A_0(2) \\ + \theta \kappa_1 A_1(2) \varphi_e (\sigma_1 - \sigma_2) e(1) (1 - \pi_2) + \theta \kappa_1 [A_0(1) - A_0(2)] (1 - \pi_2) \\ + \theta \kappa_1 [A_1(1) - A_1(2)] \varphi_e (\sigma_1 - \sigma_2) e(1) (1 - \pi_2)^2 \\ + \theta \kappa_1 [A_1(1) - A_1(2)] \varphi_e \sigma_{\varepsilon, e} ((\sigma_1 - \sigma_2) (1 - \pi_2) + \sigma_2) + \frac{1}{2} \Sigma \end{array} \right) \\
& + \left( \begin{array}{l} -\theta A_1(2) + \left( \theta - \frac{\theta}{\psi} \right) + \theta \kappa_1 A_1(2) [(\rho_1 - \rho_2) (1 - \pi_2) + \rho_2] \\ + \theta \kappa_1 [A_1(1) - A_1(2)] (\rho_1 - \rho_2) (1 - \pi_2) \varepsilon(1) \\ \theta \kappa_1 [A_1(1) - A_1(2)] [(\rho_1 - \rho_2) (1 - \pi_2)^2 + \rho_2 (1 - \pi_2)] \end{array} \right) x_t \\
& + \left( \begin{array}{l} \theta [A_0(2) - A_0(1)] + \left( \theta - \frac{\theta}{\psi} \right) (\sigma_1 - \sigma_2) \eta(1) (\pi_1 + \pi_2 - 1) \\ + \theta \kappa_1 A_1(2) \varphi_e (\sigma_1 - \sigma_2) e(1) (\pi_1 + \pi_2 - 1) \\ + \theta \kappa_1 [A_0(1) - A_0(2)] (\pi_1 + \pi_2 - 1) \\ + 2\theta \kappa_1 [A_1(1) - A_1(2)] \varphi_e (\sigma_1 - \sigma_2) e(1) (1 - \pi_2) (\pi_1 + \pi_2 - 1) \\ + \theta \kappa_1 [A_1(1) - A_1(2)] \varphi_e \sigma_{\varepsilon, e} (\sigma_1 - \sigma_2) (\pi_1 + \pi_2 - 1) \end{array} \right) p_t \\
& + \left( \begin{array}{l} \theta [A_1(2) - A_1(1)] + \theta \kappa_1 A_1(2) (\rho_1 - \rho_2) (\pi_1 + \pi_2 - 1) \\ + \theta \kappa_1 [A_1(1) - A_1(2)] [2(1 - \pi_2) (\pi_1 + \pi_2 - 1) (\rho_1 - \rho_2) + \rho_2 (\pi_1 + \pi_2 - 1)] \\ + \theta \kappa_1 [A_1(1) - A_1(2)] (\rho_1 - \rho_2) \varepsilon(1) (\pi_1 + \pi_2 - 1) \end{array} \right) p_t x_t \\
& + \theta \kappa_1 [A_1(1) - A_1(2)] (\pi_1 + \pi_2 - 1)^2 (\rho_1 - \rho_2) p_t^2 x_t \\
& + \theta \kappa_1 [A_1(1) - A_1(2)] \varphi_e (\sigma_1 - \sigma_2) e(1) (\pi_1 + \pi_2 - 1)^2 p_t^2 \\
& = 0
\end{aligned}$$

We approximate the above expression to order  $x_t$ ,  $p_t$ , and  $p_t x_t$ . Therefore, we expand the term  $p_t^2$  as a Taylor series to first order around the unconditional mean,  $\bar{p}$ , of  $p_t$ . We note that  $\bar{p} = \frac{1 - \pi_2}{2 - \pi_1 - \pi_2}$ . We obtain the following:

$$\begin{aligned}
p_t^2 &\approx \bar{p}^2 + 2\bar{p}(p_t - \bar{p}) \\
&= -\bar{p}^2 + 2\bar{p}p_t
\end{aligned}$$

Since the Euler equation holds for all observable states  $(x_t, p_t)$ , we obtain the following 4 parameter restrictions:

Constant:

$$\left( \begin{array}{l}
\theta \log \delta + \theta \kappa_0 - \theta A_0(2) + \left( \theta - \frac{\theta}{\psi} \right) (\mu + (\sigma_1 - \sigma_2) \eta(1)(1 - \pi_2)) + \theta \kappa_1 A_0(2) \\
+\theta \kappa_1 A_1(2) \varphi_e (\sigma_1 - \sigma_2) e(1)(1 - \pi_2) + \theta \kappa_1 [A_0(1) - A_0(2)] (1 - \pi_2) \\
+\theta \kappa_1 [A_1(1) - A_1(2)] \varphi_e (\sigma_1 - \sigma_2) e(1)(1 - \pi_2)^2 \\
+\theta \kappa_1 [A_1(1) - A_1(2)] \varphi_e \sigma_{\varepsilon, e} ((\sigma_1 - \sigma_2) (1 - \pi_2) + \sigma_2) + \frac{1}{2} \Sigma \\
-\theta \kappa_1 [A_1(1) - A_1(2)] \varphi_e (\sigma_1 - \sigma_2) e(1)(\pi_1 + \pi_2 - 1)^2 \bar{p}^2
\end{array} \right) = 0$$

Coefficient of  $x_t$ :

$$\left( \begin{array}{l}
-\theta A_1(2) + \left( \theta - \frac{\theta}{\psi} \right) + \theta \kappa_1 A_1(2) [(\rho_1 - \rho_2) (1 - \pi_2) + \rho_2] \\
+\theta \kappa_1 [A_1(1) - A_1(2)] (\rho_1 - \rho_2) (1 - \pi_2) \varepsilon(1) \\
\theta \kappa_1 [A_1(1) - A_1(2)] [(\rho_1 - \rho_2) (1 - \pi_2)^2 + \rho_2 (1 - \pi_2)] \\
-\theta \kappa_1 [A_1(1) - A_1(2)] (\pi_1 + \pi_2 - 1)^2 (\rho_1 - \rho_2) \bar{p}^2
\end{array} \right) = 0$$

Coefficient of  $p_t$ :

$$\left( \begin{array}{l}
\theta [A_0(2) - A_0(1)] + \left( \theta - \frac{\theta}{\psi} \right) (\sigma_1 - \sigma_2) \eta(1)(\pi_1 + \pi_2 - 1) \\
+\theta \kappa_1 A_1(2) \varphi_e (\sigma_1 - \sigma_2) e(1)(\pi_1 + \pi_2 - 1) \\
+\theta \kappa_1 [A_0(1) - A_0(2)] (\pi_1 + \pi_2 - 1) \\
+2\theta \kappa_1 [A_1(1) - A_1(2)] \varphi_e (\sigma_1 - \sigma_2) e(1)(1 - \pi_2)(\pi_1 + \pi_2 - 1) \\
+\theta \kappa_1 [A_1(1) - A_1(2)] \varphi_e \sigma_{\varepsilon, e} (\sigma_1 - \sigma_2) (\pi_1 + \pi_2 - 1) \\
+2\theta \kappa_1 [A_1(1) - A_1(2)] \varphi_e (\sigma_1 - \sigma_2) e(1)(\pi_1 + \pi_2 - 1)^2 \bar{p}
\end{array} \right) = 0$$

Coefficient of  $p_t x_t$ :

$$\left( \begin{array}{l}
\theta [A_1(2) - A_1(1)] + \theta \kappa_1 A_1(2) (\rho_1 - \rho_2) (\pi_1 + \pi_2 - 1) \\
+\theta \kappa_1 [A_1(1) - A_1(2)] [2(1 - \pi_2)(\pi_1 + \pi_2 - 1) (\rho_1 - \rho_2) + \rho_2 (\pi_1 + \pi_2 - 1)] \\
+\theta \kappa_1 [A_1(1) - A_1(2)] (\rho_1 - \rho_2) \varepsilon(1)(\pi_1 + \pi_2 - 1) \\
+2\theta \kappa_1 [A_1(1) - A_1(2)] (\pi_1 + \pi_2 - 1)^2 (\rho_1 - \rho_2) \bar{p}
\end{array} \right) = 0$$

The 4 linear equations can be solved to obtain the 4 parameters  $A_0(1)$ ,  $A_0(2)$ ,  $A_1(1)$ , and  $A_1(2)$ .

### A.2.2 Dividend Claim

The market portfolio is defined as the claim to the aggregate dividend stream. We rely on the log-linear approximation for the continuous return on the aggregate dividend claim,  $r_{m,t+1}$ ,

$$r_{m,t+1} = \kappa_{0,m} + \kappa_{1,m}z_{m,t+1} - z_{m,t} + \Delta d_{t+1},$$

where  $z_{m,t}$  is the market-wide log price-dividend ratio. We conjecture that the log price-dividend ratio at date  $t$  takes the form,

$$z_{m,t} = p_t [A_{0,m}(1) + A_{1,m}(1)x_t] + (1 - p_t) [A_{0,m}(2) + A_{1,m}(2)x_t].$$

The Euler equation for the dividend claim is,

$$E [\exp (m_{t+1} + r_{m,t+1}) | F(t)] = 1. \quad (41)$$

Substituting the expression for  $m_{t+1}$  from (14) into (41), we have,

$$E \left[ \exp \left( \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1)r_{c,t+1} + r_{m,t+1} \right) | F(t) \right] = 1, \quad (42)$$

which implies:

$$E \left[ \exp \left( \begin{array}{l} \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1)(\kappa_0 + \kappa_1 z_{t+1} - z_t + \Delta c_{t+1}) \\ + \kappa_{0,m} + \kappa_{1,m} z_{m,t+1} - z_{m,t} + \Delta d_{t+1} \end{array} \right) | F(t) \right] = 1.$$

Simplifying the above expression gives:

$$E \left[ \exp \left( \begin{array}{l} \theta \log \delta + \left( -\frac{\theta}{\psi} + \theta - 1 \right) \Delta c_{t+1} + \Delta d_{t+1} + (\theta - 1)\kappa_0 + \kappa_{0,m} \\ - (\theta - 1)z_t - z_{m,t} + (\theta - 1)\kappa_1 z_{t+1} + \kappa_{1,m} z_{m,t+1} \end{array} \right) | F(t) \right] = 1.$$

Performing a Taylor series expansion upto quadratic terms gives,

$$\begin{aligned} & \theta \log \delta + (\theta - 1)\kappa_0 + \kappa_{0,m} - (\theta - 1)z_t - z_{m,t} + \left( -\frac{\theta}{\psi} + \theta - 1 \right) E [\Delta c_{t+1} | F(t)] \\ & + E [\Delta d_{t+1} | F(t)] + (\theta - 1)\kappa_1 E [z_{t+1} | F(t)] + \kappa_{1,m} E [z_{m,t+1} | F(t)] \\ & + \frac{1}{2} var \left( \left( -\frac{\theta}{\psi} + \theta - 1 \right) \Delta c_{t+1} + \Delta d_{t+1} + (\theta - 1)\kappa_1 z_{t+1} + \kappa_{1,m} z_{m,t+1} | F(t) \right) \\ & = 0. \end{aligned}$$

We approximate the conditional variance,

$var \left( \left( -\frac{\theta}{\psi} + \theta - 1 \right) \Delta c_{t+1} + \Delta d_{t+1} + (\theta - 1)\kappa_1 z_{t+1} + \kappa_{1,m} z_{m,t+1} | F(t) \right)$ , with the constant,  $\Sigma'$  and write the above equation as:

$$\begin{aligned}
& \theta \log \delta + (\theta - 1)\kappa_0 + \kappa_{0,m} - (\theta - 1)z_t - z_{m,t} + \left(-\frac{\theta}{\psi} + \theta - 1\right) E[\Delta c_{t+1}|F(t)] \\
& + E[\Delta d_{t+1}|F(t)] + (\theta - 1)\kappa_1 E[z_{t+1}|F(t)] + \kappa_{1,m} E[z_{m,t+1}|F(t)] + \frac{1}{2}\Sigma' \\
= & 0
\end{aligned} \tag{43}$$

We calculate  $E[\Delta d_{t+1}|F(t)]$  as follows:

$$\begin{aligned}
& E[\Delta d_{t+1}|F(t)] \\
= & \mu_d + \phi x_t + \varphi_d E[\sigma_{s_{t+1}} u_{t+1}|F(t)] \\
= & \mu_d + \phi x_t + \varphi_d (\sigma_1 - \sigma_2) u(1) f(p_t).
\end{aligned}$$

Similar calculations as in Appendix A.2.1 give the following expression for  $E[z_{m,t+1}|F(t)]$ :

$$\begin{aligned}
& E[z_{m,t+1}|F(t)] \\
= & [A_{0,m}(1) - A_{0,m}(2) + (A_{1,m}(1) - A_{1,m}(2)) \rho_2 x_t] f(p_t) \\
& + A_{0,m}(2) + A_{1,m}(2) x_t (f(p_t) \rho_1 + (1 - f(p_t)) \rho_2) \\
& + (A_{1,m}(1) - A_{1,m}(2)) (\rho_1 - \rho_2) \varepsilon(1) x_t f(p_t) \\
& + (A_{1,m}(1) - A_{1,m}(2)) \varphi_e \sigma_{\varepsilon,e} \{\sigma_1 f(p_t) + \sigma_2 (1 - f(p_t))\} \\
& + (A_{1,m}(1) - A_{1,m}(2)) [(\rho_1 - \rho_2) x_t + \varphi_e (\sigma_1 - \sigma_2) e(1)] \{f(p_t)\}^2 \\
& + \varphi_e (\sigma_1 - \sigma_2) e(1) A_{1,m}(2) f(p_t).
\end{aligned}$$

Therefore, the Euler equation (43) may be written as:

$$\begin{aligned}
& \theta \log \delta + (\theta - 1)\kappa_0 + \kappa_{0,m} - (\theta - 1) \{p_t [A_0(0) + A_1(0)x_t] + (1 - p_t) [A_0(1) + A_1(1)x_t]\} \\
& - \{p_t [A_{0,m}(0) + A_{1,m}(0)x_t] + (1 - p_t) [A_{0,m}(1) + A_{1,m}(1)x_t]\} \\
& + \left( -\frac{\theta}{\psi} + \theta - 1 \right) (\mu + x_t + (\sigma_1 - \sigma_2) \eta(1)f(p_t)) + \mu_d + \phi x_t + \varphi_d (\sigma_1 - \sigma_2) u(1)f(p_t) + \frac{1}{2} \Sigma' \\
& + (\theta - 1)\kappa_1 \left( \begin{array}{l} [A_0(1) - A_0(2) + (A_1(1) - A_1(2)) \rho_2 x_t] f(p_t) \\ + A_0(2) + A_1(2)x_t (f(p_t)\rho_1 + (1 - f(p_t)) \rho_2) \\ + (A_1(1) - A_1(2)) (\rho_1 - \rho_2) \varepsilon(1)x_t f(p_t) \\ + (A_1(1) - A_1(2)) \varphi_e \sigma_{\varepsilon,e} \{ \sigma_1 f(p_t) + \sigma_2 (1 - f(p_t)) \} \\ + (A_1(1) - A_1(2)) [(\rho_1 - \rho_2) x_t + \varphi_e (\sigma_1 - \sigma_2) e(1)] \{f(p_t)\}^2 \\ + \varphi_e (\sigma_1 - \sigma_2) e(1) A_1(2) f(p_t) \end{array} \right) \\
& + \kappa_{1,m} \left( \begin{array}{l} [A_{0,m}(1) - A_{0,m}(2) + (A_{1,m}(1) - A_{1,m}(2)) \rho_2 x_t] f(p_t) \\ + A_{0,m}(2) + A_{1,m}(2)x_t (f(p_t)\rho_1 + (1 - f(p_t)) \rho_2) \\ + (A_{1,m}(1) - A_{1,m}(2)) (\rho_1 - \rho_2) \varepsilon(1)x_t f(p_t) \\ + (A_{1,m}(1) - A_{1,m}(2)) \varphi_e \sigma_{\varepsilon,e} \{ \sigma_1 f(p_t) + \sigma_2 (1 - f(p_t)) \} \\ + (A_{1,m}(1) - A_{1,m}(2)) [(\rho_1 - \rho_2) x_t + \varphi_e (\sigma_1 - \sigma_2) e(1)] \{f(p_t)\}^2 \\ + \varphi_e (\sigma_1 - \sigma_2) e(1) A_{1,m}(2) f(p_t) \end{array} \right) \\
& = 0
\end{aligned}$$

Simplifying, we obtain:

$$\begin{aligned}
& \left( \begin{aligned} & \theta \log \delta + (\theta - 1)\kappa_0 + \kappa_{0,m} - (\theta - 1)A_0(2) - A_{0,m}(2) + \frac{1}{2}\Sigma' \\ & \left(-\frac{\theta}{\psi} + \theta - 1\right) (\mu + (\sigma_1 - \sigma_2) \eta(1)(1 - \pi_2)) + \mu_d + \varphi_d (\sigma_1 - \sigma_2) u(1)(1 - \pi_2) \\ & + (\theta - 1)\kappa_1 A_1(2) \varphi_e (\sigma_1 - \sigma_2) e(1)(1 - \pi_2) + (\theta - 1)\kappa_1 [A_0(1) - A_0(2)] (1 - \pi_2) \\ & + (\theta - 1)\kappa_1 [A_1(1) - A_1(2)] \varphi_e (\sigma_1 - \sigma_2) e(1)(1 - \pi_2)^2 + (\theta - 1)\kappa_1 A_0(2) \\ & + (\theta - 1)\kappa_1 [A_1(1) - A_1(2)] \varphi_e \sigma_{\varepsilon,e} [(\sigma_1 - \sigma_2) (1 - \pi_2) + \sigma_2] \\ & \quad \kappa_{1,m} [A_{0,m}(1) - A_{0,m}(2)] (1 - \pi_2) + \kappa_{1,m} A_{0,m}(2) \\ & + \kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)] \varphi_e \sigma_{\varepsilon,e} [(\sigma_1 - \sigma_2) (1 - \pi_2) + \sigma_2] \\ & \quad + \kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)] \varphi_e (\sigma_1 - \sigma_2) e(1)(1 - \pi_2)^2 \\ & \quad + \kappa_{1,m} A_{1,m}(1) \varphi_e (\sigma_1 - \sigma_2) e(1)(1 - \pi_2) \end{aligned} \right) \\
& + \left( \begin{aligned} & -(\theta - 1)A_1(2) - A_{1,m}(2) + \left(-\frac{\theta}{\psi} + \theta - 1\right) + \phi \\ & (\theta - 1)\kappa_1 A_1(2) [(\rho_1 - \rho_2) (1 - \pi_2) + \rho_2] \\ & + (\theta - 1)\kappa_1 [A_1(1) - A_1(2)] (\rho_1 - \rho_2) (1 - \pi_2) \varepsilon(1) \\ & (\theta - 1)\kappa_1 [A_1(1) - A_1(2)] [(\rho_1 - \rho_2) (1 - \pi_2)^2 + \rho_2(1 - \pi_2)] \\ & \quad + \kappa_{1,m} A_{1,m}(2) [(\rho_1 - \rho_2) (1 - \pi_2) + \rho_2] \\ & \quad + \kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)] (\rho_1 - \rho_2) (1 - \pi_2) \varepsilon(1) \\ & + \kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)] [(\rho_1 - \rho_2) (1 - \pi_2)^2 + \rho_2(1 - \pi_2)] \end{aligned} \right) x_t \\
& + \left( \begin{aligned} & (\theta - 1) [A_0(2) - A_0(1)] + [A_{0,m}(2) - A_{0,m}(1)] \\ & + \left(-\frac{\theta}{\psi} + \theta - 1\right) (\sigma_1 - \sigma_2) \eta(1)(\pi_1 + \pi_2 - 1) + \varphi_d (\sigma_1 - \sigma_2) u(1)(\pi_1 + \pi_2 - 1) \\ & + (\theta - 1)\kappa_1 A_1(2) \varphi_e (\sigma_1 - \sigma_2) e(1)(\pi_1 + \pi_2 - 1) \\ & + (\theta - 1)\kappa_1 [A_0(1) - A_0(2)] (\pi_1 + \pi_2 - 1) \\ & + 2(\theta - 1)\kappa_1 [A_1(1) - A_1(2)] \varphi_e (\sigma_1 - \sigma_2) e(1)(1 - \pi_2)(\pi_1 + \pi_2 - 1) \\ & + (\theta - 1)\kappa_1 [A_1(1) - A_1(2)] \varphi_e \sigma_{\varepsilon,e} (\sigma_1 - \sigma_2) (\pi_1 + \pi_2 - 1) \\ & \quad + \kappa_{1,m} A_{1,m}(2) \varphi_e (\sigma_1 - \sigma_2) e(1)(\pi_1 + \pi_2 - 1) \\ & \quad + \kappa_{1,m} [A_{0,m}(1) - A_{0,m}(2)] (\pi_1 + \pi_2 - 1) \\ & + 2\kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)] \varphi_e (\sigma_1 - \sigma_2) e(1)(1 - \pi_2)(\pi_1 + \pi_2 - 1) \\ & \quad + \kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)] \varphi_e \sigma_{\varepsilon,e} (\sigma_1 - \sigma_2) (\pi_1 + \pi_2 - 1) \end{aligned} \right) p_t \\
& + \left( \begin{aligned} & (\theta - 1) [A_1(2) - A_1(1)] + [A_{1,m}(2) - A_{1,m}(1)] \\ & (\theta - 1)\kappa_1 A_1(2) (\rho_1 - \rho_2) (\pi_1 + \pi_2 - 1) \\ & + (\theta - 1)\kappa_1 [A_1(1) - A_1(2)] [2(1 - \pi_2)(\pi_1 + \pi_2 - 1) (\rho_1 - \rho_2) + \rho_2(\pi_1 + \pi_2 - 1)] \\ & + (\theta - 1)\kappa_1 [A_1(1) - A_1(2)] (\rho_1 - \rho_2) \varepsilon(1)(\pi_1 + \pi_2 - 1) \\ & \quad \kappa_{1,m} A_{1,m}(2) (\rho_1 - \rho_2) (\pi_1 + \pi_2 - 1) \\ & + \kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)] [2(1 - \pi_2)(\pi_1 + \pi_2 - 1) (\rho_1 - \rho_2) + \rho_2(\pi_1 + \pi_2 - 1)] \\ & \quad + \kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)] (\rho_1 - \rho_2) \varepsilon(1)(\pi_1 + \pi_2 - 1) \end{aligned} \right) p_t x_t \\
& + (\pi_1 + \pi_2 - 1)^2 (\rho_1 - \rho_2) \{(\theta - 1)\kappa_1 [A_1(1) - A_1(2)] + \kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)]\} p_t^2 x_t \\
& + \varphi_e (\sigma_1 - \sigma_2) e(1)(\pi_1 + \pi_2 - 1)^2 \{(\theta - 1)\kappa_1 [A_1(1) - A_1(2)] + \kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)]\} p_t^2 \\
& = 0
\end{aligned}$$

As in Appendix A.2.1, we approximate the above expression to order  $x_t$ ,  $p_t$ , and  $p_t x_t$ . Therefore, we expand the term  $p_t^2$  as a Taylor series to first order around the unconditional mean,  $\bar{p}$ , of  $p_t$ .

Since the Euler equation holds for all observable states  $(x_t, p_t)$ , we obtain the fol-

lowing 4 parameter restrictions:

Constant:

$$\left( \begin{aligned} & \theta \log \delta + (\theta - 1)\kappa_0 + \kappa_{0,m} - (\theta - 1)A_0(2) - A_{0,m}(2) + \frac{1}{2}\Sigma' \\ & \left(-\frac{\theta}{\psi} + \theta - 1\right) (\mu + (\sigma_1 - \sigma_2) \eta(1)(1 - \pi_2)) + \mu_d + \varphi_d (\sigma_1 - \sigma_2) u(1)(1 - \pi_2) \\ & + (\theta - 1)\kappa_1 A_1(2) \varphi_e (\sigma_1 - \sigma_2) e(1)(1 - \pi_2) + (\theta - 1)\kappa_1 [A_0(1) - A_0(2)] (1 - \pi_2) \\ & + (\theta - 1)\kappa_1 [A_1(1) - A_1(2)] \varphi_e (\sigma_1 - \sigma_2) e(1)(1 - \pi_2)^2 + (\theta - 1)\kappa_1 A_0(2) \\ & + (\theta - 1)\kappa_1 [A_1(1) - A_1(2)] \varphi_e \sigma_{\varepsilon,e} [(\sigma_1 - \sigma_2) (1 - \pi_2) + \sigma_2] \\ & \quad \kappa_{1,m} [A_{0,m}(1) - A_{0,m}(2)] (1 - \pi_2) + \kappa_{1,m} A_{0,m}(2) \\ & + \kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)] \varphi_e \sigma_{\varepsilon,e} [(\sigma_1 - \sigma_2) (1 - \pi_2) + \sigma_2] \\ & + \kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)] \varphi_e (\sigma_1 - \sigma_2) e(1)(1 - \pi_2)^2 \\ & \quad + \kappa_{1,m} A_{1,m}(1) \varphi_e (\sigma_1 - \sigma_2) e(1)(1 - \pi_2) \\ & - \varphi_e (\sigma_1 - \sigma_2) e(1)(\pi_1 + \pi_2 - 1)^2 \{(\theta - 1)\kappa_1 [A_1(1) - A_1(2)] + \kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)]\} \bar{p}^2 \end{aligned} \right) = 0$$

Coefficient of  $x_t$ :

$$\left( \begin{aligned} & -(\theta - 1)A_1(2) - A_{1,m}(2) + \left(-\frac{\theta}{\psi} + \theta - 1\right) + \phi \\ & (\theta - 1)\kappa_1 A_1(2) [(\rho_1 - \rho_2) (1 - \pi_2) + \rho_2] \\ & + (\theta - 1)\kappa_1 [A_1(1) - A_1(2)] (\rho_1 - \rho_2) (1 - \pi_2) \varepsilon(1) \\ & (\theta - 1)\kappa_1 [A_1(1) - A_1(2)] [(\rho_1 - \rho_2) (1 - \pi_2)^2 + \rho_2 (1 - \pi_2)] \\ & \quad + \kappa_{1,m} A_{1,m}(2) [(\rho_1 - \rho_2) (1 - \pi_2) + \rho_2] \\ & + \kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)] (\rho_1 - \rho_2) (1 - \pi_2) \varepsilon(1) \\ & + \kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)] [(\rho_1 - \rho_2) (1 - \pi_2)^2 + \rho_2 (1 - \pi_2)] - \\ & (\pi_1 + \pi_2 - 1)^2 (\rho_1 - \rho_2) \{(\theta - 1)\kappa_1 [A_1(1) - A_1(2)] + \kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)]\} \bar{p}^2 \end{aligned} \right) = 0$$

Coefficient of  $p_t$ :

$$\left( \begin{aligned} & (\theta - 1) [A_0(2) - A_0(1)] + [A_{0,m}(2) - A_{0,m}(1)] \\ & + \left(-\frac{\theta}{\psi} + \theta - 1\right) (\sigma_1 - \sigma_2) \eta(1)(\pi_1 + \pi_2 - 1) + \varphi_d (\sigma_1 - \sigma_2) u(1)(\pi_1 + \pi_2 - 1) \\ & + (\theta - 1)\kappa_1 A_1(2) \varphi_e (\sigma_1 - \sigma_2) e(1)(\pi_1 + \pi_2 - 1) \\ & \quad + (\theta - 1)\kappa_1 [A_0(1) - A_0(2)] (\pi_1 + \pi_2 - 1) \\ & + 2(\theta - 1)\kappa_1 [A_1(1) - A_1(2)] \varphi_e (\sigma_1 - \sigma_2) e(1)(1 - \pi_2)(\pi_1 + \pi_2 - 1) \\ & + (\theta - 1)\kappa_1 [A_1(1) - A_1(2)] \varphi_e \sigma_{\varepsilon,e} (\sigma_1 - \sigma_2) (\pi_1 + \pi_2 - 1) \\ & \quad + \kappa_{1,m} A_{1,m}(2) \varphi_e (\sigma_1 - \sigma_2) e(1)(\pi_1 + \pi_2 - 1) \\ & \quad + \kappa_{1,m} [A_{0,m}(1) - A_{0,m}(2)] (\pi_1 + \pi_2 - 1) \\ & + 2\kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)] \varphi_e (\sigma_1 - \sigma_2) e(1)(1 - \pi_2)(\pi_1 + \pi_2 - 1) \\ & \quad + \kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)] \varphi_e \sigma_{\varepsilon,e} (\sigma_1 - \sigma_2) (\pi_1 + \pi_2 - 1) \\ & + 2\varphi_e (\sigma_1 - \sigma_2) e(1)(\pi_1 + \pi_2 - 1)^2 \{(\theta - 1)\kappa_1 [A_1(1) - A_1(2)] + \kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)]\} \bar{p} \end{aligned} \right) = 0$$

Coefficient of  $p_t x_t$ :

$$\left( \begin{array}{l} (\theta - 1) [A_1(2) - A_1(1)] + [A_{1,m}(2) - A_{1,m}(1)] \\ (\theta - 1) \kappa_1 A_1(2) (\rho_1 - \rho_2) (\pi_1 + \pi_2 - 1) \\ + (\theta - 1) \kappa_1 [A_1(1) - A_1(2)] [2(1 - \pi_2)(\pi_1 + \pi_2 - 1) (\rho_1 - \rho_2) + \rho_2(\pi_1 + \pi_2 - 1)] \\ + (\theta - 1) \kappa_1 [A_1(1) - A_1(2)] (\rho_1 - \rho_2) \varepsilon(1)(\pi_1 + \pi_2 - 1) \\ \kappa_{1,m} A_{1,m}(2) (\rho_1 - \rho_2) (\pi_1 + \pi_2 - 1) \\ + \kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)] [2(1 - \pi_2)(\pi_1 + \pi_2 - 1) (\rho_1 - \rho_2) + \rho_2(\pi_1 + \pi_2 - 1)] \\ + \kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)] (\rho_1 - \rho_2) \varepsilon(1)(\pi_1 + \pi_2 - 1) \\ + 2(\pi_1 + \pi_2 - 1)^2 (\rho_1 - \rho_2) \{(\theta - 1) \kappa_1 [A_1(1) - A_1(2)] + \kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)]\} \bar{p} \end{array} \right) = 0$$

The 4 linear equations can be solved to obtain the 4 parameters  $A_{0,m}(1)$ ,  $A_{0,m}(2)$ ,  $A_{1,m}(1)$ , and  $A_{1,m}(2)$ .

### A.2.3 Riskfree Rate

The risk free rate,  $r_{f,t}$ , is priced using the Euler equation,

$$E[\exp(m_{t+1} + r_{f,t}) | F(t)] = 1.$$

Hence,

$$\begin{aligned} \exp(-r_{f,t}) &= E[\exp(m_{t+1}) | F(t)] \\ &= E\left[\exp\left(\theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1) r_{c,t+1}\right) | F(t)\right] \end{aligned}$$

By Taylor series expansion up to quadratic terms, we obtain the following:

$$-r_{f,t} = \theta \log \delta + (\theta - 1) (\kappa_0 - z_t) + \left(-\frac{\theta}{\psi} + \theta - 1\right) E[\Delta c_{t+1} | F(t)] + (\theta - 1) \kappa_1 E[z_{t+1} | F(t)] + \frac{1}{2} \Sigma''$$

where we approximate the conditional variance,  $var\left(\left(-\frac{\theta}{\psi} + \theta - 1\right) \Delta c_{t+1} + (\theta - 1) \kappa_1 z_{t+1} | F(t)\right)$ , with the constant,  $\Sigma''$ .

The above expression implies:

$$\begin{aligned}
-r_{f,t} &= \theta \log \delta + (\theta - 1)\kappa_0 - (\theta - 1) \{p_t [A_0(1) + A_1(1)x_t] + (1 - p_t) [A_0(2) + A_1(2)x_t]\} \\
&\quad + \left( -\frac{\theta}{\psi} + \theta - 1 \right) (\mu + x_t + (\sigma_1 - \sigma_2) \eta(1) f(p_t)) \\
&\quad + (\theta - 1)\kappa_1 \left( \begin{array}{l} [A_0(1) - A_0(2) + (A_1(1) - A_1(2)) \rho_2 x_t] f(p_t) \\ + A_0(2) + A_1(2)x_t (f(p_t)\rho_1 + (1 - f(p_t)) \rho_2) \\ + (A_1(1) - A_1(2)) (\rho_1 - \rho_2) \varepsilon(1)x_t f(p_t) \\ + (A_1(1) - A_1(2)) \varphi_e \sigma_{\varepsilon,e} \{ \sigma_1 f(p_t) + \sigma_2 (1 - f(p_t)) \} \\ + (A_1(1) - A_1(2)) [(\rho_1 - \rho_2) x_t + \varphi_e (\sigma_1 - \sigma_2) e(1)] \{ f(p_t) \}^2 \\ + \varphi_e (\sigma_1 - \sigma_2) e(1) A_1(2) f(p_t) \end{array} \right) \\
&\quad + \frac{1}{2} \Sigma''
\end{aligned}$$

Therefore, we obtain

$$r_{f,t} = A_{0,f} + A_{1,f}x_t + A_{2,f}p_t + A_{3,f}x_t p_t$$

where

$$\begin{aligned}
-A_{0,f} &= \left( \begin{array}{l} \theta \log \delta + (\theta - 1)\kappa_0 - (\theta - 1)A_0(2) + \frac{1}{2}\Sigma'' + \left( -\frac{\theta}{\psi} + \theta - 1 \right) (\mu + (\sigma_1 - \sigma_2) \eta(1)(1 - \pi_2)) \\ + (\theta - 1)\kappa_1 A_1(2) \varphi_e (\sigma_1 - \sigma_2) e(1)(1 - \pi_2) + (\theta - 1)\kappa_1 [A_0(1) - A_0(2)] (1 - \pi_2) \\ + (\theta - 1)\kappa_1 [A_1(1) - A_1(2)] \varphi_e (\sigma_1 - \sigma_2) e(1)(1 - \pi_2)^2 + (\theta - 1)\kappa_1 A_0(2) \\ + (\theta - 1)\kappa_1 [A_1(1) - A_1(2)] \varphi_e \sigma_{\varepsilon,e} [(\sigma_1 - \sigma_2) (1 - \pi_2) + \sigma_2] \\ - \varphi_e (\sigma_1 - \sigma_2) e(1)(\pi_1 + \pi_2 - 1)^2 (\theta - 1)\kappa_1 [A_1(1) - A_1(2)] \bar{p}^2 \end{array} \right) \\
-A_{1,f} &= \left( \begin{array}{l} -(\theta - 1)A_1(2) + \left( -\frac{\theta}{\psi} + \theta - 1 \right) \\ (\theta - 1)\kappa_1 A_1(2) [(\rho_1 - \rho_2) (1 - \pi_2) + \rho_2] \\ + (\theta - 1)\kappa_1 [A_1(1) - A_1(2)] (\rho_1 - \rho_2) (1 - \pi_2) \varepsilon(1) \\ (\theta - 1)\kappa_1 [A_1(1) - A_1(2)] [(\rho_1 - \rho_2) (1 - \pi_2)^2 + \rho_2 (1 - \pi_2)] \\ - (\pi_1 + \pi_2 - 1)^2 (\rho_1 - \rho_2) (\theta - 1)\kappa_1 [A_1(1) - A_1(2)] \bar{p}^2 \end{array} \right) \\
-A_{2,f} &= \left( \begin{array}{l} (\theta - 1) [A_0(2) - A_0(1)] + \left( -\frac{\theta}{\psi} + \theta - 1 \right) (\sigma_1 - \sigma_2) \eta(1)(\pi_1 + \pi_2 - 1) \\ + (\theta - 1)\kappa_1 A_1(2) \varphi_e (\sigma_1 - \sigma_2) e(1)(\pi_1 + \pi_2 - 1) \\ + (\theta - 1)\kappa_1 [A_0(1) - A_0(2)] (\pi_1 + \pi_2 - 1) \\ + 2(\theta - 1)\kappa_1 [A_1(1) - A_1(2)] \varphi_e (\sigma_1 - \sigma_2) e(1)(1 - \pi_2)(\pi_1 + \pi_2 - 1) \\ + (\theta - 1)\kappa_1 [A_1(1) - A_1(2)] \varphi_e \sigma_{\varepsilon,e} (\sigma_1 - \sigma_2) (\pi_1 + \pi_2 - 1) \\ + 2\varphi_e (\sigma_1 - \sigma_2) e(1)(\pi_1 + \pi_2 - 1)^2 (\theta - 1)\kappa_1 [A_1(1) - A_1(2)] \bar{p} \end{array} \right) \\
-A_{3,f} &= \left( \begin{array}{l} (\theta - 1) [A_1(2) - A_1(1)] \\ (\theta - 1)\kappa_1 A_1(2) (\rho_1 - \rho_2) (\pi_1 + \pi_2 - 1) \\ + (\theta - 1)\kappa_1 [A_1(1) - A_1(2)] [2(1 - \pi_2)(\pi_1 + \pi_2 - 1) (\rho_1 - \rho_2) + \rho_2 (\pi_1 + \pi_2 - 1)] \\ + (\theta - 1)\kappa_1 [A_1(1) - A_1(2)] (\rho_1 - \rho_2) \varepsilon(1)(\pi_1 + \pi_2 - 1) \\ + 2(\pi_1 + \pi_2 - 1)^2 (\rho_1 - \rho_2) (\theta - 1)\kappa_1 [A_1(1) - A_1(2)] \bar{p} \end{array} \right)
\end{aligned}$$

#### A.2.4 Pricing Kernel

The pricing kernel is given by equation (14),

$$m_{t+1} = \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1) r_{c,t+1}.$$

Now, the log-linearization in equation (15),

$$r_{c,t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + \Delta c_{t+1},$$

along with the solution for  $z_t$ ,

$$z_t = p_t [A_0(1) + A_1(1)x_t] + (1 - p_t) [A_0(2) + A_1(2)x_t],$$

together imply that,

$$\begin{aligned} m_{t+1} &= \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1) \kappa_0 \\ &\quad + (\theta - 1) \kappa_1 p_{t+1} [A_0(1) + A_1(1)x_{t+1}] \\ &\quad + (\theta - 1) \kappa_1 (1 - p_{t+1}) [A_0(2) + A_1(2)x_{t+1}] \\ &\quad - (\theta - 1) p_t [A_0(1) + A_1(1)x_t] - (\theta - 1) (1 - p_t) [A_0(2) + A_1(2)x_t] \\ &\quad + (\theta - 1) \Delta c_{t+1}. \end{aligned}$$

Collecting terms in the above expression, we have,

$$m_{t+1} = c_0 + c_1 \Delta c_{t+1} + c_2 p_{t+1} + c_3 p_t + c_4 x_{t+1} + c_5 x_t + c_6 p_{t+1} x_{t+1} + c_7 p_t x_t,$$

where,

$$\begin{aligned} c_0 &= \theta \log(\delta) + (\theta - 1) \kappa_0 + (\theta - 1) (\kappa_1 - 1) A_0(2), \\ c_1 &= -\frac{\theta}{\psi} + \theta - 1, \\ c_2 &= (\theta - 1) \kappa_1 [A_0(1) - A_0(2)], \\ c_3 &= -(\theta - 1) [A_0(1) - A_0(2)], \\ c_4 &= (\theta - 1) \kappa_1 A_1(2), \\ c_5 &= -(\theta - 1) A_1(2), \\ c_6 &= (\theta - 1) \kappa_1 [A_1(1) - A_1(2)], \\ c_7 &= -(\theta - 1) [A_1(1) - A_1(2)]. \end{aligned}$$

This expression for the pricing kernel involves the state variables,  $x_t$  and  $p_t$ . These are latent to the econometrician. However, note that the log price-dividend ratio of the aggregate stock market,  $z_{m,t}$ , and the risk free rate,  $r_{f,t}$ , are functions only of these two latent state variables (equations (18) and (19)),

$$\begin{aligned}
z_{m,t} &= p_t [A_{0,m}(1) + A_{1,m}(1)x_t] + (1 - p_t) [A_{0,m}(2) + A_{1,m}(2)x_t] \\
&= A_{0,m}(2) + A_{1,m}(2)x_t + [A_{0,m}(1) - A_{0,m}(2)]p_t + [A_{1,m}(1) - A_{1,m}(2)]p_t x_t,
\end{aligned}$$

$$r_{f,t} = A_{0,f} + A_{1,f}x_t + A_{2,f}p_t + A_{3,f}x_t p_t.$$

Therefore, the above two equations may be inverted to express the latent state variables,  $x_t$  and  $p_t$ , as functions of the observables,  $z_{m,t}$  and  $r_{f,t}$ . In particular, (18) implies

$$x_t = \frac{r_{f,t} - A_{0,f} - A_{2,f}p_t}{A_{1,f} + A_{3,f}p_t}. \quad (44)$$

Substituting (44) into (19), and simplifying gives the following quadratic equation for  $p_t$ :

$$ap_t^2 + b_t p_t + h_t = 0, \quad (45)$$

where

$$\begin{aligned}
a &= A_{3,f} [A_{0,m}(1) - A_{0,m}(2)] - A_{2,f} [A_{1,m}(1) - A_{1,m}(2)], \\
b_t &= [A_{1,m}(1) - A_{1,m}(2)] (r_{f,t} - A_{0,f}) + A_{1,f} [A_{0,m}(1) - A_{0,m}(2)] \\
&\quad + A_{0,m}(2)A_{3,f} - A_{1,m}(2)A_{2,f} - z_{m,t}A_{3,f}, \\
h_t &= A_{1,m}(2) (r_{f,t} - A_{0,f}) + A_{0,m}(2)A_{1,f} - z_{m,t}A_{1,f}.
\end{aligned}$$

Equation (45) implies two solutions for  $p_t$  in terms of the observables,  $z_{m,t}$  and  $r_{f,t}$ , given by

$$p_t = \frac{-b_t \pm \sqrt{b_t^2 - 4ah_t}}{2a} \quad (46)$$

Substituting the solutions in (46) into (44) gives the two corresponding solutions for  $x_t$  in terms of the observables,  $z_{m,t}$  and  $r_{f,t}$ .

### A.2.5 Equity Premium

Using the log-linearized return on the market portfolio in equation (16) and noting that the log price-dividend ratio of the market is given by equation (18), we have

$$\begin{aligned}
E(r_{m,t+1}|F(t)) &= \kappa_{0,m} + \kappa_{1,m}E[z_{m,t+1}|F(t)] - z_{m,t} + E[\Delta d_{t+1}|F(t)] \\
&= B_0 + B_1 x_t + B_2 p_t + B_3 p_t x_t
\end{aligned}$$

where

$$\begin{aligned}
B_0 &= \left( \begin{array}{l} \kappa_{0,m} - A_{0,m}(2) + \mu_d + \varphi_d (\sigma_1 - \sigma_2) u(1)(1 - \pi_2) \\ +\kappa_{1,m}A_{1,m}(2)\varphi_e (\sigma_1 - \sigma_2) e(1)(1 - \pi_2) + \kappa_{1,m} [A_{0,m}(1) - A_{0,m}(2)] (1 - \pi_2) \\ +\kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)] \varphi_e (\sigma_1 - \sigma_2) e(1)(1 - \pi_2)^2 + \kappa_{1,m}A_{0,m}(2) \\ +\kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)] \varphi_e \sigma_{\varepsilon,e} [(\sigma_1 - \sigma_2) (1 - \pi_2) + \sigma_2] \\ -\varphi_e (\sigma_1 - \sigma_2) e(1)(\pi_1 + \pi_2 - 1)^2 \kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)] \bar{p}^2 \end{array} \right) \\
B_1 &= \left( \begin{array}{l} -A_{1,m}(2) + \phi + \kappa_{1,m}A_{1,m}(2) [(\rho_1 - \rho_2) (1 - \pi_2) + \rho_2] \\ +\kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)] (\rho_1 - \rho_2) (1 - \pi_2) \varepsilon(1) \\ +\kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)] [(\rho_1 - \rho_2) (1 - \pi_2)^2 + \rho_2(1 - \pi_2)] \\ -(\pi_1 + \pi_2 - 1)^2 (\rho_1 - \rho_2) \kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)] \bar{p}^2 \end{array} \right) \\
B_2 &= \left( \begin{array}{l} [A_{0,m}(2) - A_{0,m}(1)] + \varphi_d (\sigma_1 - \sigma_2) u(1)(\pi_1 + \pi_2 - 1) \\ +\kappa_{1,m}A_{1,m}(2)\varphi_e (\sigma_1 - \sigma_2) e(1)(\pi_1 + \pi_2 - 1) \\ +\kappa_{1,m} [A_{0,m}(1) - A_{0,m}(2)] (\pi_1 + \pi_2 - 1) \\ +2\kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)] \varphi_e (\sigma_1 - \sigma_2) e(1)(1 - \pi_2)(\pi_1 + \pi_2 - 1) \\ +\kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)] \varphi_e \sigma_{\varepsilon,e} (\sigma_1 - \sigma_2) (\pi_1 + \pi_2 - 1) \\ +2\varphi_e (\sigma_1 - \sigma_2) e(1)(\pi_1 + \pi_2 - 1)^2 \kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)] \bar{p} \end{array} \right) \\
B_3 &= \left( \begin{array}{l} [A_{1,m}(2) - A_{1,m}(1)] \\ \kappa_{1,m}A_{1,m}(2) (\rho_1 - \rho_2) (\pi_1 + \pi_2 - 1) \\ +\kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)] [2(1 - \pi_2)(\pi_1 + \pi_2 - 1) (\rho_1 - \rho_2) + \rho_2(\pi_1 + \pi_2 - 1)] \\ +\kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)] (\rho_1 - \rho_2) \varepsilon(1)(\pi_1 + \pi_2 - 1) \\ +2(\pi_1 + \pi_2 - 1)^2 (\rho_1 - \rho_2) \kappa_{1,m} [A_{1,m}(1) - A_{1,m}(2)] \bar{p} \end{array} \right)
\end{aligned}$$

Now, the risk free rate is given by equation (19)

$$r_{f,t} = A_{0,f} + A_{1,f}x_t + A_{2,f}p_t + A_{3,f}x_t p_t.$$

Hence, the equity premium is given by

$$E [(r_{m,t+1} - r_{f,t}) | F(t)] = E_0 + E_1x_t + E_2p_t + E_3p_t x_t,$$

where  $E_i = B_i - A_{i,f}$ ,  $i = 0, 1, 2, 3$ .

## A.2.6 Predictive Implications for Consumption and Dividend Growth

Equation (2) implies that the expected consumption growth rate is given by

$$\begin{aligned}
&E (\Delta c_{t+1} | F(t)) \\
&= \mu + x_t + E [\sigma_{s_{t+1}} \eta_{t+1} | F(t)] \\
&= \mu + x_t + (\sigma_1 - \sigma_2) \eta(1) f(p_t) \\
&= \mu + (\sigma_1 - \sigma_2) \eta(1) (1 - \pi_2) + x_t + (\sigma_1 - \sigma_2) \eta(1) (\pi_1 + \pi_2 - 1) p_t.
\end{aligned}$$

Similarly, the expected dividend growth rate is given by

$$\begin{aligned}
& E(\Delta d_{t+1}|F(t)) \\
&= \mu_d + \phi x_t + \varphi_d E[\sigma_{s_{t+1}} u_{t+1}|F(t)] \\
&= \mu_d + \phi x_t + \varphi_d (\sigma_1 - \sigma_2) u(1) f(p_t) \\
&= \mu_d + \varphi_d (\sigma_1 - \sigma_2) u(1) (1 - \pi_2) + \phi x_t + \varphi_d (\sigma_1 - \sigma_2) u(1) (\pi_1 + \pi_2 - 1) p_t.
\end{aligned}$$

Therefore, the expected consumption and dividend growth rates are both linear functions of the state variables,  $x_t$  and  $p_t$ .

Finally, the model implies that the conditional variance of the aggregate consumption and dividend growth rates are functions of the probability,  $p_t$ , alone:

$$\begin{aligned}
Var(\Delta c_{t+1}|F(t)) &= Var[\sigma_{s_{t+1}} \eta_{t+1}|F(t)] \\
&= E[\sigma_{s_{t+1}}^2 \eta_{t+1}^2|F(t)] - (E[\sigma_{s_{t+1}} \eta_{t+1}|F(t)])^2 \\
&= c_1 + d_1 p_t + e_1 p_t^2,
\end{aligned}$$

where

$$\begin{aligned}
c_1 &= \sigma_2^2 + (\sigma_1^2 - \sigma_2^2) (1 - \pi_2) - (\sigma_1 - \sigma_2)^2 \eta(1)^2 (1 - \pi_2)^2, \\
d_1 &= [(\sigma_1^2 - \sigma_2^2) (\pi_1 + \pi_2 - 1) - 2(\sigma_1 - \sigma_2)^2 \eta(1)^2 (1 - \pi_2) (\pi_1 + \pi_2 - 1)], \\
e_1 &= -(\sigma_1 - \sigma_2)^2 \eta(1)^2 (\pi_1 + \pi_2 - 1)^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
Var(\Delta d_{t+1}|F(t)) &= Var[\varphi_d \sigma_{s_{t+1}} u_{t+1}|F(t)] \\
&= \varphi_d^2 E[\sigma_{s_{t+1}}^2 u_{t+1}^2|F(t)] - \varphi_d^2 (E[\sigma_{s_{t+1}} u_{t+1}|F(t)])^2 \\
&= c_2 + d_2 p_t + e_2 p_t^2,
\end{aligned}$$

where

$$\begin{aligned}
c_2 &= \varphi_d^2 [\sigma_2^2 + (\sigma_1^2 - \sigma_2^2) (1 - \pi_2) - (\sigma_1 - \sigma_2)^2 u(1)^2 (1 - \pi_2)^2], \\
d_2 &= \varphi_d^2 [(\sigma_1^2 - \sigma_2^2) (\pi_1 + \pi_2 - 1) - 2(\sigma_1 - \sigma_2)^2 u(1)^2 (1 - \pi_2) (\pi_1 + \pi_2 - 1)], \\
e_2 &= -\varphi_d^2 (\sigma_1 - \sigma_2)^2 u(1)^2 (\pi_1 + \pi_2 - 1)^2.
\end{aligned}$$

Expanding the term  $p_t^2$  as a Taylor series to first order around the unconditional mean,  $\bar{p}$ , of  $p_t$ , we obtain equations (24) and (25) for the conditional variance of the aggregate consumption and dividend growth rates, respectively.

### A.3 Time-Series Moments

We compute the unconditional moments of the aggregate consumption and dividend growth rates. To do that, we first compute the unconditional expectations of the state variables  $x_t$  and  $p_t$ , and their cross-product  $x_t p_t$ . Note that

$$\begin{aligned}
E[x_{t+1}|F(t)] &= E[\rho_{s_{t+1}}|F(t)]x_t + \varphi_e E[\sigma_{s_{t+1}}e_{t+1}|F(t)] \\
&= [f(p_t)\rho_1 + (1-f(p_t))\rho_2]x_t + \varphi_e(\sigma_1 - \sigma_2)e(1)f(p_t) \\
&= \varphi_e(\sigma_1 - \sigma_2)e(1)(1 - \pi_2) + [(1 - \pi_2)\rho_1 + \pi_2\rho_2]x_t \\
&\quad + \varphi_e(\sigma_1 - \sigma_2)e(1)(\pi_1 + \pi_2 - 1)p_t + (\pi_1 + \pi_2 - 1)(\rho_1 - \rho_2)p_t x_t
\end{aligned}$$

Taking expectations of the two sides of the above equation gives

$$\begin{aligned}
&E(x_t)[1 - (1 - \pi_2)\rho_1 - \pi_2\rho_2] \\
&= \varphi_e(\sigma_1 - \sigma_2)e(1)(1 - \pi_2) + \varphi_e(\sigma_1 - \sigma_2)e(1)(\pi_1 + \pi_2 - 1)E(p_t) \\
&\quad + (\pi_1 + \pi_2 - 1)(\rho_1 - \rho_2)E(p_t x_t)
\end{aligned} \tag{47}$$

Also,

$$\begin{aligned}
E[x_{t+1}p_{t+1}|F(t)] &= E[(\rho_{s_{t+1}}x_t + \varphi_e\sigma_{s_{t+1}}e_{t+1})((1 - \pi_2) + (\pi_1 + \pi_2 - 1)p_t + \varepsilon_{t+1})|F(t)] \\
&= (1 - \pi_2)[(1 - \pi_2)\rho_1 + \pi_2\rho_2]x_t + (1 - \pi_2)(\pi_1 + \pi_2 - 1)(\rho_1 - \rho_2)p_t x_t \\
&\quad + \varphi_e(\sigma_1 - \sigma_2)e(1)(1 - \pi_2)^2 + (1 - \pi_2)\varphi_e(\sigma_1 - \sigma_2)e(1)(\pi_1 + \pi_2 - 1)p_t \\
&\quad + (\pi_1 + \pi_2 - 1)[(1 - \pi_2)\rho_1 + \pi_2\rho_2]p_t x_t + (\pi_1 + \pi_2 - 1)^2(\rho_1 - \rho_2)p_t^2 x_t \\
&\quad + (\pi_1 + \pi_2 - 1)\varphi_e(\sigma_1 - \sigma_2)e(1)(1 - \pi_2)p_t + \varphi_e(\sigma_1 - \sigma_2)e(1)(\pi_1 + \pi_2 - 1)^2 p_t^2 \\
&\quad + (\rho_1 - \rho_2)\varepsilon(1)(1 - \pi_2)x_t + (\rho_1 - \rho_2)\varepsilon(1)(\pi_1 + \pi_2 - 1)p_t x_t \\
&\quad + \varphi_e\sigma_{\varepsilon\varepsilon}\sigma_2 + \varphi_e\sigma_{\varepsilon\varepsilon}(\sigma_1 - \sigma_2)(1 - \pi_2) + \varphi_e\sigma_{\varepsilon\varepsilon}(\sigma_1 - \sigma_2)(\pi_1 + \pi_2 - 1)p_t
\end{aligned}$$

Taking a first order Taylor series approximation of  $p_t^2 \approx [E(p_t)]^2 + 2E(p_t)(p_t - E(p_t))$ , and then taking expectations of both sides of the above expression gives

$$\begin{aligned}
E(x_{t+1}p_{t+1}) &= \left( \varphi_e(\sigma_1 - \sigma_2)e(1)(1 - \pi_2)^2 - \varphi_e(\sigma_1 - \sigma_2)e(1)(\pi_1 + \pi_2 - 1)^2[E(p_t)]^2 \right. \\
&\quad \left. + \varphi_e\sigma_{\varepsilon\varepsilon}\sigma_2 + \varphi_e\sigma_{\varepsilon\varepsilon}(\sigma_1 - \sigma_2)(1 - \pi_2) \right) \\
&\quad + \left( (1 - \pi_2)[(1 - \pi_2)\rho_1 + \pi_2\rho_2] - (\pi_1 + \pi_2 - 1)^2(\rho_1 - \rho_2)[E(p_t)]^2 \right. \\
&\quad \left. + (\rho_1 - \rho_2)\varepsilon(1)(1 - \pi_2) \right) E(x_t) \\
&\quad + \left( 2(1 - \pi_2)\varphi_e(\sigma_1 - \sigma_2)e(1)(\pi_1 + \pi_2 - 1) + \varphi_e\sigma_{\varepsilon\varepsilon}(\sigma_1 - \sigma_2)(\pi_1 + \pi_2 - 1) \right. \\
&\quad \left. + 2(\pi_1 + \pi_2 - 1)^2\varphi_e(\sigma_1 - \sigma_2)e(1)E(p_t) \right) E(p_t) \\
&\quad + \left( (1 - \pi_2)(\pi_1 + \pi_2 - 1)(\rho_1 - \rho_2) + (\pi_1 + \pi_2 - 1)[(1 - \pi_2)\rho_1 + \pi_2\rho_2] \right. \\
&\quad \left. + 2(\pi_1 + \pi_2 - 1)^2(\rho_1 - \rho_2)E(p_t) + (\rho_1 - \rho_2)\varepsilon(1)(\pi_1 + \pi_2 - 1) \right) E(p_t x_t)
\end{aligned} \tag{48}$$

Note that  $E(p_t) = \frac{1 - \pi_2}{2 - \pi_1 - \pi_2}$ . Therefore, the equations (47) and (48) can be solved to obtain  $E(x_t)$  and  $E(p_t x_t)$ .

Now,

$$E(\Delta c_{t+1}) = \mu + E(x_t) + E[\sigma_{s_{t+1}}\eta_{t+1}] \tag{49}$$

Therefore, the unconditional mean of the consumption growth rate can be computed using the expression for  $E(x_t)$  obtained above and the expression for  $E[\sigma_{s_{t+1}}\eta_{t+1}|F(t)] = (\sigma_1 - \sigma_2)\eta(1)f(p_t)$  obtained in Appendix A.2 (implying that  $E[\sigma_{s_{t+1}}\eta_{t+1}] = (\sigma_1 - \sigma_2)\eta(1)(1 - \pi_2) + (\sigma_1 - \sigma_2)\eta(1)(\pi_1 + \pi_2 - 1)E(p_t)$ ).

Similarly, we obtain

$$E(\Delta d_{t+1}) = \mu_d + \phi E(x_t) + \varphi_d E[\sigma_{s_{t+1}}u_{t+1}] \quad (50)$$

by noting from Appendix A.2 that  $E[\sigma_{s_{t+1}}u_{t+1}|F(t)] = (\sigma_1 - \sigma_2)u(1)f(p_t)$  implying that  $E[\sigma_{s_{t+1}}u_{t+1}] = (\sigma_1 - \sigma_2)u(1)(1 - \pi_2) + (\sigma_1 - \sigma_2)u(1)(\pi_1 + \pi_2 - 1)E(p_t)$ .

Next, we compute the unconditional variances of consumption and dividend growth rates. Note that,

$$Var(\Delta c_{t+1}) = Var(x_t) + Var(\sigma_{s_{t+1}}\eta_{t+1}) + 2Cov(x_t, \sigma_{s_{t+1}}\eta_{t+1}). \quad (51)$$

Consider first the second term of equation (51):

$$\begin{aligned} Var(\sigma_{s_{t+1}}\eta_{t+1}) &= E(\sigma_{s_{t+1}}^2\eta_{t+1}^2) - \{E(\sigma_{s_{t+1}}\eta_{t+1})\}^2 \\ &= E[E(\sigma_{s_{t+1}}^2\eta_{t+1}^2|s_{t+1})] - \{E(\sigma_{s_{t+1}}\eta_{t+1})\}^2 \\ &= \frac{1 - \pi_2}{2 - \pi_1 - \pi_2}\sigma_1^2 + \frac{1 - \pi_1}{2 - \pi_1 - \pi_2}\sigma_2^2 \\ &\quad - \{(\sigma_1 - \sigma_2)\eta(1)(1 - \pi_2) + (\sigma_1 - \sigma_2)\eta(1)(\pi_1 + \pi_2 - 1)E(p_t)\}^2 \end{aligned}$$

Consider next the third term of equation (51):

$$\begin{aligned} Cov(x_t, \sigma_{s_{t+1}}\eta_{t+1}) &= E(x_t\sigma_{s_{t+1}}\eta_{t+1}) - E(x_t)E(\sigma_{s_{t+1}}\eta_{t+1}) \\ &= E[E(x_t\sigma_{s_{t+1}}\eta_{t+1}|F(t))] - E(x_t)E(\sigma_{s_{t+1}}\eta_{t+1}) \\ &= E[x_t(\sigma_1 - \sigma_2)\eta(1)f(p_t)] - E(x_t)E(\sigma_{s_{t+1}}\eta_{t+1}) \\ &= (\sigma_1 - \sigma_2)\eta(1)(\pi_1 + \pi_2 - 1)[E(p_t x_t) - E(p_t)E(x_t)] \end{aligned}$$

Finally, consider the first term of equation (51):

$$\begin{aligned} Var(x_{t+1}) &\equiv E(x_{t+1}^2) - [E(x_{t+1})]^2 \\ &= Var(\rho_{s_{t+1}}x_t) + Var(\varphi_e\sigma_{s_{t+1}}e_{t+1}) + 2Cov(\rho_{s_{t+1}}x_t, \varphi_e\sigma_{s_{t+1}}e_{t+1}) \end{aligned} \quad (52)$$

Now,

$$\begin{aligned} Var(\varphi_e\sigma_{s_{t+1}}e_{t+1}) &= \varphi_e^2 \left( \frac{1 - \pi_2}{2 - \pi_1 - \pi_2}\sigma_1^2 + \frac{1 - \pi_1}{2 - \pi_1 - \pi_2}\sigma_2^2 \right) \\ &\quad - \varphi_e^2 \{(\sigma_1 - \sigma_2)e(1)(1 - \pi_2) + (\sigma_1 - \sigma_2)e(1)(\pi_1 + \pi_2 - 1)E(p_t)\}^2 \end{aligned}$$

$$\begin{aligned}
Cov(\rho_{s_{t+1}}x_t, \varphi_e\sigma_{s_{t+1}}e_{t+1}) &= \varphi_e E(\rho_{s_{t+1}}\sigma_{s_{t+1}}e_{t+1}x_t) - E(\rho_{s_{t+1}}x_t) E(\varphi_e\sigma_{s_{t+1}}e_{t+1}) \\
&= \varphi_e E[x_t E(\rho_{s_{t+1}}\sigma_{s_{t+1}}e_{t+1}|F(t))] - E(\rho_{s_{t+1}}x_t) E(\varphi_e\sigma_{s_{t+1}}e_{t+1}) \\
&= \varphi_e E[(\sigma_1\rho_1 - \sigma_2\rho_2)e(1)f(p_t)x_t] \\
&\quad - \varphi_e E[(f(p_t)\rho_1 - (1-f(p_t))\rho_2)x_t](\sigma_1 - \sigma_2)e(1)E[f(p_t)] \\
&= \varphi_e(\sigma_1\rho_1 - \sigma_2\rho_2)e(1)(1 - \pi_1)E(x_t) \\
&\quad + \varphi_e(\sigma_1\rho_1 - \sigma_2\rho_2)e(1)(\pi_1 + \pi_2 - 1)E(p_t x_t) \\
&\quad - \varphi_e(\sigma_1 - \sigma_2)e(1)E[(f(p_t)\rho_1 - (1-f(p_t))\rho_2)x_t]E[f(p_t)]
\end{aligned}$$

$$\begin{aligned}
Var(\rho_{s_{t+1}}x_t) &= E(\rho_{s_{t+1}}^2x_t^2) - [E(\rho_{s_{t+1}}x_t)]^2 \\
&= E[(f(p_t)\rho_1^2 + (1-f(p_t))\rho_2^2)x_t^2] - \{E[(f(p_t)\rho_1 + (1-f(p_t))\rho_2)x_t]\}^2 \\
&= E[(\rho_1^2 - \rho_2^2)(1 - \pi_2)x_t^2 + (\rho_1^2 - \rho_2^2)(\pi_1 + \pi_2 - 1)p_t x_t^2 + \rho_2^2 x_t^2] \\
&\quad - \{E[(\rho_1 - \rho_2)(1 - \pi_2)x_t + (\rho_1 - \rho_2)(\pi_1 + \pi_2 - 1)p_t x_t + \rho_2 x_t]\}^2
\end{aligned}$$

Approximating  $E(p_t x_t^2) \approx E(p_t)E(x_t^2)$ , we solve equation (52) for  $E(x_t^2)$ . Substituting the expressions for  $Var(x_t)$ ,  $Var(\sigma_{s_{t+1}}\eta_{t+1})$ , and  $Cov(x_t, \sigma_{s_{t+1}}\eta_{t+1})$  into equation (51) gives the unconditional variance of consumption growth. Similarly, the unconditional variance of the dividend growth rate may be obtained as:

$$Var(\Delta d_{t+1}) = \phi^2 Var(x_t) + \varphi_d^2 Var(\sigma_{s_{t+1}}u_{t+1}) + 2\phi\varphi_d Cov(x_t, \sigma_{s_{t+1}}u_{t+1}). \quad (53)$$

Finally, we have

$$\begin{aligned}
Cov(\Delta c_{t+1}, \Delta d_{t+1}) &= Cov(x_t + \sigma_{s_{t+1}}\eta_{t+1}, \phi x_t + \varphi_d \sigma_{s_{t+1}}u_{t+1}) \\
&= \phi Var(x_t) + Cov(x_t, \varphi_d \sigma_{s_{t+1}}u_{t+1}) + Cov(\sigma_{s_{t+1}}\eta_{t+1}, \phi x_t) \\
&\quad + Cov(\sigma_{s_{t+1}}\eta_{t+1}, \varphi_d \sigma_{s_{t+1}}u_{t+1})
\end{aligned} \quad (54)$$

where  $Cov(x_t, \varphi_d \sigma_{s_{t+1}}u_{t+1}) = \varphi_d \{(\sigma_1 - \sigma_2)u(1)(\pi_1 + \pi_2 - 1)[E(p_t x_t) - E(p_t)E(x_t)]\}$ ,  
 $Cov(x_t, \phi \sigma_{s_{t+1}}\eta_{t+1}) = \phi \{(\sigma_1 - \sigma_2)\eta(1)(\pi_1 + \pi_2 - 1)[E(p_t x_t) - E(p_t)E(x_t)]\}$ , and  
 $Cov(\sigma_{s_{t+1}}\eta_{t+1}, \varphi_d \sigma_{s_{t+1}}u_{t+1}) = \varphi_d p \eta(1)u(1) \left[ \sigma_1^2 - \sigma_2^2 \frac{p}{1-p} \right] - \varphi_d E(\sigma_{s_{t+1}}\eta_{t+1}) E(\sigma_{s_{t+1}}u_{t+1})$ .

**Table 1: Parameter Estimates, 1930-2009**

$\mu$	$\mu_d$	$\phi$	$\varphi_d$	$\rho_1$	$\rho_2$	$\varphi_e$	$\sigma_1$	$\sigma_2$	$\pi_1$	$\pi_2$
0.005 (0.101)	-0.044 (0.330)	3.5 (0.15)	2.0 (0.26)	0.94 (0.46)	0.6 (0.90)	0.46 (0.90)	0.005 (0.073)	0.035 (0.080)	0.95 (0.23)	0.85 (0.63)
$\eta(1)$	$e(1)$	$u(1)$	$\varepsilon(1)$	$\sigma_{\varepsilon\varepsilon}$	$\delta$	$\gamma$	$\psi$			
0 (0.72)	-0.25 (0.20)	0 (0.39)	0.02 (0.17)	-0.5 (0.39)	0.976 (0.342)	12 (20.2)	0.9 (0.044)			
	<i>Data</i>	<i>Model</i>					<i>Data</i>	<i>Model</i>		
$E(r_f)$	0.008 (0.008)	0.018 [0.011,0.026]					$E(\Delta c)$	0.015 (0.003)	0.014 [-0.01,0.029]	
$\sigma(r_f)$	0.050 (0.008)	0.011 [0.004,0.016]					$sd(\Delta c)$	0.025 (0.004)	0.036 [0.012,0.050]	
$E(r_m - r_f)$	0.058 (0.020)	0.040 [-.01,0.099]					$E(\Delta d)$	0.017 (0.013)	0.018 [-0.05,0.088]	
$\sigma(r_m - r_f)$	0.199 (0.021)	0.234 [0.105,0.350]					$sd(\Delta d)$	0.117 (0.020)	0.118 [0.042,0.172]	
$E(p/d)$	3.377 (0.082)	3.187 [2.88,3.49]					$\rho_{\Delta c, \Delta d}$	0.59 (0.26)	0.50 [0.067,0.792]	
$\sigma(p/d)$	0.450 (0.054)	0.421 [0.17,0.63]								
$E(R_s - R_b)$	0.094 (0.043)	0.083								
$E(R_v - R_g)$	0.073 (0.027)	0.037								

The table reports GMM estimates (asymptotic standard errors in parentheses) of the model parameters defined in Section 2.1. It also reports the model-implied moments, obtained from a single simulation of 1 million observations, and the 95% confidence interval (in square brackets), obtained through 10000 simulations of the same length as the historical data, and the sample values (asymptotic standard errors in parentheses) of the mean and volatility of the risk free rate, price-dividend ratio, equity, size, and value premia, and unconditional moments of the consumption and dividend growth rates.

**Table 2: Calibrated Parameter Values**

$\mu$	$\mu_d$	$\phi$	$\varphi_d$	$\rho_1$	$\rho_2$	$\varphi_e$	$\sigma_1$	$\sigma_2$	$\pi_1$	$\pi_2$
0.025	0.017	7.5	2.0	0.90	0.60	0.46	0.005	0.040	0.95	0.85
$\eta(1)$	$e(1)$	$u(1)$	$\varepsilon(1)$	$\sigma_{\varepsilon e}$	$\delta$	$\gamma$	$\psi$			
0.50	-0.05	0.30	0.068	-0.20	0.9989	12.0	1.5			

The table reports the calibrated values of the model parameters defined in Section 2.1.

**Table 3: Summary Statistics in the Two Regimes, 1931-2009**

	<i>Panel A: Regime 1</i>				<i>Panel B: Regime 2</i>			
	<i>Data</i>		<i>Model</i>		<i>Data</i>		<i>Model</i>	
	<i>E(.)</i>	<i>sd(.)</i>	<i>E(.)</i>	<i>sd(.)</i>	<i>E(.)</i>	<i>sd(.)</i>	<i>E(.)</i>	<i>sd(.)</i>
$\Delta c$	0.016 (0.002)	0.017 (0.002)	0.025 [0.013,0.036]	0.020 [0.009,0.034]	0.007 (0.011)	0.044 (0.008)	-0.020 [-0.044,0.004]	0.049 [0.034,0.062]
$\Delta d$	0.036 (0.011)	0.067 (0.008)	0.024 [-0.05,0.103]	0.084 [0.034,0.138]	-0.063 (0.047)	0.204 (0.038)	0.000 [-0.133,0.122]	0.185 [0.110,0.243]
$\Delta gdp$	0.040 (0.005)	0.032 (0.004)			0.019 (0.017)	0.064 (0.007)		
<i>Inflation</i>	0.031 (0.005)	0.032 (0.004)			0.018 (0.024)	0.084 (0.016)		
$\log(P/D)$	3.474 (0.090)	0.432 (0.056)	3.23 [2.89,3.60]	0.385 [0.156,0.632]	3.047 (0.092)	0.349 (0.060)	3.066 [2.705,3.420]	0.492 [0.264,0.723]
$r_f$	0.020 (0.003)	0.021 (0.002)	0.015 [0.009,0.022]	0.007 [0.003,0.012]	-0.033 (0.026)	0.088 (0.020)	0.028 [0.016,0.038]	0.015 [0.008,0.020]
$r_m$	0.053 (0.023)	0.185 (0.018)	0.040 [-0.01,0.080]	0.145 [0.085,0.263]	0.111 (0.054)	0.233 (0.058)	0.144 [-0.076,0.337]	0.393 [0.250,0.559]
$r_m - r_f$	0.033 (0.022)	0.182 (0.017)	0.025 [-0.03,0.060]	0.144 [0.084,0.264]	0.144 (0.059)	0.235 (0.068)	0.086 [-0.095,0.304]	0.390 [0.251,0.552]
$r_s - r_b$	0.007 (0.033)	0.210 (0.022)			0.182 (0.067)	0.253 (0.024)		
$r_v - r_g$	0.024 (0.029)	0.201 (0.018)			0.116 (0.046)	0.218 (0.030)		

Panel A reports the sample mean and volatility (asymptotic standard errors in parentheses) of consumption, dividend, and GDP growth rates, the rate of inflation, log price-dividend ratio, risk free rate, market return, and equity, size, and value premia in the first regime. It also reports the model-implied values, obtained from a single simulation of 1 million observations, and the 95% confidence interval (in square brackets), obtained through 10000 simulations of the same length as the historical data, of the mean and volatility of consumption and dividend growth rates, the log price-dividend ratio, risk free rate, market return, and equity premium in the first regime, obtained through 10000 simulations. Panel B reports the corresponding moments in the second regime.

**Table 4: Forecastability of the Equity, Size, and Value Premia, 1931-2009**

<i>Panel A: Equity Premium</i>							
<i>const.</i>	<i>x</i>	<i>p</i>	<i>xp</i>	<i>log(P/D)</i>	<i>r<sub>f</sub></i>	<i>log(P/D) r<sub>f</sub></i>	<i>Adjusted-R<sup>2</sup></i>
0.13 (0.05)	-0.80 (0.83)	-0.10* (0.06)	1.79 (1.81)				0.074
0.36 (0.17)				-0.09* (0.05)			0.028
0.33 (0.17)				-0.08 (0.05)	-0.60 (0.45)		0.038
0.40 (0.19)				-0.10* (0.06)	-3.41 (3.61)	0.96 (1.23)	0.034
<i>Panel B: Size Premium</i>							
<i>const.</i>	<i>x</i>	<i>p</i>	<i>xp</i>	<i>log(P/D)</i>	<i>r<sub>f</sub></i>	<i>log(P/D) r<sub>f</sub></i>	<i>Adjusted-R<sup>2</sup></i>
0.20 (0.05)	1.13 (0.91)	-0.21*** (0.06)	-5.45*** (1.98)				0.143
0.28 (0.20)				-0.07 (0.06)			0.006
0.25 (0.19)				-0.06 (0.06)	-0.86* (0.52)		0.028
0.02 (0.21)				0.02 (0.06)	9.11** (4.07)	-3.42** (1.38)	0.089
<i>Panel C: Value Premium</i>							
<i>const.</i>	<i>x</i>	<i>p</i>	<i>xp</i>	<i>log(P/D)</i>	<i>r<sub>f</sub></i>	<i>log(P/D) r<sub>f</sub></i>	<i>Adjusted-R<sup>2</sup></i>
0.15 (0.05)	1.43* (0.86)	-0.14** (0.06)	-2.69* (1.87)				0.048
0.17 (0.18)				-0.04 (0.05)			-0.007
0.16 (0.18)				-0.03 (0.05)	-0.06 (0.48)		-0.020
0.07 (0.20)				-0.003 (0.06)	4.25 (3.85)	-1.48 (1.31)	-0.016

Panels A, B, and C report results from forecasting regressions for the equity, size, and value premia, respectively. The first row of each panel reports the regression coefficients along with the associated standard errors in parentheses, and the adjusted- $R^2$  from the forecasting regression of the realized premium on  $x$ ,  $p$ , and  $xp$ . The second, third, and fourth rows report, respectively, the corresponding results when the set of predictor variables consists of the lagged aggregate log price-dividend ratio, the price-dividend ratio and log risk free rate, and the price-dividend ratio, risk free rate, and their product. One, two, and three asterisks signify statistical significance at the 10%, 5%, and 1% levels, respectively.

**Table 5: In- and Out-of-Sample Forecastability of Equity, Size, and Value Premia**

<i>Panel A: Equity Premium, 1976-2009</i>								
<i>const.</i>	<i>x</i>	<i>p</i>	<i>xp</i>	<i>log (P/D)</i>	<i>r<sub>f</sub></i>	<i>log (P/D) r<sub>f</sub></i>	<i>Adjusted-R<sup>2</sup></i>	<i>R<sup>2</sup><sub>OOS</sub></i>
0.09 (0.06)	2.01 (2.36)	-0.02 (0.07)	1.32 (3.77)				0.085	0.052
0.36 (0.23)				-0.08 (0.06)			0.022	-0.046
0.35 (0.25)				-0.08 (0.07)	0.06 (1.10)		-0.010	-0.025
0.80 (0.31)				-0.21*** (0.08)	-20.5** (9.64)	5.82** (2.72)	0.095	-0.029
<i>Panel B: Size Premium, 1976-2009</i>								
<i>const.</i>	<i>x</i>	<i>p</i>	<i>xp</i>	<i>log (P/D)</i>	<i>r<sub>f</sub></i>	<i>log (P/D) r<sub>f</sub></i>	<i>Adjusted-R<sup>2</sup></i>	<i>R<sup>2</sup><sub>OOS</sub></i>
0.09 (0.06)	-1.57 (2.37)	-0.09 (0.07)	-6.51* (3.78)				0.255	0.226
-0.03 (0.29)				0.01 (0.08)			-0.030	-0.149
0.19 (0.27)				-0.02 (0.07)	-3.39*** (1.20)		0.155	-0.068
0.08 (0.37)				0.006 (0.099)	1.50 (11.2)	-1.39 (3.17)	0.133	0.053
<i>Panel C: Value Premium, 1976-2009</i>								
<i>const.</i>	<i>x</i>	<i>p</i>	<i>xp</i>	<i>log (P/D)</i>	<i>r<sub>f</sub></i>	<i>log (P/D) r<sub>f</sub></i>	<i>Adjusted-R<sup>2</sup></i>	<i>R<sup>2</sup><sub>OOS</sub></i>
0.15 (0.06)	3.45 (2.59)	-0.09 (0.08)	-8.49** (4.14)				0.059	-0.003
0.22 (0.28)				-0.04 (0.08)			-0.022	-0.114
0.29 (0.29)				-0.05 (0.08)	-1.05 (1.29)		-0.033	-0.119
-0.09 (0.38)				0.05 (0.10)	16.4 (11.7)	-4.96 (3.30)	0.008	-0.103

Panels *A*, *B*, and *C* report in-sample and out-of-sample forecasting results for the equity, size, and value premia, respectively. The first row of each panel reports the in-sample regression coefficients along with the standard errors in parentheses, and the adjusted- $R^2$  from the forecasting regression of the realized premium on  $x$ ,  $p$ , and  $xp$ . It also reports the out-of-sample  $R^2$  from rolling predictive regressions on  $x$ ,  $p$ , and  $xp$ . The second, third, and fourth rows report, respectively, the corresponding results when the set of predictor variables consists of the lagged aggregate log price-dividend ratio, the price-dividend ratio and log risk free rate, and the price-dividend ratio, risk free rate, and their product. One, two, and three asterisks signify statistical significance at the 10%, 5%, and 1% levels, respectively.

**Table 6: Forecastability of the Consumption Growth Rate**

<i>Panel A: Consumption Growth, 1931-2009</i>						
<i>const.</i>	<i>x</i>	<i>p</i>	<i>log(P/D)</i>	<i>r<sub>f</sub></i>	<i>Adjusted - R<sup>2</sup></i>	
0.004 (0.006)	-0.233*** (0.088)	0.012* (0.007)			0.080	
-0.039 (0.021)			0.016*** (0.006)		0.068	
-0.043 (0.021)			0.017*** (0.006)	-0.086 (0.056)	0.084	
<i>Panel B: Consumption Growth, 1947-2009</i>						
<i>const.</i>	<i>x</i>	<i>p</i>	<i>log(P/D)</i>	<i>r<sub>f</sub></i>	<i>Adjusted - R<sup>2</sup></i>	
0.006 (0.006)	0.302*** (0.089)	0.008 (0.006)			0.214	
-0.025 (0.016)			0.011** (0.005)		0.067	
-0.014 (0.015)			0.007* (0.004)	0.201*** (0.052)	0.239	
<i>Panel C: Consumption Growth, 1976-2009</i>						
<i>const.</i>	<i>x</i>	<i>p</i>	<i>log(P/D)</i>	<i>r<sub>f</sub></i>	<i>Adjusted - R<sup>2</sup></i>	<i>R<sup>2</sup><sub>OOS</sub></i>
0.011 (0.004)	0.269** (0.122)	0.006 (0.005)			0.156	0.007
-0.009 (0.020)			0.006 (0.005)		0.009	-0.959
-0.025 (0.018)			0.009* (0.005)	0.264*** (0.081)	0.236	-1.232

Panels A and B report in-sample forecasting results for consumption growth over 1931–2009 and 1947–2009, respectively. Panel C reports in-sample forecasting and out-of-sample predictive results over 1976–2009. One, two, and three asterisks signify statistical significance at the 10%, 5%, and 1% levels, respectively.

**Table 7: Forecastability of the Dividend Growth Rate**

<i>Panel A: Dividend Growth, 1931-2009</i>						
<i>const.</i>	<i>x</i>	<i>p</i>	<i>log(P/D)</i>	<i>r<sub>f</sub></i>	<i>Adjusted - R<sup>2</sup></i>	
-0.071 (0.027)	-0.347 (0.401)	0.111*** (0.032)			0.117	
-0.256 (0.097)			0.080*** (0.029)		0.080	
-0.251 (0.099)			0.078*** (0.029)	0.119 (0.263)	0.070	
<i>Panel B: Dividend Growth, 1947-2009</i>						
<i>const.</i>	<i>x</i>	<i>p</i>	<i>log(P/D)</i>	<i>r<sub>f</sub></i>	<i>Adjusted - R<sup>2</sup></i>	
0.013 (0.030)	1.077** (0.448)	0.015 (0.032)			0.087	
-0.002 (0.078)			0.007 (0.022)		-0.015	
0.037 (0.076)			-0.007 (0.022)	0.738*** (0.269)	0.084	
<i>Panel C: Dividend Growth, 1976-2009</i>						
<i>const.</i>	<i>x</i>	<i>p</i>	<i>log(P/D)</i>	<i>r<sub>f</sub></i>	<i>Adjusted - R<sup>2</sup></i>	<i>R<sup>2</sup><sub>OOS</sub></i>
-0.000 (0.024)	0.536 (0.734)	0.041 (0.032)			0.029	0.044
-0.022 (0.114)			0.013 (0.031)		-0.026	-0.602
-0.089 (0.113)			0.025 (0.030)	1.068** (0.502)	0.076	-0.603

Panels A and B report in-sample forecasting results for dividend growth over 1931–2009 and 1947–2009, respectively. Panel C reports in-sample forecasting and out-of-sample predictive results over 1976–2009. One, two, and three asterisks signify statistical significance at the 10%, 5%, and 1% levels, respectively.

**Table 8: Forecastability of Variance of Market Return and Growth Rates, 1931-2009**

<i>Panel A: Variance of Market Return</i>							
<i>const.</i>	<i>x</i>	<i>p</i>	<i>xp</i>	<i>log (P/D)</i>	<i>r<sub>f</sub></i>	<i>log (P/D) r<sub>f</sub></i>	<i>Adjusted-R<sup>2</sup></i>
0.003 (0.001)	0.016 (0.012)	-0.002** (0.001)	-0.031 (0.026)				0.021
0.005 (0.002)				-0.001 (0.001)			0.010
0.005 (0.002)				-0.001 (0.001)	0.003 (0.006)		0.001
0.005 (0.003)				-0.001 (0.001)	-0.022 (0.052)	0.009 (0.018)	-0.009
<i>Panel B: Variance of Consumption Growth, <math>(\Delta c_{t+1} - E[\Delta c_{t+1} F(t)])^2</math></i>							
<i>const.</i>	<i>x</i>	<i>p</i>	<i>xp</i>	<i>log (P/D)</i>	<i>r<sub>f</sub></i>	<i>log (P/D) r<sub>f</sub></i>	<i>Adjusted-R<sup>2</sup></i>
0.0014 (0.0003)		-0.0011*** (0.0003)					0.127
0.0014 (0.0003)	0.002 (0.004)	-0.0011*** (0.0003)					0.119
0.003 (0.001)				-0.0006 (0.0004)			0.027
0.002 (0.001)				-0.0005 (0.0003)	0.0008 (0.0028)		0.006
<i>Panel C: Variance of Dividend Growth, <math>(\Delta d_{t+1} - E[\Delta d_{t+1} F(t)])^2</math></i>							
<i>const.</i>	<i>x</i>	<i>p</i>	<i>xp</i>	<i>log (P/D)</i>	<i>r<sub>f</sub></i>	<i>log (P/D) r<sub>f</sub></i>	<i>Adjusted-R<sup>2</sup></i>
0.036 (0.007)		-0.032*** (0.008)					0.155
0.036 (0.007)	-0.043 (0.105)	-0.032*** (0.008)					0.146
0.061 (0.025)				-0.014** (0.007)			0.034
0.062 (0.025)				-0.015** (0.007)	0.016 (0.068)		0.024

Panels A, B, and C report results of forecasting regressions for the variance of the market return and consumption and dividend growth rates, respectively. One, two, and three asterisks signify statistical significance at the 10%, 5%, and 1% levels, respectively.

**Figure 1: The Probability of the First Regime ( $p$ ) as a Function of  $\log(P/D)$  and  $R_f$**

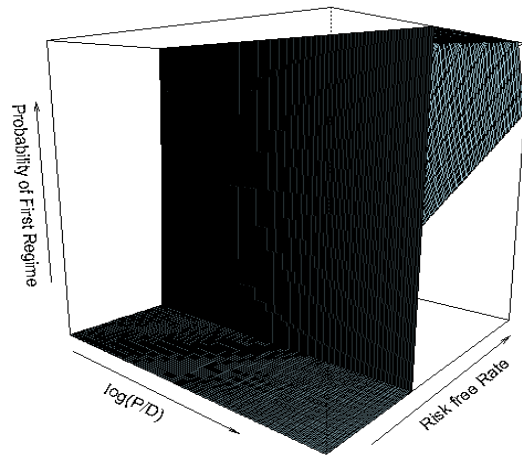


Figure 1: The figure plots the probability of being in the first regime against the price-dividend ratio and risk free rate.

**Figure 2: The State Variable  $x$  as a Function of  $\log(P/D)$  and  $R_f$**

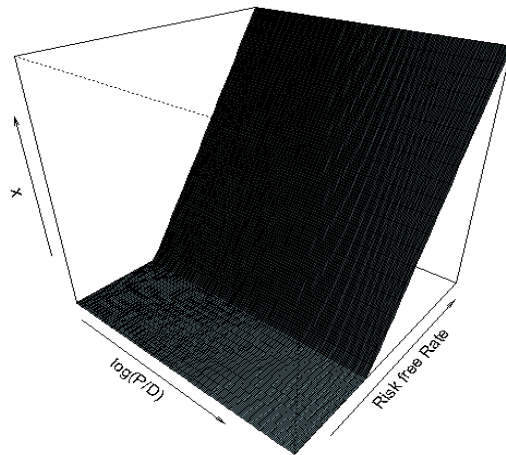


Figure 2: The figure plots the state variable  $x$  against the price-dividend ratio and risk free rate.

**Figure 3: Time Series of the Probability of Being in the First Regime**

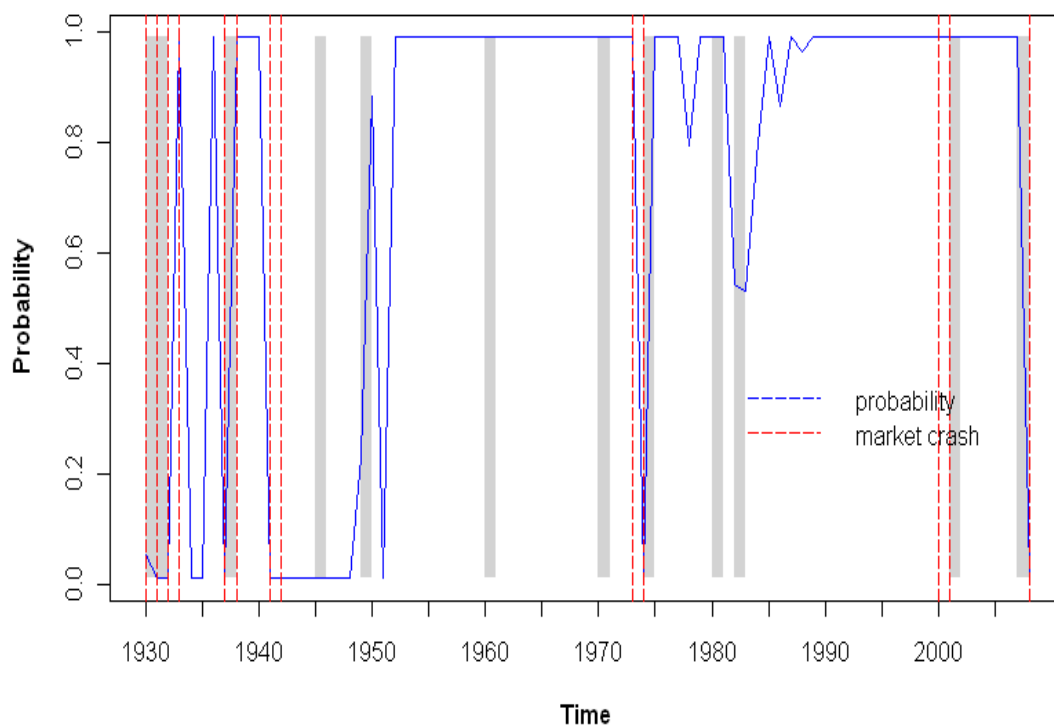


Figure 3: The Figure presents the time series of the probability of being in the first regime over 1930 – 2008 along with the NBER recessions (shaded columns) and the major stock market downturns (vertical dashed lines) identified in Barro and Ursua (2009).

**Figure 4: Time Series of the State Variable  $x$**

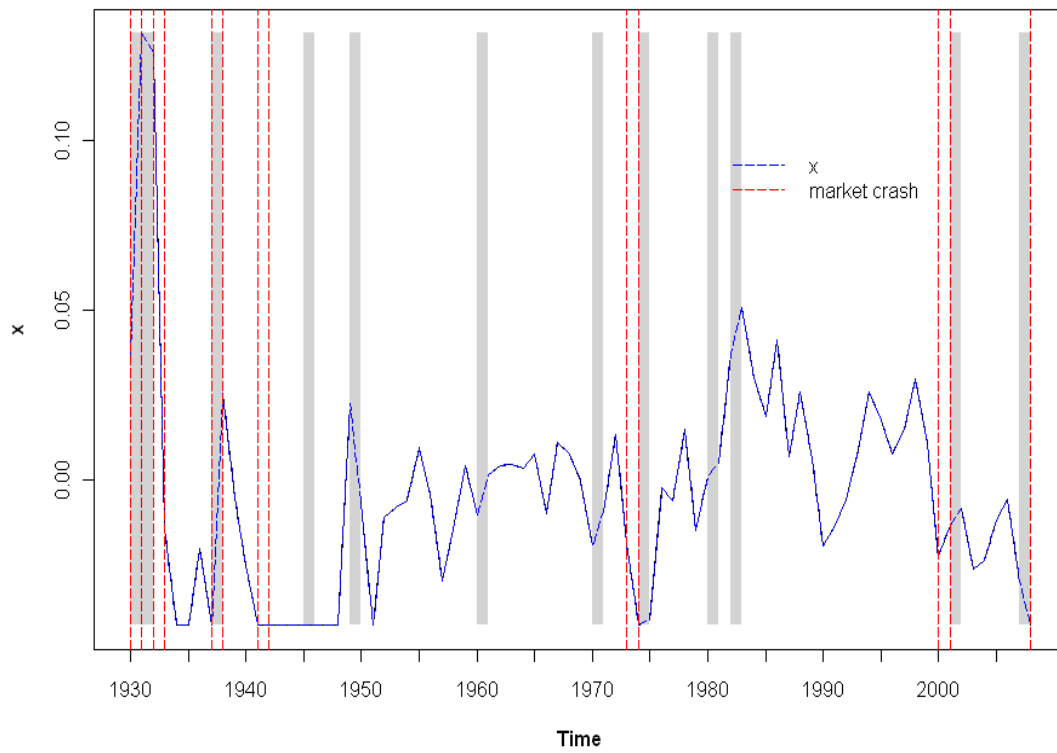


Figure 4: The Figure presents the time series of the state variable  $x$  over 1930 – 2008 along with the NBER recessions (shaded columns) and the major stock market downturns (vertical dashed lines) identified in Barro and Ursua (2009).

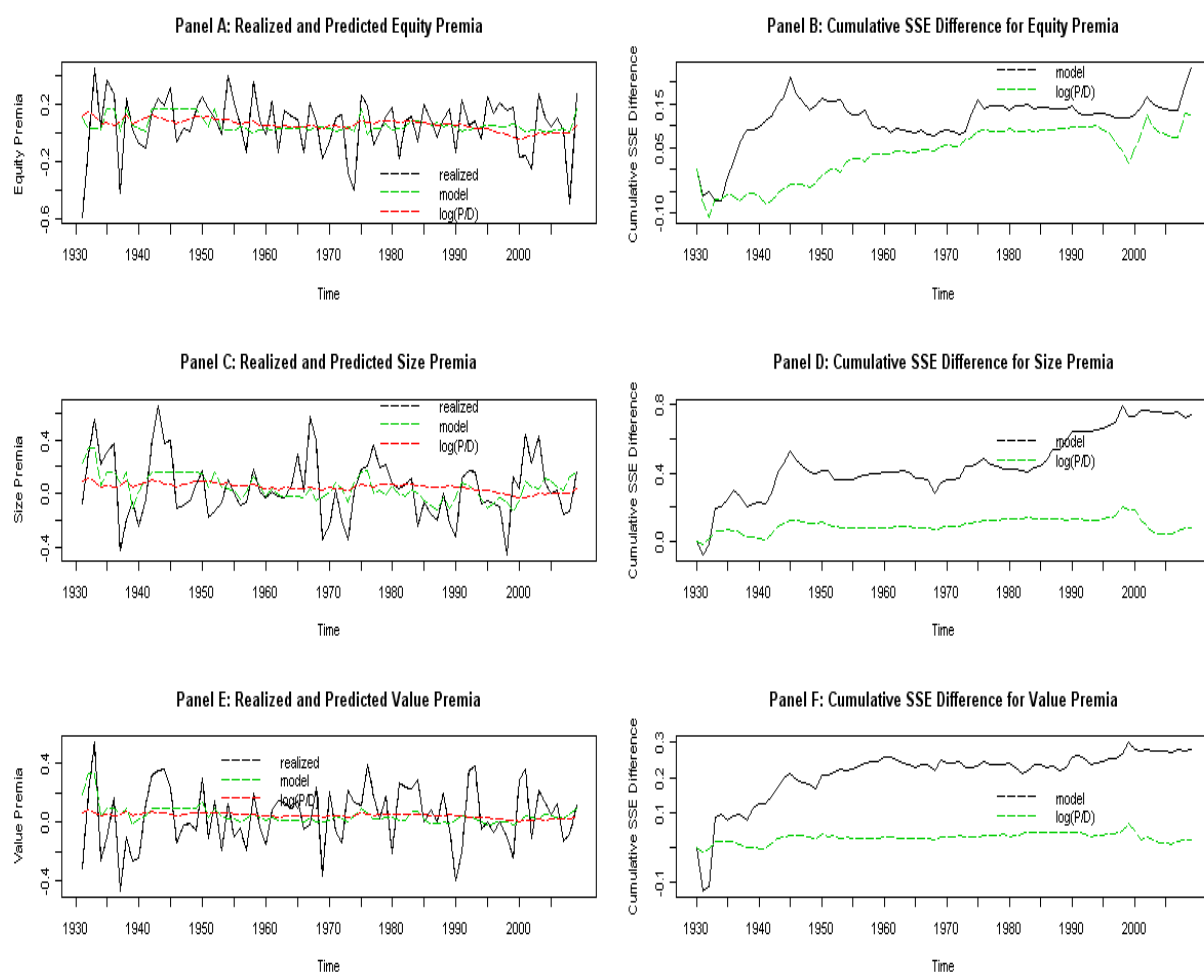


Figure 5: Panels *A*, *C*, and *E* display the realized equity, size, and value premia (black solid line), respectively, along with their forecasted values from the regressions implied by the regime shift model (green dotted line) and linear regressions using the log market-wide price-dividend ratio as a predictor variable (red dashed line). Panels *B*, *D*, and *F* display the cumulative squared demeaned equity, size, and value premia, respectively, minus the cumulative squared regression residual from the alternative forecasting regression specifications: the forecasting regression implied by the model (black solid line) and a linear forecasting regression with the log price-dividend ratio as a predictor variable (red dashed line). An increase in a line indicates better performance of the named model relative to the historical mean of the premia while a decrease in a line indicates better performance of the historical mean.

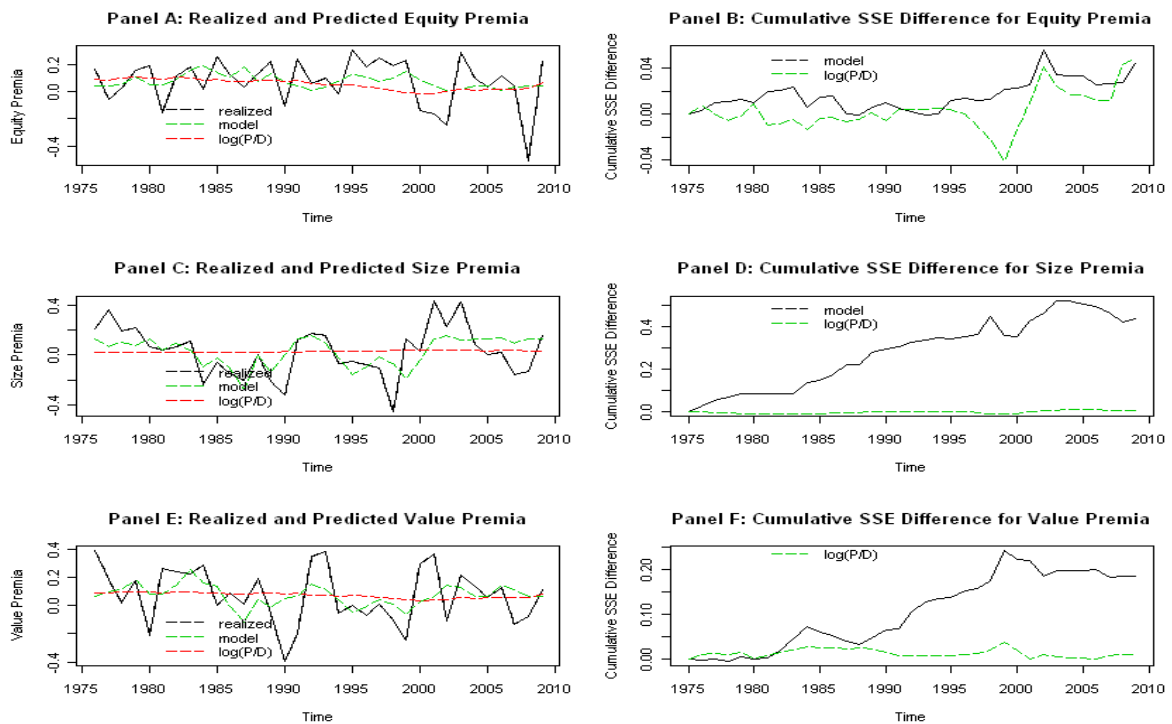


Figure 6: Panels *A*, *C*, and *E* display the realized equity, size, and value premia (black solid line), respectively, along with their forecasted values from the regressions implied by the regime shift model (green dotted line) and linear regressions using the log market-wide price-dividend ratio as a predictor variable (red dashed line). Panels *B*, *D*, and *F* display the cumulative squared demeaned equity, size, and value premia, respectively, minus the cumulative squared regression residual from the alternative forecasting regression specifications: the forecasting regression implied by the model (black solid line) and a linear forecasting regression with the log price-dividend ratio as a predictor variable (red dashed line). An increase in a line indicates better performance of the named model relative to the historical mean of the premia while a decrease in a line indicates better performance of the historical mean.

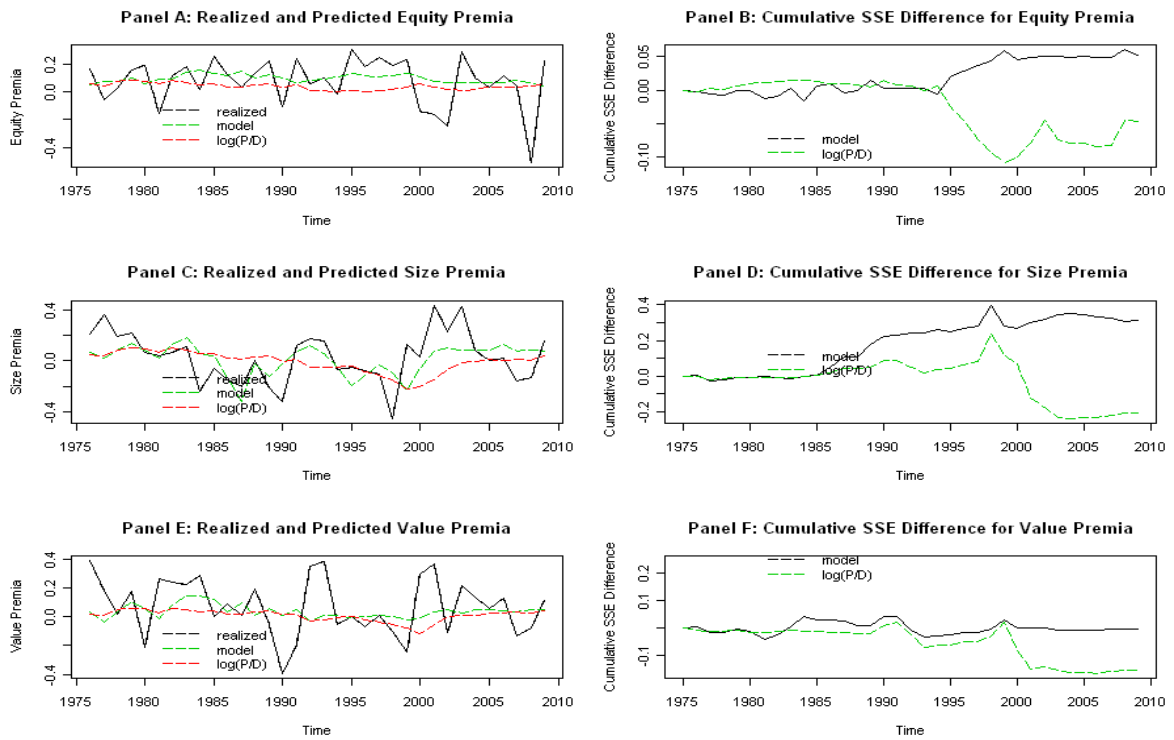


Figure 7: Panels *A*, *C*, and *E* display the realized equity, size, and value premia (black solid line), respectively, along with their predicted values from the rolling out-of-sample predictive regressions implied by the regime shift model (green dotted line) and linear regressions using the log market-wide price-dividend ratio as a predictor variable (red dashed line). Panels *B*, *D*, and *F* display the cumulative squared demeaned equity, size, and value premia, respectively, minus the cumulative squared regression residual from the alternative forecasting regression specifications: the forecasting regression implied by the model (black solid line) and a linear forecasting regression with the log price-dividend ratio as a predictor variable (red dashed line). An increase in a line indicates better performance of the named model relative to the historical mean of the premia while a decrease in a line indicates better performance of the historical mean.

**Figure 8: Realized and Predicted Market Variance, 1931-2009**

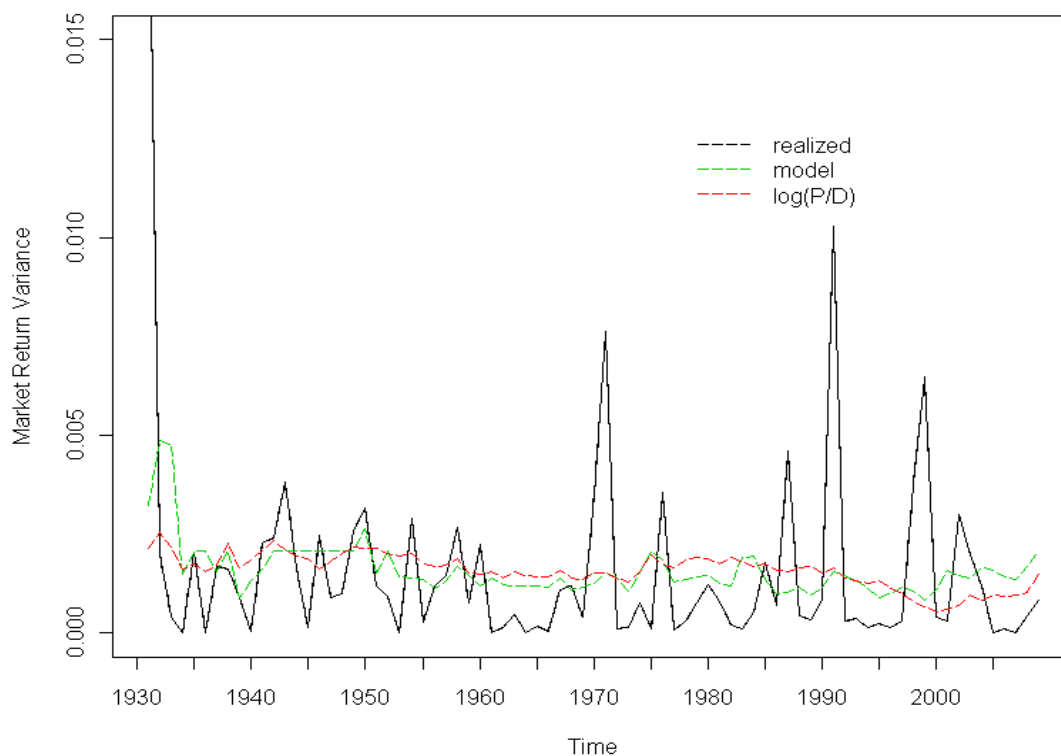


Figure 8: The figure plots the realized market variance along with its model-predicted value. The black solid line plots the annual realized variance that is calculated as the sum of squares of the monthly log returns. The green dotted line plots the model-predicted variance and is obtained as the fitted value from a regression of the realized variance on the two state variables and their product. The red dashed line shows the predictive power of the log price-dividend ratio by plotting the fitted value from a regression of the realized variance on the lagged log price-dividend ratio.