SOME VIEWS ON MATHEMATICAL MODELS AND MEASUREMENT THEORY

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We shall undertake first to review the role of mathematical models in a science and then briefly discuss the models used in classical measurement theory. This will be followed by a generalization of measurement models. Illustrations will be introduced when needed to clarify the concepts discussed.

THE ROLE OF MATHEMATICAL MODELS

We shall use the terms physical objects, real world, and object system synonymously to signify that which the empirical scientist seeks to study, including such objects as opinions or psychological reactions. The scope and content of a domain is selected by the scientist with the intent of discovering laws which govern it, of making predictions about it, or of controlling or at least influencing it.

There are potentially at least as many ways of dividing up the world into object systems as there are scientists to undertake the task. Just as there is a potential variety of object systems, so also is there a potential variety of mathematical systems. Let us describe the nature of a mathematical system: A mathematical system consists of a set of assertions from which consequences are derived by mathematical (logical) argument. The assertions are referred to as the axioms or postulates of a mathematical system. They always contain one or more primitive terms which are undefined and have no meaning in the mathematical system. The axioms of the mathematical system will usually consist of statements about the existence of a set of elements, relations on the elements, properties of the relations, operations on the elements, and the properties of the operations. Particular mathematical systems differ in the particular postulates which form their bases. It is evident then that the variety of mathematical systems is limited only by the ability of man to construct them.

Our view of the role that mathematical models play in a science is illustrated in Fig. 1. With some segment of the real world as his starting point, the scientist, by means of a process we shall call abstraction (A), maps his object system into one of the mathematical systems or models. By mathematical argument (M) certain mathematical conclusions are arrived at as necessary (logical) consequences of the postulates of the system. The mathematical conclusions are then...
converted into physical conclusions by a process we shall call interpretation (I).

Let us start with a specific real-world situation \((RW)_1\) and by process \(A\) map it into a mathematical system \((MS)_1\). We can look at \((MS)_1\) as a model of \((RW)_1\). Looked at in reverse, we can start with consideration of \((MS)_1\) and then \((RW)_1\) can be viewed as a model of \((MS)_1\), and the process of going from \((MS)_1\) to \((RW)_1\) we call “realization.” Thus, “realization” is the converse of “abstraction.” Now, given \((MS)_1\) we might be able to find a real-world situation \((RW)_2\) such that by assigning meanings to the undefined terms of the mathematical system, the assertions about “sets of elements,” “relations,” and “operations” in \((MS)_1\) become identified with objects or concepts about \((RW)_2\). That is, \((RW)_2\) may be another model of \((MS)_1\), and the process of going from \((RW)_1\) to \((MS)_1\) to \((RW)_2\) often indicates subtle analogies between systems such as \((RW)_1\) and \((RW)_2\). To the mathematician who often starts with an abstract system the model is a concrete analogue of the abstract system. To the social scientist who starts with phenomena in the real world the model is the analogue in the abstract system.

In establishing a model for a given object system one of the most difficult tasks is to attempt a division of the phenomenon into two parts; namely that part which we abstract \((A)\) into the basic assumptions or axioms of the abstract system, and that part which we relegate to the physical conclusions and which we reserve as a check against the interpretations from the abstract system. In a given object system there is no unique partition of the phenomena, and which partition is made depends on the creative imagination of the model builder. Indeed, there are models in the physical and biological sciences for which there are no experimentally verified or verifiable correlates in the real world for the undefined terms, relations, and operations in the abstract model. A similar situation prevails on the side of the abstract system; i.e., it is often possible in a given abstract system to interchange the roles of certain axioms and theorems. Thus, in a given system there is no unique method of splitting the mathematical propositions into axioms and theorems. In going from the abstract system to the object system we have the parallel processes of realization and interpretation. It is quite common to consider these synonymous; however, we prefer in this discussion to reserve the word “interpretation” for the process which maps the mathematical conclusions (rather than the axioms) into the object system.

Let us summarize briefly up to this point. Beginning with a segment of the real world, the scientist, by an entirely theoretical route, has arrived at certain conclusions about the real world. His first step is a process of abstraction from the real world, then a process of logical argument to an abstract conclusion, then a return to the real world by a process of interpretations yielding conclusions with physical meaning. But there is an alternative route to physical conclusions and this is by way of working with the object system itself. Thus, the scientist may begin with the real-
world segment in which he is interested and proceed directly to physical conclusions by a process of observation or experiment \((T)\).

The path \((T)\) (experimentation) from the real world to the physical conclusions needs further scrutiny. Usually in theory construction the scientist embarks on model building after he has many facts at his disposal. These facts he partitions into two parts—one part serves as a springboard for the abstraction process \((A)\); the other part serves as a check on the model by making comparisons with these initial facts and the interpretations \((I)\) stemming from the model. If a specific interpretation is not at variance with a fact in the initial reservoir, but at the same time not corroborated by our a priori notions of the object system, then the model perhaps has contributed to our knowledge of the object system. The scientist next tests this tentative conclusion by setting up a plan of experimental verification, if this is possible. Often direct verification may not be possible, and corroboration stems from examination of experimental evidence which supports claims of the model quite indirectly. That is, motivated by interpretations of the model, the scientist sets up an experimental design, obtains observations by experimentation, makes a statistical interpretation of these observations into physical conclusions, and compares the conclusions with those of the abstract route in order to appraise the model. As suggested by Frederick Mosteller\(^4\) of Harvard University, it would be appropriate to generalize Fig. 1 as shown in Fig. 2. The route \(A_2EI_2\) in Fig. 2 is summarized by the route \(T\) in Fig. 1. If the physical conclusions of the process \(A_1MI_1\) are at variance with the a priori facts or with conclusions arrived at via \(A_2EI_2\) (and if more confidence is placed in the experimental route than in the theoretical route), then the suitability of the model is suspect.

The task of a science looked at in this way may be seen to be the task of trying to arrive at the same conclusions about the real world by two different routes: one is by experiment and the other by logical argument; these correspond, respectively, to the

\[^4\text{Personal communication.}\]

![Fig. 2. A generalization of Fig. 1](image_url)
There is no natural or necessary order in which these routes should be followed. The history of science is replete with instances in which physical experiments have suggested axiom systems to the mathematicians and, thereby, contributed to the development of mathematics. On the other hand, mathematical systems developed under such stimulation, in turn, have suggested experiments. And there have been many instances of mathematical systems developed without reference to any known reality which subsequently filled a need of theoretical scientists. The direction that mathematics has taken is in considerable part due to its interaction with the physical sciences and the problems arising therein.

It is illuminating here to observe the way in which the models of the mathematical theory of probability and statistics fit this picture. As in any abstract system, the mathematical theory of probability is devoid of any real-world content; and as in any other mathematical system, the axioms of probability specify interrelationships among undefined terms. It is common to let the notion of probability itself be undefined and to attempt to capture in the axiomatic structure properties of probability motivated by the interpretations we have in mind (e.g., gambling games, processes of physical diffusion, etc.). Given an association of probabilities to prescribed elementary sets, the axioms of probability dictate how one must associate probabilities with other sets. How we make these preliminary associations, provided that we have consistency, is not relevant to the purely abstract system. When we come to apply the probability model, we are confronted with the problem of identifying real events with abstract sets in the mathematical system and the measurement problem of associating probabilities to these abstract sets. Experience has taught us that if we exploit the notion of the relative frequency of occurrence of real events when making our preliminary associations, then the interpretations from the model have a similar frequency interpretation in the real world. To be sure, our rules of composition in the formal system were devised with this in mind. We associate probabilities in one way rather than another in the process $A$ so that when we generate $AMI$, our interpretations are in "close" accord with results of experimentation, $T$, when $T$ is possible. When $T$ is not possible we have to rely to a great extent on analogy.

An extremely important problem of statistics can be viewed as follows: For a priori reasons we may have a well-defined family of possible probability associations. Each element of this family, when used in the abstraction process $A$, generates by $AMI$ a probability measure having a frequency interpretation over real events. In addition, we are given a set of possible actions to be taken. Preferences for these actions depend in some way on the relative "appropriateness" of different probability associations in the abstraction process, $A$. By conducting an experiment, $T$, and noting its outcome we gain some insight into the relative "appropriateness" of the different probability associations and thus base our action accordingly. Variations of this problem, which involves the entire $AMI-T$ process, have been abstracted sufficiently so that models of mathematical statistics include counterparts of all these ingredients within the mathematical system itself.

In a given model we may be confronted with the problem of deciding
whether the AMI argument gives results "close enough" to the experimental results from \( T \). We often can view this problem involving a complete AMI-T process as the real-world phenomenon to which we apply the \( A \) process, sending it into a formal mathematical statistics system. The statistics system analyzes step \( M \), and our interpretation / takes the form of a statement of acceptance or rejection concerning the original theory.

The process of measurement, corresponding to \( A \) in Fig. 1, provides an excellent illustration of the role of mathematical models. There are many types of observations that can be called "measurement." Perhaps the most obvious are those made with yardsticks, thermometers, and other instruments, which result immediately in the assignment of a real number to the object being measured. In other cases, such as the number of correct items on a mental test or the size of a herd of cattle, the result of measurement is a natural number (positive integer). In still other cases, such as relative ability of two chess players, relative desirability of a pair of pictures, or relative hardness of two substances, the result is a dominance (or preference) relation. We might even stretch the concept of measurement to include such processes as naming each element of some class of objects, or the photographic representation of some event, or the categorization of mental illnesses or occupations.

The process of measurement may be described formally as follows. Let \( P = \{p_1, p_2, \ldots \} \) denote a set of physical objects or events. By a measurement \( A \) on \( P \) we mean a function which assigns to each element \( p \) of \( P \) an element \( b = A(p) \) in some mathematical system \( B = \{b_1, \ldots \} \). That is, to each element of \( P \), we associate an element of some abstract system \( B \) (the process \( A \) of Fig. 1). The system \( B \) consists of a set of elements with some mathematical structure imposed on its elements. The actual mapping into the abstract space \( B \) comprises the operation of measurement. The mathematical structure of the system \( B \) belongs to the formal side of measurement theory. The structure of \( B \) is dictated by a set of rules or axioms which states relationships between the elements of \( B \). However, no connotation can be given to these elements of \( B \) unless it is explicitly stated in the axioms; i.e., their labels are extraneous with respect to considerations of the structure of \( B \).

After making the mapping from \( P \) into \( B \), then one may operate with the image elements in \( B \) (always abiding by the axioms, process \( M \) of Fig. 1). Purely mathematical results obtained in \( B \) must then be interpreted back in the real world (the process \( I \) of Fig. 1) to enable one to make predictions or to synthesize data concerning set \( P \).

If the manifestations of \( P \) (as a result of the process \( T \) of Fig. 1) are in conflict with the results of process \( I \) obtained from \( B \), then one must search for a new cycle AMI. Suppose that we have a family of abstractions \( \{A_\alpha\} \) from the given situation \( P \), and suppose that \( M_\alpha, I_\alpha \) complete the cycle begun with \( A_\alpha \). Among all of the available cycles \( A_\alpha M_\alpha I_\alpha \) we seek one, say \( A_0 M_0 I_0 \), which is "closest" to \( T \) according to some criterion. Some models have a criterion built in to judge closeness, and others of a more deterministic nature require an exact fit.

The process \( T \) represents the experimental or operational part of model building; the process \( M \) represents the formal or logical aspect. The processes \( A \) and \( I \) are really the keys to the model and serve as bridges between experiment and formal reasoning.
It might be well here to draw clearly the distinction between a model and a theory. A model is not itself a theory; it is only an available or possible or potential theory until a segment of the real world has been mapped into it. Then the model becomes a theory about the real world. As a theory, it can be accepted or rejected on the basis of how well it works. As a model, it can be right or wrong only on logical grounds. A model must satisfy only internal criteria; a theory must satisfy external criteria as well.

An example of the distinction between models and theories lies in the domain of measurement. A measurement scale, such as an ordinal, interval, or ratio scale, is a model and needs only to be internally consistent. As soon as behavior or data are "measured" by being mapped into one of these scales, then the model becomes a theory about those data and may be right or wrong. Scales of measurement are only a very small portion of the many formal systems in mathematics which might serve as image spaces or models, but will be discussed here as they constitute very simple and immediate examples of the role of mathematical models. First to be discussed will be the models of conventional measurement theory, and then a generalization of these models will be presented.

**Mathematical Models of Classical Measurement Theory**

The first comprehensive classification of the mathematical models used in conventional measurement theory was made by Stevens (9). He classified scales of measurement into nominal, ordinal, interval, and ratio scales, the latter two christened by him. A more complete discussion of these scales is contained in a later work by him (10), and also in Coombs (3, 5) and Weitzenhoffer (11). Because of the available literature on these scales and because they constitute a restricted class of models, they will be briefly summarized here only to provide a basis for generalization in the next section.

The mathematical model of measurement is said to be nominal if it merely contributes a mapping $A_0$ of $P$ into $M_0$ without imposing any further structure on $M_0$. A nominal scale $M$ may be subjected to any 1-1 transformation without gain or loss in information.

An ordinal scale of measurement is implied if there is a natural ranking of the objects of measurement according to some attribute. More precisely, the ordinal scale is appropriate if the objects of measurement can be partitioned into classes in such a manner that (a) elements which belong to the same class can be considered equivalent relative to the attribute in question; (b) a comparative judgment or an order relation can be made between each pair of distinct classes (for example, class $x$ is more ____ than class $y$); (c) there is an element of consistency in these comparative judgments—namely, if class $x$ is more ____ than class $y$ and class $y$ is more ____ than class $z$, then class $x$ is more ____ than class $z$ (that is, the comparative judgment or order relation is transitive). For example, the familiar socioeconomic classes, upper-upper, lower-upper, upper-middle, lower-middle, upper-lower, and lower-lower, imply the measurement of socioeconomic status on an ordinal scale. The numbers 1, 2, 3, 4, 5, 6, or 1, 5, 10, 11, 12, 14, or the letters A, B, C, D, E, F could designate the six classes without gain or loss of information.

The measurement is said to be an interval scale when the set $M$ consists of the real numbers and any linear
transformation, \( y = ax + b \ (a \neq 0) \), on \( M \) is permissible. Measurement on an interval scale is achieved with a constant unit of measurement and an arbitrary zero. An example of an interval scale is the measure of time. That is, "physical events" can be mapped into the real numbers and all the operations of arithmetic are permissible on the differences between all pairs of these numbers.

If the set \( M \) consists of the real numbers subject only to the transformation group \( y = cx \) where \( c \) is any nonzero scalar, the scale is called a ratio scale. Measurement on a ratio scale is achieved with an absolute zero and a constant unit of measurement. The scalar \( c \) signifies that only the unit of measurement is arbitrary. In a ratio scale all the operations of arithmetic are permissible. The most familiar examples of ratio scales are observed in physics in such measurements as length, weight, and absolute temperature.

A Generalization of Measurement Models

An axiomatic basis for certain scales of measurement will be presented in this section. Other scales can be generated by forming mixtures (or composites) of these. Indeed, some of the scales listed in the diagram shown in Fig. 3 can be regarded as composites of others.

We will now list defining axioms for each of these systems and briefly discuss their roles. It is not claimed that this list is exhaustive; it is presented to illustrate certain possibilities for significant generalizations of scales used in the classical theory. The arrangement in the diagram is from top to bottom in order of increasing strength of axioms; a connecting line indicates that the lower listed system is a special case of the higher one.

![Diagram showing measurement scales](image-url)
ognize that for $R$ to constitute a very useful relation, not all possible pairs $(b, b')$ from $B_1$ can be included in the relation $R$.

With some risk of misinterpretation or distortion, these concepts might be illustrated as follows. Consider a set of persons identified by a nominal scale, $B_0$. Let us now define the relation $R$ on $B_0$ to be “loves.” Thus $R$ consists of the ordered pairs $(a, b)$ for which, $a$ loves $b$.

The particular relation used here as an illustration is one whose mathematical properties are mostly negative. We cannot conclude from $a$ loves $b$ and $b$ loves $c$ that $a$ loves $c$, or that $b$ loves $a$, or that $b$ does not love $a$. For example, if John loves Mary and if Mary loves Peter, it may well be that, far from loving him, John would like to see Peter transported to the South Pole. In the terminology to be introduced below, we would say that love is not symmetric, is not asymmetric, and is not transitive.

$B_2$, the antisymmetric relation scale. A relation $R$ on a set $B$ is said to be antisymmetric if $aRb$ and $bRa$ together imply that $a$ is identical with $b$. An example is the relation $\geq$ for real numbers. A statement such as “picture $a$ is at least as good as picture $b”$ illustrates an antisymmetric relation on a collection of pictures, provided that there are not in the collection two distinct pictures of equal merit, i.e., two pictures about which the judge is indifferent.

Closely connected to the concept of antisymmetry is that of asymmetry. A relation $R$ on a set $B$ is said to be asymmetric if $aRb$ and $bRa$ together imply that $a$ is identical with $b$. An example is the relation $\geq$ for real numbers. A statement such as “picture $a$ is at least as good as picture $b”$ illustrates an antisymmetric relation on a collection of pictures, provided that there are not in the collection two distinct pictures of equal merit, i.e., two pictures about which the judge is indifferent.

Antisymmetry and asymmetry are seen to be at the root of statements of comparison. These two classes of relations can be regarded as the most primitive types of order relations. At the opposite pole from these concepts is that of symmetry. A relation $R$ is said to be symmetric if $aRb$ implies $bRa$. For example, the relations “is a sibling of,” “is a cousin of,” and “is the same color as” are all symmetric.

If $S$ is an asymmetric relation on a set $B$, we can obtain from it an antisymmetric relation $R$ by the definition: $aRb$ means either $aSb$ or $a = b$. Conversely, if $R$ is antisymmetric and we define $aSb$ to mean $aRb$ and $a \neq b$, then $S$ is asymmetric. It is customary to use the symbols $\leq$, $\geq$ for antisymmetric relations and to use $<$, $>$ for the associated asymmetric relations.

$B_3$, the transitive relation scale. A relation $R$ is said to be transitive if $aRb$ and $bRc$ imply $aRc$. In the physical world, preference judgments which are not transitive are frequently regarded as inconsistent or irrational. However, situations such as that of three chess players, each of whom can beat one of the other two, show that transitivity is not a requirement of nature.

The chess player relation is antisymmetric but not transitive. An example of a relation that is symmetric and transitive is given by a communication system where each link is bidirectional; here $aRb$ is given the meaning “there exists a chain of links starting with $a$ and ending with $b.”$ If the links are not required to be bidirectional, the relation is still transitive but is no longer symmetric. Note that in this example it is quite possible to have $aRa$, i.e., a chain be-
ginnning at $a$ and ending at $a$. (This chain must have at least one element different from $a$.)

The relation "$a$ is the rival of $b" (say as suitors of a particular girl) is symmetric and is almost transitive. If $aRb$ and $bRc$, we can conclude $aRc$ unless $a = c$; but we can hardly regard $a$ as being his own rival. This type of relation arises frequently in studies of social structures. We say a relation $R$ is quasi-transitive if $aRb$, $bRc$, and $a \neq c$ imply $aRc$. The sibling relation is also quasi-transitive. Of course, if $R$ is quasi-transitive we can define a new relation $S$ to be the same as $R$ except that also $aRb$, $bRa$ imply $aSa$. In some instances $S$ is just as good a model as $R$, but in others the extension from $R$ to $S$ destroys the usefulness of the model.

As an example consider the structure matrix $A = ||a_{ij}||$ of some society. Thus we set $a_{ij} = 1$ if person $i$ has direct influence on person $j$ and set $a_{ij} = 0$ otherwise. One must decide in accordance with the purpose of the investigation whether or not to set the diagonal element $a_{ii}$ equal to 0 or to 1. (The relation "i has direct influence on $j" is not transitive even if we take each $a_{ii} = 1$, but this example nevertheless illustrates the kind of problem involved in the contrast between transitivity and quasi-transitivity.)

If the relation $aRb$ meant $a$ "is higher in socioeconomic status than" $b$, and this required that $a$ had more income and more education than $b$, then the relation $R$ would be asymmetric and transitive.

$B_3$, the partly ordered scale. A relation $\succeq$ which is reflexive, antisymmetric, and transitive is called a partial order. If for some pair $(a, b)$ neither of the relations $a \succeq b$ nor $b \succeq a$ holds, we say that $a$ and $b$ are incomparable relative to $\succeq$. In the case of a preference relation, incomparability is not the same thing as indifference. We call a set a poset (partly ordered set) if there is a partial order relation defined on $B$.

If $a \succeq b$, we also write $b \preceq a$; if $a \succeq b$ and $a \npreceq b$, we write $a \succ b$ or $b \prec a$.

A partial order may be illustrated as follows. Suppose that on a mental test no two individuals in a group pass exactly the same items. Now let $a \succeq b$ symbolize the relation "$a$ passed every item $b$ did and perhaps more." Then $a \succ b$ means that "$a$ passed all the items $b$ did and at least one more." This poset reflects multidimensionality of the attributes mediating the test performance, and some interesting mathematical problems arise regarding the partial order as a "product" of simple orders. The result is a nonmetric form of factor analysis with some of the same problems as factor analysis (3).

Next we consider the mental test example modified so as to allow the possibility that two individuals $a$ and $b$ pass exactly the same items. Then in the above notation we have $a \succeq b$ and $b \succeq a$, but not $b = a$. Hence, $\succeq$ no longer gives a partial order. However, if we define $a \mathrel{\parallel} b$ to mean $a \succeq b$ and $b \succeq a$, it is not hard to show that if we identify individuals with the same test performance then $\mathrel{\parallel}$ is a partial order relation. Or, alternatively, we could consider $\succeq$ as a partial order relation on the set of possible test performances. It is customary to make such identifications and speak of a partial order as if it were actually on the initial set rather than on the identified classes or on the test results.
Another example of a partial order is implicit in the treatment of the comparative efficiency of mental tests on a “cost-utility” basis (1). “Cost” is the fraction of potentially successful people who are eliminated by a test; “utility” is the fraction of potential failures who are eliminated by the test. If for their respective cutting scores one test has a higher utility and a lower cost than another it is a superior test, but if it had a higher utility and a higher cost the two tests would be incomparable unless the relative weight of excluding a potential success to including a potential failure were known.

A basic problem in the theory of testing hypotheses in statistical inference is to test a simple hypothesis, \( H_0 \) (null hypothesis), against a single alternative hypothesis, \( H_1 \), by means of experimental data. A test, \( T \), associates to each experimental outcome the decision to accept \( H_0 \) or to accept \( H_1 \) (but not both!). Each test \( T \) is appraised by a pair of numbers, namely, the probability of accepting \( H_1 \) if \( H_0 \) is true, \( P_T(H_1|H_0) \), and the probability of accepting \( H_0 \) if \( H_1 \) is true, \( P_T(H_0|H_1) \). Given two tests, \( T' \) and \( T'' \), then \( T' \) is said to be as good as \( T'' \) (\( T' \geq T'' \)) if and only if

\[
\begin{align*}
P_T(H_1|H_0) & \leq P_{T''}(H_1|H_0) \\
P_T(H_0|H_1) & \leq P_{T''}(H_0|H_1)
\end{align*}
\]

The relation \( \geq \) on the set of all tests is an example of a partial order.

A poset is said to be a lattice if, for every pair \( a, b \), both \( a \cup b \) and \( a \cap b \) exist. The lattice is an intermediate model between a partial order and a vector space.

George Miller\(^6\) (Massachusetts Institute of Technology) has recently investigated the use of a lattice theoretic treatment of information in experimental psychology. To each item of information he associates (process \( A \)) an element of a lattice. If two items of information are associated respectively with elements \( x \) and \( y \) of the lattice, then the item which consists of the information common to the original items is associated with the element \( x \cap y \), and the item which consists of the information contained in either of the original items is associated with the element \( x \cup y \). As used by Miller, an item of information might consist of a cue or a sequence of cues in an experimental situation. The common procedure is to summarize the structure of the experiment by means of a lattice, given an experimental setup. However, the

\(^6\) Personal communication.
abstract lattice in turn can motivate new types of experimental situations and indicate analogies between experimental designs which otherwise would not be apparent.

B_4, weak order. A transitive order \( \preceq \) is defined on \( B_4 \) and has the property that for every pair \( a, b \) either \( a \preceq b \) or \( b \preceq a \). If both \( a \preceq b \) and \( b \preceq a \), we say that \( a \) and \( b \) are indifferent. Indifference is an equivalence relation (i.e., is reflexive, symmetric, and transitive).

A weak ordering would be illustrated by the military ranks of second lieutenant, first lieutenant, captain, major, etc. Each of these would constitute an equivalence class, and for any two officers \( (a, b) \), either \( a \preceq b \) or \( b \preceq a \), or both.

B_5, chain. A poset in which every pair is comparable is called a chain (or simple order, or linear order, or complete order). Alternatively, a chain is a weak order in which each indifference class consists of a single element. Here every pair of elements is ordered.

The previous example of a weak ordering of military rank could be converted into a chain if date of rank, standing in class, etc. were taken into account. Then, for every two distinct elements, \( a, b \), either \( a > b \) or \( b > a \).

The ordinal scales of classical measurement theory are examples of chains.

B_6, partly ordered vector space. A special case of lattice is provided by a real vector space (or a subset of a vector space). A vector \( x = (x_1, \ldots, x_n) \) is an ordered set of \( n \) real numbers called the components of the vector. We define \( x \preceq y \) to mean that \( x_i \leq y_i \) for each component. (Here the second symbol \( \leq \) refers to the usual ordering of real numbers.) This definition makes the vector space into a poset, and this poset is a lattice which is called a partly ordered vector space.

A partly ordered vector space is illustrated by the comparability of individuals in mental abilities. Conceiving of intelligence as made up of a number of primary mental abilities, each of these constitutes a component or dimension. Then, it may be said of two individuals \( x \) and \( y \) that \( y \) is at least as intelligent as \( x \) if and only if \( y \) has as much or more of each component as \( x \) has.

The term vector is sometimes used in a more general situation. If \( C_1, \ldots, C_n \) are chain orders we may consider vectors or \( n \)-tuples \( c = (c_1, \ldots, c_n) \) where the \( i \)th component \( c_i \) lies in the chain order \( C_i \) \((i = 1, \ldots, n)\). The set \( C \) of all such vectors \( c \) is called the Cartesian product of \( C_1, \ldots, C_n \) and is denoted by \( C = C_1 \times \cdots \times C_n \). We can make \( C \) into a poset by a process analogous to that used above for real vector spaces. Note that a real vector space is the special case of a Cartesian product of \( n \) factors \( C_1, \ldots, C_n \), each equal to the set of real numbers.

B_7, simply ordered vector space, or utility space. A real vector space (or subset) in which \( x < y \) is defined lexicographically, i.e., \( x < y \) if \( x_1 = y_1, \ldots, x_{i-1} = y_{i-1}, x_i < y_i \) is a special case of simple order.

A lexicographic ordering can be illustrated by the manner in which we might expect a fortune hunter to simply order a number of unmarried women. Presumably, financial assets would be the principal component and he might construct a weak ordering of the women, into, say five classes, on this basis. Then he would turn to the second component, say beauty, and within each of the financial classes construct a simple ordering of the women on this component. Any two women \( (a, b) \) would then be simply ordered as follows:

1. If \( a \) were in a higher financial class than \( b \), \( a \) would be preferred to \( b \).
2. If \( a \) were in the same financial class as \( b \), then preference would be determined by their relation on the beauty component.

**\( B_7 \), real numbers.** The transition to scales using the real numbers has been given additional importance by the development of von Neumann-Morgenstern utilities. Even though one may wish to arrive here in order to have a simple index when a decision is to be made, it may frequently be desirable not to get here all at once, but to keep the components at a weaker level until it is necessary to map into the real numbers.

Measurement scales involving the real numbers, the interval scale, and the ratio scale have been discussed at length in the literature (3, 5, 9, 10, 11) and will not be pursued again here.

**FURTHER EXTENSIONS**

The various mathematical systems discussed here as available for measurement have been illustrated with objects of the real world mapped into the elements of an abstract system. A further level of abstraction is provided by defining a “distance function” in the abstract system, in which ordered pairs of elements in the abstract system are mapped into elements of another abstract system about which a variety of assertions may be made. In the context of measurement, these pairs of elements may correspond to “differences” between pairs of objects in the real world. These differences may themselves then be mapped into an appropriate abstract system such as one of those discussed here.

A number of these types of scales have been discussed by one of the authors (4, ch. 1). An illustration is the ordered metric scale in which the objects themselves satisfy \( B_8 \), a simply ordered scale, and ordered pairs of objects, regarded as “distances” between them, satisfy \( B_3 \), a partly ordered scale. Such scales are now being utilized for the measurement of utility and psychological probability in experiments on decision making under uncertainty (6, 7).

**SUMMARY**

One role of mathematical models is to provide a logical route to go from characteristics of the real world to predictions about it. The alternative route is by observation or experiment on the real world itself. The view expressed here is that these two routes are coordinate.

The various scales used in measurement serve as an illustration of the application of mathematical models and are subject to the same constraints as other mathematical models. That is, if the axioms underlying the scale are not satisfied by that segment of the real world which is mapped into it, then the interpretations of the mathematical conclusions may have no reality or meaning. Thus, to insist that measurement always constitutes the mapping of physical objects into the real number system is to impose on the real world an abstract theory which may be invalid.

A partial ordering of various alternative mathematical systems available for measurement has been presented with illustrations in order to reveal the relative strengths of these scales to which the real world must conform to permit their application. We make no claim to completeness in this list of models for measurement theory. Our purpose is to point out the richness of the set of possible models and to give some examples that show how the use of more general models can extend the domain of classical measurement theory.
None of the discussion here should be taken as an argument for the use of weaker scales in the place of stronger scales for their own sake. The measurement scale utilized constitutes a theory about the real world and the stronger the theory the better, so long as it is correct. The addition to a scale of axioms which are not satisfied by the real world is a step away from the path of progress.

REFERENCES

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