V Time Varying Covariance and Correlation

- DEFINITION OF CORRELATIONS
- ARE THEY TIME VARYING?
- WHY DO WE NEED THEM?
- ONE FACTOR ARCH MODEL
- DYNAMIC CONDITIONAL CORRELATIONS
- ASSET ALLOCATION
- THE VALUE OF CORRELATION INFORMATION AND ITS RELATION TO RISK MANAGEMENT

Covariances and Correlations

- CORRELATIONS MEASURE THE DEGREE TO WHICH TWO SERIES MOVE TOGETHER
- THEORETICAL DEFINITION:

Let $r_1$ and $r_2$ be mean zero random variables, then

$$E(r_1 r_2) = \rho_{1,2} \sqrt{E(r_1^2)E(r_2^2)}, \text{ and}$$

$$\rho_{1,2} = \frac{E(r_1 r_2)}{\sqrt{E(r_1^2)E(r_2^2)}}$$
Unconditional Covariances and Correlations

- Data \{r_{1,t}, r_{2,t}\}
- Calculation taking out means:

  - Covariance: \( \hat{\sigma}_{1,2} = \frac{1}{T} \sum_{t=1}^{T} (r_{1,t} - \bar{r}_1)(r_{2,t} - \bar{r}_2) \)
  
  - Correlation: \( \hat{\rho}_{1,2} = \frac{1}{\hat{\sigma}_1 \hat{\sigma}_2} \sum_{t=1}^{T} (r_{1,t} - \bar{r}_1)(r_{2,t} - \bar{r}_2) \)
10 YEARS OF LARGE CAP RETURNS

-0.15 -0.10 -0.05 0.00 0.05 0.10 0.15

MMM

-0.26

.18 .13

HD

-0.16 -0.12 -0.08 -0.04 0.00 0.04 0.08 0.12 0.16

MO

-0.15 -0.10 -0.05 0.00 0.05 0.10 0.15

AXP

-0.15 -0.10 -0.05 0.00 0.05 0.10 0.15

JPM

-0.15 -0.10 -0.05 0.00 0.05 0.10 0.15 0.20

INTC

-0.2 -0.1 0.0 0.1 0.2

MSFT

-0.15 -0.10 -0.05 0.00 0.05 0.10 0.15

MRK

-0.15 -0.10 -0.05 0.00 0.05 0.10 0.15

JPM

-0.15 -0.10 -0.05 0.00 0.05 0.10 0.15 0.20

INTC

-0.2 -0.1 0.0 0.1 0.2
ARE CORRELATIONS TIME VARYING? YES!

- Derivative prices of correlation sensitive products imply changes.
  - “Dispersion trades” on the S&P500 go long volatility on individual assets (i.e. buy straddle) and short volatility on the index (i.e. sell straddle).
  - The expected payoff is determined by the correlation.
  - The bet is on low correlation so that high volatility in the individual assets doesn’t appear in the portfolio.

- Time series estimates change. There are many varieties.
Recall that for a single asset, we defined the conditional variance as the variance of the unpredictable part.

That is, if $r_t = \mu_t + \varepsilon_t$, then the conditional variance is given by $h_t = E_{t-1}(\varepsilon_t^2)$.

Hence if there is predictability in the mean, we don’t include that variation in the conditional variance, we remove it first.

The same is true for the conditional variance covariance matrix.

We define the conditional variance covariance matrix for the part of $y_t$ that is not predictable.

The conditional variance covariance matrix is given by $\Omega_t$ and is the variance covariance matrix of $\varepsilon_t = y_t - \mu_t$.

If the mean is zero and has no dynamics then $y_t = \varepsilon_t$. 

If mean returns are not predictable then $y_t = \epsilon_t$ and a two dimensional covariance matrix looks like:

$$
\Omega_t = E_{t-1} \left( \begin{pmatrix} r_{1,t}^2 & r_{1,t} r_{2,t} \\ r_{1,t} r_{2,t} & r_{2,t}^2 \end{pmatrix} \right) = \begin{pmatrix} \sigma_{1,t}^2 & \sigma_{1,2,t} \\ \sigma_{1,2,t} & \sigma_{2,t}^2 \end{pmatrix}
$$

Conditional Covariance: $E(r_{1,t} r_{2,t} | F_{t-1})$

Conditional Variances

**CONDITIONAL CORRELATIONS**

Conditional correlations are then given by

$$
\rho_{1,2,t} = \frac{\sigma_{1,2,t}}{\sigma_{1,t} \sigma_{2,t}} = \frac{E_{t-1}(r_{1,t} r_{2,t})}{\sqrt{E_{t-1}(r_{1,t}^2) E_{t-1}(r_{2,t}^2)}}
$$
ESTIMATION

- **HISTORICAL CORRELATIONS**
  - Use a rolling window of $N$ observations for both covariances and variances.

- **EXPONENTIAL SMOOTHING**
  - Use an exponential smoother for both covariances and variances using the same smoothing parameter.

100 day historical correlations between AXP and GE
WHY DO WE NEED CORRELATIONS?

- CALCULATE PORTFOLIO RISK
- FORM OPTIMAL PORTFOLIOS
- PRICE HEDGE AND TRADE DERIVATIVES
PORTFOLIO VARIANCE WITH TWO ASSETS

- HISTORICAL VARIANCE
  - Look at historical variance of $w_1 r_{1t} + w_2 r_{2t}$

- CALCULATE VARIANCE
  $$\sigma_{\text{portfolio}}^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1w_2 \sigma_{1,2}$$

- CALCULATE DYNAMIC VARIANCE
  $$\sigma_{\text{portfolio},t}^2 = w_1^2 \sigma_{1,t}^2 + w_2^2 \sigma_{2,t}^2 + 2w_1w_2 \rho_{1,2,t} \sigma_{1,t} \sigma_{2,t}$$

FINDING THE OPTIMAL PORTFOLIO

- Minimize portfolio variance subject to a required return. “Markowitz”

  With covariance matrix $\Omega$ and expected excess returns above a riskless rate of $\mu$

  $$\min_{s.t. w' \mu \geq \mu_0} w' \Omega w$$

  $$w = \frac{\Omega^{-1} \mu}{\mu' \Omega^{-1} \mu} \mu_0$$
In a time varying setting conditionally optimal weights look like:

\[ w_t = \frac{\Omega_t^{-1} \mu}{\mu' \Omega_t^{-1} \mu} \mu_0 \]
SOME MODELS

- ONE FACTOR MODEL
- MANY FACTOR MODEL
- MULTIVARIATE GARCH
- DYNAMIC CONDITIONAL CORRELATION

ONE FACTOR ARCH

- One factor model such as CAPM
- There is one market factor with fixed betas and constant variance idiosyncratic errors independent of the factor. The market has some type of GARCH with variance $\sigma_{m,t}^2$.

\[
\begin{align*}
    r_{i,t} &= \beta_i r_{m,t} + e_{i,t} \\
    \sigma_{i,t}^2 &= \beta_i^2 \sigma_{m,t}^2 + \nu_i
\end{align*}
\]

- If the market has asymmetric volatility, then individual stocks will too.
CORRELATIONS

- Between stock i and stock j assuming idiosyncracies are uncorrelated.
  \[ \sigma_{i,j,t} = \beta_i \beta_j \sigma_{m,t}^2 \]
  \[ \rho_{i,j,t} = \frac{\beta_i \beta_j \sigma_{m,t}^2}{\sqrt{\left(v_i + \beta_i^2 \sigma_{m,t}^2\right)\left(v_j + \beta_j^2 \sigma_{m,t}^2\right)}} \]

- Assuming betas are both positive, correlations range from zero to one and increase with market volatility.
- Returns will have lower tail dependence if the market is negatively skewed.

CALCULATIONS

- VOLATILITY OF INDIVIDUAL STOCK = 
  \[ \sigma_{i,t}^2 = \beta_i^2 \sigma_{m,t}^2 + v_i \]
  Notice that we can do this either in daily vols or annualized vols. Whatever we use for inputs, we will get the same for an answer.

- CORRELATION = 
  \[ \rho_{i,j,t} = \frac{\beta_i \beta_j \sigma_{m,t}^2}{\sqrt{\left(v_i + \beta_i^2 \sigma_{m,t}^2\right)\left(v_j + \beta_j^2 \sigma_{m,t}^2\right)}} \]
CALCULATION

- Find the correlation between two stocks with betas and annualized idiosyncratic volatilities given by:
  - Beta1=1, Beta2=2
  - Sig1=20%, Sig2=30%

- When Market vol = 10%
  - Volatility 1 =
  - Volatility 2 =
  - Correlation =

- When Market vol = 40%
  - Volatility 1 =
  - Volatility 2 =
  - Correlation =

PORTFOLIO VARIANCE AND VaR

- For a $1,000,000 portfolio with 60% in the first asset and 40% in the second find the portfolio volatility and 1% VaR for each of these market volatilities. Assume normality for the VaR.

- When market vol = 10%
- When market vol = 40%
HOW TO ESTIMATE A ONE FACTOR MODEL

- Fit the volatility of the market portfolio
- Estimate the betas of the stocks and the variance of the idiosyncracies
- Calculate the time varying correlations
- Calculate the volatility and VaR

MARKET VOLATILITY

![Market Volatility Chart](image)
CALCULATE DYNAMIC CORRELATIONS

\[ \rho_t = \frac{\beta_1 \beta_2 h_t}{\sqrt{(\beta_1^2 h_t + \sigma_1^2)(\beta_2^2 h_t + \sigma_2^2)}} \]

AXP AND GE AGAIN
**Dynamic Conditional Correlation**

- DCC is a new type of multivariate GARCH model that is particularly convenient for big systems. See Engle(2002) or Engle(2005).
- Engle’s text came out in 2009.

**DYNAMIC CONDITIONAL CORRELATION OR DCC**

1. Estimate volatilities for each asset and compute the *standardized residuals* or *volatility adjusted returns*.
2. Estimate the time varying covariances between these using a maximum likelihood criterion and one of several models for the correlations.
3. Form the correlation matrix and covariance matrix. They are guaranteed to be positive definite.
HOW IT WORKS

- When two assets move in the same direction, the correlation is increased slightly.
- This effect may be stronger in down markets (asymmetry in correlations).
- When they move in the opposite direction it is decreased.
- The correlations often are assumed to only temporarily deviate from a long run mean.

CORRELATIONS UPDATE LIKE GARCH

Approximately,

\[ \rho_t = \omega + \alpha z_{1,t-1} z_{2,t-1} + \beta \rho_{t-1} \]

\[ \bar{\rho} = \frac{\omega}{1 - \alpha - \beta} \]  
Unconditional correlation

\[ z_{it} = \frac{r_{it}}{\sqrt{h_{it}}} \]  
This is the standardized residual for asset i.
An asymmetric model allows correlations to increase more when both prices move down together (like our asymmetric GARCH models).

\[ \rho_t = \omega + \alpha z_{1,t-1} z_{2,t-1} + \gamma z_{1,t-1} z_{2,t-1} (I_{z_{1,t} < 0}) (I_{z_{2,t} < 0}) + \beta \rho_{t-1} \]
Application:
Asymmetric Dynamic Correlations of Global Equity and Bond Returns

Lorenzo Capiello, Robert Engle and Kevin Sheppard
Data

- Weekly $ returns Jan 1987 to Feb 2002 (785 observations)
- 21 Country Equity Series from FTSE All-World Index
- 13 Datastream Benchmark Bond Indices with 5 years average maturity

Europe
- AUSTRIA*
- BELGIUM*
- DENMARK*
- FRANCE*
- GERMANY*
- IRELAND*
- ITALY
- THE NETHERLANDS*
- SPAIN
- SWEDEN*
- SWITZERLAND*
- NORWAY
- UNITED KINGDOM*

Australasia
- AUSTRALIA
- HONG KONG
- JAPAN*
- NEW ZEALAND
- SINGAPORE

Americas
- CANADA*
- MEXICO
- UNITED STATES*

*with bond returns
GARCH Models
(asymmetric in orange)

- GARCH
- AVGARCH
- NGARCH
- EGARCH
- ZGARCH
- GJR-GARCH
- APARCH
- AGARCH
- NAGARCH
AVERAGE EMU COUNTRY BOND RETURN CORRELATION
RESULTS

- Asymmetric Correlations – correlations rise after negative returns
- Shift in level of correlations with formation of Euro
- Correlations are rising not just within EMU
- EMU Bond correlations are especially high