Are correlations/covariances time varying?

- Yes Correlations are time varying
  - Derivative prices of correlation sensitive products imply changes (dispersion trade).
  - Derivatives on correlation now are traded (correlation swaps).
  - Time series estimates change. There are many varieties.
How do we define time varying correlations?

- Recall that for a single asset, we defined the conditional variance as the variance of the unpredictable part.
- That is, if \( y_t = \mu_t + \varepsilon_t \) then the conditional variance is given by \( h_t = E(\varepsilon_t^2 | F_{t-1}) \).
- Hence if there is predictability in the mean, we don’t include that variation in the conditional variance, we remove it first.

The same is true for the conditional variance covariance matrix.

- We define the conditional variance covariance matrix for the part of \( y_t \) that is not predictable.
- The conditional variance covariance matrix is given by \( \Omega_t = E(\varepsilon_t \varepsilon_t' | F_{t-1}) \) where \( \varepsilon_t = y_t - \mu_t \).
- If there are no dynamics in the mean then \( y_t = \varepsilon_t \)
• If mean returns are not predictable then $y_t = \varepsilon_t$ and a two dimensional covariance matrix looks like:

$$\Omega_t = E(y_t y'_t | F_{t-1}) = E(r_{1,t} r_{2,t} | F_{t-1}) = \begin{pmatrix} \sigma_{1,t}^2 & \sigma_{1,2,t} \\ \sigma_{1,2,t} & \sigma_{2,t}^2 \end{pmatrix}$$

**Conditional Covariance:** $E(r_{1,t} r_{2,t} | F_{t-1})$

**Conditional Variances**

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**CONDITIONAL CORRELATIONS**

• Conditional correlations are then given by

$$\rho_{1,2,t} = \frac{\sigma_{1,2,t}}{\sigma_{1,t} \sigma_{2,t}} = \frac{E_{t-1}(r_{1,t} r_{2,t})}{\sqrt{E_{t-1}(r_{1,t}^2)} E_{t-1}(r_{2,t}^2)}$$
ESTIMATION

- HISTORICAL CORRELATIONS

\[ \rho_{i,j,t} = \frac{\sum_{m=1}^{s} r_{i,t-m} r_{j,t-m}}{\sqrt{\sum_{m=1}^{s} r_{i,t-m}^2 \sum_{m=1}^{s} r_{j,t-m}^2}} \]

- Exponential smoother:

\[ q_{i,j,t} = \lambda r_{i,t-1} r_{j,t-1} + (1 - \lambda) q_{i,j,t-1} \]

- Then

\[ \rho_{i,j,t} = \frac{q_{i,j,t}}{\sqrt{q_{i,i,t} q_{j,j,t}}} \]

- Use the same value for \( \lambda \) for each series
100 day historical correlations between AXP and GE

MULTIVARIATE MODELS
SOME MODELS

- ONE FACTOR MODEL
- ONE FACTOR MODEL WITH GARCH IDEOSYNCRATIC ERRORS
- MANY FACTOR MODEL
- MULTIVARIATE GARCH
- DYNAMIC CONDITIONAL CORRELATION

ONE FACTOR ARCH

- One factor model such as CAPM
- There is one market factor with fixed betas and constant variance idiosyncratic errors independent of the factor. The market has some type of ARCH with variance $\sigma^2_{m,t}$.
  
  \[ r_{i,t} = \beta_i r_{m,t} + e_{i,t} \]
  \[ \sigma_{i,i,t} = \beta_i^2 \sigma^2_{m,t} + \nu_i \]
- If the market has asymmetric volatility, then individual stocks will too.
CORRELATIONS

- Between stock $i$ and stock $j$ assuming idiosyncracies are uncorrelated.
  \[ \sigma_{i,j,t} = \beta_i \beta_j \sigma_{m,t}^2 \]

- Assuming betas are both positive, correlations range from zero to one and increase with market volatility.
  \[ \rho_{i,j,t} = \frac{\beta_i \beta_j \sigma_{m,t}^2}{\sqrt{(\nu_i + \beta_i^2 \sigma_{m,t}^2)(\nu_j + \beta_j^2 \sigma_{m,t}^2)}} \]

HOW TO ESTIMATE A ONE FACTOR MODEL

- FIT THE VOLATILITY OF THE MARKET
- ESTIMATE THE BETAS OF THE STOCKS AND THE VARIANCE OF THE IDIOSYNCRACIES
MARKET VOLATILITY

CALCULATE DYNAMIC CORRELATIONS

\[ \rho_t = \frac{\beta_1 \beta_2 h_t}{\sqrt{\left(\beta_1^2 h_t + \sigma_1^2\right) \left(\beta_2^2 h_t + \sigma_2^2\right)}} \]
The multivariate factor model

- Consider the case of $n$ assets and let $\beta$ is the vector of betas for the $n$ stocks.

- Then $\Omega_t = \beta \beta' h_{m,t} + V_n$

- Where $V_n$ is the diagonal matrix with the idiosyncratic variances ($v_i$'s) on the diagonal.
The correlations are given by:

\[ R_t = \text{diag} \left( \Omega_t^{-1/2} \right) \Omega_t \text{diag} \left( \Omega_t^{-1/2} \right) \]

**FACTOR DOUBLE ARCH**

- Idiosyncratic errors also follow a GARCH models

\[ \Omega_t = \beta \beta' h_{w,t} + \mathbf{V}_t^2, \quad \mathbf{V}_t \sim \text{diagonal} \{ \text{garch std} \} \]

- where

\[
\mathbf{V}^2_t = \begin{bmatrix}
    h_{1t} & 0 \\
    \vdots & \ddots \\
    0 & h_{mt}
\end{bmatrix}
\]

where \( h_{jt} = \omega_j + \alpha_j \epsilon_{t-j}^2 + \beta_j h_{t-j} \)
Multifactor models

Initially, suppress the time subscript and consider our K factor model for asset i

\[ r_i = \alpha_i + \beta_{i1}F_1 + \beta_{i2}F_2 + \ldots + \beta_{ik}F_k + e_i = \alpha_i + \sum_{j=1}^{K} \beta_{ij}F_j + e_i \]

Where \( E(e_i e_j) = \begin{cases} 0 & \text{if } i \neq j \\ v_j^2 & \text{if } i = j \end{cases} \)

It is convenient to write in matrix notation:

\[ r_i = \alpha_i + B_i^T\bar{F} + e_i \]

If the factors are mean zero and the idiosyncratic errors are uncorrelated with the factors we get that the variance of asset i is:

\[
E((r_i - \alpha_i)^2) = E\left[ (B_i^T\bar{F} + e_i)(B_i^T\bar{F} + e_i) \right] \\
= E\left[ (B_i^T\bar{F} + e_i)(F_i^T\bar{B}_i + e_i) \right] \\
= E\left( B_i^T\bar{F}F_i + 2E[B_i^T\bar{F}e_i] + E(e_i^2) \right) \\
= B_i^T\Theta B_i + v_i
\]

\( \Theta = E(FF^T) \) is the covariance matrix of the factors.
The idiosyncratic errors are uncorrelated across assets so that the covariance of asset $i$ with asset $j$ is

$$E[(r_i - \alpha_i)(r_j - \alpha_j)] = E[(B'_iF + e_i)(B'_jF + e_j)]$$

$$= E[(B'_iF + e_i)(F'B_j + e_j)]$$

$$= E(B'_iFF'B_j) + B'_iE[F'e_j] + E[e_iF']B'_j + E(e_i e_j)$$

$$= B'_i \Theta B_j$$

Now let $\mathbf{r}$ denote a vector of $n$ asset returns.

$$\mathbf{r} = \mathbf{a} + B' \mathbf{F} + \mathbf{e}$$

$$E[(r - \mathbf{a})(r - \mathbf{a})'] = E[(B'F + e)(B'F + e)'] = E[(B'F + e)(F'B + e')]$$

$$= E[(B'FF'B + ee')] = B'E(F'F)B + E(ee') = B' \Theta B + V$$

Where $V$ is a $nxn$ matrix with diagonal element $i$ equal to $\nu_i$. $B'$ is a matrix with $i^{th}$ row given by $B_i$.
Now let’s extend this to the time varying case:

\[ E \left[ (r - \alpha)(r - \alpha)'^\top \right] = B'E (FF')B + E \left( ee' \right) = B'\Theta B + V \]

Can be made time varying as in:

\[ E_{t-1} \left[ (r_t - \alpha)(r_t - \alpha)' \right] = B'E_{t-1} \left( F_{t} F_{t}' \right) B + E_{t-1} \left( ee' \right) = B'\Theta_{t} B + V \]

where \( \Theta_{t} = E_{t-1} \left( F_{t} F_{t}' \right) \)

If the factors are observable (like portfolios), we can apply a model for time varying (co)variances to the factors.

\( \Theta_{t} = E_{t-1} \left( F_{t} F_{t}' \right) \)
Principal Components

- Can we identify factors from the returns data rather than specifying them?
- Recall that the variance covariance matrix of the K factor model returns are given by:
  \[ \Omega = E\left[(r - \alpha)(r - \alpha)\right]' = B'\Theta B + V \]
  where \[ \Theta = E\left(FF'\right) \]

- With the factors unknown, B is identified only up to an orthogonal transformation. To see this, let G be ANY KxK matrix such that \[ GG' = I \] then \[ B^* = B'G \] and \[ F^* = G'F \] also yields the same variance covariance structure since the matrix
  \[ E\left(B'^*F^*F^*B^*\right) + V = B'GG'\Theta GG'B + V \]
  \[ = B'\Theta B + V \]
This lack of identification can be solved by imposing some condition (if we don’t have names on the factors we don’t care which of the equivalent models we choose)!

Typically, we impose that the factors are uncorrelated. This yields:

$$\Omega = B^T D B + V$$

where $D$ is a diagonal matrix

We can find a representation of the factors by finding the linear combination of the data that has the highest variance. In practice, we solve $\max_{x_i} \left( x_i' \hat{\Omega} x_i \right)$ subject to $x_i' x_i = 1$

where $\hat{\Omega}$ is the estimated variance covariance matrix of the returns.
The solution is that $x_1$ will be the eigenvector associated with the largest eigenvalue of $\hat{\Omega}$.
Next we simply normalize $x_1$ to sum to one so that it is a valid vector of portfolio weights.
The second principal component solves the problem

$$\max_{x_2} \left( x_2' \hat{\Omega} x_2 \right) \text{ subject to } x_2' x_2 = 1 \text{ and } x_1' \hat{\Omega} x_2 = 0$$

The new set of portfolio weights correspond to the portfolio that has the highest variance but is uncorrelated with the first portfolio.
The solution is that $x_2$ will be the eigenvector associated with the second largest eigenvalue of $\hat{\Omega}$.
Again, the weights $x_2$ can be normalized to sum to 1.
This process can be repeated \( K \) times until we have weights \( w_1, w_2, \ldots, w_K \) which are the normalized eigenvectors of \( K \) largest eigenvectors of the variance covariance matrix of returns.

Once the weights are known, we can construct the portfolios and treat the factors as if they are observed.

How many factors?

- The largest eigenvalues should be associated with the common components.
- The smaller eigenvalues will be associated with the ideosynratic variances starting with the largest and then becoming smaller.
- When \( n \) gets large, the ideosyncratic variances become an arbitrarily small fraction of the overall variance. The factors do not go away.
Hence in large samples we can identify the K factors up to an orthogonal transformation.

Often we look at the “scree” plot of the eigenvalues from largest to smallest. Number of factors is determined by the drop off.

Looks like 4 factors here:
**Dynamic Conditional Correlation**

- DCC is a new type of multivariate GARCH model that is particularly convenient for big systems. See Engle(2002) or Engle(2005).
- Engle’s new text came out recently.

**DYNAMIC CONDITIONAL CORRELATION OR DCC**

1. Estimate volatilities for each asset and compute the *standardized residuals* or *volatility adjusted returns*.
2. Estimate the time varying covariances between these using a maximum likelihood criterion and one of several models for the correlations.
3. Form the correlation matrix and covariance matrix. They are guaranteed to be positive definite.
HOW IT WORKS

- When two assets move in the same direction, the correlation is increased slightly.
- This effect may be stronger in down markets (asymmetry in correlations).
- When they move in the opposite direction it is decreased.
- The correlations often are assumed to only temporarily deviate from a long run mean.

CORRELATIONS UPDATE LIKE GARCH

- Approximately,
  \[ \rho_t = \omega + \alpha z_{1,t-1}z_{2,t-1} + \beta \rho_{t-1} \]
  \[ \bar{\rho} = \frac{\omega}{1 - \alpha - \beta} \]
  \[ \rho_t = \bar{\rho} (1 - \alpha - \beta) + \alpha z_{1,t-1}z_{2,t-1} + \beta \rho_{t-1} \]
An Asymmetric model allows correlations to increase more when both prices move down together (like our asymmetric GARCH models).

\[ \rho_t = \omega + \alpha z_{1,t-1} z_{2,t-1} + \gamma z_{1,t-1} z_{2,t-1} (I_{z_{1,t-1}<0})(I_{z_{2,t-1}<0}) + \beta \rho_{t-1} \]
A simple approach to estimating and forecasting large covariance matrices


\[ \Omega_t = H_t = D_t R_t D_t, \]

\( D_t \) is diagonal matrix of conditional standard deviations
\( R_t \) is correlation matrix

Based on “standardized returns” \( z \)

\[ z_t \equiv D_t^{-1} r_t, \quad V_{t-1}(z_t) = R_t \]
The general, multivariate version of the DCC model looks like:

\[ R_t = \text{diag} \left( Q_t \right)^{-\frac{1}{2}} Q_t \text{diag} \left( Q_t \right)^{-\frac{1}{2}} \]

\[ Q_t = S (1 - \alpha - \beta) + \alpha z_{t-1}' z_{t-1}' + \beta Q_{t-1} \]

where \( \alpha \) and \( \beta \) are scalars. Hence the dynamics are the same for all series. S is the unconditional correlation matrix.

Estimation

\[
L = -\frac{1}{2T} \sum_{t=1}^{T} \left[ \log |H_t| + r_t H_t^{-1} r_t \right]
\]

\[
= -\frac{1}{T} \sum_{t=1}^{T} \left[ \log |D_t| + r_t D_t^{-1} r_t \right] - \frac{1}{2T} \sum_{t=1}^{T} \left[ \log |R_t| + z_t R_t^{-1} z_t \right] + \frac{1}{T} \sum_{t=1}^{T} z_t z_t'^t
\]

- The log likelihood can be written as the sum of a part that depends on the variances and a part that depends on the correlations.
- It is additive and can be separated into two estimation problems.
Bivariate Estimation of correlation parameters

\[
L = -\frac{1}{2} \sum_{t=1}^{T} \left[ \log(2\pi) + \log\left(1 - \rho_{i,j,t}^2\right) + \frac{z_{i,t}^2 + z_{j,t}^2 - 2\rho_{i,j,t}z_{i,t}z_{j,t}}{1 - \rho_{i,j,t}^2} \right]
\]

\[
q_{i,j,t} = (1 - \alpha - \beta)\bar{r}_{i,j} + \alpha z_{i,t-1}z_{j,t-1} + \beta q_{i,j,t-1}
\]

\[
\rho_{i,j,t} = \frac{q_{i,j,t}}{\sqrt{q_{i,j,t}q_{i,j,t}}}
\]

- Engle proposes estimating a model for all univariate pairs and then averaging parameter estimates (easy, but not statistically sound).
- An alternative approach is to treat the pairs as a panel dataset and simultaneously estimate DCC parameters (inference works here).

Asymmetric Dynamic Correlations of Global Equity and Bond Returns

Lorenzo Capiello, Robert Engle and Kevin Sheppard
Data

- Weekly $ returns Jan 1987 to Feb 2002 (785 observations)
- 21 Country Equity Series from FTSE All-World Index
- 13 Datastream Benchmark Bond Indices with 5 years average maturity

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GARCH Models
(asymmetric in orange)

- GARCH
- AVGARCH
- NGARCH
- EGARCH
- ZGARCH
- GJR-GARCH
- APARCH
- AGARCH
- NAGARCH

- 3EQ,8BOND
- 0
- 1BOND
- 6EQ,1BOND
- 8EQ,1BOND
- 3EQ,1BOND
- 0
- 1EQ,1BOND
- 0
AVERAGE EMU COUNTRY BOND RETURN CORRELATION
RESULTS

- Asymmetric Correlations – correlations rise after negative returns
- Shift in level of correlations with formation of Euro
- Correlations are rising not just within EMU
- EMU Bond correlations are especially high