1. Consider the conditionally Normal GARCH(p,q) model where \( f(r_t | F_{t-1}) = N(\mu_t, h_t) \).
   
a. Write down the log likelihood for the model when \( p=0 \) and \( q=1 \), and \( \mu=0 \).  
   conditional on \( r_1 \) and \( h_1 \). In other words, you can do conditional (on the first observation) maximum likelihood estimation.
   
b. Find the first order conditions for \( \omega \) and \( \alpha \).
   
c. Now, consider the model where \( p=0, q=1, \) and \( \mu_t = \beta_0 + \beta_1 r_{t-1} \). Find the first order conditions for all the parameters. Recall how we saw that MLE is the same as OLS for the parameters of the AR model. What is the difference between the first order conditions for the mean parameters when \( q=1 \) vs when there is no GARCH so that \( p=q=0 \). They are both least squares estimators for the mean parameters but one is …. 
   
d. Write down the log likelihood as a function for the model when \( p=1 \) and \( q=1 \), and \( \mu=0 \) conditional on \( r_1 \) and \( h_1 \).
   
e. Find the first order conditions. They will take a recursive form.
   
f. Show that the information matrix is block diagonal in the parameters for the mean and the parameters for the variance for the model in part c.
   
g. Now, consider the model where \( p=0, q=1, \) and \( \mu_t = \beta_0 + \beta_1 r_{t-1} + \gamma r_{t-1}^2 \). What is the information matrix for this model? Is the information matrix block diagonal in the parameters for the mean and parameters for the variance in this model?

2. Consider the GARCH(1,1) model.

   a. Write the k-step ahead forecast \( h_k^t = E(r_{t+k}^2 | F_t) \).

   b. The cumulative return over k periods can be written as \( R_k^t = \sum_{j=1}^{k} r_{t+j} \). Assuming the returns are uncorrelated, find the variance of the k-period return.

   c. Annualize the cumulative return variance. What does it converge to for large k?

3. For this problem, download recent price data from Yahoo Finance. Pick your favorite index and download 20 years worth of daily data for your favorite index. Pick your favorite stock that has been around for at least 20 years. Be sure that the data is in ascending chronological order, not reversed.

   a. Report the skewness and the kurtosis for both the index and the stock. What do these numbers tell you about the distribution of the returns? Report the
autocorrelations through lag 20 for the squared returns. What does this tell you about the time varying volatility in each series?

b. Fit a GARCH(1,1) model to the index and the stock you chose and report the results.

c. From the last day in the sample, build the sequence of daily forecasts from one-day ahead through 60 days (1 quarter). Plot the forecasts for both the index and the stock.

d. Cumulate the forecasts to get the forecasted variance for the next quarter for both the index and the stock. How do these forecasts compare to the unconditional variance? (just give a general interpretation)

e. Repeat part c. and d. for the one year ahead forecast.

f. We can use the standardize residuals to check if our GARCH model fits the data well. From part d, we have \( \frac{r_i}{\sqrt{h_i}} = z_i \) so that \( \left( \frac{r_i}{\sqrt{h_i}} \right)^2 = z_i^2 \). Since \( z_i \) should have variance of 1, there should be no correlation in the \( z_i \)'s but also, there should be no correlation in the squared \( z_i \)'s. That’s because if \( h_i \) is the correct conditional variance then dividing each return by its conditional standard deviation \( \sqrt{h_i} \) should make result in a series with constant variance (constant variance of 1). Only if we divide by the correct conditional standard deviation will this be true. Hence, we can examine the autocorrelation in the squared \( z_i \)'s to check to see if our fitted GARCH model fits the data well. Examine the autocorrelations for the squared standardized residuals out through lag 20 for each series. Has the time varying volatility been adequately captured by your GARCH model? If not, try a higher order GARCH model, and examine the autocorrelation in the (squared) standardized residuals again.

g. Fit a TARCH(1,1) model (a threshold ARCH model). What does the asymmetric term tell you about the dynamics of the volatility?