1. Write down the exact log likelihood for an AR(2) model.

2. Write down the conditional log likelihood for an AR(2) model.

3. The Autoregressive Conditional Duration (ACD) model (Engle and Russell 1998) is a model for waiting times (durations) between events like financial asset transactions. Since the data do not arrive in fixed intervals, it is interesting to ask how long we have to wait until the next trader occurs. Let \( x_i \) denote the time between the \( i \)th trade and the \((i-1)\)th trade – the duration. The ACD model assumes that \( \frac{x_i}{\psi_i} = \varepsilon_i \) is iid. That is, if we divide the duration by \( \psi_i \), we get an iid series. Equivalently, \( x_i = \psi_i \varepsilon_i \). \( \varepsilon_i \) must be a positive valued distribution and we also impose the condition that \( E(\varepsilon_i) = 1 \) so that \( \psi_i \) becomes the conditional expected duration \( \psi_i = E(x_i | F_{i-1}) \). If we assume an exponential distribution we get the Exponential ACD model. The conditional density function for an model with conditionally exponentially distributed durations is given by: \( f(x_i | F_{i-1}, \Theta) = \frac{1}{\psi_i} \exp \left( -\frac{x_i}{\psi_i} \right) \). A simple ACD model parameterizes the conditional expected duration by \( \psi_i = \omega + \alpha x_{i-1} \).

   a. Write down the log likelihood function for a sample with \( N \) observations.
   b. Take the derivative of the log likelihood function with respect to \( \omega \) and \( \alpha \) to find the first order conditions.
   c. Express the first order conditions in terms of the \( \varepsilon \)’s and provide an interpretation of the first order conditions. The constant should be identifying the level and the slope coefficient solves an orthogonality condition. Notice that the first order conditions only use two conditions that do not hinge on the exponential distribution assumption. Since the moments don’t involve the distributional assumptions the estimator will provide a consistent estimate of the parameter values even if the exponential distribution assumption is wrong. Hence, the same way that the Normal distribution still provides consistent estimates even if the assumption is wrong, here the exponential distribution plays the same role.
   d. Take the second derivative of the log likelihood function to get the second derivative estimate of the information matrix. You will have 4 second derivatives that you can put in a matrix

\[
\begin{pmatrix}
\frac{d^2 L}{d \omega d \omega} & \frac{d^2 L}{d \omega d \alpha} \\
\frac{d^2 L}{d \alpha d \omega} & \frac{d^2 L}{d \alpha d \alpha}
\end{pmatrix}
\]

Let \( \hat{\omega} \) and \( \hat{\alpha} \) denote the parameter values that
Express \( \hat{I}_{2d} = \frac{1}{T} \begin{bmatrix} \frac{d^2L}{d\omega d\omega} & \frac{d^2L}{d\omega d\alpha} \\ \frac{d^2L}{d\alpha d\omega} & \frac{d^2L}{d\alpha d\alpha} \end{bmatrix} \). Each element of the matrix will be a sum of terms evaluated at the maximum likelihood estimates \( \hat{\omega} \) and \( \hat{\alpha} \).

4. Construct the absolute value of the daily S&P500 returns from the earlier assignment.
   a. Plot the absolute value of the s&P500 returns.
   b. What do these autocorrelations imply about volatility? Explain.
   c. What is the kurtosis of the returns (not absolute value). What does the kurtosis tell you about the distribution of the returns relative to a Normal distribution? Plot and interpret the quantile quantile plot as well and provide an interpretation.

5. Construct a moving average of the squared S&P500 returns. Use the k most recent observations to get an estimate of volatility at time t (you can’t do this for the first k observations).
   a. Do this for k=100 and plot the annualized daily volatility series.
   b. Do this for k=252 and plot the annualized daily volatility series.
   c. Construct the Riskmetrics exponential smoother for the daily S&P500 returns. Initiate the first variance (\( \sigma^2 \)) to the unconditional sample variance. Then construct the series for each period after the first.
   d. Do this for \( \lambda = .95 \) and plot the annualized daily volatility series.
   e. Do this for \( \lambda = .8 \) and plot the annualized daily volatility series.
   f. Do this for \( \lambda = .1 \) and plot the annualized daily volatility series.

6. Consider the GARCH(1,1) model for annual returns where
   \( h_t = .00045 + .04r_{t-1}^2 + .94h_{t-1} \). This is a stationary GARCH model (we’ll see why later).
   a. What is conditional variance at time t if \( r_{t-1} = .04 \) and \( h_{t-1} = .03 \)?
   b. What is the conditional variance at time t if \( r_{t-1} = .01 \) and \( h_{t-1} = .03 \)?
   c. What is the unconditional variance of \( r_t \) (take unconditional expectations of both sides of the equation).

7. Use the S&P500 daily returns data. Eviews will output the “likelihood” associated with a model estimate.
   a. Estimate an ARCH(1) model and report the Likelihood
   b. Estimate an ARCH(9) model and report the Likelihood
c. Estimate a GARCH(1,1) model and report the Likelihood

d. Estimate a GARCH(2,2) model and report the Likelihood

e. Which model has the highest likelihood?

8. For your preferred GARCH model from problem 7:

a. Calculate the sequence of conditional daily variance for each day in the sample.

b. Using the conditional standard deviation (\(\sqrt{h_t}\)), build a 2 standard deviation interval for the return in period t. Calculate the fraction of the daily returns fall into these intervals.

c. Calculate the time series of annualized volatilities and plot them.