Operations Research: The Legacy of George Bernard Dantzig for Today and Tomorrow

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Outline

• GBD’s Legacy

• Model classes
  – Uncertainty
  – Dynamics
  – Competition

• Conclusions
GBD Chronology

• Mathematical legacy
• UMd/UMich/UCalBerkeley
• “Urban Myths”
• The birth of OR/Mathematical Programming
  – Air Force
  – JvN
  – RAND
RAND and Beyond

• LP/Simplex Method (Air Force – Computing)
• Degeneracy
• TSP
• Stochastic Programming
• Generalized programming
• Networks/MaxFlow/MinCut
• Equilibrium (Chemical Equilibrium/Control)
• QP/LCP
• Decomposition
• Integer
GBD Academic:
UC/Berkeley       Stanford
Student Thesis Topics

- LP: 17
- Stochastic: 9
- Integer/Networks: 8
- LCP/QP: 5
- Nonlinear: 3
- Other: 9
GBD’s Big Three Contributions

• The Simplex Method
• Systems with *inequalities*
• An *objective function*

• Note: All motivated by practical problems
Models for Practice

• Differentiating models
  Certain v. Uncertain
  Static v. Dynamic
  Non-competitive v. Competitive

• Best for optimizers
  Certain, Static, Non-competitive

• For Practice
  Uncertain, Dynamic, Competitive
Dealing with Uncertainty

• The Classic Model (GBD): Stochastic Linear Programming (SLP)

\[
\begin{align*}
\min & \quad c x_1 + \sum_{i=1}^{N} p_i^2 q x_2^i \\
\text{s. t.} & \quad A x_1 = b, \quad x_1 \geq 0 \\
& \quad T x_1 + W x_2^i = \xi_2^i, \quad x_2^i \geq 0
\end{align*}
\]

How to solve efficiently?
• Decomposition (GBD, more later..)
• Sampling (GBD et al…)
• Structure…

What can SLP do?
Applications

- Transportation - fleet assignment (GBD, generalized network)
- Production - supply chains
- Communications – network design
- Finance – portfolios

...  

- Are there limits to applications? Problem sizes? How to treat samples?
General Stochastic Program

• Formulation:
  \[
  \min_x \left\{ E_\xi[f(x,\xi)] \mid g(x,\xi) \leq 0 \text{ a.s.} \right\}
  \]

• Alternatives?
  – Suppose the only information for \( \xi \) is through samples:
    \( \xi_1, \ldots, \xi_\nu \)
  – What can we say about sampled problems?
    \[
    \min_{x \in X} \left( 1/ \nu \right) \sum_{i=1}^{\nu} f(x,\xi_i) \text{ s.t. } g(x,\xi_i) \leq 0
    \]
    and also
    \[
    \min_{x \in X} f(x, (1/ \nu) \sum_{i=1}^{\nu} \xi_i) \text{ s.t. } g(x, (1/ \nu) \sum_{i=1}^{\nu} \xi_i) \leq 0
    \]
  – What are the best ways to use those samples?
General Sampling Result

Suppose $x^\nu$ solves:

$$\min_{x \in X} \left( \frac{1}{\nu} \sum_{i=1}^{\nu} f(x, \xi^i) \right) \text{ s.t } g(x, \xi^i) \leq 0$$

then, under a suitable set of conditions, we can find a random vector, $u$, that solves another optimization problem such that $\nu^{0.5}(x^\nu - x^*)$ converges to $u$

Similar to a Central Limit Theorem but maybe even better.
Observations: The Good News and What’s Next?

• Asymptotic distribution of optimal solution of sampled problem is:
  – Sometimes multivariate normal
  – Sometimes projection of multivariate normal onto constraints
  – Sometimes an atom at a single point

• What about large parameter sets?
  – When do we start to observe the asymptotic behavior?
  – How big must \( \nu \) (no. of samples) be?
Overall Result and Challenge

- By batching means of the samples, we can often reduce required number of samples for given a confidence interval
- Can optimize over k to find best batching for given confidence interval

**Challenge:** How in general to characterize very large optimization models with sampled parameters? What is the best way to model them?
Dynamic Models

• Discrete Form (Stochastic Model):

\[
\text{minimize} \quad E_p \left[ \sum_{t=1}^{T} f_t(x_t, x_{t+1}, p) \right] \\
\text{s.t.} \quad x_t \in X_t, \ x_t \text{ nonanticipative} \\
p \text{ in } P \text{ (distribution class)}
\]

• Dynamic Program (MDP..)

What is the value function? How to find that?
DYNAMIC PROGRAMMING FORM

- **STAGES**: $t=1,\ldots,T$
- **STATES**: $x_t$
- **VALUE FUNCTION**:

$$Q_t(x_t) = E[Q_t(x_t, \xi_t)]$$ where

$\xi_t$ is the random element and

$$Q_t(x_t, \xi_t) = \min f_t(x_t, x_{t+1}, \xi_t) + Q_{t+1}(x_{t+1})$$

s.t. $x_{t+1} \in X_{t+1,t}(\xi_t)$
How to Find the Value Function?

• Use structure
  – Decomposition
  – Cutting planes
  – Sampling again

• Result:
  – Can find improving bounds on $Q_{t+1}(x_{t+1})$
    (Easiest with serial independence for sampling but converging bounds for many stages)
DECOMPOSITION METHODS

– FORM AN OUTER LINEARIZATION OF $Q_t$

– ADD CUTS ON FUNCTION:

Feasible region

$Q_t$

(linearization at iteration $k$)

[min at $k : < Q_t$]

(optimality cut)

(new cut)

(feasibility cuts)
Opportunities and Challenges in the Dynamic Case

• Can adapt methods to infinite horizons/specifics of production/inventory systems

• What about relationship to continuous time case?
  – Suppose decisions can change at any time?
  – Bounds from discrete case?
  – Semi-Markov analogs?
  – Analogs of FEM-type error bounds?
  – Specifics for relevant applications? (Portfolios?)
Competitive Situations

• In reality, decisions must involve competitors and their reactions
• Example: energy industry – electric power competition (price bids)
• Models of interest: Optimize own decisions subject to others choosing optimally (and all are subject to uncertainty and dynamics!)
Formulations (SMPEC)

- Stochastic Mathematical Programs with Equilibrium Constraints (SMPEC):

\[
\text{minimize } \mathbb{E}_p \left[ \sum_{t=1}^T f_t(x_t, x_{t+1}, p) \right] \\
\text{s.t. } x_t \in X_t, \ x_t \text{ nonanticipative} \\
\quad H(x_t, x_{t+1}) \in N(x_t) \text{ a.s.}
\]

If the last constraint has good properties, and so does the objective, then all can be solved but…

What happens when good conditions for equilibria do not exist?
General Issue

• Standard Equilibrium Results
  – Concave utility functions for agents
  – Consistent information sets
  – Unique equilibrium with strict concavity

• Realistic Markets
  – Market mechanisms (and other things) negate concavity assumptions
  – Inconsistent and varying information sets
  – Multiple, disconnected equilibria (or disequilibrium)

• Goal: Find the Set of Equilibria (Worst Case?)

• Example: Electric Power Market
Competitive Electric Power Markets

$N$ Suppliers (bidders),
Each submits bid price and quantity

Power Exchange Market

Consumer

Demand

Supply bids
Market Clearing Process

Demand is 10

Supplier 1: 5 MWh @ $10
Supplier 2: 10 MWh @ $15
Supplier 3: 10 MWh @ $20

Problem: find optimal bidding strategies and the resulting MCP
Results

• Multiple equilibria
• Known demand:
  – At the highest MCP equilibrium point, every bidder $i$ bids at $\lceil c_i \rceil$, except the marginal bidder $j$ who bids at $\lfloor c_j \rfloor$
  – Unique marginal bidder if partially dispatched

• Stochastic demand:
  – Single marginal $i$
    • At any demand, bids at $\lceil p_j - \varepsilon \rceil$
    • At the highest demand, bids at $\lfloor c_j \rfloor$
  – Two marginal bidders, $i, j$
    • They bid just above the cost of the bidder with a lower quantity $p_i = p_j = \lceil c_j \rceil$ where $x_j \leq x_i$
Example Comparison of Payoffs

Case 1: Algorithm (worst equilibrium), MCP = 9.75
Case 2: at next higher bidder's cost, MCP = 8
Case 3: at cost, MCP = 6.51
Challenges and Opportunities with Competition

- Multiple equilibria, non-convexities
- In some cases, can find highest market clearing price (equilibrium point)
- How to do this efficiently?
- When is this possible in general?
- How to design market toward “socially optimal” points?
Conclusions

- GBD provided foundation for OR and optimization
- Motivation of practical decisions that involve uncertainty, dynamics, and competition
- Challenges remain as our computational power and our desire to reach broader issues increases