Optimizing Portfolios with Liquidity Risk

John R. Birge
The University of Chicago
(Some of work joint with Amit Bhandari, Northwestern U., and Gongyun Zhao, National University of Singapore)
Background

• Alternative investments offer diversification with additional risk
• Liquidity risk may arise in meeting liabilities and re-balancing
• Optimization models can capture liquidity risk through objective and constraints
• Finding optimal stationary policies possible with cutting plane method
• Results from models ignoring liquidity risk may be substantially different from models that include liquidity risk
OUTLINE

• Dynamic model framework
• Liquidity risk factors
• Strategies for including liquidity into models
• Impact on optimized portfolio
• Long-term (infinite horizon) strategy solutions
• Conclusions
Why Model Dynamically?

• Alternative: static mean-variance (efficient) portfolio
  – Requires constant re-balancing to maintain efficiency
  – Does not adjust for objective (or liquidity needs)

• Three potential reasons:
  – Changing return distributions
  – Possibilities for market timing
  – Capture liquidity with transaction costs (taxes)
  – Include path-dependent objective (e.g., coherent risk/recursive utility, previous benchmark, high-water mark)
Continuous-Time Models

• Advantages
  – Provides structure of optimal solutions
  – Policy adjusts for changing conditions

• Disadvantages
  – Hard to include liquidity constraints
  – Policy complex in higher dimensions

=> Consider general, discrete-time dynamic model
Dynamic Programming Approach

- State: $x_t$ corresponding to positions in each asset plus possibly some history, prices, economic, and other factors
- Value function: $V_t(x_t)$
- Actions: $u_t$
- Constraints: $u_t \in K_t(x_t)$
- Possible events $s_t$, probabilities $p_{st}$
- Find:

$$V_t(x_t) = \max_{u_t \in K_t(x_t)} c_t(u_t, x_t) + PV_{x_t,t} \left( \sum_{s_t} p_{st} V_{t+1}(x_{t+1}(x_t, u_t, s_t)) \right)$$

**Advantages:** general, dynamic, can limit types of policies

**Disadvantages:** Dimensionality, approximation of $V$ at some point needed, limited policy set may be needed, accuracy hard to judge
Stochastic Programming Models

Asset-Liability Management Formulation

Scenarios $s$, Assets $k$, Liabilities $L$, Transaction Costs $(\alpha^+, \alpha)$:

$$\max \sum_t \sum_s p_{st} \beta_{st} \left(c_t(u_t(s), x_t(s)) \right)$$

s.t. (for all $s$, $k$, $t$):

$$r_t(k,s) x_t(k,s) + (1-\alpha^+) u_t^+(k,s) - (1+\alpha) u_t^-(k,s) = x_{t+1}(k,s)$$

$$\sum_k (-u_t^+(k,s) + u_t^-(k,s)) = L_t(s), \; u_t^+ \geq 0, \; u_t^- \geq 0$$

**Nonanticipativity:**

$$u_t(k,s) - u_t(k,s') = 0 \text{ if } s, s' \text{ have same history at } t.$$ 

**Advantages:**

General model, includes transaction costs, (can include tax lots), additional constraints.

**Disadvantages:** Size of model, insight
Incorporating Liquidity Risk

• Sources
  – Trading effects in limited market
  – Limited marking to market
  – Direct constraints on trades (e.g., lockout period, trading delays)
  – Hard constraint on liability
  – Limited borrowing ability

• General approach: add to objective and constraints
Adding in Liquidity Constraints

- Market impact effects
  - Suppose convex cost: effective share price increases in quantity bought and decreases in quantity sold
  - Represent as convex (piecewise linear) function:

\[
\begin{align*}
\alpha_1^- & \quad \alpha_1^+ \\
\alpha_2^- & \quad \alpha_2^+ \\
\alpha_3^- & \quad \alpha_3^+
\end{align*}
\]

\[
\begin{align*}
-\alpha_2^- & \quad -\alpha_1^- \\
\quad u_1^- & \quad u_1^+ \\
\quad u_2^- & \quad u_2^+ \\
\quad u_3^- & \quad u_3^+
\end{align*}
\]

**Constraints:**

\[
\begin{align*}
& r_t(k,s) x_t(k,s) + \sum_i [(1-\alpha_i^+) u_{it}^+(k,s) - (1+\alpha_i^-) u_{it}^-(k,s)] = x_{t+1}(k,s) \\
& \sum_{ik} (-u_{it}^+(k,s) + u_{it}^-(k,s)) = L_t(s), \quad u_{it}^+ \geq 0, \quad u_{it}^- \geq 0
\end{align*}
\]
Cost Representation

• Degree of effect on market can vary with asset type, time, and other factors
• Convexity ensures that optimization is not affected
• Does not include effect of $x_t$ position on market impact
• Similar form can include concave utility (or coherent risk measure) for exceeding liability (e.g., consumption)
Liquidity Trading Constraints

- Situation: alternative investments (e.g., hedge funds, private equity) with restrictions on sales
- Include as constraints (e.g., $u_t(k)=0$ at ineligible $t$)
- Additional variables $y_t(k,s)^+, y_t(k,s)^-$ represent commitment for trade at $t+\Delta$, new constraints: $u_{t+\Delta}(k,s)=y_t(k,s)$
- Result: maintain convex optimization, solve with same method
Test: Comparison of Fixed Trading Interval to Continuous-time Model

- Consumption utility: value in each period
- Liquidity constraints: consumption becomes fixed liability for next period (i.e., no decreases)
- Trading constraint: trades only occur at fixed time intervals in discrete-time model
- Solve the discrete time model
- Evaluate the effect of trading restrictions
Example Results

- Situation: 3 periods, 3 assets: Riskfree, Risky Market, Risky Private (Illiquid)
- Market transaction costs: 1%, 1% increasing to 5%, 5%
- Potential liability in period 2: 0, 30%, 60%
- Trade-ability of private asset: on or off
Trade/No Trade Differences

- Observations: Inability to trade private asset can lead to less investment in market asset
Differences with Trans. Cost/Liability

- Observation: Increasing transaction costs can increase risky investments overall/sometimes decreases private investment

![Chart showing differences with and without transaction costs, with asset allocations for different percentage of costs and no trade/no transaction scenarios.](chart.png)
Extending to Long Term (Infinite Horizon)

• (Very) long-term investor (e.g., university endowment)
• Annual payout from portfolio necessary to maintain operations, employees
• Decisions:
  – How much to payout (consume)?
  – How to invest in asset categories?
• Complication from alternative assets (e.g., lock-out periods, advance notice, quarterly trade dates)
Infinite Horizon Formulation

• Notation:

\[ x \] – current state \((x \in X)\)

\[ u \text{ (or } u_x) \] – current action given \(x\) \((u \text{ (or } u_x) \in U(x); \text{ may depend on previous periods for liquidity constraints})\)

\[ \delta \] – single period discount factor

\[ P_{x,u} \] – probability measure on next period state \(y\) depending on \(x\) and \(u\)

• Problem: Find \(V\) such that

\[
V(x) = \max_{u \in U(x)} \left\{ c(x,u) + \delta E_{P_{x,u}}[V(y)] \right\} (= TV)
\]

for all \(x \in X\).
Formulation Issues (not covered in this talk)

- **Time aggregation (discretization)**
  - What interval to use when some assets can be traded continuously?
  - How close is a discrete-time solution to the continuous-time solution?

- **Asset aggregation**
  - How accurate is it to represent asset as “classes” and not individual assets?
Issues for Discrete-time Form: Alternative Approaches

- Direct dynamic programming
  - Problem: curse of dimensionality

- State space approximation
  - Reduce dimensionality with limited state space
  - Simplest: wealth; next: previous consumption
  - Difficulty for transaction costs

- Policy approximation
  - Optimize over policy parameters (low dimension)

- Value-function approximation
  - Approximate dynamic programming (ADP)
Approximate Dynamic Programming Format

• Use LP solution of dynamic (Bellman) equation:
  \[ \max (d, V) \text{ s.t. } TV \geq V \text{ for distribution } d \text{ on } x \]

• Approximate \( V \) with finite set of basis functions \( \Phi_j \), weights \( \lambda_j \)

• LP for finite set becomes: Find \( \lambda \) to
  \[ \max (d, \Phi \lambda) \text{ s.t. } T \Phi \lambda \geq \Phi \lambda \]
Solving ADP Form

• Bounds available (Tsitsiklis, Van Roy, De Farias)

• State discretizations:
  – Use structure to reduce constraint set

• Use duality
  – Dual Form:
    \[ \min_{\mu} \max_{\lambda} (d, \Phi\lambda) + (\mu, T\Phi\lambda - \Phi\lambda) \]
Successive Linear Approximation Approach

• Define an upper bound on the value function

\[ V^0(x) \geq V(x) \quad \forall x \in X \]

*Here: Assume highest expected asset rate with certainty.*

• Iteration \( k \): upper bound \( V^k \)

Solve for some \( x^k \)

\[ TV^k(x^k) = \max_u c(x^k,u) + \delta E_{P^{x_k,u}}[V^k(y)] \]

Update to a better upper bound \( V^{k+1} \)

• Update uses an outer linear approximation on \( V^k \)
Successive Outer Approximation

\[ V^0 \]

\[ V^1 \]

\[ TV^0 \]

\[ V^* \]

\[ x^0 \]
Properties of Approximation

• $V^* \leq TV^k \leq V^{k+1} \leq V^k$

• Contraction
  $$\| TV^k - V^* \|_\infty \leq \delta \| V^k - V^* \|_\infty$$

• Unique Fixed Point
  $TV^* = V^*$

  $\Rightarrow$ if $TV^k \geq V^k$, then $V^k = V^*$. 
Convergence

• Value Iteration
  \( T^k V^0 \to V^* \)

• Distributed Value Iteration
  If you choose every \( x \in X \) infinitely often, then \( V^k \to V^* \).
  (Here, random choice of \( x \), use concavity.)

• Deepest Cut
  Pick \( x^k \) to maximize \( V^k(x) - TV^k(x) \)
  DC problem to solve
  Convergence again with continuity (caution on boundary of domain of \( V^* \))
Two-asset Portfolio with Consumption Constraint

• Determine asset allocation and consumption policy to maximize the expected discounted utility of spending
  – State and Action
    \[ x = (\text{cons}, \text{risky}, \text{wealth}) \quad u = (\text{cons}_\text{new}, \text{risky}_\text{new}) \]
  – Two asset classes
    • Risky asset, with lognormal return distribution
    • Riskfree asset, with given return \( r_f \)
  – Power utility function
    \[ c(\text{cons}_\text{new}) = \frac{\text{cons}_\text{new}^{1-\gamma}}{1-\gamma} \]
  – Consumption rate constrained to be non-decreasing
    \[ \text{cons}_\text{new} \geq \text{cons} \]
Continuous-time Results

• Dybvig ’95*
  • Consumption rate remains constant until wealth reaches a new maximum
  • The risky asset allocation $\alpha$ is proportional to $w-c/r_f$, which is the excess of wealth over the perpetuity value of current consumption
  • $\alpha$ decreases as wealth decreases, approaching 0 as wealth approaches $c/r_f$ (which is in absence of risky investment sufficient to maintain consumption indefinitely).

• Dybvig ’01
  • Considered similar problem in which consumption rate can decrease but is penalized (soft constrained problem)

• Rogers (and Zane ’98, ’01)
  • “Relaxed investor” but no consumption constraint

Goals in Comparison to Continuous-time Model

• Solve the discrete time model (and verify convergence to continuous time results)
• Evaluate the effect of trading restrictions in terms of discrete period lengths without trading in restricted liquidity assets
• Consider additional problem features
  – Transaction Costs
  – Multiple risky assets
Results – Non-decreasing Consumption

As number of time periods per year increases, solution converges to continuous time solution
Results – Non-Decreasing Consumption with Transaction Costs
Observations

• Effect of Trading Restrictions
  – Continuously traded risky asset: 70% of portfolio for 4.2% payout rate
  – Quarterly traded risky asset: 32% of portfolio for same payout rate

• Transaction Cost Effect
  – Small differences in overall portfolio allocations
  – Optimal mix depends on initial conditions
Algorithm Convergence

- Sometimes convergence is fast (with sufficiently high discounting)
Convergence

• Sometimes convergence is slow for very small discount factors

• (Note: ADP results also depend on discount factor)
High-dimensional Performance

• Stochastic linear-quadratic control

\[
V^*(x) = \min_{y \sim \mathcal{N}(m,\Sigma)} \mathbb{E} \left[ \sum_{t=0}^{\infty} \delta^t \left( x_t^T Q x_t + y_t^T R y_t \right) \right]
\]

s.t.

\[x_{t+1} = A x_t + B y_t + b,\]

for \( t = 0,1,2,\ldots,\)

\[x_0 = x, \quad x_t \in \mathbb{R}^n, \quad y_t \in \mathbb{R}^m\]
Performance Observations

• Convergence
  – Early iterations:
    • Rapid approximate results
    • Little initial deterioration with increasing dimension
  – Later iterations:
    • Convergence consistent with high discounting and lower dimensions
    • Slow convergence with low discounting and high dimensions
Challenges

- Constrain feasible region to obtain convergence results
- Accelerate the DC search problem to find the deep cut
- Accelerate overall algorithm using:
  - multiple simultaneous cuts?
  - nonlinear cuts?
  - bundles approach?
Extensions

• Soft constraint on decreasing consumption
  – Allow some decreases with some penalty

• Lag on sales
  – Waiting period on sale of risky assets (e.g., 60-day period)

• Multiple assets
  – Allocation bounds
Broader Optimization Issues

• Possible extensions of iterative value function approximation
  – Depends here on convexity (concavity)
  – General ADP assumes no structure (relies on state approximation)
  – Possible to combine approaches (constructively)?
• How to combine with continuous-time solutions and have overall error bounds?
Conclusions

- Liquidity risk (esp. trading restrictions can change optimal portfolio characteristics)
- Solution of infinite-horizon investment problem with cutting plane method
- Convergence with some conditions
- More to do on improving convergence behavior of algorithm