Liquidity Risk in Dynamic Portfolio Optimization

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Background

- Alternative investments offer diversification with additional risk
- Liquidity risk may arise in meeting liabilities and rebalancing
- Optimization models can capture liquidity risk through objective and constraints
- Finding optimal stationary policies possible with cutting plane method
- Results from models ignoring liquidity risk may be substantially different from models that include liquidity risk
OUTLINE

• Dynamic model framework
• Liquidity risk factors
• Strategies for including liquidity into models
• Impact on optimized portfolio
• Conclusions
Dynamic Optimization Model
Asset-Liability Management Formulation

Scenarios $s$, Assets $k$, Liabilities $L$, Transaction Costs $(\alpha^+, \alpha)$:

$$\max \sum_t \sum_s p_{st} \beta_{st} \left( c_t(u_t(s), x_t(s)) \right)$$

s.t. (for all $s$, $k$, $t$):

$$r_t(k,s) x_t(k,s) + (1-\alpha^+) u_t^+(k,s) - (1+\alpha^-) u_t^-(k,s) = x_{t+1}(k,s)$$

$$\sum_k (-u_t^+(k,s) + u_t^-(k,s)) = L_t(s), \ u_t^+ \geq 0, \ u_t^- \geq 0$$

**Nonanticipativity:**

$$u_t(k,s) - u_t(k,s') = 0 \text{ if } s, s' \text{ have same history at } t.$$ 

Advantages:

General model, includes transaction costs, (can include tax lots), additional constraints.

Disadvantages: Size of model, insight
Incorporating Liquidity Risk

• Sources
  – Trading effects in limited market
  – Limited marking to market
  – Direct constraints on trades (e.g., lockout period, trading delays)
  – Hard constraint on liability
  – Limited borrowing ability

• General approach: add to objective and constraints
Adding in Liquidity Constraints

• Market impact effects
  – Suppose convex cost: effective share price increases in quantity bought and decreases in quantity sold
  – Represent as convex (piecewise linear) function:

\[
\begin{align*}
  r_t(k,s) x_t(k,s) + \sum_i [(1 - \alpha_i^+) u_{it^+}(k,s) - (1 + \alpha_i^-) u_{it^-}(k,s)] &= x_{t+1}(k,s) \\
  \sum_{ik} (-u_{it^+}(k,s) + u_{it^-}(k,s)) &= L_t(s), \quad u_{it^+} \geq 0, \quad u_{it^-} \geq 0
\end{align*}
\]
Cost Representation

- Degree of effect on market can vary with asset type, time, and other factors
- Convexity ensures that optimization is not affected
- Does not include effect of $x_t$ position on market impact
- Similar form can include concave utility (or coherent risk measure) for exceeding liability (e.g., consumption)
Liquidity Trading Constraints

• Situation: alternative investments (e.g., hedge funds, private equity) with restrictions on sales

• Include as constraints (e.g., $u_t(k) = 0$ at ineligible $t$)

• Additional variables $y_t(k,s)^+, y_t(k,s)^-$ represent commitment for trade at $t+\Delta$, new constraints:
  $u_{t+\Delta}(k,s) = y_t(k,s)$

• Result: maintain convex optimization, solve with same method
Test: Comparison of Fixed Trading Interval to Continuous-time Model

- Consumption utility: value in each period
- Liquidity constraints: consumption becomes fixed liability for next period (i.e., no decreases)
- Trading constraint: trades only occur at fixed time intervals in discrete-time model
- Solve the discrete time model
- Evaluate the effect of trading restrictions
Example Results

- Situation: 3 periods, 3 assets: Riskfree, Risky Market, Risky Private (Illiquid)
- Market transaction costs: 1%, 1% increasing to 5%, 5%
- Potential liability in period 2: 0, 30%, 60%
- Trade-ability of private asset: on or off

Period 1: Initial Allocation

Period 2: Re-allocate if trade-able; Pay liability if any

Period 3: Pay liability, Measure utility
Trade/No Trade Differences

- Observations: Inability to trade private asset can lead to less investment in market asset
Differences with Trans. Cost/Liability

• Observation: Increasing transaction costs can increase risky investments overall/sometimes decreases private investment
Extending to Long Term (Infinite Horizon)

• (Very) long-term investor (e.g., university endowment)

• Annual payout from portfolio necessary to maintain operations, employees

• Decisions:
  – How much to payout (consume)?
  – How to invest in asset categories?

• Complication from alternative assets (e.g., lock-out periods, advance notice, quarterly trade dates)
Infinite Horizon Formulation

• Notation:
  \( x \) – current state \((x \in X)\)
  \( u \) (or \( u_x \)) – current action given \( x \) \((u \) or \( u_x \) \( \in U(x); \) may depend on previous periods for liquidity constraints\))
  \( \delta \) – single period discount factor
  \( P_{x,u} \) – probability measure on next period state \( y \) depending on \( x \) and \( u \)

• Problem: Find \( V \) such that

\[
V(x) = \max_{u \in U(x)} \{ c(x,u) + \delta E_{P_{x,u}}[V(y)] \} \text{ (= } TV) \text{ for all } x \in X.
\]
Results – Non-decreasing Consumption

As number of time periods per year increases, solution converges to continuous time solution.
Results – Non-Decreasing Consumption with Transaction Costs
Observations

• Effect of Trading Restrictions
  – Continuously traded risky asset: 70% of portfolio for 4.2% payout rate
  – Quarterly traded risky asset: 32% of portfolio for same payout rate

• Transaction Cost Effect
  – Small differences in overall portfolio allocations
  – Optimal mix depends on initial conditions
Extensions

• Soft constraint on decreasing consumption
  – Allow some decreases with some penalty

• Lag on sales
  – Waiting period on sale of risky assets (e.g., 60-day period)

• Multiple assets
  – Allocation bounds
Conclusions

• Liquidity risk (esp. trading restrictions can change optimal portfolio characteristics)
• Solutions possible with form of dynamic programming approximation
• Possible extensions to general formulations