Effectively Managing Liquidity Risk in Dynamic Asset-Liability Optimization

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Background

• Alternative investments offer diversification with additional risk
• Liquidity risk may arise in meeting liabilities and re-balancing
• Optimization models can capture liquidity risk through objective and constraints
• Results from models ignoring liquidity risk may be substantially different from models that include liquidity risk
OUTLINE

• Dynamic model framework
• Liquidity risk factors
• Strategies for including liquidity into models
• Impact on optimized portfolio
Why Model Dynamically?

• Alternative: static mean-variance (efficient) portfolio
  – Requires constant re-balancing to maintain efficiency
  – Does not adjust for objective (or liquidity needs)

• Three potential reasons:
  – Changing return distributions
  – Possibilities for market timing
  – Capture liquidity with transaction costs (taxes)
  – Include path-dependent objective (e.g., previous benchmark, high-water mark)
Continuous-Time Models

- **Advantages**
  - Provides structure of optimal solutions
  - Policy adjusts for changing conditions

- **Disadvantages**
  - Hard to include liquidity constraints
  - Policy complex in higher dimensions

=> Consider general, discrete-time dynamic model
Dynamic Programming Approach

- State: $x_t$ corresponding to positions in each asset plus possibly some history, prices, economic, and other factors
- Value function: $V_t(x_t)$
- Actions: $u_t$
- Constraints: $u_t \in K_t(x_t)$
- Possible events $s_t$, probabilities $p_{st}$
- Find:
  $$V_t(x_t) = \max_{u_t \in K_t(x_t)} c_t(u_t, x_t) + PV_{x_t,t}(\sum_{s_t} p_{st}V_{t+1}(x_{t+1}(x_t, u_t, s_t)))$$

Advantages: general, dynamic, can limit types of policies

Disadvantages: Dimensionality, approximation of $V$ at some point needed, limited policy set may be needed, accuracy hard to judge
Stochastic Programming Models

Asset-Liability Management Formulation

Scenarios $s$, Assets $k$, Liabilities $L$, Transaction Costs $(\alpha^+, \alpha)$:

$$\max \sum_t \sum_s p_{st} \beta_{st} (c_t(u_t(s), x_t(s)))$$

s.t. (for all $s, k, t$):

$$r_t(k,s) x_t(k,s) + (1-\alpha^+) u_t^+(k,s) - (1+\alpha) u_t^-(k,s) = x_{t+1}(k,s)$$

$$\sum_k (-u_t^+(k,s)+u_t^-(k,s))=L_t(s), \ u_t^+ \geq 0, \ u_t^- \geq 0$$

**Nonanticipativity:**

$$u_t(k,s) - u_t(k,s') = 0 \text{ if } s, s' \text{ have same history at } t.$$  

Advantages:

General model, includes transaction costs, (can include tax lots), additional constraints.

**Disadvantages:** Size of model, insight
Incorporating Liquidity Risk

• Sources
  – Trading effects in limited market
  – Limited marking to market
  – Direct constraints on trades (e.g., lockout period, trading delays)
  – Hard constraint on liability
  – Limited borrowing ability

• General approach: add to objective and constraints
Adding in Liquidity Constraints

• Market impact effects
  – Suppose convex cost: effective share price increases in quantity bought and decreases in quantity sold
  – Represent as convex (piecewise linear) function:

\[
\begin{align*}
  r_t(k,s) x_t(k,s) + \sum_i [(1-\alpha_i^+) u_{it}^+(k,s) - (1+\alpha_i^-) u_{it}^-(k,s)] &= x_{t+1}(k,s) \\
  \sum_{ik} (-u_{it}^+(k,s) + u_{it}^-(k,s)) &= L_t(s), \quad u_{it}^+ \geq 0, \quad u_{it}^- \geq 0
\end{align*}
\]
Cost Representation

• Degree of effect on market can vary with asset type, time, and other factors
• Convexity ensures that optimization is not affected
• Does not include effect of $x_t$ position on market impact
• Similar form can include concave utility (or coherent risk measure) for exceeding liability (e.g., consumption)
Liquidity Trading Constraints

• Situation: alternative investments (e.g., hedge funds, private equity) with restrictions on sales
• Include as constraints (e.g., \( u_t(k) = 0 \) at ineligible \( t \))
• Additional variables \( y_t(k,s)^+, y_t(k,s)^- \) represent commitment for trade at \( t + \Delta \), new constraints:
  \[ u_{t+\Delta}(k,s) = y_t(k,s) \]
• Result: maintain convex optimization, solve with same method
Solution Approach: Decomposition

– Form an outer linearization of value $-V_t$
– add cuts on function:

Feasible region

(new cut (optimality cut))

LINEARIZATION AT ITERATION $k$

min at $k : < -V_t$
Method Enhancement: Abridged Nested Decomposition*

- Incorporates sampling into the general framework of nested decomposition for stochastic programs
- Form of approximate dynamic programming
- Samples both the subproblems to solve and the solutions to continue from in the forward pass of nested decomposition
- Consistent test for optimality on each iteration
- Can also extend approach to long horizons

Test: Comparison of Fixed Trading Interval to Continuous-time Model

- Consumption utility: value in each period
- Liquidity constraints: consumption becomes fixed liability for next period (i.e., no decreases)
- Trading constraint: trades only occur at fixed time intervals in discrete-time model
- Continuous-time solution: Dybvig (Review Econ Studies, 1995)
- Solve the discrete time model (verify convergence to continuous time results)
- Evaluate the effect of trading restrictions in terms of discrete period lengths without trading in restricted liquidity assets
Algorithm

Convergence: Infinite Horizon

• Convergence depends on discounting
• Fast initially then reduced
Results – Non-decreasing Consumption

As number of time periods per year increases, solution converges to continuous time solution.
Observations

- **Effect of Trading Restrictions**
  - Continuously traded risky asset: 70% of portfolio at 4.2% payout rate
  - Quarterly traded risky asset: 32% of portfolio for same payout rate

- **Transaction Cost Effect**
  - Small differences in overall portfolio allocations
  - Optimal mix depends on initial conditions
Conclusions

• Liquidity effects can be included in overall asset-liability portfolio optimization
• Key in implementation is to ensure convexity
• Solutions available using decomposition (and Monte Carlo) methods
• Results indicate potential for significant asset-mix differences with trading restrictions and hard liability constraints