Alternative Value-Function Approximations in Portfolio Optimization

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Issues

Approximation drivers in portfolio optimization:

– Numbers of assets, re-balancing periods
– Transaction costs (and taxes)
– Path-dependent utilities
– Investment/consumption constraints
– Liquidity costs

• Questions:
  – What bounds are available from different approximation forms?
  – What forms of approximation work best? When?
Themes

- **Bounds**
  - Problem structure (convexity and objective limits) can provide bounds

- **Efficiency**
  - Simplifications (e.g., continuous-time friction-free relaxation) can provide template for constrained optimization
Outline

• General formulation
• Linearization methods
• General function approximation and FEM
• Use of relaxed problem structure
• Conclusions
General Formulation

• Notation:
  - $x$ – current state (prices/other factors/previous consumption/cash)
  - $y_t$ – portfolio allocations (risky assets)
  - $\tau$ – stopping time of next re-balance
  - $c_t$ – consumption (from cash) at $t$
  - $U_t$ – utility of consumption at $t$ [bequest at $T$]
  - $Y$ – set of attainable portfolios

• Problem: Find $V(V(0,x_0,y_0))$ such that

$$V(t,x,y) =$$

$$\sup_{c \in C(x), \tau, y' \in Y(x,c,\tau,y)} E_{t,x} \left[ \int_t^\tau e^{-\int_t^s r(u,X(u))du} U_s(c_s)ds ight. + e^{-\int_t^\tau r(u,X(u))du} V(\tau,X,\tau,y') \bigg],$$

with $V(T,x,y) = U(T,x,y)$. 
Alternative Approximations

• Simplify conditions (e.g., no transaction costs, no constraints)
• Assume a policy (e.g., trade at boundaries of no-trade region)
• Discretize (aggregate) time (and assets)
• Simplify function form (before time discretization)
Issues for Discrete-time Form: Alternative Approaches

• Direct dynamic programming
  – Problem: curse of dimensionality

• State space approximation
  – Reduce dimensionality with limited state space
  – Simplest: wealth; next: previous consumption
  – Difficulty for transaction costs

• Policy approximation
  – Optimize over policy parameters (low dimension)

• Value-function approximation
  – Approximate dynamic programming (ADP)
Simplifying Functions

• Suppose a (finite) set of basis functions, \( \phi_i \)
• Write \( V_h(t,x,y) = \sum \lambda_i(t) \phi_i(x,y) \)
• If no re-balancing:
  – Solving for \( V_h \) gives differential equations in \( \lambda \)
  – Now, can discretize in time for \( \lambda_i^n \) (and \( V_h^n \))
  – Can obtain error bounds on \( ||V-V_h^n|| \)
• With re-balancing,
  – Form of variational inequality:
    \[
    \frac{\partial V}{\partial t} + Gu - ru \leq 0 \\
    (\frac{\partial V}{\partial t} + Gu - ru)(V - V') = 0 \text{ over } [0,T] \times \mathbb{R}^n
    \]
    where \( V' \) is best attainable value at re-balancing.
Solving the Variational Inequality

Iterative Method:

Consider American option problem to find best re-balancing point.
Repeat process to find additional re-balancing points.
Sub-problems as linear complementarity problems.
Method 2

• Sequential Method over Time
  – Solve from period $n$ to $n+1$ in time
  – At each $n$, update to find $V'$ using optimization method (e.g., Newton’s method)

• Result:
  – Solutions with bounds provided by approximating basis functions
  – Time discretization bounds from time derivative (bounds)
  – Overall, error in time and space (possible higher-order accuracy with higher-order polynomial basis functions)
Computational Issues

- Convection effect generally dominates diffusion in portfolio problem – creating oscillations in numerical schemes
- Local discontinuous Galerkin method applies to the convection-diffusion equation – allowing error bounds and efficient computation
LDG Results

LDG avoids oscillations and fits at boundary.
Portfolio Extensions

• How to handle additional portfolio constraints:
  – Consumption restrictions (e.g., no lower than in past, “habit formation”)
  – Benchmark objectives
  – Dependence of utility on state

• Approach: look at continuous-time simplification and try to emulate.
Example: 2-asset Portfolio-Consumption Constraint

• Determine asset allocation and consumption policy to maximize the expected discounted utility of spending
  – State and Action
    \[ x=(\text{cons, risky, wealth}) \quad u=(\text{cons}_\text{new}, \text{risky}_\text{new}) \]
  – Two asset classes
    • Risky asset, with lognormal return distribution
    • Riskfree asset, with given return \( r_f \)
  – Power utility function
    \[ c(\text{cons}_\text{new}) = \frac{\text{cons}_\text{new}^{1-\gamma}}{1-\gamma} \]
  – Consumption rate constrained to be non-decreasing
    \[ \text{cons}_\text{new} \geq \text{cons} \]
Continuous-time Results

• Dybvig ’95*
  • Consumption rate remains constant until wealth reaches a new maximum
  • The risky asset allocation $\alpha$ is proportional to $w-c/r_f$, which is the excess of wealth over the perpetuity value of current consumption
  • $\alpha$ decreases as wealth decreases, approaching 0 as wealth approaches $c/r_f$ (which is in absence of risky investment sufficient to maintain consumption indefinitely).

• Dybvig ’01
  • Considered similar problem in which consumption rate can decrease but is penalized (soft constrained problem)

• Rogers (and Zane ’98, ’01)
  • “Relaxed investor” but no consumption constraint

Goals in Comparison to Continuous-time Model

- Solve the discrete time model (and verify convergence to continuous time results)
- Evaluate the effect of trading restrictions in terms of discrete period lengths without trading in restricted liquidity assets
- Use the form of the objective from continuous-time to guide a discrete-time solution
Continuous-Time Emulation

- Continuous-time solution:
  \[ V(c, w) = \frac{u(\max(c, r*w))}{r} + \delta(\max(c, r*w))^\gamma((w/\max(c, r*w))-(1/r))^\gamma^* \]
  where \( r^*, \delta, \gamma^* \) are parameters

Approach: fit discrete-time solution to same form where parameters may vary with \( c / w \)
Comparison to Continuous Time

gammaStar for different deltaT

- 0.685
- 0.68
- 0.675
- 0.67
- 0.665
- 0.66
- 0.655
- 0.65
- 0.645

- 0.00%
- 0.40%
- 0.80%
- 1.20%
- 1.60%
- 2.00%
- 2.40%
- 2.80%
- 3.20%
- 3.60%
- 4.00%

- continuous
- 8 yrs
- 4 yrs
- 2 yrs
- 1 yr
- 0.5 yrs
- 0.25 yrs
Constrained Consumption Extensions

• Next: include multiple assets with varying potential re-balancing points (e.g., hedge funds v. market assets)
• Construct multi-dimensional continuous template for discrete-time solution
• Provide overall error bounds
Conclusions

• Continuous time methods can provide computational methods and templates for discrete-time approximations
• FEM approaches provide error bounds (including time discretization)
• Approximating polynomial structure can lead to close approximations