Optimization Models in Financial Engineering and Modeling Challenges

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Introduction

• History of financial engineering
  – Rapid expansion of derivative market (total now greater than global equity)
  – Rise in successful quantitative investors (e.g., hedge funds)
  – Applications in asset management and risk management
  – Dot-com boom market
  – Securitization, housing bubble, and current crisis

• Current situation
  – Overall consolidation in the industry
  – Maintaining asset management and risk management interest
Presentation Outline

• Selected applications
  • Option pricing
  • Portfolio/asset-liability models
  • Tracking and trading
  • Securitization and its role in the crisis
  • Risk management/real options and going forward

• Future potential
Option Models

• “Derivative” securities
  – Example: Call: Buy a share at a given price at a specific time (European)
    • If by a specific time - American
  – Put: Sell; Straddle: Buy or sell

• Why?
  – Reduce risk (hedge)
  – Speculate
  – Arbitrage

• Original analysis - L. Bachelier (1900 - Brownian motion)
Results on European Options

• Black-Scholes-Merton formula
• Put-call parity for exercise price \( K \) and expiration \( T \)

\[
C_t - P_t = S_t - e^{-r(T-t)}K
\]

American options:

• Can exercise before \( T \)
• No parity
• Calls not exercised early if no dividend
• Puts have value of early exercise
American Option Complications

- American options
  - Decision at all $t$ - exercise or not?
- Find best time to exercise (optimize!)
American Options

• Difficult to value because:
  – Option can be exercised at any time
  – Value depends on entire sample path not just state (current price)

• Model (stopping problem):
  \[ \max_{0 \leq t \leq T} e^{-rt} V_t(S_{0t}) \]

• Approaches:
  – Linear programming, linear complementarity, dynamic programming
Formulating as Linear Program

• At each stage, can either exercise or not

\[ V_t(S) \geq K-S \text{ and } e^{-r\delta} (pV_{t+\delta}(uS)+(1-p) V_{t+\delta}(dS)) \]

If minimize over all \( V_t(S) \) subject to these bounds, then find the optimal value.

• Linear program formulation (binomial model)

\[
\begin{align*}
\min & \sum_t \sum_{k,t} V_{t,k,t} \\
\text{s.t.} & V_{t,k,t} \geq K-S_t,k_t, t=0,\delta,2\delta,\ldots,T; V_{T,k,T} \geq 0 \\
& V_{t,k,t} \geq e^{-r\delta} \left(pV_{t+\delta,k,U(t)}+(1-p) V_{t+\delta,k,D(t)}\right) \\
& t=0,\delta,2\delta,\ldots,T-1; k_t=1,\ldots,t+1; S_{t+\delta,U(k_t)}=uS(k_t); \\
& S_{t+\delta,D(k_t)}=dS(k_t); S_{0,1}=S(0).
\end{align*}
\]

Result: can find the value in a single linear program
Extensions of LP Formulation

• General model:
  – Find a value function $v$ to
    $$\min \langle C, V \rangle \text{ s.t. } V_t(S_t) \geq (K-S_t)^+,\quad -\mathcal{L}V + \frac{\partial V}{\partial t} \geq 0,$$
    $$V_T(S_T) = (K-S_T)^+$$
  where $C>0$ and $\mathcal{L}$ denotes the Black-Scholes operator for price changes on a European option.

• Can consider in linear complementarity framework

• Solve with various discretizations
  – Finite differences
  – Finite element methods
General Option Pricing
Applications: Implied Trees

• Basic Idea:
  – Assume a discrete representation of the price dynamics (often binomial) but not with associated probabilities
  – Observe prices of all assets associated with this tree of sample paths (and imply probabilities)
  – Find price for new claim (or check on consistency of option in market)

• Methodology:
  – Minimize deviations in prices or maximize/minimize price subject to fitting different set of prices (linear programming)
Finding Implied Trees

- Given call prices \((\text{Call}(K_i, T_i))\) at exercise prices \(K_i\) and maturities \(T_i\) (assuming risk-neutral pricing)
- Find probabilities \(P_j\) on branches \(j\) to:

\[
\begin{align*}
\min \sum_i (u_i^+ + u_i^-) \\
\text{s.t. } \sum_j P_j (S_j - K_i)^+ + u_i^+ - u_i^- = \text{FV}(\text{Call}(K_i, T_i)) \\
\sum_j P_j S_j = \text{FV}(S_t) \\
\sum_j P_j = 1, P_j \geq 0.
\end{align*}
\]
OUTLINE

• Applications
  • Option pricing
    • Portfolio/asset-liability models
  • Tracking and trading
  • Securitization
  • Risk management/Real options

• Future Potential
Overview of Approaches

• General problem
  – How to allocate assets (and accept liabilities) over time?
  – Uses: financial institutions, pensions, endowments

• Methods
  – Static methods and extensions:
  – Dynamic extensions of static
  – Portfolio replication (duration matching)
  – DP policy based
  – Stochastic program based
Static Portfolio Model

Traditional model

– Choose portfolio to minimize risk for a given return
– Find the efficient frontier

Quadratic program (Markowitz):

find investments \( x=(x(1),\ldots,x(n)) \) to

\[
\min x^T Q x
\]

s.t. \( r^T x = \text{target} \), \( e^T x = 1 \), \( x \geq 0 \).
Static Model Results

For a given set of assets, find
  – fixed percentages to invest in each asset
  – maintain same percentage over time
  – implies trading but gains over “buy-and-hold”

Needs
  – rebalance as returns vary
  – cash to meet obligations

Problems
  - transaction costs
  - cannot lock in gains
  - tax effects
Static Asset and Liability Matching: Duration +

• Idea: Find a set of assets to match liabilities (often WRT interest rate changes)
  – Duration (first derivative) and convexity (second derivative) matching

• Formulation:
  Given duration $d$, convexity $v$ and maturity $m$ of target security or liability pool, find investment levels $x_i$ in assets of cost $c_i$ to:

$$\min \sum_i c_i x_i$$

s.t. \(\sum_i d_i x_i = d; \sum_i v_i x_i = v; \sum_i m_i x_i = m; x_i \geq 0, i = 1...n\)

• Extensions:
  – Put in scenarios for the durations.. extend their application

• Problems:
  – Maintaining position over time
  – Asymmetry in reactions to changing (non-parallel yield curve shifts)
  – Assumes assets and liabilities face same risk
Extension to Liability Matching

• Idea (Black et al.)
  – Best thing is to match each liability with asset
  – Implies bonds for matching pension liabilities
• Formulation:
  Suppose liabilities are $l_t$ at time and asset $i$ has cash flow $f_{it}$ at time, then the problem is:

$$\min \sum_i c_i \ x_i$$

$$\text{s.t. } \sum_i f_{it} \ x_i = l_t \ \text{all } t; \ x_i \geq 0, \ i = 1 \ldots n$$

• Advantages:
  – Liabilities matched over time
  – Can respond to changing yield curve
• Disadvantages
  – Still assumes same risk exposure
  – Does not allow for mix changes over time
Further Extensions to Liability Matching

• Include scenarios $s$ for possible future liabilities and asset returns

• Formulation:

$$\min \Sigma_i c_i x_i$$

s.t. $\Sigma_i f_{it_s} x_i = l_{ts}$ all $t$ and $s$; $x_i \geq 0$, $i = 1\ldots n$

• If not possible to match exactly then include some error that is minimized.

• Allows more possibilities in the future, but still not dealing with changing mixes over time.

• Also, does not consider possible gains relative to liabilities which can be realized by rebalancing and locking in
Extended Policies – Dynamic Programming Approaches

• Policy in static approaches
  – Fixed mix or fixed set of assets
  – Trading not explicit

• DP allows broader set of policies

• Problems: Dimensionality, Explosion in time

• Remedies: Approximate (Neuro-) DP

• Idea: approximate a value-to-go function and possibly consider a limited set of policies
Dynamic Programming Approach

• State: $x_t$ corresponding to positions in each asset (and possibly price, economic, other factors)
• Value function: $V_t(x_t)$
• Actions: $u_t$
• Possible events $s_t$, probability $p_{st}$
• Find:

$$V_t(x_t) = \max -c_t u_t + \sum_{s_t} p_{st} V_{t+1}(x_{t+1}(x_t, u_t, s_t))$$

Advantages: general, dynamic, can limit types of policies

Disadvantages: Dimensionality, approximation of $V$ at some point needed, limited policy set may be needed, accuracy hard to judge
General Methods

• Basic Framework: Stochastic Programming
  – Allows general policies

• Model Formulation:

\[
\max \sum_{\sigma} p(\sigma) \left( U(W(\sigma, T)) \right)
\]

s.t. (for all \(\sigma\)):
\[
\begin{align*}
\sum_k x(k,1, \sigma) &= W(0) \text{ (initial)} \\
\sum_k r(k,t-1, \sigma) x(k,t-1, \sigma) - \sum_k x(k,t, \sigma) &= 0, \text{ all } t > 1; \\
\sum_k r(k,T-1, \sigma) x(k,T-1, \sigma) - W(\sigma, T) &= 0, \text{ (final)}; \\
x(k,t, \sigma) &\geq 0, \text{ all } k,t;
\end{align*}
\]

Nonanticipativity:
\[
x(k,t, \sigma') - x(k,t, \sigma) = 0 \text{ if } \sigma', \sigma \in S_i \text{ for all } t, i, \sigma', \sigma
\]

This says decision cannot depend on future.

Advantages: General model, can handle transaction costs, include tax lots, etc.

Disadvantages: Size of model, computational capabilities, insight into policies.
General Model Properties

- Assume possible outcomes over time
  - discretize generally
- In each period, choose mix of assets
- Can include transaction costs and taxes
- Can include liabilities over time
- Can include different measures of risk aversion
Example: Investment to Meet Goal

- Proportion in stock versus bonds depends on success of market (no fixed fraction)

![Graph showing investment outcomes](graph.png)
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Tracking a Security/Index

• **GOAL:** Create a portfolio of assets that follows another security or index with maximum deviation above the underlying asset

![Chart showing deviation](chart.png)
Asset Tracking Decisions

• Pool of Assets:
  – TBills
  – GNMAs, Other mortgage-backed securities
  – Equity issues

• Underlying Security:
  – Mortgage index
  – Equity index
  – Bond index

• Decisions:
  – How much to hold of each asset at each point in time?
Traditional Approach

• **MODEL:** variant of Markowitz model
• **SOLUTION:** Nonlinear optimization
• **PROBLEMS:**
  – Must rebalance each period
  – Must pay transaction costs
  – May pay taxes
  – Reward on beating target?
• **RESOLUTION:**
  – Make transaction costs explicit
  – Include in dynamic model
Trading and Pricing

• Situation:
  – A can borrow 7% fixed or LIBOR+3%
  – B can borrow 6.5% fixed or LIBOR+2%
  – Dealer offers a swap of fixed interest rate for floating (LIBOR)

• Questions
  – How to price? Who pays what?
  – How to trade? How to identify partners?
Dynamic Trading Formulation

- **PRICES**: $p(i)$ for asset $i$ with future cash flows $c(i,t,s)$ under scenario $s$; required cash flow of $b(t,s)$;
- Pay $x(i)$ now (and perhaps in future);
- **PRICING MODEL** (like liability matching):

$$\min \sum_i p(i) x(i)$$

s.t. (for all $s$): $\sum_i c(i,t,s) x(i) = b(t,s)$ all $t,s$.

**Extensions**
- Different maturity on the securities
- Maintain hedge over time
- Trade securities and match as closely as possible
- Again, can include transaction costs.
Real-time Trading

- Arbitrage searching:
  - Assume a set of prices $p_{ijk}$ for asset $i$ to asset $j$ trade in market $k$ (e.g., currency)
  - Start with initial holdings $x(i)$ and maximize output $z$ from asset 1 over trades $y$

$$\max z(1)$$

$$\text{s.t. } x(i) - \sum_{jk} p_{ijk} y_{ijk} + \sum_{jk} p_{jik} y_{jik} = z(i)$$

$$y \geq 0, \quad z \geq 0$$

(Generalized network: want to find negative cycles)
Trading and Market Impact

- Suppose goal is to purchase $Q$ shares.
- The transaction cost of trading increases in the amount of each trade by going through order book.
- Objective: break $Q$ into $q_1, \ldots, q_N$ to minimize transaction cost.

Order book: list of limit orders to buy or sell at a given price.
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Securitization

- Suppose you hold a collection of assets (loans, royalties, real properties) with different credit worthiness, maturities, and chance for early return of principal
- Idea: divide cash flows into marketable slices with different ratings, maturities
- Maximize value of division of asset cash flows:

$$\max \sum_i p(i) x(i)$$

s.t. (for all s): $$\sum_i c(i,t,s) x(i) = b(t,s)$$ all t,s.
Securitized Products

- Collateralized Debt Obligations (CDOs):
  Re-organize debt by losses due to default

Promised payments

CDO Tranches:
- First 3% of losses: Equity
- 3-7% of losses: 1st Mezzanine
- 7-10% of losses: 2nd Mezzanine
- 10-15% of losses: Senior
- 15-30% of losses: Super Senior

Some may default, then collect collateral.
Extensions and Implications of CDOs

• Synthetic CDOs: Instead of actual loans, make payments based on other party’s credit quality (or an index)
  – Funding requirement: Issuer buys credit default swap (CDS) to insure payments on the CDO
  – Requires credit worthiness of CDS counter-party

• CDO-squared: CDO composed of other CDOs
Key Assumptions for Valuing CDOs

- Known credit quality of original loans (often assumed homogeneous)
- Correlation structure of defaults
- Valuation of collateral
- Credit quality of counterparty for CDO (and their CDS counterparty)
Implications of Models: Multiple Interconnections

CDO Issuer

CDS Issuer

CDO Issuer

CDS Issuer

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Sequence of Events

• Interest rate rise ⇒ Defaults ↑
  Collateral value ↓
  High correlation
• Defaults↑/Collateral ↓ ⇒ Multiple CDO tranches
  ↓
  ⇒CDS counterparty stretched
  ⇒Liquidations to meet obligations
  ⇒More defaults/counterparty defaults and repetition
  ⇒No confidence in prices and credit quality
Problems for Models

- How to assess the credit worthiness of multiple inter-connected obligations?
- What is the impact of multiple guarantees on a single asset?
- What happens with “agency issues”?
- How to structure products that can be properly valued and restore liquidity?
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Real Options for Comprehensive Risk Management

- Use real option approach to risks of the firm
- Combine operational and financial decisions
- Set levels for risk (insurance from buy and sell sides)
- Use of stochastic models on several levels and distributed optimization
Future Possibilities and Needs

• Better discretization methods (FEM v. finite differences)
• On-line (continual) optimization for real-time applications
• Inclusion of incomplete markets – distributed optimization
• Consideration of taxes – nonconvex and discrete optimization
• Integration of stochastic model/simulation and optimization
Conclusions

• Analysis and optimization bring value to financial engineering
• Existing implementations in multiple areas of financial industry
• Current crisis partly caused by inability to assess higher-level complexity of interactions
• Potential for resolution with comprehensive risk management models requiring research, theory, methodology, and implementation in real options, incomplete markets, and broader pricing issues