Dynamic Portfolio Optimization with Stochastic Programming

John R. Birge
The University of Chicago Graduate School of Business

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Background

• Interest in asset-liability management
  – Investment holdings with multiple objectives
• Why dynamic?
  – Change circumstances over time
• Why use stochastic programming?
  – Comprehensive and customizable
• Issues in models and methods
OUTLINE

• Motivation for dynamics
• Overview of approaches
• Building consistent models
• Enabling efficient methods
• Extensions
Why Model Dynamically?

• Three potential reasons:
  – Market timing
  – Reduce transaction costs (taxes) over time
  – Maximize wealth-dependent objective

• Example
  – Suppose major goal is $100K down-payment for house in 2 years
  – Start with $82K; Invest in stock (annual vol=18.75%, annual exp. Return=7.75%); bond (Treasury, annual vol=0; return=3%)
  – Can we make the down payment? How likely?
Alternatives

• Markowitz (mean-variance) – Fixed Mix
  – Pick a portfolio on the efficient frontier
  – Maintain the ratio of stock to bonds to minimize expected shortfall

• Buy-and-hold (Minimize expected loss)
  – Invest in stock and bonds and hold for 2 years

• Dynamic (stochastic program)
  – Allow trading before 2 years that might change the mix of stock and bonds
Efficient Frontier

- Some mix of risk-less and risky asset
- For 2-year returns:
Best Fixed Mix and Buy-and-Hold

- Fixed Mix: 27% in stock
  - Make the down payment 25% of time (with binomial model)

- Buy-and-Hold: 25% in stock
  - Make the down payment 25% of time
Best Dynamic Strategy

- Start with 57% in stock
- If stocks go up in 1 year, shift to 0% in bond
- If stocks go down in 1 year, shift to 91% in stock
- Make the down-payment 75% of time
Advantages of Dynamic Mix

- Able to lock in gains
- Take on more risk when necessary to meet targets
- Respond to individual utility that depends on level of wealth
Approaches for Dynamic Portfolios

- **Static extensions**
  - Can re-solve (but hard to maintain consistent objective)
  - Solutions can vary greatly
  - Transaction costs difficult to include

- **Dynamic programming policies**
  - Approximation
  - Restricted policies (optimal – feasible?)
  - Portfolio replication (duration match)

- **General methods (stochastic programs)**
  - Can include wide variety
  - Computational (and modeling) challenges
Dynamic Programming Approach

• State: $x_t$ corresponding to positions in each asset (and possibly price, economic, other factors)
• Value function: $V_t (x_t)$
• Actions: $u_t$
• Possible events $s_t$, probability $p_{st}$
• Find:

$$V_t (x_t) = \max -c_t u_t + \sum_{st} p_{st} V_{t+1} (x_{t+1}(x_t,u_t,s_t))$$

**Advantages:** general, dynamic, can limit types of policies

**Disadvantages:** Dimensionality, approximation of $V$ at some point needed, limited policy set may be needed, accuracy hard to judge

**Consistency questions:** Policies optimal? Policies feasible? Consistent future value?
Other Restricted Policy Approaches

• Kusy-Ziemba ALM model for Vancouver Credit Union

• Idea: assume an expected liability mix with variation around it; minimize penalty to meet the variation

• Formulation:

\[
\begin{align*}
\min & \sum_i c_i x_i + \sum_{st} p_{st} (q_{st}^+ y_{st}^- + q_{st}^- y_{st}^-) \\
\text{s.t.} & \sum_i f_{its} x_i + y_{st}^+ - y_{st}^- = l_{ts} \text{ all } t \text{ and } s; \quad x_i, y \geq 0, \quad i = 1 \ldots n
\end{align*}
\]

Problems: Similar to liability matching.

Consistency questions: Possible to purchase insurance at cost of penalties? Best possible policy?
General Methods

• Basic Framework: Stochastic Programming

• Model Formulation:

$\max \sum_{\sigma} p(\sigma) \left( U(W(\sigma, T)) \right)$

s.t. (for all $\sigma$):

$\sum_{k} x(k,1, \sigma) = W(o) \text{ (initial)}$

$\sum_{k} r(k,t-1, \sigma) x(k,t-1, \sigma) - \sum_{k} x(k,t, \sigma) = 0, \text{ all } t > 1;$

$\sum_{k} r(k,T-1, \sigma) x(k,T-1, \sigma) - W(\sigma, T) = 0, \text{ (final)};$

$x(k,t, \sigma) \geq 0, \text{ all } k,t;$

Nonanticipativity:

$x(k,t, \sigma') - x(k,t, \sigma) = 0 \text{ if } \sigma', \sigma \in S_i \text{ for all } t, i, \sigma', \sigma$

This says decision cannot depend on future.

Advantages:

General model, can handle transaction costs, include tax lots, etc.

Disadvantages: Size of model, insight

Consistency questions: Price dynamics appropriate?

objective appropriate? Solution method consistent?
Model Consistency

• Price dynamics may have inherent arbitrage
  – Example: model includes option in formulation that is not the present value of future values in model (in risk-neutral prob.)
  – Does not include all market securities available

• Policy inconsistency
  – May not have inherent arbitrage but inclusion of market instrument may create arbitrage opportunity
  – Skews results to follow policy constraints

• Lack of extreme cases
  – Limited set of policies may avoid extreme cases that drive solutions
Objective Consistency

• Examples with incoherent objectives
  – Mean and variance
  – Probability of beating benchmark
• Coherent measures of risk (Heath et al.)
  – Can lead to piecewise linear utility function forms
  – Expected shortfall, downside risk, or conditional value-at-risk (Uryasiev and Rockafellar)
Model and Method Difficulties

• Model Difficulties
  – Arbitrage in tree
  – Loss of extreme cases
  – Inconsistent utilities

• Method Difficulties
  – Deterministic incapable on large problems
  – Stochastic methods have bias difficulties
    • Particularly for decomposition methods
    • Discrete time approximations
  – Stopping rules and time hard to judge
Resolving Inconsistencies

• Objective: Coherent measures
• Model resolutions
  – Construction of no-arbitrage trees (Klaassen)
  – Extreme cases (Generalized moment problems and fitting with existing price observations)
• Method resolutions
  – Use structure for consistent bound estimates
  – Decompose for efficient solution
Model Consistency

• Construct consistent scenarios with observed prices

• Find prices and scenarios to fit observed data and include extreme events (e.g., max probability of large decline)

• Format of general moment problem:

\[
\max \sum_{\Xi} g(\xi) \ P(d\xi) \\
\text{over probability measures } P \text{ s.t.} \\
\sum_{\Xi} v_i(\xi) \ P(d\xi) \leq \alpha_i, \ i=1,\ldots,s, \\
\sum_{\Xi} v_i(\xi) \ P(d\xi) = \beta_i, \ i=s+1,\ldots,M
\]

where \( M \) is finite and the \( v_i \) are bounded, continuous functions.
Extremal Probabilities

- Problem: find maximum (risk-neutral equivalent) probability of price above 55 given observed call premia \( C \):

\[
\text{Max } \sum_j \mathbb{1}_{S_j \geq 55} p_j \\
\text{s.t. } \sum_j p_j = 1 \\
\sum_j p_j (S_j - K_i)^+ = FV(C(K_i, T)) \\
\sum_j p_j S_j = FV(S_t), \ p_j \geq 0
\]

For example, suppose \( S_j = 30, 35, 40, 45, 50, 55, 60 \) and Call values: \( C(35)=10.3, C(40)=5.5, C(45)=2, C(50)=0.5 \)

Result:

\[
\text{Prob}(S_T \geq 55)=0.10
\]

Extend to find sets of probabilities and ranges
Method Consistency: Abridged Nested Decomposition

- Incorporates sampling into the general framework of nested decomposition for stochastic programs
- View as approximate dynamic programming
- Samples both the subproblems to solve and the solutions to continue from in the forward pass of nested decomposition
- Eliminates inconsistency by use of deterministic lower bound and re-sampled upper bound (consistent check of optimality on each iteration)
Decomposition Methods

- Benders idea
  - Form an outer linearization of value $V_t$
  - add *cuts* on function:

![Diagram](image)

LINEARIZATION AT ITERATION $k$

Feasible region

$V_t$

$(feasibility cuts)$

min at $k$ : $< V_t$

new cut

(optimality cut)
Abridged Nested Decomposition

Forward Pass
1. Solve root node subproblem
2. Sample Stage 2 subproblems and solve selected subset
3. Sample Stage 2 subproblem solutions and branch in Stage 3 only from selected subset (i.e., nodes 1 and 2)
4. For each selected Stage t-1 subproblem solution, sample Stage t subproblems and solve selected subset
5. Sample Stage t subproblem solutions and branch in Stage t+1 only from selected subset
Abridged Nested Decomposition

Backward Pass
1. Starting in first branching node of Stage $t = N-1$, solve all Stage $t+1$ descendant nodes and construct new optimality cut for all stage $t$ subproblems. Repeat for all sampled nodes in Stage $t$, then repeat for $t = t - 1$

Consistent Convergence Test
1. Randomly select $HN$-Stage scenarios. For each sampled scenario, solve subproblems from root to leaf to obtain total objective value for scenario
2. Calculate statistical estimate of the first stage objective value $\bar{z}$
   - algorithm terminates if current first stage objective value $c_jx_j + \theta_j$ is within a specified confidence interval of $\bar{z}$ else, a new forward pass begins
Additional Features for Portfolio Problems

• Serial independence
  – If increments are serially independent, formulation is directly applicable

• Using structure to relax serial independence
  – Can still use structure but assume some serial correlation
  – Define a state space determining future price trajectory
Sample Computational Results

CPU Time (seconds)

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Summary of Extreme Probability Modeling and AND

- Finding extreme probabilities allows for ranges in sensitivity analysis over distributions and reduced model risk

- Combinations of nested decomposition with outer linearization and sampling allows:
  - Reduction from exponential to linear effort in number of re-balance points
  - Confidence intervals on overall value
  - Efficient solution relative to alternatives
Challenges

- Extensions for serial correlation
- Testing for early termination
- Bounds on time-discretization effects
- Effective methods for taxable portfolios and non-convexities (e.g., short-term, long-term)
Conclusions

- Dynamic models offer advantages for portfolios with transaction costs, serial dependence and wealth-dependent objectives
- Stochastic programs provide a general and customizable framework
- Care required in modeling due to arbitrage, coverage of paths, objective consistency and method consistency
- With some effort, models and methods can become consistent
- Efficiency possible with optimization based on structure