Bounds and Comparisons of Quasi-Monte Carlo Methods in Option Pricing

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Outline

- Motivation - Story
- Integration
- Error analysis
- Option model
- Sample Results
- Why it works?
**Motivation**

**Challenge:** Find $x$ to

$$\min_{x \in X} \int_{\Xi} \phi(x, \xi) P(d\xi),$$

where $\phi$ is found by optimization for each $\xi$.

**Fundamental Problem:** Evaluate the integral, in general, find

$$\int_{\Xi} \phi(x, \xi) P(d\xi),$$

where the dimension of $\Xi$ is large (100’s or 1000’s).
Alternatives

Simplified Problem: Find

\[ \int_{[0,1]^n} f(x) \, dx. \]

Options:

- **Quadrature**
  - Requires smooth \( f \)
  - Low dimension

- **Bounding Inequalities**
  - Require some properties (e.g., \( f \) convex)
  - Generally quite loose

- **Monte Carlo Sampling**
  - Confidence intervals
  - Independent of dimension
  - Limited to \( \frac{1}{\sqrt{N}} \) error bounds for \( N \) samples
Monte Carlo Method

Method: Choose random samples \( x_1, \ldots, x_N \) and estimate

\[
I = \int_{x \in [0,1]^n} f(x) \, dx
\]

by

\[
\hat{I}(N) = \frac{1}{N} \sum_{i=1}^{N} f(x_i).
\]

Result: Assuming independent samples, finite variance \( V_f \), use Central Limit Theorem to find, with probability \( \alpha \):

\[
I \in [\hat{I}(N) - \sqrt{\frac{V_f}{N}} \Phi^{-1}\left(\frac{1-\alpha}{2}\right), \hat{I}(N) + \sqrt{\frac{V_f}{N}} \Phi^{-1}\left(\frac{1-\alpha}{2}\right)],
\]

where \( \Phi \) is the standard normal cumulative.

Note: in optimization, this becomes projection on feasible set, sometimes better asymptotic results.

Problem 1: Without reduced region, cannot avoid error declining as \( 1/\sqrt{N} \).

Problem 2: Use pseudo-random points - problems in high dimensions (often collinear unless care is taken in generating these points).
Alternative Strategies

Question: Can you do better if you choose where to put your sample points, $x_1, \ldots, x_N$?

Alternative analysis: Numerical instead of statistical (quasi-random)

Example: Consider integration on the unit interval ($n = 1$) where $f$ has bounded total variation ($TV_f$). What is the error in using $\hat{I}(N)$?

Approach: Integrate over $f$ and count discrepancy, $D_N$ of $x_i$ measure relative to natural (uniform), where

$$D_N(x_1, \ldots, x_N) = \sup_{0 \leq x \leq 1} \sum_{i=1}^{N} \frac{1_{[0,x)}(x_i)}{N} - x.$$
**Error Bound Result**

**Result:** Assume continuity and use integration by parts.

\[
|I - \hat{I}(N)| = |\int_{[0,1]} f(x) dx - \frac{1}{N} \sum_{i=1}^{N} f(x_i)|
\]

\[
= |f(1) - \int_{[0,1]} x df(x) - f(1) + \frac{1}{N} \sum_{i=1}^{N} (f(1) - f(x_i))|
\]

\[
= | - \int_{[0,1]} x df(x) + \frac{1}{N} \sum_{i=1}^{N} \int_{[0,1]} (1_{[0,x]}(x_i)) df(x)|
\]

\[
= | \int_{[0,1]} (\frac{1}{N} \sum_{i=1}^{N} 1_{[0,x]}(x_i) - x) df(x) |
\]

\[
\leq |\sup_{J} (\frac{1}{N} \sum_{i=1}^{N} (1_{[0,x_j]}(x_i) - x)(f(x_{j+1}) - f(x_j)))| \text{ for partitions } J
\]

\[
\leq D_N TV_f.
\]

**Impact:** Error depends on discrepancy and characteristics of \(f\). Also, generalizes to higher dimension (independent of \(n\)).
Finding Low Discrepancy Points

Question: Where is the best place to put your sample points, \(x_1, \ldots, x_N\)?

Example: On the unit interval, how well can you do?

0 \[\begin{array}{cccc}
1
\end{array}\]

Result: Best place is \(x_1 = 1/2N, x_2 = 3/2N, \ldots, x_N = \frac{2^{N-1}}{2N}\) so \((1/6, 1/2, 5/6)\) for \(N = 3\).

Higher dimensions: Want to avoid collinearity and also allow \(N\) to change (without shifting all the points).

Alternatives:

- **Alpha**: Based on irrationals, such as the roots of the first \(n\) primes.
- **Halton**: Based on van der Korput sequence.
- **Faure
- **Sobol**
\textbf{\(\alpha\)-Sequence}

- \(\{x\} = x - \lfloor x \rfloor, \ x \in \mathbb{R}\)

- \(n_1, \ldots, n_s\): the first \(s\) prime numbers

- \(\alpha\) sequence: \(x_0, x_1, \ldots, x_n, \ldots\) with

\[
x_n = (\{n\sqrt{n_1}\}, \ldots, \{n\sqrt{n_s}\}) \in I^s
\]

for all \(n \geq 1\).
Van der Corput sequence

- $b$: $b \geq 2$, base of the Van der Corput sequence
- $Z_b = \{0,1,\ldots,b-1\}$: the least residue system mod $b$
- $n = \sum_{i=0}^{L(n)} a_i(n) b^i$, where $L(n) = \lfloor \log_b n \rfloor$, $n \geq 0$.
- Radical-inverse function (Glasserman [2004])
  \[ \phi_b(n) = \sum_{i=0}^{L(n)} a_i(n) b^{-i-1}; \]
- Van der Corput sequence in base $b$: $x_0, x_1, \ldots, x_n, \ldots$ with
  \[ x_n = \phi_b(n) \]
  for all $n \geq 0$. 
Halton sequence

- $b_1, \ldots, b_s$: the first $s$ prime numbers
- **Halton sequence**: $x_0, x_1, \ldots$ with
  $$x_n = (\phi_{b_1}(n), \ldots, \phi_{b_s}(n)) \in I^s$$
  for all $n \geq 0$.
- Note: The Halton sequence is acceptably uniform for lower dimensions, up to about $s = 10$. 
Faure sequence

- $p$: the smallest prime number, $p \geq s$ and $p \geq 2$
- $x_n^1 = \phi_p(n) = \sum_{i=0}^{L(n)} a_i(n)p^{-i-1}$ where
  $$n = \sum_{i=0}^{L(n)} a_i(n)p^i$$

- $x_n^k = \phi_p^k(n) = \sum_{i=0}^{L(n)} a_i^k(n)p^{-i-1}$ $k = 1, \ldots, s$ where
  $$a_i^k(n) = \sum_{j=i}^{L(n)} C_j^i a_i^{k-1}(n) \mod p, \quad C_j^i = \frac{j!}{i!(j-i)!}.$$  

- **Faure sequence**: $x_0, x_1, \ldots, x_n, \ldots$ where
  $$x_n = (x_n^1, x_n^2, \ldots, x_n^s)$$
Sobol’ sequence

• \{v_i\}: “direction numbers,” \( v_i = \frac{m_i}{2^i} \)
• \( m_i \): odd positive integers, \( m_i \leq 2^i \)
• Primitive polynomial (Knuth [1981]):
  \[
P = x^d + a_1 x^{d-1} + \ldots + a_{d-1} x + 1
  \]
• recurrence formula for \( m_i \) and \( v_i \):
  \[
v_i = a_1 v_{i-1} \oplus a_2 v_{i-2} \oplus \ldots \oplus a_{d-1} v_{i-d+1} \oplus v_{i-d} \\
  \oplus \left[ v_{i-d}/2^d \right], \ i > d,
  \]
  \[
m_i = 2a_1 m_{i-1} \oplus 2^2 a_2 m_{i-2} \oplus \ldots \oplus 2^{d-1} a_{d-1} m_{i-d+1} \\
  \oplus 2^d m_{i-d} \oplus \left[ m_{i-d}/2^d \right], \ i > d,
  \]
Sobol’ sequence (Cont’d)

• 1-dimensional Sobol’ sequence: \(x_0, x_1, \ldots, x_n, \ldots\) where

\[
x_n = b_1 v_1 \oplus b_2 v_2 \oplus \ldots, \\
n = \sum_{i=1}^{[\log_2 n]} b_i 2^i
\]

• \(s\)-dimensional Sobol’ sequence:

  – choose \(s\) different primitive polynomials to calculate \(s\) different sets of direction numbers
Error Bounds in Option Pricing

- Traditional bounded variation bounds do not apply (Koksma-Hlawka Inequality)

\[
\frac{1}{N} \sum_{n=1}^{N} f(x_n) - \int f d\mu \leq V(f) D_N^*(P)
\]

Owen(2004): \( C(X), P(X) \) are not BVHK.

- Seeking error bounds that apply to option pricing problems
Does Quasi-Random Work Well for Options and (if so) Why?

Works Well

- Standard European options results
- American option results

Towards Understanding

- Dimension is effectively lower than $n$
- Functions in applications are well-behaved
- Discrepancy only matters on certain sets
European Option Results - Short Horizon

Figure 1
European Option Results - 2

Figure 2
European Option Results - 3

Percent Errors: 180 days, sigma=0.3

Figure 3
European Option Results -4
Error Bounds \( s=1 \)

- Consider a European call

\[
c = e^{-rT} E_{ST} [(S_T - K)^+]
\]

\[
= e^{-rT} \int_0^{+\infty} (S_T - K)^+ d\nu(S_T)
\]

\[
= e^{-rT} \int_0^{+\infty} (S_T - K)^+ g(S_T) dS_T,
\]

- Black-Scholes model:

\[
S_T = S_0 e^{(r - \frac{\sigma^2}{2})T + \sigma\epsilon\sqrt{T}},
\]

\[
c = e^{-rT} \int_{-\infty}^{+\infty} (S_0 e^{(r - \frac{\sigma^2}{2})T + \sigma\epsilon\sqrt{T}} - K)^+ g(\epsilon) d\epsilon,
\]
• QMC approximation:

\[ c \approx \frac{1}{N} \sum_{i=1}^{N} e^{-rT} (S_0 e^{(r - \frac{\sigma^2}{2})T + \sigma \epsilon_i \sqrt{T}} - K)^+ \]

where \( \epsilon_i = G^{-1}(z_i) \), \( G(\epsilon) = \int_{-\infty}^{\epsilon} g(x) \, dx \)

• Error of approximation (in 1 dimension):

\[ |e^{-rT} \int_{-\infty}^{+\infty} (S_0 e^{(r - \frac{\sigma^2}{2})T + \sigma \epsilon \sqrt{T}} - K)^+ g(\epsilon) \, d\epsilon \]

\[ -\frac{1}{N} \sum_{i=1}^{N} e^{-rT} (S_0 e^{(r - \frac{\sigma^2}{2})T + \sigma \epsilon_i \sqrt{T}} - K)^+ | \]
Let $M$ to be a smallest number such that all $\epsilon_i, i = 1, \ldots, N$ are located in $[-M, M]$. Then, we define a truncated function. Let

$$Q(\epsilon) = \begin{cases} 
(S_0e^{(r - \frac{\sigma^2}{2})T + \sigma M\sqrt{T}} - K)^+ & \text{if } \epsilon > M, \\
(S_0e^{(r - \frac{\sigma^2}{2})T + \sigma \epsilon \sqrt{T}} - K)^+ & \text{if } \epsilon \in [-M, M], \\
(S_0e^{(r - \frac{\sigma^2}{2})T - \sigma M\sqrt{T}} - K)^+ & \text{otherwise}; 
\end{cases}$$
Error Bounds (s=1) (Cont’d)

Then the error is

\[
|e^{-rT} \int_{-\infty}^{+\infty} (S_0 e^{(r-\frac{\sigma^2}{2})T + \sigma \epsilon \sqrt{T}} - K)^+ g(\epsilon) d\epsilon - \frac{1}{N} \sum_{i=1}^{N} e^{-rT} Q(\epsilon_i)|
\]

\[
\leq |e^{-rT} \int_{-\infty}^{+\infty} (S_0 e^{(r-\frac{\sigma^2}{2})T + \sigma \epsilon \sqrt{T}} - K)^+ g(\epsilon) d\epsilon - e^{-rT} \int_{-\infty}^{+\infty} Q(\epsilon) g(\epsilon) d\epsilon|
\]

\[
+ |e^{-rT} \int_{-\infty}^{+\infty} Q(\epsilon) g(\epsilon) d\epsilon - \frac{1}{N} \sum_{i=1}^{N} e^{-rT} Q(\epsilon_i)|.
\]
Proof: The proof of Theorem 1 is based on the following lemma for \( s = 1 \) and \( 2 \).

Lemma 0.1. If \( M \geq 0 \), then
\[
\int_{M}^{+\infty} e^{-\frac{x^2}{2}} \, dx \leq \sqrt{\frac{\pi}{2}} e^{-\frac{M^2}{2}}.
\]

Lemma 0.2. If \( g(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \) and \( M \geq \sigma \sqrt{T} \), then
\[
\int_{M}^{+\infty} (S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma x \sqrt{T}} - K)^+ g(x) \, dx \leq \frac{1}{2} S_0 \: e^{rT} \: e^{-\frac{(M-\sigma \sqrt{T})^2}{2}}.
\]

Lemma 0.3. If \( M \) is assumed as above, then
\[
|e^{-rT} \int_{-\infty}^{+\infty} (S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma \epsilon \sqrt{T}} - K)^+ g(\epsilon) \, d\epsilon - e^{-rT} \int_{-\infty}^{+\infty} Q(\epsilon) g(\epsilon) \, d\epsilon|
\leq S_0 e^{-\frac{(M-\sigma \sqrt{T})^2}{2}}.
\]
Error Bounds (s=1) (Cont’d)

Lemma 0.4. If $M > 0$, $G(x) = \int_{-\infty}^{x} g(\epsilon) d\epsilon$ where 
\[ g(\epsilon) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\epsilon^2}{2}}, \]
then there exists an $\alpha = \sqrt{2\pi} e^{\frac{M^2}{2}}$ such that 
\[ G^{-1}(G(M) - \frac{1}{N}) \geq M - \alpha \frac{1}{N}. \]

Lemma 0.5. If $M$ is assumed as above, and $P$ is a $(N, \nu)$-uniform point set such that $N$ includes subintervals $[\frac{i-1}{N}, \frac{i}{N})$, $i = 1, \ldots, N$, then 
\[ |e^{-rT} \int_{-\infty}^{+\infty} Q(\epsilon) g(\epsilon) d\epsilon - \frac{1}{N} \sum_{i=1}^{N} e^{-rT} Q(\epsilon_i)| \leq \sqrt{2\pi T} \sigma S_0 e^{-\frac{\sigma^2}{2} T} e^{\sigma \sqrt{T M + \frac{M^2}{2}}} \frac{1}{N}, \]
where $Q(x)$ is defined as above.
Theorem 0.6. Assuming a uniform point set as given in Lemma 0.5, there exists a number $N_0$ and $C$ such that for all $N > N_0$,

$$|e^{-rT} \int_{-\infty}^{+\infty} (S_0e^{(r-\frac{\sigma^2}{2})T+\sigma\epsilon\sqrt{T}} - K)^+ g(\epsilon) d\epsilon - \frac{1}{N} \sum_{i=1}^{N} e^{-rT} (S_0e^{(r-\frac{\sigma^2}{2})T+\sigma\epsilon_i\sqrt{T}} - K)^+ | \leq CN^{-\frac{1}{3}},$$

where $C$ is a constant that only depends on $S_0$, $\sigma$ and $T$. 
Error Bounds(s=1)(Cont’d)

Corollary 0.1. Under the uniform point set conditions as given in Lemma 0.5, for any $k \in \mathbb{N}$, there exists a number $N_0$ such that for all $N > N_0$,

\[
|e^{-rT} \int_{-\infty}^{+\infty} (S_0 e^{(r - \frac{\sigma^2}{2})T + \sigma \epsilon \sqrt{T}} - K)^+ g(\epsilon) \, d\epsilon
\]

\[- \frac{1}{N} \sum_{i=1}^{N} e^{-rT} (S_0 e^{(r - \frac{\sigma^2}{2})T + \sigma \epsilon_i \sqrt{T}} - K)^+ | \leq CN^{-\frac{k}{2k+1}},
\]

where $C$ is a constant that only depends on $S_0$, $\sigma$ and $T$. 
Corollary 0.2. Under the uniform point set conditions as given in Lemma 0.5, given any $\delta > 0$, there exists a number $N_0$ such that for all $N > N_0$,

\[ |e^{-rT} \int_{-\infty}^{+\infty} (S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma \epsilon \sqrt{T}} - K)^+ g(\epsilon) \, d\epsilon - \frac{1}{N} \sum_{i=1}^{N} e^{-rT} (S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma \epsilon_i \sqrt{T}} - K)^+ | \leq CN^{-\frac{1}{2}+\delta}, \]

where $C$ is a constant that only depends on $S_0$, $\sigma$ and $T$. 
Simulating the path followed by the price process $S$,

$$S(t + \frac{T}{s}) = S(t)e^{(r - \frac{\sigma^2}{2})\frac{T}{s} + \sigma \epsilon_t \sqrt{\frac{T}{s}}}, \quad t = 0, \frac{T}{s}, \ldots, \frac{(s - 1)T}{s},$$

where $\epsilon_t \sim \phi(0, 1)$.

The European call price is then given by

$$c = e^{-rT} E_{S_T}[(S_T - K)^+]$$

$$= e^{-rT} \int_{\mathbb{R}^s} (S_T - K)^+ d\nu(S_T).$$
Error Bounds (s=n) (Cont’d)

Denote $X = R^s$, $\epsilon^i = (\epsilon^i_1, \epsilon^i_2, ..., \epsilon^i_s)$. The error of the approximation is then

$$|e^{-rT} \int_{\mathbb{R}^s} \int_{...} (S_T - K)^+ \, d\nu(S_T)$$

$$- \frac{1}{N} \sum_{i=1}^{N} e^{-rT} (S_0 e^{(r - \frac{\sigma^2}{2})T + \sigma \sqrt{\frac{T}{s}}} (\sum_{j=1}^{s} \epsilon^i_j) - K)^+|.$$

We select $M$ to be such that all $(\epsilon^i_1, ..., \epsilon^i_s), i = 1, ..., N$ are located in $[-M, M]^s$.

Let $\epsilon = (\epsilon_1, ..., \epsilon_s)$. From now on, we assume $M$ is large enough that $S_0 e^{(r - \frac{\sigma^2}{2})T - sM \sqrt{\frac{T}{s}}} \leq K.$
Error Bounds (s=n) (Cont’d)

We can then define the truncated function as below:

\[
Q(\epsilon) = \begin{cases} 
(S_0 e^{(r - \frac{\sigma^2}{2}) T + s M N - \frac{T}{s} - K})^+ & \text{if } \sum_{j=1}^{s} \epsilon_j > s M, \\
(S_0 e^{(r - \frac{\sigma^2}{2}) T + \sum_{j=1}^{s} \epsilon_j N - \frac{T}{s} - K})^+ & \text{if } -s M \leq \sum_{j=1}^{s} \epsilon_j \leq s M, \\
0 & \text{otherwise.}
\end{cases}
\]

The error of the approximation is then

\[
|e^{-rT} \int \cdots \int_{\mathbb{R}^s} (S_T - K)^+ \, d\nu(S_T) - \frac{1}{N} \sum_{i=1}^{N} e^{-rT} Q(\epsilon^i)| \\
\leq |e^{-rT} \int \cdots \int_{\mathbb{R}^s} (S_T - K)^+ \, d\nu(S_T) - e^{-rT} \int \cdots \int_{\mathbb{R}^s} Q(\epsilon) \, d\nu(\epsilon)| \\
+ |e^{-rT} \int \cdots \int_{\mathbb{R}^s} Q(\epsilon) \, d\nu(\epsilon) - \frac{1}{N} \sum_{i=1}^{N} e^{-rT} Q(\epsilon^i)|.
\]
Error Bounds (s=n) (Cont’d)

Note that

\[ | \int \ldots \int_{\mathbb{R}^s} (S_T - K)^+ \, d\nu(S_T) - \int \ldots \int_{\mathbb{R}^s} Q(\epsilon) \, d\nu(\epsilon) | \]

\[ \leq \int \ldots \int_{\mathbb{R}^s \setminus [-M,M]^s} (S_T - K)^+ \, d\mu(S_T). \]

**Lemma 0.7.** There exists a constant \( L \) that only depends on \( S_0, r, T, \sigma, \) and \( s \) such that

\[ e^{-rT} \int \ldots \int_{\mathbb{R}^n \setminus [-M,M]^s} (S_T - K)^+ \, d\mu(S_T) \leq L e^{\sigma M \sqrt{\frac{T}{s}} - \frac{M^2}{2}}. \]
Let $b_1, \ldots, b_{s-1}$ be the first $s - 1$ prime numbers. For the $s$-dimensional case here, we use the following construction (see, e.g., Deák(1990)), where

$$x^i = (\phi_{b_1}(i), \ldots, \phi_{b_{s-1}}(i), \frac{i}{N} - \delta),$$

and $0 < \delta < \frac{1}{N}$ is an arbitrary parameter to maintain the sequence in $[0, 1)$.

**Lemma 0.8.** For the quasi-random sequence defined above, there exists a constant $L'$ that only depends on $S_0$, $r$, $T$, $\sigma$, and $s$ such that

$$|e^{-rT} \int \cdots \int_{\mathbb{R}^s} Q(\epsilon) \, d\nu(\epsilon) - \frac{1}{N} \sum_{i=1}^{N} e^{-rT} Q(\epsilon^i)| \leq L' \frac{e^{s \sigma M \sqrt{T_s + \frac{M^2}{2}}}}{N}.$$
Combining Lemma 0.7 and Lemma 0.8, we have the following theorem.

**Theorem 0.9.** For the quasi-random sequence defined above, given any $\delta > 0$, there exists a number $N_0$ such that for all $N > N_0$,

$$|e^{-rT} \int \cdots \int_{\mathbb{R}^n} (S_T - K)^+ d\mu(S_T) - \frac{1}{N} \sum_{i=1}^{N} e^{-rT}(S_0 e^{\xi_i} - K)^+| \leq CN^{-\frac{1}{2}+\delta},$$

where $C$ only depends on $S_0$, $r$, $\sigma$ and $s$. 

Theorem 0.10. If a derivative has a payoff function $g(S_T)$ such that $\exists$ a constant $K_0$ s.t. $|g(S_T)| < K_0S_T$ and, over any $H_j = \Pi_i[a_i^j, b_i^j] \in \mathcal{H}$ in the uniform point set condition,

$G_j(g(S_T)) - g_j(g(S_T)) \leq K_0(S_T(b^j) - S_T(a^j))$, then, for the sequence defined above, given any $\delta > 0$, there exists a number $N_0$ such that for all $N > N_0$

$$|e^{-rT} \int \ldots \int_{\mathbb{R}^n} g(S_T) \, d\mu(S_T) - \frac{1}{N} \sum_{i=1}^{N} e^{-rT} g(S_0e^{\epsilon_i})| \leq C N^{-\frac{1}{2}} + \delta,$$

where $C$ is a constant that only depends on $K_0$, $S_0$, $r$, $\sigma$, $T$ and $s$. 
Corollary 0.3. If a derivative has a payoff function $g(S_T)$ such that $\exists$ a constant $K_0$ s.t. $|g(S_T)| < K_0 S_T$ and, over any $H_j = \Pi_i[a_i^j, b_i^j] \in \mathcal{N}$, $G_j(g(S_T)) - g_j(g(S_T)) \leq K_0(S_T(b^j) - S_T(a^j))$, then, for a quasi-random sequence with discrepancy $D_N^* < (R(N)/N)$ decreasing in $N$, given any $\delta > 0$, there exists a number $N_0$ such that, for all $N > N_0$,

$$|e^{-rT} \int_{\mathbb{R}^N} g(S_T) d\mu(S_T) - \frac{1}{N} \sum_{i=1}^{N} e^{-rT} g(S_0 e^{\epsilon_i})| \leq C \left[ \frac{N}{R(N)} \right]^{-\frac{1}{2} + \delta},$$

where $C$ is a constant that only depends on $K_0$, $S_0$, $r$, $\sigma$, $s$, and $T$. 

Error Bounds(s=n)(Cont’d)
Numerical Experiment

• Model: duality approach to American option valuation
  – Rogers (2001)
  – Haugh & Kogan (2001)
  – Andersen & Broadie (2001)

• Numerical comparisons
Duality approach to American option valuation
— Rogers and Haugh & Kogan (2001)

• Economy : \((\Omega, \mathcal{F}, \mathcal{Q})\)

• State : \(\{X_t \in \mathbb{R}^d : t \in [0, T]\}\)

• Payoff of the option: \(h_t = h(X_t)\)

• Primal problem:
  − Price of American option:

\[
V_0 = \sup_{\tau \in \Gamma} E_\tau \left[ \frac{h_\tau}{B_\tau} \right]
\]
Duality approach (Cont’d)

- Dual function: \( F(t, M) \)

\[
\frac{F(t, M)}{B(t)} = E\left[ \max_{t \leq \tau \leq T} (Z(\tau) - M(\tau)) \right] + M(t)
\]

- Dual problem:

\[
U(0) = \inf_{M} F(0, M) = \inf_{M} E\left[ \max_{t \leq \tau \leq T} (Z(\tau) - M(\tau)) \right] + M(0)
\]

- Upper bound

\[
F(0, M) = E\left[ \max_{0 \leq t \leq T} (Z(t) - M(t)) \right] + M(0) \geq V(0)
\]

\[
M(t) = B(t)^{-1} V_{euro}[S(t), K, \sigma, T - t, r] - V_{euro}[S(0), K, \sigma, T, r]
\]
Numerical Comparisons

Simulation prices of standard American puts. Parameter values were \( K = 100, r = 0.06, T = 0.5, \) and \( \sigma = 0.4. \) \( N = 50,000. \) (50 batches with batch size 1000)

<table>
<thead>
<tr>
<th>( S_0 )</th>
<th>True Price</th>
<th>Pseudo</th>
<th>alpha</th>
<th>Halton</th>
<th>Faure</th>
<th>Sobol’</th>
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<td>18.1202</td>
<td>18.1195</td>
<td>18.1207</td>
<td>18.1205</td>
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<td>8.04813</td>
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<td>8.04494</td>
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<tr>
<td>115</td>
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<td>5.13394</td>
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<td>Avg. RE</td>
<td>0.00%</td>
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<td>0.29%</td>
<td>0.31%</td>
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Numerical Comparisons (Cont’d)

Standard errors of simulations.

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<tr>
<th>$S_0$</th>
<th>Pseudo</th>
<th>alpha</th>
<th>Halton</th>
<th>Faure</th>
<th>Sobol’</th>
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<td>Avg. &gt;</td>
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<td>Min.</td>
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<td>65.41%</td>
<td>169.71%</td>
<td>1.44%</td>
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Numerical Comparisons (Cont’d)

Simulation times (seconds).

<table>
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<th>alpha</th>
<th>Halton</th>
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<th>Sobol’</th>
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<td>36.28%</td>
<td>349.29%</td>
<td>648.84%</td>
<td>993.15%</td>
</tr>
</tbody>
</table>
Conclusion

• Error bounds
  – Deterministic bounds at order of Monte Carlo method without bounded variation property

• Numerical experiment:
  – alpha and Sobol’ most consistent results with low variation
  – alpha more efficient to simulate