A Primer on Finance for MS Teachers: What Do You Need to Know?

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 THEMES

• A few good ideas go a long way
• Trust the Force
• Know the sound of one hand clapping…

• Answer the 3 questions….
  – Supreme being?
  – Source of evil?
  – Life after death?

Almost…
Big Three Questions of Financial-osophy

How much should you pay?
What should you buy?
How much can you lose?

Main application areas:
• Pricing: How much should you pay?
• Portfolio Optimization: What should you buy?
• Risk Management: How much can you lose?
Where Does ORMS Come in?

- Optimization is a key part of the fundamental questions
- Simulation can help with many difficult financial problems
- Duality and optimal solution properties provide a foundation for finding answers
- The answers are critical in answering operational as well as financial questions
Outline

• Financial kindergarten:
  – Learn to play well with others: Basic code of conduct
  – Hear the questions

• Elementary school:
  – Learn enough to hurt yourself: Toy tools
  – Ask the questions

• High school:
  – Learn how not to hurt yourself: Hand tools
  – Know the questions

• College:
  – Learn how to help someone else: Power tools
  – Answer (some of) the questions
Financial Kindergarten

• Always look both ways…
  – Don’t take unnecessary risks.
  – You have to pay people a premium to take on risk.

• Don’t do it just because he did it…
  – You can avoid unique risk with diversification.
  – The only remaining risk is the market.
Financial Kindergarten II

• Don’t do unto others what they can do unto themselves.
  – Investors can diversify on their own.
  – Firms should only diversify to protect against bankruptcy.

• Seen one; seen them all.
  – Every asset is only valued as contribution to an overall portfolio.
  – Each can be duplicated (if the market is “big enough”).
Financial Kindergarten III

• If it walks like a duck and talks like a duck, it’s a duck…
  – If you can replicate a cash flow with assets in the market, their price is price of whatever asset produces that cash flow.

• If it looks too good to be true, it probably is.
  – Trust the market – you cannot earn positive net present value on trade-able financial instruments.
Financial Kindergarten IV

• You can price anything, if you just believe..
  – and can find cash flow correlation to the market – the Capital Asset Pricing Model (CAPM).

• All roads lead to Rome and, when in Rome, do as the Romans do…
  – Without arbitrage, there is a risk-neutral equivalent probability (a risk-neutral world, “Rome”); and you can find the price of any asset using these probabilities – Fundamental Theorem of Asset Pricing.
Financial Kindergarten V

• Build a better mousetrap and the world will beat a path to your door.
  – Anyone should be willing to finance a project with positive net present value.

• We’re only human.
  – Results only apply up to a point; in reality: incomplete markets, friction, barriers to entry, agency, asymmetric information, irrationality
Lessons from Kindergarten

- Lesson 1: Everything you really need to know you learned in kindergarten
- Lesson 2: Use common sense
- Lesson 3: Trust the market….

----------unless you’re very very sure of yourself.
Financial First Grade

• Diversification leads to:
  – CAPM
  – Neutral to unique risk (ideal world)

• Debt (leverage) leads to:
  – Increased return
  – No effect on investment (ideal world)

• Replication leads to:
  – Pricing anything (in ideal world)

Caveat: *Watch out for what can hurt you!*
Financial First Grade: Diversification Effects

• Why not consider unique risk?
  – We only consider investors’ valuation.
  – They can diversify away unique risk.
  – The only part they cannot diversify is what is correlated with the overall market.

• Model for future return on asset $i$

\[ Return(i) = \text{Risk-free return} + \beta_i \times \text{Market-premium} + \text{Unique-return}(i) \]

where \( E[\text{Unique-return}(i)] = 0 \)

or \( r_i = r_f + \beta_i \delta + \epsilon_i \)
Portfolio Effects

- Suppose we invest $x_i$ in each project $i$
- Total return:
  $$\sum_i r_i x_i = \sum_i (r_f + \beta_i \delta + \epsilon_i) x_i = \sum_i (r_f + \beta_i \delta) x_i + \sum_i \epsilon_i x_i$$
- To the investor, the $\epsilon_i x_i$ part does not matter if number of investments is large
- Investors just care about the $\beta_i x_i$ – contribution to the portfolio risk (through $\delta$)
- Question:
  How many investments needed to make this work?
  (Answer: depends on how small you want $\sum_i \epsilon_i x_i$; Can simulate to find distribution or adjust to IID to use version of CLT)
Implications

• The firm has no incentive to diversify (if we do not consider the costs of bankruptcy and financial distress)

• CAPM: Can discount future cash flow from $i$ by discounting with the expectation of:
  $r_i = r_f + \beta_i \cdot \delta$

For any investment, we just need to find the correlation with the market.
Financial First Grade: Debt Effects

• Suppose borrowing/lending possible at riskfree rate
• Consider investment x in market and 1-x in riskfree borrowing/lending

\[
\text{Return} = r_m x + r_f (1-x) \\
\text{Var} = \sigma_m^2 x^2 + \text{Var}(r_f) (1-x)^2 + 2 \text{Cov}(r_m, r_f) x (1-x) \\
\Rightarrow \text{Vol} = \sigma_m x (> \sigma_m \text{ if } x > 1) \\
\Rightarrow \text{Borrowing increases risk (which increases required return)}
\]
Debt and Investments: Miller-Modigliani (MM) Theory

- Suppose a project with input $x$ earns 1-year future output $y$ with market correlation $\beta$
- All equity: put in $x$
  \[ NPV(\text{No debt}) = -x + \frac{1}{1 + r_f + \beta \delta} E[y] \]
- Debt $D$: put in $x-D$ & pay back loan at rate $r_D$
  \[ Payback = \max (y - (1 + r_D)D, 0); \text{ Market corr.: } \beta \]
  \[ NPV(\text{equity}) = -x + D + \frac{1}{1 + r_f + \delta} E[(y - (1 + r_D)D)^+] \]
  \[ NPV(\text{debt}) = 0 - D + \frac{1}{1 + r_f + \delta} E[(y - (y - (1 + r_D)D)^+] \]
  \[ NPV(\text{equity} + \text{debt}) = NPV(\text{No debt}), NPV(\text{equity}) = NPV(\text{No debt}) \]
  \[ \Rightarrow \text{Capital structure (debt) is irrelevant} \]
  \[ \Rightarrow \text{Investment and financing decision separate} \]
  \[ \Rightarrow \text{Investors can borrow as well as firm} \]

**But:** What about taxes and bankruptcy?

*Can model these and then have optimization of debt (capital structure)*
Financial First Grade: Pricing with Replication

- Example: asset with price $S$ can go up to $uS$ or down to $dS$ with some probability ($p$) in next time interval (binomial model)
- Call option at exercise (strike) price $K$ ($>dS, <uS$) $dS$
- Call pays $uS-K$ if price goes to $uS$, 0 if to $dS$
- What is the value of the call now?
Call Valuation by Replication

- Consider position that borrows $PV(L)$ and buys $\delta$ of the stock

Future payoffs of loan and stock:
\[
\begin{align*}
\text{if } uS: & \delta(uS)-L \\
\text{if } dS: & \delta(dS)-L
\end{align*}
\]

Future payoff of call:
\[
\begin{align*}
\text{if } uS: & (uS-K) \\
\text{if } dS: & 0
\end{align*}
\]

If $\delta(uS)-L = uS-K$ and $\delta(dS)-L = 0$
or $\delta = (uS-K)/(uS-dS)$, $L = \delta(dS)$,
then $-PV(L) + \delta S = Call$;

Notice: Never used $p$!

Can do this over (and over) again for multiple periods
Replication Pricing Example

Data:
\[ S=1, \ uS=1.3, \ dS=0.9, \ r=5\%, p=0.5, K=1.1 \]

Formulas:
\[ \delta = \frac{(uS-K)}{(uS-dS)} = \frac{(1.3-1.1)}{(1.3-0.9)} = 0.5 \]
\[ L = \delta(dS) = 0.5(0.9) = 0.45 \]

Result:
\[
\text{Call}(1.1) = -PV(L) + \delta S \\
= -(1/1.05)*(0.45) + (0.5)*1 \\
= 1/14 = 0.07143
\]
Lessons from First Grade

• Investors can do it themselves
  – No need for firm to diversify
  – Only correlation with the market matters

• With no taxes or bankruptcy cost:
  – Debt does not affect an investment
  – Investing and financing separate

• You can price anything that you can replicate with market instruments
Financial High School

- How to capitalize on the “any $p$” valuation?
  - risk-neutral pricing
- How to extend when exercise happens at any time?
  - American option
- How to tell what to buy?
  - Portfolio optimization
- How to tell how much you can lose?
  - Value at Risk
Axiom of the Market

• **Market Axiom:** *There is no free lunch*
  – The market does not allow *arbitrage*
  – No one can trade assets, never lose money, and sometimes make a profit

• How to write this mathematically?

• Assume prices $S_t(1), ..., S_t(n)$ for $n$ assets at times $t$

• Own (owe) $x_t = x_t(1), ..., x_t(n)$ shares of each

• Trades at $t$ change our position from $x_{t-1}$ to $x_t$ and must satisfy *conservation of funds*:

\[
\sum_{i=1}^{n} S_t(i) \cdot x_{t-1}(i) = \sum_{i=1}^{n} S_t(i) \cdot x_t(i) \quad \text{or} \quad S_t \cdot x_{t-1} = S_t \cdot x_t
\]
Linear Program for “No Free Lunch”

• Prices and share decisions are random variables, some distribution on events $P$
• No losses means $S_T x_T \geq 0$ almost surely;
• No positive profits without losses means:

\[0 \geq \max E_P [S_T x_T]\]

s.t. \[S_0 x_0 = 0, \quad S_T x_T \geq 0 \quad (a.s.)\]
\[S_t x_{t-1} = S_t x_t, \quad t = 1, \ldots, T\]
No Free Lunch Example

• Suppose invest $B$ in government bond and $S$ in stock
• Current price of bond and stock is $1$
• Future price:
  - Bond: $1.05$
  - Stock: w.pr. $\frac{1}{2}$: $1.30$
    - w.pr. $\frac{1}{2}$: $0.90$
  - Exp. stock: $1.10$

• What does “no free lunch” mean?
Example (2)

• Linear program:

\[
0 \geq \max \left( \frac{1}{2}(1.05B + 1.3S) + \frac{1}{2}(1.05B + 0.9S) \right)
\]

s.t. \( B + S = 0 \)

\[
1.05B + 1.3S \geq 0
\]

\[
1.05B + 0.9S \geq 0
\]

\[\Rightarrow\] Dual:

\[
\lambda + 1.05\pi_1 + 1.05\pi_2 = 1.05
\]

\[
\lambda + 1.3\pi_1 + 0.9\pi_2 = 1.3\left(\frac{1}{2}\right) + 0.9\left(\frac{1}{2}\right)
\]

\(\pi_1, \pi_2 \geq 0\)

Dual Solution:

\[
1.05 \left(1 - \pi_1 - \pi_2\right)
\]

\[= 1.3\left(\frac{1}{2} - \pi_1\right) + 0.9\left(\frac{1}{2} - \pi_2\right)\]

or

\[
1.05 =
\]

\[
1.3 \left(\frac{1}{2} - \pi_1\right) / \left(1 - \pi_1 - \pi_2\right)
\]

\[+ 0.9 \left(\frac{1}{2} - \pi_2\right) / \left(1 - \pi_1 - \pi_2\right)\]

Example: \(\pi_1 = 1/4, \pi_2 = 1/12\)

\[1.05 = 1.3\left(\frac{3}{8}\right) + 0.9\left(\frac{5}{8}\right)\]
General Linear Program

• Primal problem:
  \[ 0 \geq \max c^T x \]
  s.t. \( A x = 0, B x \geq 0 \)

• Dual problem:
  \[ \exists \pi, \rho \text{ s.t. } \pi^T A - \rho^T B = c^T, \rho \geq 0 \]

What does that mean for no-arbitrage problem?
Dual for No Arbitrage

- \( \pi^T A + \rho^T B = c^T \) becomes

\[
\pi_0 S_0 - \sum_{i=1}^{N1} \pi_1(i) S_1(i) = 0
\]

\[
\pi_t(i) S_t(i) - \sum_{j \in D(i,t)} \pi_{t+1}(j) S_{t+1}(j) = 0, \quad t=1..T-1
\]

\[
\pi_T(i) S_T(i) - \rho_T(i) S_T(i) = p(i,T) S_T(i)
\]

- Suppose Asset 1 is a mattress (riskfree investment – Treasury bonds), price of Asset 1 is \( S_t(1,i) = 1 \) for all \( t,i \)

This means: \( \pi_t(i) = \sum_{j \in D(i,t)} \pi_{t+1}(j), \quad t=0..T-1 \)

Let \( q_{t+1}(j) = \sum_{j \in D(i,t)} \pi_{t+1}(j) / \pi_t(i) \) then

\[
S_t(i) = \sum_{j \in D(i,t)} q_{t+1}(j) S_{t+1}(j) = E_Q[S_{t+1}], \quad t=1..T-1
\]

where \( Q \) is called a risk-neutral equivalent measure to \( P \)
Fundamental Theorem of Asset Pricing

• The absence of arbitrage is equivalent to the presence of a risk-neutral equivalent measure such that the expected return on all assets is the same with respect to this measure

• $Q$ is also called a *martingale* measure

• Can price any asset (in fact, derivative) using $Q$  
  *(i.e., find the price in Rome and it’s the same price wherever you go)*
Using the Risk-Neutral Probabilities

- Example: Stock to $S_T=1.3$ or $0.9$
- Natural probability: $\frac{1}{2}$ for 1.3 and $\frac{1}{2}$ for 0.9
- Found risk-neutral (same expected return as risk-free asset): $\frac{3}{8}$ for $S_T=1.3$ and $\frac{5}{8}$ for $S_T=0.9$
- How much premium for a Call with strike at 1.1?
- Use risk-neutral probabilities:

\[
Call(1.1) = \frac{1}{1/(1+r_f)} \cdot E_Q[\text{Max}(S_T-1.1,0)] = (1/1.05) \cdot (3/8(0.2)+5/8(0)) = 1/14 = 0.07143
\]

(Note: Same as the replication price – they are equivalent methods.)
Results on European Options

• Black-Scholes-Merton formula can be found using $Q$

• Find values of Calls and Puts for buying and selling at $K$ at $T$:

\[ Call = e^{-rT}E_Q[(S_T-K)^+] \]

Call - Put

K
Option Pricing: GBM

If prices follow geometric Brownian motion (GBM):

\[ dS = (r - \sigma^2/2) S \, dt + \sigma S \, dz \]

where \( dz \) is a Wiener process,

then \( E[S_t] = S_0 \, e^{rt} \)

For risk-neutral, we change the drift term, so that

\[ E_Q[S_t] = S_0 \, e^{r_ft} \]

(i.e., subtract risk premium \((r-r_f)\) from drift)
Option Pricing in Risk-neutral World

• Assume the risk-neutral process, then we have a lognormal price at T (maturity):

\[
\text{Call} = e^{-rfT} E_Q[(S_T - K)^+] \\
= e^{-rfT} \int_{[K, \infty)} (S_T - K) f(S_T/S_0) dS_T \\
= S \Phi((\ln(S_0/K) + (rf + \sigma^2/2)T)/\sigma \sqrt{T}) \\
- Ke^{-rfT} \Phi((\ln(S_0/K) + (rf - \sigma^2/2)T)/\sigma \sqrt{T})
\]

where \( \Phi \) is the standard normal cumulative.
Extending: American Options

- American options
  - Decision at all $t$ - exercise or not?

- Find best time to exercise Put (optimize!)
American Options

• Difficult to value because:
  – Option can be exercised at any time
  – Value depends on entire sample path not just state (current price)

• Model (stopping problem):
  \[
  \sup_{0 \leq t \leq T} e^{-rt} V_t(S_0) 
  \]

• Approaches:
  – Linear programming, linear complementarity, dynamic programming, duality
Formulating as Linear Program

• At each stage, can either exercise or not

• Suppose price goes from \( S \) to \( uS \) or \( dS \) each \( \delta \) time step (binomial model):

\[
V_t(S) \geq K - S \quad \text{and} \quad e^{-r\delta}(pV_{t+\delta}(uS) + (1-p)V_{t+\delta}(dS))
\]

If minimize over all \( V_t(S) \) subject to these bounds, then find the optimal value.

• Linear program formulation (binomial model)

\[
\begin{align*}
\text{min} & \quad \sum_t \sum_{kt} V_{t,kt} \\
\text{s. t.} & \quad V_{t,kt} \geq K - S_{t,kt}, \ t=0, \delta, 2\delta, ..., T; \ V_{T,kT} \geq 0 \\
& \quad V_{t,kt} \geq e^{-r\delta}(pV_{t+\delta,kU(kt)} + (1-p)V_{t+\delta,kD(kt)}) \\
& \quad t=0, \delta, 2\delta, ..., T-1; \ kt=1, ..., t+1; S_{t+\delta,kU(kt)} = uS(kt); \\
& \quad S_{t+\delta,kD(kt)} = dS(kt); S_{0,1} = S(0).
\end{align*}
\]

Result: can find the value in a single linear program
Setup for American Put LP

- Set up branches of the binomial at each point
- Find price (and probability) under each condition
- Find value to exercise now and value to wait
- Changing cells which will pick the maximum of current exercise or waiting
Solver Parameters

- Objective: Minimize sum of the value cells
- Decisions: The value cells
- Constraints: Be above current exercise and holding
Solution Observations

- American value exceeds the European value
- Exercise early after first period
- Last period: exercise whenever the option has value
- Can vary exercise price and other characteristics
What to Buy?
Maximize utility for buy-and-hold portfolio

• Situation:
  – Invest in 5 asset classes from static portfolio
  – Fund a fixed liability payment each year
  – Simulate returns over times
  – Find utility in each of simulated scenarios
  – Maximize expected utility for investments in each of the asset classes (compare to Markowitz mean-variance portfolio)
Year 1 Sample

- First pick uniform random, then standard normal, then transform, then give resulting output

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<thead>
<tr>
<th>Sample</th>
<th>Year 1</th>
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<tbody>
<tr>
<td>1</td>
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<td>22</td>
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<tr>
<td>24</td>
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- Repeat for each of years and each sample path
Optimization Form

- Add Investment Amounts and ensure outputs from one year carry over to start of next
- Subtract payment in proportion to investments
- Take returns at end of 5 years
- Take utility and expectation (divide by number of samples)
Solver Parameters

- Maximize expected utility
- Investments sum to 1
- (Non-negative investments)
- (non-linear)
Solution to LN Utility

- Maximize long-run growth (logarithm of return)

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<tr>
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<th>C</th>
<th>D</th>
<th>E</th>
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- Low Sharpe ratio (ratio of excess return to risk)
  - try maximizing it?
Solution to Max Sharpe Ratio

• Different allocations

<table>
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<tr>
<th>Investment Amounts</th>
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<td>B</td>
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<td>FixedIncome</td>
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<td>IntFixedInc</td>
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• Other objectives?

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Alternative Objectives: Penalize Underperforming a Benchmark

- Variables: Return(1), Return(2), ...
- Constraints on Return(i) for Outcome (Scenario) i=1, 2, ..

- Objective for Outcome(i):

  ![Graph showing objectives and constraints]

Return(i) vs Benchmark
Objective Variables

- Define Under(i) for amount below Benchmark (if any) and Over(i) for amount above Benchmark (if any)
- Suppose at most one of Under(i), Over(i) can be >0 and the other must be 0
- Constraint: Return(i)+Under(i)-Over(i)=Benchmark

\[ \text{Under}(i) \geq 0, \quad \text{Over}(i) \geq 0, \quad \text{Return}(i) \]
Overall Objective

• Put penalty (-10) on Under(i) and reward (1) on Over(i)
• Weight each equally

Goal = \sum_{\text{over } i} \text{Prob}(i) \times (-10 \times \text{Under}(i) + 1 \times \text{Over}(i))

to Maximize

Note: Maximizing forces Under(i) or Over(i) to 0 (Example: Return(i)=5% or 1%)
Observations on Portfolio Choice

• Objectives change the portfolio
• Can model dynamics of investment over time with optimization and simulation
• Can optimize choice and observe various performance metrics
• Can show tradeoffs between various measures
How Much Can You Lose? Value-at-Risk (\textit{VaR})

- \( \text{VaR} = \) Maximum loss in a given period (\(T\)) with a given probability (1-\(\alpha\))

\[
\text{VaR} = \max_W \{ W_0 - W \mid P\{W_0 - W_T \leq W_0 - W\} \geq 1 - \alpha \}
\]

(i.e., difference between \(W_0\) and \(\alpha\)-quantile of \(W_T\))
Estimating VaR

• Parametric models
  – Assume a distribution
  – Fit parameters
  – Make estimate

• Normal easy to use but aren’t prices non-negative?
  Difference with log-normal?
  – Future value is weighted sum of log-normals (no easy closed form)
  ⇒ Use Monte Carlo simulation
VaR Simulation

• **Objective:**
  – Compare normal approximation to log-normal using simulation in Crystal Ball
  – Find weights on assets to minimize VaR

• **Assets:**
  – 6 asset categories (DomComm, SmallCap, IntComm, FI, IntFI, Cash)
  – Start with equal weightings, $T=1$, $\alpha=5\%$
VaR Sheet Setup

• Assumption cells:
  – Returns for each asset
  – Log-normal assumed with given total returns and standard deviations
  – Include correlations

• Forecasts:
  – Simulated return

• Decisions:
  – Investment in each asset class
VaR Observations

• Differences from the normal approximation
  – Log-normal gives lower VaR here
  – Could it be worse?

• Optimizing portfolio
  – Can use OptQuest to find low VaR portfolios
  – Compare to a Markowitz (mean-variance) portfolio
  – Compare to normal approximation
  – Impose constraints on mean return
Nonparametric VaR: Implied (Binomial) Trees

• Assume that you have a set of option prices:
  Example: Share price = 45, r = 5%, T - t = 56/365
  Observe: Call(45, T) = 2, Call(40, T) = 5.5
  Assume binomial with ending prices: 52, 45, 39
  What are the probabilities with these branches?

\[
\begin{align*}
P_2 & \quad P_0 + P_1 + P_2 = 1 \\
P_1 & \quad P_1(5) + P_2(12) = e^{r(T-t)} \cdot 5.5 \\
P_0 & \quad P_2(7) = e^{r(T-t)} \cdot 2
\end{align*}
\]
Implied Trees Example

- Here: \( P_2 = .28, \) \( P_1 = .43, \) \( P_0 = .29 \)
- Can also go back to find probabilities on each branch
- Can check consistency with market price?
  \[ (.29(39) + .43(45) + .28(52))e^{-0.05(56/365)} = 44.9 \]
- In general, might have more options than branches
- Fit the observed prices as closely as possible
General Implied Tree Method

• Basic Idea:
  – Assume a discrete representation of the price dynamics (often binomial) but not with associated probabilities
  – Observe prices of all assets associated with this tree of sample paths (and imply probabilities)
  – Find price range (or probability range) for new claim (or check on consistency of option in market)

• Methodology:
  – Minimize deviations in prices or maximize/minimize price or probability subject to fitting different set of prices
Solving for Implied Trees

• Given call prices \((Call(K_i, T_i))\) at exercise prices \(K_i\) and maturities \(T_i\) (assuming risk-neutral pricing)

• Find probabilities \(P_j\) on branches \(j\) to:

\[
\begin{align*}
\min & \sum_i (u_i^+ + u_i^-) \\
\text{s.t.} & \sum_j P_j (S_j - K_i) + u_i^+ - u_i^- = FV(Call(K_i, T_i)) \\
& \sum_j P_j S_j = FV(S_t) \\
& \sum_j P_j = 1, P_j \geq 0.
\end{align*}
\]
Implied Tree Setup

- Suppose the following Call prices are observed

<table>
<thead>
<tr>
<th>Strike</th>
<th>Premium</th>
</tr>
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<tbody>
<tr>
<td>65</td>
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</tr>
<tr>
<td>60</td>
<td>1.25</td>
</tr>
<tr>
<td>55</td>
<td>2.25</td>
</tr>
<tr>
<td>50</td>
<td>4.25</td>
</tr>
<tr>
<td>45</td>
<td>7.15</td>
</tr>
<tr>
<td>40</td>
<td>11</td>
</tr>
<tr>
<td>35</td>
<td>15.5</td>
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</table>

Current price: $50, Riskfree rate: 5%, Vol=0.30, $T=90$ days
Worksheet Setup

- Put in the binomial prices (can actually choose any set of price paths)
- Changing cell for probability on each path
- Changing cells for being below or above the given call price
- Calculate each call price as consistent with the model and probabilities

<table>
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<tr>
<th>Days</th>
<th>Rate</th>
<th>Days</th>
<th>Rate</th>
<th>Days</th>
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<td>30</td>
<td>0.05</td>
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<td>Std Binomial</td>
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<td>38.62891</td>
<td>1</td>
<td>0</td>
<td>1</td>
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</tr>
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</table>
Lessons from High School

- With no arbitrage, there is a risk-neutral equivalent distribution (by LP)
- Can price anything (in complete market) with risk-neutral equivalence
- Can use LP for American options
- Can find optimal portfolios with multiple different objectives
- Can use simulation to estimate Value-at-Risk for difficult distributions
- Can find range of implied VaR, probabilities (risk-neutral), and prices using market prices and optimization
Financial College
(Discrete to Continuous Time)

• Why use continuous time?
  – Solving for the option value with a partial differential equation

• What happens for portfolio optimization in continuous time?
  – Bellman equation (dynamic programming form) to Bellman-Hamilton-Jacobi equation
Key “College” Points

• Continuous time can:
  – make analytical solutions easier
  – make new numerical tools available

• Continuous time requires:
  – Ito derivatives that include a second-order term for the diffusion
  – DP form that optimizes a (special) differential of the value function plus action cost
Conclusions

• You can ask/know/begin to answer the 3 questions (price?/buy?/loss?)
• You can (did?) learn enough not to hurt yourself in finance
• You can add to finance through your knowledge of ORMS
• Financial principles can help your teaching and research in ORMS
Further Reading: Examples

- Bodie/Merton, Finance
- Duffie, Dynamic Asset Pricing
- Karatzas/Shreve, Lessons of Mathematical Finance
- Luenberger, Investment Science
- McDonald, Derivative Markets
- Class examples: Winston, Financial Models using Simulation and Optimization