Asset-Liability Management

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Overview

- Portfolio optimization involves:
  - Modeling
  - Optimization
  - Estimation
  - Dynamics

- Key issues:
  - Representing utility (or risk and reward)
  - Choosing distribution classes (and parameters)
  - Building consistent models
  - Solving the resulting problems
  - *Implementing solutions over time with non-stationary processes, transaction costs, taxes, and uncertain future regulations*
Outline

- Introduction
- Modeling
- Portfolio basics
- Additions of assets and liabilities
- Dynamics
- Methods
- Conclusion
Background on ALM

- Manage a set of assets to meet a stream of liabilities over time
  - Pension funds
  - Insurance companies
  - Banks

- Differences from standard portfolio optimization
  - Dynamics of liabilities/nonlinearities
  - State-dependent utility/contingencies
Basic Problem Setup

Start: basic portfolio: Choose an allocation $x \in \mathbb{R}^n$ across $n$ assets (classes) to maximize expected utility at time $T$: $E[u_T(x)]$

Add: liabilities to meet; intermediate goals

utility may be function of the path

Note: $x$ may be a process $x_t$
Model

\[
\max E\left[\sum_{t=1}^{T} u_t(x_t)\right]
\]

s. t. \(x_{t+1} = r_t x_t + b_t - s_t - l_t,\)

\((-\tau^+(b_t) + \tau^-(s_t)) = 0,\)

\(x_t \in N_t,\)
Model Construction

Asset returns: estimation issues, factor models, etc. to capture asset behavior

Liabilities:

- Actuarial conditions
- Losses due to claims
- Losses to default
- Relationships to asset trajectories (e.g., wages to market return)
Example: bank model – rates for assets (loans), liabilities (deposits), losses (charge-offs) (B., Judice)
Additional Issues

Non-normal distributions (Chavez-Bedoya/B.):
• Mean-variance may be far from optimizing utility
• For exponential utility, can use generalized hyperbolic distributions – closed form for some examples
• Mean-variance can be close (but only if the risk-aversion parameter is chose optimally)

Additional approaches:
• Non-linear functions of Gaussian distributions
• Can use polynomial approximations and higher moments to obtain optimal solutions for these non-normal cases
Transaction Costs/Taxes and Dynamics

**Transaction costs:**
Each trade has some impact (e.g., bid-ask spread plus commission). Large trades may have long-term impacts.

**Taxes:**
Taxes depend on the basis and vintage of an asset and involve alternative selling strategies (LIFO, FIFO, lowest/highest price).
Why Model Dynamically?

Three potential reasons:

- Market timing
- Reduce transaction costs (taxes) over time
- Maximize wealth-dependent objectives

Example

Suppose major goal is $100MM to pay pension liability in 2 years
Start with $82MM; Invest in stock (annual vol=18.75%, annual exp. Return=7.75%); bond (Treasury, annual vol=0; return=3%)
Can we meet liability (without corporate contribution)?
How likely is a surplus?
Alternatives

Markowitz (mean-variance) – Fixed Mix
   Pick a portfolio on the efficient frontier
   Maintain the ratio of stock to bonds to minimize expected shortfall

Buy-and-hold (Minimize expected loss)
   Invest in stock and bonds and hold for 2 years

Dynamic (stochastic program)
   Allow trading before 2 years that might change the mix of stock and bonds
Efficient Frontier

Some mix of risk-less and risky asset

For 2-year returns:
Best Dynamic Strategy

Start with 57% in stock

If stocks go up in 1 year, shift to 0% in bond
If stocks go down in 1 year, shift to 91% in stock
Meet the liability 75% of time
Advantages of Dynamic Mix

Able to lock in gains
Take on more risk when necessary to meet targets
Respond to individual utility that depends on level of wealth

\[
\text{Shortfall} \quad \text{Target}
\]
Approaches for Dynamic Portfolios

Static extensions
  Can re-solve (but hard to maintain consistent objective)
  Solutions can vary greatly
  Transaction costs difficult to include

Dynamic programming policies
  Approximation
  Restricted policies (optimal – feasible?)
  Portfolio replication (duration match)

General methods (stochastic programs)
  Can include wide variety
  Computational (and modeling) challenges
Basic Model with Transaction Costs

• Basic setup:

Find \( x(t), b(t), s(t) \) to maximize \( E(u(x(T))) \) subject to \( x(0) \):

\[
\begin{align*}
    e^T x^+(t) &= e^T x(t) - \tau^T b(t) - \tau^T s(t), \\
    e^T (b(t) + s(t)) &= 0, \\
    x^+(t) + (I + \text{diag}(\tau))s(t) - (I - \text{diag}(\tau))b(t) &= x(t),
\end{align*}
\]

where \( \tau \) represents transaction costs and \( x(0) \) gives initial conditions and, without control, \( x(t) \) follows geometric Brownian motion \( dx(t) = x(t)(\mu(t) + \Sigma(t)^{1/2}dW(t)) \) where \( W(t) \) represents \( n \) independent Brownian motions.
Continuous-Time Results


Results: No trading in a region $H$; boundary at some distance from optimal no-transaction-cost point (for CRRA utility:

$$x^* = \frac{1}{\gamma} \Sigma^{-1} (\mu - r), \text{ Merton line}$$

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General Result

\[ x_1(t) \]

Merton line

No-trade region

Time \[ T \]

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Equivalence in Discrete Time

General observation: The continuous time solution is (approximately) equal to a discrete-time problem with a fixed boundary.

$x_1(t)$

Merton line

No-trade region

Boundary here: same as for one period to $T^*$.

Time $T$
Dynamic Programming Approach

State: \( x_t \) corresponding to positions in each asset (and possibly price, economic, other factors)

Value function: \( V_t(x_t) \)

Actions: \( u_t \)

Possible events \( s_t \), probability \( p_{st} \)

Find:

\[
V_t(x_t) = \max -c_t u_t + \sum_{st} p_{st} V_{t+1}(x_{t+1}(x_t, u_t, s_t))
\]

Advantages: general, dynamic, can limit types of policies

Disadvantages: Dimensionality, approximation of \( V \) at some point needed, limited policy set may be needed, accuracy hard to judge

General Form in Discrete Time

Find \( x = (x_1, x_2, \ldots, x_T) \) and \( p \) (allows for “robust formulation”) to

\[
\text{minimize } E_p \left[ \sum_{t=1}^{T} f_t(x_t, x_{t+1}, p) \right]
\]

s.t. \( x_t \in X_t, \ x_t \) nonanticipative, \( p \in P \) (distribution class)

\[
P[ h_t(x_t, x_{t+1}, p_t) \leq 0 ] \geq a \text{ (chance constraint)}
\]

General Approaches:
Simplify distribution (e.g., sample) and form a mathematical program:
• Solve step-by-step (dynamic program)
• Solve as single large-scale optimization problem
Use iterative procedure of sampling and optimization steps
What about Continuous Time?

Sometimes very useful to develop overall structure of value function

May help to identify a policy that can be explored in discrete time (e.g., portfolio no-trade region)

Analysis can become complex for multiple state variables

Possible bounding results for discrete approximations (e.g., FEM approach)
Restricted Policy and ADP Approaches

Restricted Policy Approaches:

1. Fixed proportions
2. Fixed function of factors/state variables
3. Contingent functions

ADP Approaches:
Approximate value function $V_t(x_t)$ by a combination of basis functions:

$$V_t(x_t) = \sum_i \lambda_i \phi_i(x_t)$$

and optimize over weights $\lambda$. 

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Large-Scale Optimization

Basic Framework: Stochastic Programming

Model Formulation:

\[
\begin{align*}
\text{max} & \quad \sum_{\sigma} p(\sigma) \left( U(W(\sigma, T)) \right) \\
\text{s.t. (for all } \sigma) & \quad \sum_{k} x(k, 1, \sigma) = W(o) \quad \text{(initial)} \\
& \quad \sum_{k} r(k, t-1, \sigma) x(k, t-1, \sigma) - \sum_{k} x(k, t, \sigma) = 0, \text{ all } t > 1; \\
& \quad \sum_{k} r(k, T-1, \sigma) x(k, T-1, \sigma) - W(\sigma, T) = 0, \text{ (final)}; \\
& \quad x(k, t, \sigma) \geq 0, \text{ all } k, t; \\
\end{align*}
\]

Nonanticipativity:

\[x(k, t, \sigma') - x(k, t, \sigma) = 0 \text{ if } \sigma', \sigma \in S^t_i \text{ for all} \]

Advantages:

General model, can handle transaction costs, include tax lots, etc.

Disadvantages: Size of model, insight
Simplified Finite Sample Model

Assume $p$ is fixed and random variables represented by sample $\xi_t$ for $t=1,2,..,T$, $i=1,...,N_t$ with probabilities $p_t^i$, $a(i)$ an ancestor of $i$, then model becomes (no chance constraints):

$$\text{minimize} \quad \Sigma_{t=1}^T \Sigma_{i=1}^{N_t} p_t^i f_t(x^{a(i)}_t, x_t^i, \xi_t)$$

s.t. $x_t^i \in X_t^i$

Observations?

- Problems for different $i$ are similar – solving one may help to solve others
- Problems may decompose across $i$ and across $t$ yielding
  - smaller problems (that may scale linearly in size)
  - opportunities for parallel computation.
Model Consistency

Price dynamics may have inherent arbitrage

Example: model includes option in formulation that is not the present value of future values in model (in risk-neutral prob.)

Does not include all market securities available

Policy inconsistency

May not have inherent arbitrage but inclusion of market instrument may create arbitrage opportunity

Skews results to follow policy constraints

Lack of extreme cases

Limited set of policies may avoid extreme cases that drive solutions
Objective Consistency

Examples with non-coherent objectives

- Value-at-Risk
- Probability of beating benchmark

Coherent measures of risk

- Can lead to piecewise linear utility function forms
- Expected shortfall, downside risk, or conditional value-at-risk (Uryasiev and Rockafellar)
Model and Method Difficulties

Model Difficulties
- Arbitrage in tree
- Loss of extreme cases
- Inconsistent utilities

Method Difficulties
- Deterministic incapable on large problems
- Stochastic methods have bias difficulties
  - Particularly for decomposition methods
  - Discrete time approximations
- Stopping rules and time hard to judge
Resolving Inconsistencies

Objective: Coherent measures (& good estimation)

Model resolutions

Construction of no-arbitrage trees (e.g., Klaassen)

Extreme cases (Generalized moment problems and fitting with existing price observations)

Method resolutions

Use structure for consistent bound estimates

Decompose for efficient solution
Abridged Nested Decomposition
(B., Donohue)

Incorporates sampling into the general framework of Nested Decomposition
Assumes relatively complete recourse and serial independence
Samples both the sub-problems to solve and the solutions to continue from in the forward pass through sample-path tree
Dual/Lagrangian-based Approaches

General idea:

- Relax nonanticipativity (or perhaps other constraints)
- Place in objective
- Separable problems

\[
\min \mathbb{E} \left[ \sum_{t=1}^{T} f_t(x_t, x_{t+1}) \right]
\]
\[\text{s.t. } x_t \in X_t\]
\[x_t \text{ nonanticipative} \]

\[
\min \mathbb{E} \left[ \sum_{t=1}^{T} f_t(x_t, x_{t+1}) \right]
+ \mathbb{E}[w^T x] + \frac{r}{2}||x-x^*||^2
\]

Update: \( w_t \); Project: \( x \) into \( N \) - nonanticipative space as \( x \)

Convergence: Convex problems
(Rockafellar and Wets);
In portfolios (Haugh, Kogan, Wang/Brown, Smith
Advantage: Maintain problem structure (e.g., network)
Summary Observations

• Asset-Liability Management involves all of the issues of dynamic portfolio optimization plus:
  • Modeling of the liability and asset relationships (not simple linear forms)
  • Path-dependent utilities
  • Care to avoid arbitrage in model

• Solution methods involve some form of approximation
  • Price paths, Time/cost to no-trade
  • Discrete with value function, state, and path decomposition
  • Dualization
Thank you!