The Values of Information and Solution in Stochastic Programming

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Themes

• The values of information and of solution provide measure of maximum to invest in:
  – Increased for resolving uncertainty
  – Increased effort for modeling and optimizing

• Both goals can guide computational methods that may combine optimization and resolving uncertainty

• Can unify the concepts by viewing optimization as resolving uncertainty over optimal decisions
Outline

• Basics of EVPI and VSS
• Relevance in dynamic stochastic optimization
• Using measures to guide computation
• Unifying views with optimization as uncertainty resolution
• Results for dynamic Gaussian problems
• Extensions to Gaussian mixtures
Dynamic Model

• Discrete Form (Stochastic Model):

\[ E[z(x^*, \xi)] = \text{minimize} \quad E_p \left[ \sum_{t=1}^{T} f_t(x_t, x_{t+1}, \xi) \right] \]
\[ \text{s.t.} \quad x_t \in X_t, x_t \text{ nonanticipative} \]
\[ \xi \sim P \text{ (distribution)} \]

• EVPI: \[ E[z(x^*, \xi)] - E[\min_x z(x, \xi)] \]

• VSS: \[ E[z(x(E\xi), \xi)] - E[z(x^*, \xi)] \]

where \( x(E\xi) \in \arg\min \sum_{t=1}^{T} f_t(x_t, x_{t+1}, E\xi) \)
Example: Routing Problem

- Serve clients at A, B, C, and D from depot with vehicle of capacity 10. Demand=2 at A, B, C; Demand=1 or 7 at C

Depot
Routing Observations

- Expected cost of expected value solution (ABCD) depends on direction (14 or 13.5) \( E[z(x(E\xi),\xi)] = 14 \) or 13.5
- Best recourse solution (CBAD, RP= \( E[z(x^*,\xi)] = 12.5 \)) is not optimal for any fixed demand
- Note: For high reliability, return to depot. )
Values of Information

- Definition: \( z(x, \xi) = c^T x + Q(x, \xi) \)
  subject to \( Ax = b, \ x \geq 0 \)

- Wait-and-See: (EVWPI):
  \( WS = E_\xi[\min_x z(x, \xi)] = 12. \)

- Recourse Problem:
  \( \min_x E_\xi[z(x, \xi)] = 12.5. \)
EVPI and VSS

- EVPI = RP - WS = 0.5 for Routing.
- Mean Value Problem:
  EV = \min_x z(x, E(\xi)) = 10 for Routing.
- Expectation of EV problem solution:
  EEV = E_\xi[z(x_{EV}, \xi)] = 14 or 13.5 for Routing.
- Value of the Stochastic Solution:
  VSS = EEV - RP = 1.5 or 1 for Routing
  (note: 2 or 3 \times EVPI)
Relationships

- \( \text{EVPI} = \text{RP} - \text{WS} \geq 0 \)
- \( \text{VSS} = \text{EEV} - \text{RP} \geq 0 \)
- \( \text{EVPI} \leq \text{EEV} - \text{EV} \)
- \( \text{VSS} \leq \text{EEV} - \text{EV} \)

But, examples:
\( \text{EVPI} = 0, \text{VSS} \neq 0; \text{VSS} = 0, \text{EVPI} \neq 0 \)

Note: If \( \text{EEV} \gg \text{WS} \), then potential for improvement.
Computational Implication

• At any branch of a decision tree, can compare the value of expectation over a given policy (e.g., EEV) and the expectation for a clairvoyant policy (e.g., WS)

• A large difference indicates a value from:
  – Resolving the uncertainty or
  – Improving the solution
EVPI Processes and Computation

• From MAHD+ (1981, 1988, 1999+)
• Consider the EVPI at each node:
  \[ \eta_t(\omega_t) \]
  with marginal process \( \rho_t(\omega_t) \)
• For a given sample tree, estimate \( \eta_t(\omega_t) \)
• Use importance sampling over distribution on value
• More samples where \( \eta_t \) high (fewer where low)
Computational Goals

• Additional sampling effort yields:
  – Better resolution of uncertainty
  – Better computation

• But, while correlated, not necessarily the same

• Ideally, sampling should improve both at the same time

• Strategy: put sampling and optimization together
Optimization via Sampling

**Problem:** find \( x \) to

\[
\min_x f(x) \text{subject to } g(x) \leq 0
\]

Pincus (1968): enough to sample from:

\[
\pi_\kappa(x) \propto \exp\{-\kappa(f(x))\}
\]

with constraints (e.g., Geman/Geman (1985)):

\[
\pi_{\kappa,\lambda}(x) \propto \exp\{-\kappa(f(x) + \lambda \kappa g(x))\}
\]

**Extension:** Augment with latent variables \((\lambda, \omega)\) then we can write

\[
\pi_{\kappa,\lambda}(x) \propto \exp\{-\kappa(f(x) + \lambda \kappa g(x))\}
\]

\[
= e^{ax} E_{\omega} \{e^{-\frac{\omega^2}{2}}\} e^{bx} E_{\lambda} \{e^{-\frac{\lambda^2}{2}}\}
\]

where the parameters \((a, b)\) depend on \((\kappa, \lambda \kappa)\) and the functional forms of \((f(x), g(x))\).
Basic Optimization via Simulation

• Pincus results:
• Model: \( \min F(x) \)
• Create a distribution \( P_\lambda(x) \propto \exp(-\lambda F(x)) \)
• Then:

\[
\lim_{\lambda \to \infty} E_{P_\lambda}(x) = x^*
\]
Markov Chain Simulation

• From $x_t$ to $x_{t+1}$
• Sample over
  $$\exp(-\lambda F(\mu x_t + (1-\mu) x_{t+1}))$$
• Choose directions to obtain mixing in the region
Example

- **Rosenbrock function:**

\[
\phi(x = (x_1, x_2)) = (1 - x_1)^2 + 100(x_2 - x_1^2)^2,
\]

which has a global minimum at \(x^* = (1, 1)\).

Specification for \(f\):

\[
x_t = \begin{cases} 
  u & \text{if } v \leq e^{-\kappa(\phi(u)/\phi(x_{t-1}))}; \\
  x_{t-1} & \text{otherwise,}
\end{cases}
\]

where \(u \sim U([-4, 4]^2)\), a uniform distribution over \([-4, 4] \times [-4, 4]\), and \(v \sim U([0, 1])\), a standard uniform draw.

Note: corresponds to the Metropolis-Hastings method of Markov Chain Monte Carlo using the distribution proportional to \(e^{-\kappa \phi(x)}\) for the acceptance criterion.
Rosenbrock Graphs

\( \kappa = 2 \quad 10 \quad 100 \)
Particle Samples and Optimization

• Suppose $N$ particles are used
• Consider a 2-stage problem
• $f(x, \xi) = c(x) + Q(x, y, \xi)$
• Distribution on $x, y_i, \xi$:  

$$P(x, y_1, \ldots y_N, \xi) \propto \prod_{i=1}^{N} (c(x) + Q(x, y_i, \xi_i))$$

Result: $E_P (x) \rightarrow x^*$
Implementation

- Propagation forward on y’s:
- \( q(\xi_j | x_1) \propto \{c(x) + Q(\xi_j, y_j, x)\} p(\xi_j) \)
- where \( y_j = \text{argmin}(c(x) + Q(\xi_j, y_j, x)) \)
- \( p(x | \xi_1, y_1, \ldots, \xi_N y_N) \)
  \( \propto \prod_{j=1}^N \{c(x) + Q(\xi_j, y_j, x)\} p(\xi_j) \)

The particles concentrate the distribution on \( x^* \) as in the deterministic optimization.
Example

• From B/Louveaux, Chapter 1 (Farmer):
Dynamic Models

• General Idea:

Create a distribution over decision variables and random variables

Use MCMC to sample – with sufficient number of particles or “annealing parameter” – marginal mean on optimum

• Extension for dynamics:

Keep constant sample particle numbers

Convergence in fixed number of particles per stage?
Example: Linear-Quadratic Gaussian

- All distributions are available in closed form

\[ x_t = Ax_{t-1} + B_t u_{t-1} + v_t \]

- All normal distributions with risk-sensitive objective:

\[ -\log E(\exp(-\frac{1}{2}(x_2^TQx_2+u_1^TRu_1))) \]

where \( u_1 \) is control in the first period

Note: Equivalent to robust formulation

- Now, all can be found in closed form
Results for LQG

• Mean of $u$: $m_{u_1}$
• $m_{u_1} = L_1(\theta) x_1$,
where $L_1$ is the result from the Kalman filter
• Variance of $u$: $V_u(N)$ where $V_{u_1}(N) \to 0$ as $N \to \infty$
$m_{u_1}(\theta) \to u_1^*$ as $\theta \to 0$
where $u_1^*$ is the two-stage risk-neutral solution.
Extensions to T Stages

- Can derive $S_t^\theta$ recursively to show that:

\[
p(x_t|x_{t-1},u_{t-1}) = K \exp\left(-\frac{1}{2}\right) \times \left(\sum_{s=1}^{t-1}(x_t^TQ_sx_s + u_s^TR_su_s) + x_t^TS_t^\theta x_t\right) + (x_t-Ax_{t-1}-Bu_{t-1})^T V_{t-1} (x_t-Ax_{t-1}-Bu_{t-1})\]

For particle methods, $j=1\ldots N$:

Each $x_t^j$ drawn consistently from this distribution.
General Method Structure

Propogate

Re-sample:

- $w_t^1, x_t^1 \rightarrow x_{t+1}^1$
- $w_t^2, x_t^2 \rightarrow x_{t+1}^2$
- $\ldots$
- $w_t^N, x_t^N \rightarrow x_{t+1}^N$

- $w'_t^1, x_t^1$
- $w'_t^2, x_t^2$
- $\ldots$
- $w'_t^N, x_t^N$
General Result

• If the propagation step is unbiased, then
  \[ E(u_1) \rightarrow u_1^* \]

• The unbiased result is possible in LQG from the recursive structure.

• Harder to obtain in more general systems (without additional future sampling)
Two-Stage: Compare Direct MC

Low Risk Aversion  High Risk Aversion
Three Stage Example

\[
\theta=20, \text{ Samples}=10000: \text{Marginal Joint Distribution } \pi(x_1, u_j) \text{ Defining Solution to } u_j=-L_j.
\]
Generalizing Objectives

• Direct: risk-sensitive exponential utility:
  \[ \exp(-\theta Q(x,y,\xi)) - Q \text{ quadratic} \]

• Objective: \( e^{-\theta |y-d|} \)

\[ E_\lambda [\exp((-\theta/2 \lambda_j)(y_j-d)^2)] \]

where \( \lambda \sim \text{Exp}(2) \)

Also, use:

\[ E[ e^{-\theta|y-d|} ] \approx 1 + \theta |y-d| + O(1) \]
Further Generalizations

• Use objective approximations that preserve normal forms
• Extend LQG-like results to this general framework
• Find other frameworks where unbiased future samples are available
• Obtain sensitivity results on bias from “imperfect” future sampling
• Use optimization to pick ML instead
Conclusions

- EVPI and VSS indicate the values in resolving uncertainty and optimizing
- Can use importance sampling based on EVPI to guide samples for optimization (but may have some lost efficiency)
- Combining optimization and sampling results in consistent importance sampling for uncertainty resolution and optimization
- Convergence possible for symmetric problems (LQG) and maybe others
Thank you!

• Questions?