Convergence of SAA Approaches for Stochastic MPCC with Expectation Constraints

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Themes

• Stochastic programs with equilibrium constraints arise in a variety of areas
• Conditions can appear with constraints on expectations of the equilibrium values
• These constraints can complicate convergence results for sample-average approximation (SAA), but results are still possible with mild assumptions
Outline

• Forms of stochastic MPECs
• Example in electric power markets
• Challenges and results for SAA
• Conclusions
SPs and Equilibrium Problems

• References: Shapiro (2006), Birbil et al. (2004), Lin et al. (2003), Patriksson and Wynter (1999)
• General idea: decisions depend on reactions of others in market; assumption of some form of equilibrium
• Types of equilibrium:
  – Wait-and-see: lower level decisions taken after observing \( \omega \); equilibrium for each \( \omega \)
  – Here-and-now, almost sure equilibrium: lower level decisions taken before seeing \( \omega \); equilibrium for each \( \omega \)
  – Here-and-now, expectation equilibrium: lower level decisions taken before seeing \( \omega \); equilibrium in expectation
  – Our focus: Wait-and-see with here-and-now decisions depending on expectations of the wait-and-see solutions
General Results

• Patriksson and Wynter: existence in wait-and-see case
• Lin et al.: equivalence to MPEC and solution algorithm
• Birbil et al.: Conditions for sample path average approaches to converge in here-and-now
• Shapiro: Convergence results in the wait-and-see case
Basic Formulations

• Upper level: $x$; Lower level: $y$
• Wait-and-see:
  Min $E[\phi(x,y,\omega)]$
  s.t. $x, y \in Z$; $y \geq 0$, $F(x,y(\omega),\omega) \geq 0$ a.s.; $y(\omega)^T F(x,y(\omega),\omega) = 0$
  ($\Gamma(x, y(\omega), \omega) = 0$)
  Here-and-now: $y = E(y(\omega))$
  In expectation: $E(F(x,y(\omega),\omega)) \geq 0$ a.s.; $E(y^T F(x,y(\omega),\omega)) = 0$
  Lin et al.: $F(x,y,\omega) + z(\omega) \geq 0$ a.s.; $y^T (F(x,y(\omega),\omega) + z(\omega)) = 0$
  $z(\omega) \geq 0$

With expectation constraints: $E[\psi(x,y(\omega),\omega))] = 0$
Example for Electric Power

• Values \( y(\omega) \) contain future spot prices for electricity
• To ensure availability of capacity, ISO pays producers in an opening round (day-ahead auction)
• Payment generally based on a quantile but this may result in insufficient capacity
• Payment on the expectation of future prices may encourage more participation and greater reliability
Basic Questions for SMPEC with Expectation Constraints

\[ \min_{x \in K_x, y(\omega) \in K_y} \quad \mathbb{E}_\omega [\phi(x, y(\omega), \omega)] \]

such that

\[ \mathbb{E}_\omega [\psi(x, y(\omega), \omega)] = 0. \]

\[ \Gamma(x, y(\omega), \omega) = 0, \forall \omega \in \Omega \]

• Does SAA converge to a solution of this problem?

• Most theory that we are aware of is done for either
  • expectations only in the objective function or
  • VIs with expectations everywhere, using perturbation analysis of finite dimensional VIs
The mixed minimization-nonlinear equations problems

\[
\begin{align*}
E_\omega [\psi(x, y(\omega), \omega)] &= 0 \\
x, y(\omega) &= \text{arg} \min_{x \in K_x, y(\omega) \in K_y} E_\omega [\phi(x, y(\omega), \omega)] + \lambda^T E_\omega [\psi(x, y(\omega), \omega)]. \\
&\text{such that} \quad \Gamma(x, y(\omega), \omega) = 0, \forall \omega \in \Omega
\end{align*}
\]

- Use a Lagrangian Relaxation approach to eliminate the expectation constraints with the extra variables $\lambda$
- Enforce the expectation constraints on the outer loop.
SAA-friendly reformulation

\[ \tilde{y}(\omega, \lambda, x) = \min_{y(\omega) \in K_y} \phi(x, y(\omega), \omega) + \lambda^T \psi(x, y(\omega), \omega) \]  
\[ \text{such that } \Gamma(x, y(\omega), \omega) = 0, \forall \omega \in \Omega \]

Assume that the solution, \( y \), is unique. Then the solution of the following mixed nonlinear equation-optimization problem solves the original optimization problem with expectation constraints.

\[
\left\{ \begin{array}{l}
E_\omega [\psi(x, \tilde{y}(\omega, x, \lambda), \omega)] = 0 \\
x = \arg\min_{x \in K_x} \left[ E_\omega [\phi(x, y(\omega, x, \lambda), \omega)] + \lambda^T E_\omega [\psi(x, y(\omega, x, \lambda), \omega)] \right].
\end{array} \right.
\]
Key result--

Lemma Consider the coupled minimization-nonlinear equation problem

$$\min_{x \in X} f(x, \lambda); \quad g(x, \lambda) = 0.$$ 

Denote by $S(\lambda)$ the solution sets of the first problem and by $F$ the ones of the second problem. Assume that

i. $\cup_{\lambda} \{S(\lambda), \lambda\} \cap F$ has a unique solution point $(x^*, \lambda^*) \in C_X \times C_{\Lambda}$. Here the sets $C_X$ and $C_{\Lambda}$ are closed and compact.

ii. The sequences $f^N$ and $g^N$ converge to $f$ and $g$ uniformly on $C_X \times C_{\Lambda}$.

iii. The coupled problem

$$\min_{x \in X} f^N(x, \lambda); \quad g^N(x, \lambda) = 0.$$ 

has a solution $(x^N, \lambda^N) \in C_X \times C_{\Lambda}$ with probability 1.

Then $(x^N, \lambda^N) \rightarrow (x^*, \lambda^*)$ with probability 1.
Result

Corollary Assume that the problem

$$\arg\min_{y(\omega) \in K_y} \phi(x, y(\omega), \omega) + \lambda^T \psi(x, y(\omega), \omega)$$

such that
$$\Gamma(x, y(\omega), \omega) = 0, \forall \omega \in \Omega$$

has a solution $\tilde{y}(\omega, \lambda, x)$ that is unique in $x$, $\lambda$ for almost all $\omega$. Under the assumptions of Lemma, the SAA approximation of the problem

$$\begin{cases}
E_\omega [\psi(x, \tilde{y}(\omega, x, \lambda), \omega)] = 0 \\
x = \arg\min_{x \in K_x} E_\omega [\phi(x, y(\omega, x, \lambda), \omega)] + \lambda^T E_\omega [\psi(x, y(\omega, x, \lambda), \omega)].
\end{cases}$$

has a solution $(x^N, \lambda^N)$ that converges to the solution $(x^*, \lambda^*)$ almost surely.

.. and, thus, to the $x$-value of the problem with expectation constraints.
Conclusions

• SMPECs with expectation constraints may be useful in coordinating markets and avoiding costly losses

• Expectation constraints present challenges for SAA convergence

• Convergence issues can be resolved with assumptions of uniqueness using a Lagrangian form
• Thank you and questions?