Continuum Approximation of Multiple-Asset Taxable Portfolios

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Motivation

• Capital gain taxes create difficulties for portfolio optimization
• Increasing assets creates greater difficulties in optimizing but may reduce taxes
• In the US, re-setting the tax basis at death also avoids capital gains
• When can portfolios with many assets avoid (or significantly reduce) capital gains taxes for investor lifetimes?
  – What is the incremental value of holding individual assets over an index?
Outline

• Background
• Asset continuum model
• Bounding approximations
• Example results
• Extensions and conclusions
Literature

• Pricing with taxes

• Portfolio problems
General Results

• If no transaction costs (and allowed wash sales):
  – Capital gains taxes may be avoided (with costs to sell assets or options short)
  – Losses should be realized as they occur and gains deferred

• Exact basis may not have large advantage over average basis

• Even models with limited number of assets become quite complex
Avoiding Capital Gains

- **Short selling**
  - Open short position
  - Buy insurance against forced liquidation
  - Borrow to finance short position margin

- **Call option trading**
  - Open three (short and long) call positions to balance cash flow
  - Realize losses (reinvest) and defer gains

- **Problems: Continuous trading – high cost**
Multiple Asset Advantage

Example

• Suppose invest $3 in equal-weight index of 3 assets or $1 in each individually

• Starting price of each is $S_0(i) = \$1, i = 1, 2, 3$

• In one period, $S_1(1) = \$1.30, S_1(2) = \$1.10, S_1(3) = \$0.90, \text{Index}_1 = \$1.10$

• Need $2 for consumption at Time 1

• How much is left with index or individual assets?
Example after Taxes

• Remaining portfolio with index
  – Sell $2+a (where $a=\text{cap gain tax})$
  – Capital gain = \$2+a-\left(\frac{2+a}{1.1}\right)=\$(2+a)*(0.091)
  – $a=(0.15*0.091)\$(2+a)=\$0.027+0.014*a$
  – $a=\$0.028$; Portfolio value=$3.30-2.028=\$1.27$

• Remaining portfolio with individual assets
  – Sell $1.10$ of Asset 2 and $0.90$ of Asset 3
  – Capital gain=$((1.10-1.00)+(0.90-1.00))=\$0$
  – Cap gain tax = $\$0$
  – Portfolio value = $3.30-2.00=\$1.30$

• Increase with individual assets $0.03/\$1.30=2.1\%$

• Questions: Can this be sustained? How do increased numbers of assets improve this value?
Extreme Case: Continuum of Assets

- Portfolio of assets indexed by $\theta$
- Assume single tax basis for all assets of $S_0(\theta) = $1 (always maintained)
- Prices follow market and individual factors:
  \[ \ln\left(\frac{S_t(\theta)}{S_{t-1}(\theta)}\right) = \mu + \sigma_0 W_0 + \sigma W_{\theta} \]
  where $W_0$ and $W_{\theta}$ are independent standard normal and $W_{\theta}$ and $W_{\theta'}$ are independent, $\theta \neq \theta'$

What happens to portfolio prices after one period?
One Period Result

- Period 0 Values:

- Period 1 Values (lognormal, mean=market*e^{\sigma^2/2}):
General Approach

- Maintain tax basis of $1 for all assets
- Realize losses in each period
- Liquidate any additional amounts to maintain consumption as fixed proportion of portfolio
- Calculate taxes if any, consume remainder, and carry over losses
- Find consumption compared to index portfolio with same bequest amount (so, difference is only in consumption)
Liquidation Process

- If liquidation proportion $\alpha V_t < E_p[S_t|S_t<1]$, sell all to $\$1$ and carryover
- If $\alpha V_t > E_p[S_t|S_t<1]$, sell lowest priced $\alpha V_t$ of portfolio
- What is the distribution of $S_t$’s?
Distribution of Period $t$ Prices

• General:
  – Infinite sum (integral) of lognormal r.v.’s
  – May have analytical form (reciprocal Gamma) but difficult to evaluate
  – Alternative bounding approximation

• Bounding intervals
  – Divide prices (above $1$ or $\alpha V_t$) into intervals
  – Assume point for each interval representation (and lose potential losses from right tail)
Bounding Intervals

Upper bound on consumption (assume lower bound on prices; maintain value)

Lower bound on consumption (assume upper prices; no losses from right tail)
Example Comparison

• Suppose the following parameters:
  \[ \mu=0.02, \sigma_0=0.2, \sigma=0.4, \alpha=0.05, T=25 \]

• Notes:
  – This implies the log of expected market return is 0.12 with volatility 0.20
  – The idiosyncratic volatility of each asset is twice the market (to match observations in Campbell, Lettau, Malkiel, Xu (JF, 2001) that idiosyncratic vol is \( \sim 70\% \) of total individual asset volatility)
Example Results

- Average consumption (normalized to 1 for No-Tax case)
- Consumption increase for average case over index: 7%
- Tax year decrease for average over index: 90%
Improvements (in Process)

• Better bounds
  – Use arithmetic-geometric mean inequality and extensions
  – Dynamically choose intervals

• Transaction costs
  – Optimize over when to take losses
  – Demonstrating bounds

• Cash in-flows
Extensions (TBD)

- How does finite asset number converge to asset continuum? (I.e., how many assets needed to approximate continuum behavior?)
- What are the effects of portfolio of assets (factors of market, industry, and individual)? (When to re-balance?)
- What other questions might be amenable to analysis with asset continuum?
Conclusions

- Multiple assets may be modeled as continuum of assets
- Bounds found on distributions over time
- Capital gain taxes reduced substantially over index portfolio in asset-continuum case
- Potential extensions for portfolios and more detailed models