Optimizing Portfolios with Non-normal Distributions

John Birge
University of Chicago
Booth School of Business

Luis Chávez-Bedayo
ESAN Graduate School of Business
General Theme

• Asset returns generally have non-normal characteristics:
  • Skewness
  • Heavy tails
  • Clustered correlation
• Generalized hyperbolic distributions capture many of these empirical observations
• Analytical results are available for exponential utilities that can substantially improve on the use of normal approximations
Agenda

• Motivation
• Problem description
• Basic problem
• Risk-return interpretation
• Active portfolio management interpretation
• Conclusions
Motivation

• Characterize portfolios with non-elliptical distributions
• Present mean-variance adjustments and relationships
• Relate this approach to consensus and forecasted (view) in active management
• Compare various portfolio rules
Portfolio Problem

We are basically going to study:

Maximize \[ \mathbb{E} \left[ - \exp \left\{ -aW_0 \left( 1 + (1 - x^T 1) r_f + x^T R \right) \right\} \right] \]

subject to \[ x \in S \]

- \( x \in \mathbb{R}^N \) portfolio weights on the risky assets
- \( 1 - x^T 1 \) weight on the risk-free asset
- Investor with CARA utility, \( a > 0 \), \( W_0 > 0 \)
Normal Mean-Variance Mixtures

The random vector $\mathbf{R}$ has a multivariate normal mean-variance mixture (in $N$ dimensions) if

$$\mathbf{R} = \mu + Y\gamma + \sqrt{Y}\mathbf{A}\mathbf{Z}$$

- $\mathbf{Z} \sim N_k(0, I_k)$
- $Y \geq 0$ is a non-negative scalar-valued random variable independent of $\mathbf{Z}$
- $\mathbf{A} \in \mathbb{R}^{N \times k}$, $\text{rank}(\mathbf{A}) = N \leq k$
- $\mu$ and $\gamma$ belong to $\mathbb{R}^N$
Generalized Hyperbolic Distribution

Generalized Hyperbolic (GH) Distribution
Barndorff-Nielsen (1977)

When $Y \sim GIG(\lambda, \chi, \psi)$, $R$ is said to have a GH distribution:

$$R \sim GH_N(\lambda, \chi, \psi, \mu, \Sigma = AA^T, \gamma)$$

Parameters:
- $\chi > 0, \psi \geq 0$ if $\lambda < 0$
- $\chi > 0, \psi > 0$ if $\lambda = 0$
- $\chi \geq 0, \psi > 0$ if $\lambda > 0$
Portfolio Property

*Portfolio Property*

If \( \mathbf{R} \sim \text{GH}_N(\lambda, \chi, \psi, \mu, \Sigma, \gamma) \) and

\[
R_p = \mathbf{x}^T \mathbf{R}
\]

where \( \mathbf{x} \in \mathbb{R}^N \), then

\[
R_p \sim \text{GH}(\lambda, \chi, \psi, \mathbf{x}^T \mu, \mathbf{x}^T \Sigma \mathbf{x}, \mathbf{x}^T \gamma)
\]
Special Cases

- If $\lambda = -\frac{1}{2}$, $\psi > 0$ and $\chi > 0 \rightarrow$ Normal Inverse Gaussian (NIG) distribution

- If $\lambda > 0$ and $\chi = 0 \rightarrow$ Variance-Gamma (VG) distribution

- If $\lambda = -\frac{1}{2} \nu$, $\chi = \nu$ and $\psi = 0 \rightarrow$ Skewed t distribution
GH distribution recently has received attention in the portfolio optimization literature

- Hellmich and Kassberger (2009)
- Mencía and Sentana (2009)
- Madan and Yen (2008)
Equivalent Portfolio Problem

Maximize \( a \times b \times c \)

\[
a = -e^{-aW_0(1+(1-x^T1)r_f+x^T\mu)}
\]

\[
b = \left( \frac{\psi}{\psi + aW_0x^T(2\gamma - aW_0\Sigma x)} \right)^{\frac{1}{2}}
\]

\[
c = \frac{K_{\lambda} \left( \sqrt{\chi(\psi + aW_0x^T(2\gamma - aW_0\Sigma x))} \right)}{K_{\lambda}(\sqrt{\chi\psi})}
\]
Optimal Portfolio Weights

They have the following general form

$$x^* = \frac{1}{aW_0} \left( L(x^*)\Sigma^{-1} \left( \mu - r_f 1 \right) + \Sigma^{-1} \gamma \right)$$

- \( L(x^*) \in \mathbb{R} \), and \( 0 \leq L(x^*) \leq \sqrt{\frac{\psi + A}{C}} \) with
  
  \[ A = \gamma^T \Sigma^{-1} \gamma \]
  \[ C = \left( \mu - r_f 1 \right)^T \Sigma^{-1} \left( \mu - r_f 1 \right) \]
NIG Example

NIG distribution ($\lambda = -\frac{1}{2}$)

$$x^* = \frac{1}{aW_0} \left( \sqrt{\frac{A + \psi}{C + \chi}} \sum^{-1}(\mu - r_f 1) + \sum^{-1} \gamma \right)$$
Portfolio of Portfolios

In the case of $Y^T(\mu - r_f 1) \neq 0$,

$$x(\tau^*) = \frac{1}{aw_0} \left( \delta Y \hat{\beta}_\mu + Y \hat{\beta}_\gamma \right).$$

The vector $\hat{\beta}_\mu$ minimizes

$$Q_\mu = \left( \Sigma^{-1}(\mu - r_f 1) - Y \beta_\mu \right)^T \Sigma \left( \Sigma^{-1}(\mu - r_f 1) - Y \beta_\mu \right).$$
Three-Fund Rule

If \( b \in \mathbb{R}^N \) and \( b \neq 0 \), we define

\[
x_{mv}(\eta_1(b), \eta_2(b)) = \frac{1}{aW_0} (\eta_1(b) \text{Cov}(R)^{-1} \mathbb{E}[R - r_f 1]) + \eta_2(b) \text{Cov}(R)^{-1} b)
\]

with \( \eta_1(b) \) and \( \eta_2(b) \) in \( \mathbb{R} \).

We want to choose these quantities optimally.
Risk-Return Analysis

Given a feasible portfolio $x$, the Extended Sharpe Ratio (ESR) of $x$ is defined as

$$\frac{1}{2} \text{ESR}(x) = aW_0x^T(\mu - r_f 1) - g(x)$$

$$g(x) = \ln \left( \left( \frac{\psi}{\psi - (a^2W_0^2x^T\Sigma x - 2aW_0x^T\gamma)} \right)^{\frac{1}{2}} \times \frac{\zeta}{K_\lambda \left( \sqrt{\chi \psi} \right)} \right)$$
Extended Sharpe Usage

Given a feasible portfolio $\mathbf{x}$, the Extended Sharpe Ratio (ESR) of $\mathbf{x}$ is defined as

$$\frac{1}{2} \text{ESR}(\mathbf{x}) = aW_0 \mathbf{x}^\top (\mu - r_f \mathbf{1}) - g(\mathbf{x}).$$

If $\mathbf{x}^1$ and $\mathbf{x}^2$ are feasible portfolios, then

$$\mathbb{E}[U(\mathbf{x}^1)] \geq \mathbb{E}[U(\mathbf{x}^2)] \iff \text{ESR}(\mathbf{x}^1) \geq \text{ESR}(\mathbf{x}^2).$$

Similar to the Generalized Sharpe ratio of Hodges (1998)

*NIG Distribution* ($\lambda = -\frac{1}{2}$)

If $\chi = \psi$, then

$$\lim_{\psi \to \infty} \text{ESR}(\mathbf{x}^*) = \left(\mu + \gamma - r_f \mathbf{1}\right)^\top \Sigma^{-1} \left(\mu + \gamma - r_f \mathbf{1}\right)$$

Optimal Sharpe ratio squared
ESR – Risk/Return

If \( x \) is a feasible portfolio:

\[
Q(x) = aW_0 x^T (\mu - r_f 1)
\]

\[
KE(x) = (aW_0)^2 x^T \Sigma x - 2aW_0 x^T \gamma
\]

It is possible to associate \( Q(x) \) with “return”, and \( KE(x) \) with “risk”

\[
\frac{1}{2} ESR(Q, KE) = Q - \ln \left( \left( \frac{\psi}{\psi - KE} \right)^{\frac{1}{2}} K_\lambda \left( \frac{\sqrt{\chi(\psi - KE)}}{K_\lambda(\sqrt{\chi}\psi)} \right) \right)
\]

- \( \frac{\partial ESR}{\partial Q} > 0 \)
- \( \frac{\partial ESR}{\partial KE} < 0 \)
Combining Portfolios

If $\mu - r_f 1 \neq 0$ and $\gamma \neq 0$, any optimal portfolio $x^*$ can be expressed as an affine combination of the following two feasible portfolios:

$$x^{KE} = \frac{1}{aW_0} \Sigma^{-1} \gamma$$

$$x^{Q} = \frac{1}{aW_0} \left( \sqrt{\frac{A}{C}} \Sigma^{-1} (\mu - r_f 1) + \Sigma^{-1} \gamma \right)$$
Min-Risk/Max-Return

Minimum “risk” portfolio:

\[ x^{KE} = \arg \min_{x \in S} \{ KE(x) \} \]

Maximum “return” portfolio (at some specific level of “risk”):

\( x^Q \) is the solution to:

Maximize \( Q(x) \)

subject to \( KE(x) = 0 \)

Then, the optimal portfolio \( x^* \) satisfies

\[ x^* = \alpha^* x^Q + (1 - \alpha^*) x^{KE} \]

where \( \alpha^* \geq 0 \) is a function of the parameters of the corresponding GH distribution
Numerical Results

- The optimal portfolio and the optimal MV portfolio tend to be very similar in the elliptical and non-elliptical NIG cases.

- The three-fund rules that account for estimation error (Kan and Zhou and Jorion) tend to perform “better” even under non-elliptical assumptions.

- Corroborate the fact that the estimation of the mean vector have more impact than the estimation of the covariance matrix in portfolio selection.
Consensus and Forecasted Excess Returns

Let $\mu_B > 0$ be the expected excess return of the benchmark portfolio $B$, and $\beta \neq 0$ the $N$-element vector of asset betas with respect to $B$

Consensus excess returns: $R^e_C \sim N(\mu_B \beta, V)$

Forecasted excess returns: $R^e_F \sim N(\alpha + \mu_B \beta, V)$ with $\alpha \neq 0$
Optimal Portfolio

If the investment manager maximizes a particular linear trade-off between alpha and residual risk, the optimal portfolio is

$$h_F = \left( \frac{IR_A^2}{2\lambda_R} \right)^{\text{Max. Information Ratio}} h_A + h_B$$

- $IR_A$ is the information ratio of portfolio $A$
- $\lambda_R$ measures the aversion to residual risk
Confidence in Consensus

We introduce a non negative random variable $Y$ and a random vector $\mathbf{R}$ such that

$$\mathbf{R} | Y = y \sim N(\alpha + y\mu_B\beta, y\mathbf{V})$$

If the investment manager is a MV investor

$$h_{MV} = \frac{1}{\varrho} \mathbf{V}^{-1} \left( \frac{\alpha}{\gamma} + \frac{\mu B}{\beta} \right)$$

If we assume

$$\frac{1}{\varrho} = \frac{\sigma_B^2}{\mu_B} \quad \text{and} \quad y \to \infty,$$

then portfolio $h_{MV}$ approaches the benchmark portfolio $B$
Confidence Example

If $Y \sim IG(\chi, \psi)$, then $R$ follows a Normal Inverse Gaussian (NIG) distribution.

We can solve our original portfolio problem under the specified NIG distribution with

$$\frac{1}{aW_0} = \frac{\sigma^2}{\mu_B}$$

Under a particular choice $\psi^*$, the optimal portfolio is

$$h^* = \left( \frac{IR_A^2}{2\lambda^*_R(\chi)} \right) h_A + h_B,$$

where $\lambda^*_R(\chi)$ is an increasing function of $\chi$ and its minimum value is $\lambda_R$.
Conclusions

- Analytically tractable expressions for the portfolio optimization problem under GH distribution and exponential utility
- Risk-return tradeoff similar to the one of MV portfolio optimization
- Conditions to have a two-fund rule under non-elliptical distributions
- New measures of portfolio performance
- Sensitivity analysis and the relationship with tail behavior
- Alternative way to incorporate the confidence in the consensus distribution into the framework of an active portfolio manager
- Testing of typical mean-variance portfolio rules and some empirical properties of the optimal portfolios
Thank you!

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