Dynamic Portfolio Optimization with Transaction Costs

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General Theme

- Dynamic portfolio optimization with explicit consideration of transaction costs is challenging
- Several alternatives are possible including:
  - Discrete (coarse) multistage approximation (stochastic programming)
  - Discrete two-stage with transaction cost adjustment
  - Continuous ellipsoidal no-trade region approximation
  - Continuous no-trade boundary approximation
- Each appears to have advantages for different situations
- Results are sensitive to parameter combinations
Basic Model

• Basic setup:

Find \( x(t), b(t), s(t) \) to maximize \( E(u(x(T))) \) subject to \( x(0) \):

\[
e^T x^+(t) = e^T x(t) - \tau^T b(t) - \tau^T s(t),
\]

\[
e^T (b(t) + s(t)) = 0,
\]

\[
x^+(t) + (I + diag(\tau)) s(t) - (I - diag(\tau)) b(t) = x(t),
\]

where \( \tau \) represents transaction costs and \( x(0) \) gives initial conditions and, without control, \( x(t) \) follows geometric Brownian motion

\[
dx(t) = x(t)(\mu(t) + \Sigma(t)^{1/2} dW(t))
\]

where \( W(t) \) represents \( n \) independent Brownian motions.
Continuous-Time Results


Results: No trading in a region $H$; boundary at some distance from optimal no-transaction-cost point (for CRRA utility: $x^* = (1/\gamma) \sum^1 (\mu-r)$, Merton line)
General Result

\[ x_1(t) \]

Merton line

No-trade region

Time \[ T \]
Alternative Approaches
Alternative Approaches

- Multistage discrete time (stochastic programming)
- Two (single)-stage discrete (approximate transaction cost)
- Continuous ellipsoidal approximation
- Continuous boundary approximation
General observation: The continuous time solution is (approximately) equal to a discrete-time problem with a fixed boundary.

Boundary here: same as for one period to $T^*$.  

$x_1(t)$  

Merton line  

No-trade region  

Time $T$
Find \( x, b, s \) to minimize \( E[u(x(t + T^*))] \) s.t.

\[
e^T x(t) = e^T x_0(t) - \tau^T b(t) - \tau^T s(t),
\]

\[
e^T (b(t) + s(t)) = 0,
\]

\[
x(t) + (I + \text{diag}(\tau))s(t) - (I - \text{diag}(\tau))b(t) = x_0,
\]

where \( e^T x_0 = w(t) \).

Challenge: How to find \( T^* \)?
Effective Result in Terms of Average Number of Re-balances

Observation: $T^*$ is approximately the average time between re-balances or $1/T^*$ is approximately the expected number of re-balances in a single period.

- Can normalize to a single period and use $\pi/T^*$ for transaction cost.

- (Note: can learn $T^*$ along with $\mu$, $\Sigma$)
Ellipsoidal Region Approximation

Idea (Morton/Pliska): suppose each rebalancing has the same charge $\epsilon$ regardless of amount to re-balance

Result: Rebalance only if solution $x_t$ leaves $x^* + \epsilon^{0.25} C$
Empirical Setup

Testing results:
- Simulate portfolios over time
- Vary $T^*$
- Observe final objectives and number of re-balances as a function of $T^*$

$\mu = 0.08$, $\Sigma = 0.04*I$, $\tau = 0.01$.

100 trials for each run; 100 periods.
Comparisons on final Sharpe ratio
Empirical Results

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Adjustment for Different Volatilities

- The value of $T^*$ will vary depending on $\mu$ and $\Sigma$ (which may vary over time)
- An approximation to deal with variation is to assign $T^*$ as a function (e.g., a multiple $\lambda$) of the portfolio volatility $(x^T \Sigma x)^{1/2}$
- Now, the learning can occur on $\mu$, $\Sigma$, $\lambda$
Conclusions

• Transaction costs create difficulties for portfolio optimization
• Finding the no-trade region is difficult in higher dimensions
• Finding an effectively equivalent single-period formulation with appropriate modification of the transaction cost can approximate the continuous-time solution
Thank you!

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