Motivation: The control of energy systems, particularly electric power networks, involves the coordination of multiple agents who choose both discrete and continuous actions that may depend on unobserved properties of their environment.

Goals:
- Define a continuous-variable model that closely approximates the discrete choices in stochastic unit commitment and related equilibrium problems.
Overview

- Power generation schedules are set by unit commitment models that involve discrete variables for generation assets for on/off; start-up/shutdown; minimum load...
- Smart grid may also include active transmission controls
- Including integer restrictions leads to improved decisions but creates a computational cost
- Lack of direct dual information for prices can create inefficiencies, especially with equilibrium characterizations of real-time operations

**Approach:**
- Interpret on/off decision as Bernoulli random variables
- Constrain variance of the random variables to approximate integrality

**Result:**
- Tighter relaxations
- Approximation of reliability constraints
Outline

- Problem formulation
- Integrality issues
- Equilibrium issues
- Probability interpretation
- Relaxation implications
- Reliability interpretation
- Conclusions
Stochastic IP and Unit Commitment Problem

Basic Two-stage Stochastic Integer Program:

$$\min f(x) + E[g(x, y(\omega), \omega)]$$
$$\text{s. t. } x \in X, y(\omega) \in Y(x, \omega), \text{ a. s.}$$

where $f$, $g$, $X$, and $Y$ are convex $x_l \in \{0, 1\}$, and $y_J \in \{0, 1\}$ for some sets $I$ and $J$ of the variables.

Unit Commitment Setup:

- $x$ - initial states $x_l$ and generation and load at each node (can represent day-ahead market)
- $y(\omega)$ - scenario dependent states ($y_J(\omega)$) and generation/load amounts (can represent real-time market)
- Transmission: also potentially part of the decision set and constraints
- Costs: $f$, $g$ - represents potentially nonlinear increases in cost with generation (or load reductions)
- Equilibrium: cost information for the real-time and possibly day-ahead market must also be consistent with maximizing contribution
Solution Methods: Deterministic vs. Stochastic

- **Deterministic: Mixed integer programs**
  - **Benefits:**
    - Gains over previous Lagrangian+heuristic solutions
  - **Issues:**
    - Does not capture uncertainty in demand and supply
    - Cannot reconcile price and quantities between day-ahead and real-time
    - Creates greater opportunities for market power exploitation
    - Leads to overall inefficiencies relative to a welfare-maximizing stochastic solution

- **Stochastic: Progressive hedging**
  - **Benefits:**
    - Addresses issues of the deterministic solutions
    - Methodology converging efficiently with aid of heuristics
  - **Issues:**
    - Implementation in bidding systems not obvious
    - Losses relative to optimality remain difficult to judge
    - Resulting prices and effects on behavior may be difficult to assess and may still lead to inefficiencies
Stochastic Method

Progressive Hedging Method: Augmented Lagrangian/Proximal Point Method:

- Idea:
  - Write $x$ as $x(\omega)$ with $x - E[x(\omega)] = 0$ a.s. (nonanticipativity)
  - Relax nonanticipativity constraint
  - Update multipliers and sequentially fix $x^k(\omega)$ to a feasible solution $\hat{x}^k = E[x^k(\omega)]$.

- Iteration $k + 1$: find $x^{k+1}(\omega), y^{k+1}(\omega)$ that solve:

  $$
  \min E[f(x(\omega)) + g(x(\omega), y(\omega), \omega) + \rho^k(\omega)(\hat{x}^k - x(\omega)) + \frac{\gamma}{2}\|\hat{x}^k - x(\omega)\|^2]
  $$

  s. t. $x(\omega) \in X, y(\omega) \in Y(x, \omega)$. a. s.

- Update $\hat{x}^{k+1} = E[x^{k+1}(\omega)]; \rho^{k+1}(\omega) = \rho^k(\omega) + \gamma(\hat{x}^{k+1} - x^{k+1}(\omega))$.

Observations/Issues:

- Can solve separately for each $\omega$.
- No convergence guarantee when integrality is forced in the subproblems.
- Resulting multipliers (prices) may vary / invalidate alternatives.
Probabilistic Methodology

- Basic idea: Treat binary commitment decisions as parameters of Bernoulli random variables - the probability \( p \) that a given unit is committed and available for generation
  - Allows for inclusion of reliability directly in the model without additional safety stock constraints
  - By constraining the variance \((p(1-p) \approx 0)\), integrality can be enforced
  - Provides a mechanism for Progressive Hedging (and other algorithms) to converge to an equilibrium without additional heuristic procedures
  - Gives price information that covers the individual rationality constraints not currently captured in existing systems

- Approach
  - Replace \( x_i \in \{0,1\} \) with \( 0 \leq x_i(\omega) \leq 1 \) and \( \text{Var}(x_i) \approx 0 \)
  - Similar to constraint \( x_i (1-x_i) \approx 0 \).
  - Idea: use a convex relaxation of this nonlinear constraint.
Solution Methodology:

- **Progressive Hedging Form:**
  \[
  \rho(\omega)^T(x(\omega) - E(x(\omega))) + \frac{\gamma}{2}\|x(\omega) - \bar{x}^\nu\|^2,
  \]
  where \(\gamma\) is a parameter and \(\bar{x}^\nu\) is the nonanticipative projection of the previous solution.

- **Probabilistic modification:**
  Suppose constraints:
  \[Tx + Wy(\omega) \geq d(\omega).\]
  Interpret \(\tilde{x}_i(x_i), i = 1, \ldots, n\) as the Bernoulli random variable with probability \(x_i\), the binary constraints become:
  \[
  P(T\tilde{x}_i(x_i) + Wy(\omega) \geq d(\omega)) \geq 1.
  \]
  Relax as:
  \[
  P(T\tilde{x}_i(x_i) + Wy(\omega) \geq d(\omega)) \geq \alpha
  \]
  for some \(\alpha < 1\), or, more generally,
  \[
  E[R((T\tilde{x}_i(x_i) + Wy(\omega) - d(\omega))^-)].
  \]
Demand Constraint Specification:

Suppose feasibility is given by aggregate capacity across the network as:

$$
\sum_{i=1}^{n} k_i x_i(x_i) \geq d_0(\omega),
$$

where $k_i$ is the capacity of plant $i$ and $d_0$ is aggregate demand. Assuming independence among plants, the left-hand side is binomial (precisely for $k_i = k_j$ for all $i$ and $j$). Using the normal approximation of the binomial, the left-hand side is then approximately

$$
\mathcal{N}\left(\sum_{i=1}^{n} k_i x_i, \sum_{i=1}^{n} k_i x_i(1 - x_i)\right),
$$

the normal with mean $\sum_{i=1}^{n} k_i x_i$ and variance $(\sum_{i=1}^{n} k_i x_i(1 - x_i))$. This yields a deterministic approximation:

$$
\sum_{i=1}^{n} k_i x_i + z_{1-\alpha} \left(\sum_{i=1}^{n} k_i x_i(1 - x_i)\right)^{0.5} \geq d_0(\omega),
$$

where $z_{1-\alpha} = \Phi^{-1}(1 - \alpha)$ for $\Phi$ the standard normal cumulative.
Additional Relaxation

- Features: typically non-convex (for $z_{1-\alpha} < 0$ or $\alpha > 0.5$), but a semi-definite relaxation can be applied
- Penalty versions (for $\alpha > 0.5$):

\[
\left( \frac{k^T xx^T k}{z_{1-\alpha}^2} \right) - 2 \frac{d_0 k^T x}{z_{1-\alpha}^2} + \frac{d_0^2}{z_{1-\alpha}^2} - (k^T V k) \geq 0,
\]

or, with the second approximation with $Z$,

\[
k^T (Z - z_{1-\alpha}^2 V) k - 2d_0 k^T x + d_0^2 \geq 0,
\]

where now restrictions can be placed on $V$ and $Z$. 
Constraint Specification and Semi-definite Relaxations

- **Form of the constraints:**

  For binary problems with variance matrix, $V \succeq 0$, we can assume that $v_{ij} = 0$ and use a matrix $Z \approx xx^T$, where $z_{ii} = x_i$ (for binary $x$) and use the relaxation $Z \succeq xx^T$ (or equivalently $\begin{pmatrix} 1 & x^T \\ x & Z \end{pmatrix} \succeq 0$), stronger than the linear program relaxation.

  In addition, we might take constraints of the form:

  $$a^T x \geq b,$$

  and re-write them by multiplying to $x_i$ and $(1 - x_i)$ to obtain:

  $$\sum_{j=1}^{n} a_j Z_{ij} \sqcap bZ_{ii}; \sum_{j=1}^{n} a_j (Z_{jj} - Z_{ij}) \sqcap b(1 - Z_{ii}),$$

  to yield tighter inequalities.
Computational Results

Generators: $G_1, G_2, G_3$ - Effect of increasing $z_{1-\alpha}^2$ reliability

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Table: Solution of the NLP relaxation converges to the MILP solution as $z_{1-\alpha}^2 \to \infty$
Additional Modifications

Restrictions on relaxed solutions:

- Decompose demand into equal segments of load, $\delta = d_j, j = 1, \ldots, M$.
- Coordinate generation: $x_{ijk} = \sum_j x_{ijk}$ corresponding to the probability of satisfying demand at $j$ with the $k$th ($\delta$) capacity unit ($k = 1, \ldots, k_i/\delta$) from $i$.
- In this case, if the demand at $j$ must be satisfied with probability 1, then

$$\sum_i \sum_k x_{ijk} = 1$$

for each $j$ and

$$\sum_i \sum_k y_{ijk} \geq d_j = \delta,$$

where $y_{ijk} \leq \delta x_{ijk}$ is the production from $i$ and capacity unit $k$ to meet demand at $j$.

- To ensure that capacity is equally committed

$$\sum_j x_{ijk} - \sum_j x_{ijk'} = 0,$$

for all $i$ and $k$. 
Permutation Matrix Result

- Double-stochastic property
  - Add fictitious demand nodes to absorb all excess supply,
  - Implication:
    \[ \sum_j x_{ijk} = 1 \]
  - so that \( x \) is doubly stochastic (with columns correspond to \( i \) and \( k \) and rows for \( j \)).

- Use Birkhoff-von Neumann Theorem: any doubly stochastic matrix can be written as a convex combination of permutation matrices;

- Result: all solutions correspond to a mixture of feasible dispatches.

- By restricting the reliability level as well, can make these dispatch scenarios compatible with each unit’s overall availability.
Conclusions

- We can obtain stronger relaxations using the nonlinear constraints and semidefinite form

Challenges:
- The general SDP form may be more computationally intensive than previous linear/convex programming based methods
- The overall approximation may not yield close approximations to the original binary form
- Additional constraints may still be required to obtain sufficient reliability
- Prices may still not be able to provide sufficient support for all committed units

Further work:
- Quantify the approximation in the SDP relaxation of the general variance constraint
- Find additional constraint to approximate the conditions and obtain convergence
- Discover minimal representations of prices
- Bound divergence from equilibrium behavior and overall losses of efficiency