Joint Production and Financing Decisions: Modeling and Analysis

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Abstract

This paper develops models to make production and financing decisions simultaneously in the presence of demand uncertainty and market imperfections. While the Modigliani and Miller propositions demonstrate that a firm’s investment and financing decisions can be made independently in a perfect capital market, our models illustrate how a firm’s production decisions are affected by the existence of financial constraints. We analyze the interactions between a firm’s production and financing decisions as a tradeoff between the taxes benefits of debt and financial distress costs. Our numerical examples illustrate that a traditional all-equity manufacturing company can improve its performance significantly by making real and financial decisions together. The results illustrate greater firm value sensitivity to production decisions than to financing decisions and that low-margin producers face significant risk in not coordinating production and financing decisions.

Key Words: Production Decisions; Financial Constraints; Capital structure; Debt Capacity

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1 Introduction

How much should a firm produce and what should be its optimal capital structure? While these two decisions are often independent, the first one made by a production manager or operating officer and the second one under the responsibility of a financial officer, the firm’s production/investment decisions and capital leverage are closely related. The operations management literature, however, tends to focus on the areas of capacity expansion, inventory control, or supply chain management without considering the effects of financial constraints or capital structure on the firm’s operating decisions; on the other hand, while financial economists have long considered the capital structure of a company, they usually assume that the investment or production decisions are exogenously determined. Recently, a growing trend in the operations and financial economic communities aims to unite these two views of the firm by analyzing interactions between production and financing decisions. These studies follow two main directions: a focus in the product-market literature on how capital structure influences a firm’s incentive to compete in the product market, and an emphasis in the market imperfection literature concentrating on the effect of financial leverage on the cost of production, and, hence, the company’s output decisions.

Three major theories appear in the product-market category. According to the strategic commitment theory of Brander and Lewis (1986), a company’s decision to use debt works as a commitment to more aggressive behavior in the product market. Because debt alters the ownership of residual cash-flows, this threat becomes credible, which induces a firm’s un-levered rival to reduce output. A second theory is the long purse or predation argument formalized by Bolton and Scharfstein (1990), wherein rivals may increase output to drive down prices and cause the highly-leveraged firm to exit the industry. Finally, in the industrial equilibrium model of Williams (1995), a firm’s financial contract is jointly determined by various industry characteristics, such as the number of firms, project riskiness, and technologies. Williams’ model supports the coexistence of profitable
firms that are large, make more fixed-capital investment, and use more debt, with smaller firms that are less profitable and use less debt.

The seminal work by Modigliani and Miller (1958) shows that a firm’s investment and financial decisions can be made separately within a perfect capital market. Due to market imperfections, such as taxes, agency costs, and asymmetric information, however, the choice of a firm’s capital structure may in fact be closely related to its production decisions (see Harris and Raviv (1991) for a general review on the theory of capital structure). Discussions of market imperfection effects mainly focus on the tradeoffs among tax benefits, financial distress costs, and the effect of personal tax. Modigliani and Miller’s (1963) traditional tradeoff model indicates that the tax benefit of debt is the tax saving that results from deducting interest from taxable operating income; the prime costs are those associated with financial distress. Miller (1977) points out that the traditional approach ignores personal tax because the interest income of bond-holders is taxed at the personal income level while equity holdings are subject to corporate tax and then personal tax according to realized capital gains.

Hite (1977) made an early study of the interdependence of real and financial decisions taking a market-imperfection approach. He shows that an increase in financial leverage leads to an increase in the optimal output level of the firm in a situation with riskless debt (i.e., no bankruptcy). In the DeAngelo and Masulis (1980) state-preference model, non-debt-related tax shields, such as depreciation deductions, investment tax credits, and asymmetric corporate tax code, are introduced into Miller’s framework. Dotan and Ravid (1985) treat debt as an independent decision variable, include possible bankruptcy, and explicitly model a firm’s investment as a depreciation-related tax shield. They show that high production capacity should be financed by less debt, while increasing debt usage reduces optimal capital investments, thus, reversing Hite’s conclusion. Our paper follows a similar treatment to Dotan and Ravid’s, but we model future demand uncertainty instead of price
uncertainty to place our model in the context of operations models for firms that set prices and
determine production volume before demand is realized.

Several other studies in the literature examine interactions between capital structure and invest-
ment decisions in contingent claims models. Brennan and Schwartz (1984) consider firm valuation
in a setting in which bond covenants restrict financial policy and influence investment policy. Mello
and Parsons (1992) compare the operating decisions of a mine under all-equity financing to those
when the mine is partially debt financed and maximizes leveraged equity value. In Mauer and Tri-
antis (1994)' analysis, the firm has the flexibility to shut down or reopen its production facility in
response to price fluctuations. Although these analysis are valuable in understanding the dependent
mechanism between financing and investment decisions, they usually include strong assumptions
on the firm’s production or financing flexibility, limiting their applicability. For example, Mauer
and Triantis assume the firm continuously produces an infinitely divisible commodity at the rate
of one unit per year if the production facility is in operation, while, in reality, companies make
varying quantitative production decisions according to realized demands.

Recently, several studies in the operations management community have addressed the interface
of production and financial decisions. Among these analysis, Lederer and Singhal (1994) consider
joint financing and technology choices when making manufacturing investments and show that
considerable value can be added to investments through financing decisions. Birge and Zhang
(1999) seek to use option theory to introduce risk into inventory management. In another example,
Birge (2000) adapts contingent claim pricing methods to incorporate risk into a capacity planning
model. Babich and Sobel (2004) examine the relationship between operational decisions and timing
of IPO of a startup firm. The paper by Buzacott and Zhang (2004) also considers the connection
between production and finance for a company with financial constraints. These studies do not,
however, consider optimal capital structure or discuss debt capacity.
This paper addresses the shortcoming of previous analyses by considering a firm’s capital structure in a model which characterizes production input and financial policy as endogenous decision variables; we maintain analytical tractability by limiting decisions to a single period and obtain results on the relationship between production decisions and capital structures choices. While we sacrifice fidelity to the dynamics of investment over time, our models’ consideration of coupled investment decisions and revenue realizations provides insight into practical management decisions.

We find that financial constraints play an important role in determining the firm’s production decisions because the internal cash position of the constrained company may not be able to support its optimal output level. Our research indicates that the constrained firm can improve its profit by issuing debt in the financial market. We further show that the firm’s optimal production decision is negatively related to its financial decisions and that misidentifying the company’s optimal leverage ratio or output decreases firm value; hence, the firm’s production and financial decisions should be made jointly.

We also demonstrate that the firm’s optimal debt level is less than its debt capacity, i.e., it is not optimal to borrow up to the debt limit. Our numerical examples illustrate that a firm’s investment and financing decisions are both sensitive to critical operating parameters, such as profit margin, demand volatility, corporate tax rate, and bankruptcy recovery rate. These sensitivity analyses provide useful managerial insights for different types of companies. The results illustrate, for example, greater firm value sensitivity to production decisions than to financing decisions and that low-margin producers face significant risk in not coordinating production and financing.

The paper is organized as follows. In Section 2 we show that the firm’s production decision does not depend on its capital structure in a perfect capital market. We also find that, without market imperfection, the value of the company is independent of its capital structure as long as its optimal production level can be fully supported. Our debt valuation method is presented in
Section 3. With the assumption of a risk-neutral equivalent measure, we show the existence of debt capacity and illustrate the relationship between credit risk and face value of debt. To fully understand the interactions between investment and capital structure decisions, Section 4 models debt leverage ratio as an independent variable in the presence of market imperfections. Finally, Section 5 contains results and analysis of a numerical example.

2 The simple model

This section first shows the effects of financial constraints on a firm’s production decision, particularly that financial constraints can reduce the value of the firm. The firm can, therefore, improve its performance by raising debt from financial market. We then point out that, without market imperfection, the value of the company is independent of its capital structure as long as its optimal production level can be fully supported.

To analyze the relationships between production and financing decisions, we first employ a simple model, which is essentially a classical news vendor problem with financial constraints \( k \). The assumptions of the model are as follows: the firm is in a quantity competitive industry, makes a single type of product, and only operates for one period within an equivalent risk-neutral world. The stochastic demand \( s \), realized at the end of the operating period, has a risk-neutral equivalent cumulative distribution function \( F \) and density function \( f \). We also assume \( F \) is continuous, differentiable, and strictly increasing. At the beginning of the period, the company produces \( x \) units of product at a constant cost of \( c \) dollars per unit so that \( x \) is used as the capacity constraint on production. The firm then sells \( \min(x, s) \) units of product at a fixed price \( p \geq c(1+r_f) \), where \( r_f \) is the risk-free interest rate, and then liquidates the remaining inventory. To simplify the problem, we assume the firm produces perishable or fashion goods with no salvage value.

The risk-neutral equivalence assumption represents a transformation from a nominal probabil-
ity distribution that is usually considered in studies of the news vendor model and extensions. By transforming to a risk-neutral measure, the optimal production decision for the all-equity firm considers market risk and is equivalent to an evaluation with the capital asset-pricing model (CAPM). Singhal (1998) derives conditions for the all-equity news vendor decision under the risk-neutral measure or CAPM in a general case. Birge and Zhang (2000) give an explicit solution to the all-equity news vendor decision under risk-neutral equivalence and log-normal demand distribution and show that this solution equals the standard news vendor solution under the original demand distribution times the market premium for the risk of the product’s overall market.

To find the optimal production decision, $x^*$, we have

$$
\begin{align*}
\text{maximize} & \quad p \left( \int_0^x s \, dF(s) + x \int_x^\infty dF(s) \right) - cx(1 + r_f) \\
\text{subject to} & \quad 0 \leq cx \leq k,
\end{align*}
$$

(1)

Let $\hat{x}$ be the solution to $F(x) = \frac{p - c(1 + r_f)}{p}$, then the optimal production policy for the financially constrained company is $\hat{x} = \min (k/c, \hat{x})$. If $k/c < \hat{x}$, the financial constraint is redundant and has no effect on the firm’s production decision; however, if internal cash is not sufficient to support optimal production, i.e., $k/c > \hat{x}$, the firm cannot achieve its optimal output level and incurs a loss in market value. Model (1) illustrates that the firm’s financial situation may affect its production by capping its initial investment. Depending on the tightness of the constraints, the effects of the financial constraints can be significant.

In reality, many companies face financial constraints and raise funds to support production in multiple ways. In our case, we assume the firm can issue a single homogeneous class of zero-coupon discount bond with price $D$ and interest rate $r$. Let $s^b = \frac{D(1 + r)}{p}$ be the amount of demand for which the end-of-period revenues are just sufficient to cover promised payments to bondholders. If demand falls below $s^b$, the company is forced into bankruptcy where the payoff to debt-holders is $ps$; otherwise, the company pays debt in full. Under the risk-neutral equivalence assumption, the
investors are indifferent to the risk. The payoff of the risk-free account and the zero coupon bond are identical; hence, the interest rate of the bond is given by the following equation:

\[ D(1 + r_f) = \int_0^b ps \, dF(s) + D(1 + r) \int_{s_b}^\infty dF(s). \]

The following model then gives the optimal production decision including financial considerations:

\[
\begin{align*}
\text{maximize} & \quad p \left( \int_0^x s \, dF(s) + x \int_x^\infty dF(s) \right) - cx(1 + r_f) \\
\text{subject to} & \quad 0 \leq cx \leq k + (1 + r_f)^{-1} \left( \int_0^{s_b} ps \, dF(s) + D(1 + r) \int_{s_b}^\infty dF(s) \right),
\end{align*}
\]

where \( k \) represents the initial cash position of the company.

If the firm’s optimal investment level \( cx^* \) is less than its initial cash position \( k \), the above model is simply the traditional news vendor problem. From now on, we assume the firm’s internal cash position is not sufficient to support its optimal investment, i.e., \( cx^* > k \). Because of interest cost, the financial constraint in Model (2) is tight. Denote \( V \) as the value of the firm. Substituting \((cx - k)(1 + r_f) = \int_0^{s_b} ps \, dF(s) + D(1 + r) \int_{s_b}^\infty dF(s)\) into the objective function and rearranging terms, (2) becomes,

\[
\begin{align*}
\text{maximize} & \quad V(x, L) = \int_x^{s_b} (ps - L) \, dF(s) + (px - L) \int_x^\infty dF(s), \\
\text{subject to} & \quad \text{nonnegative constraints on } x \text{ and } L \text{ respectively, where } L = D(1 + r) \text{ is the face value of debt.}
\end{align*}
\]

Substituting \( dx/dL = \frac{1}{c(1 + r_f)} \int_{s_b}^\infty dF(s) \) into \( dV/dL = \int_x^\infty (px - 1) \, dF(s) \), we have

\[
dV/dL = \left( \frac{p}{c(1 + r_f)} \int_x^\infty dF(s) - 1 \right) \int_{s_b}^\infty dF(s).
\]

From (3), we have that the capital structure of the company has no influence on the production decision in a perfect market (as also follows from the Modigliani-Miller (MM) theory).

**Proposition 2.1** In a perfect market, the firm’s production decision is independent of its capital structure whenever its optimal investment level can be fully financed. The optimal production and
financial decisions have the following properties:

(1) the single period production policy is to produce up to level \( x^* \) without considering financial constraints, i.e., \( x^* = F^{-1} \left( \frac{p - c(1 + r_f)}{p} \right) \);

(2) the optimal amount of debt raised is \( \max(0, cx^* - k) \).

3 Debt Valuation and Corporate Debt Capacity

The conclusions in Proposition 2.1 only hold in perfect capital markets. This section shows the effects of bankruptcy costs on debt pricing. If the firm decides to finance part of its investment by selling corporate debt at time zero, the end-of-period market value of the debt is uncertain because it depends on the market demand for the firm’s product. If operating income is insufficient to repay the debt, debt-holders take ownership of the firm, pay bankruptcy costs, and acquire the residual value of the company. Bankruptcy costs include administrative expenses, such as fees paid to lawyers, trustees, auctioneers, and accountants, and indirect costs due to financial distress.

Similar to Leland (1994), our paper takes a proportional form with bankruptcy cost represented as \((1 - \alpha) ps \ \forall \ s < s^b\), where \( s^b = L/p \) is the bankruptcy point in terms of sales and \( 0 < \alpha < 1 \) represents the asset recovery rate after bankruptcy. If bankruptcy occurs, a fraction \( 1 - \alpha \) of the operating income represents the loss due to bankruptcy costs. From the analysis above and our bankruptcy form, the end-of-period payoff to debt-holders is:

\[
Y_D(x, D) = \begin{cases} 
D(1 + r(D)) & \text{if } s \geq s^b, \\
\alpha ps & \text{if } s^b > s,
\end{cases}
\]

where \( r(D) \) is the nominal interest charged by debt-holders for lending \( D \); this rate depends on the risk characteristics of the market, such as the demand distribution, the profit margin, and the amount of debt. Because of market uncertainty and bankruptcy costs, the debtholders’ actual income may be less than the firm’s promised payment. Following Dotan and Ravid (1985), we
again assume an equivalent risk-neutral measure of future demand, so that we can analyze optimal
decisions as if the firm is in a risk-neutral world; hence, the interest rate paid to bondholder
must guarantee that the expected payment equals the return obtained at the risk-free rate, i.e.,
\[ E(Y_D) = D(1 + r_f). \]
The explicit form is
\[ D(1 + r_f) = D(1 + r) \int_{s^b}^{\infty} f(s)ds + \alpha \int_{s^b}^{\infty} psf(s)ds. \] (4)
We also assume that, in the single period model, no additional debt can be issued; thus, Equation (4)
effectively prevents stock-holders from transferring wealth away from debt-holders and guarantees
that bond-holders’ rights are not violated.

The debt-raising ability of a company depends on the willingness of debt-holders to extend
credit. Given the existence of bankruptcy costs, the lenders require an expected rate of return to
compensate for the risk of default and other associated costs. After the debt amount reaches a
certain level, the risk premium increment cannot provide a market equilibrium return because of
the company’s profitability characteristics. Since other investment opportunities are available in
the capital market, lenders invest no more than the maximum level.

Corporate debt capacity is defined as this maximum amount that a firm can borrow. Unless
the firm has already reached its debt capacity, the firm can borrow more by promising to pay
more to lenders. While a firm may attempt to increase its debt usage by increasing the promised
future repayment, debt capacity can be reached where increasing the promised repayment does not
increase the market value of debt. In other words, once the company reaches its debt capacity,
it can borrow no more regardless of how much it promises to pay. Let \( \bar{D} \) denote corporate debt
capacity and \( \bar{L} = \bar{D}(1 + \bar{r}(\bar{D})) \) represent the amount the firm promises its debt holders to reach \( \bar{D} \).
The existence of debt capacity means that \( \frac{d^2D}{dL^2} \bigg|_{L=\bar{L}} < 0 \) and \( \frac{dD}{dL} \bigg|_{L=\bar{L}} = 0 \); hence, debt capacity
can be determined by setting \( dD/dL = 0 \), and solving for \( D \) and \( L \). Conditioning on the demand
distribution function, the following proposition shows the existence of debt capacity \( \bar{D} \).
Proposition 3.1 If the product market demand distribution function satisfies
\[ \frac{s^b f'(s^b)}{f(s^b)} < \frac{2 - \alpha}{1 - \alpha}, \]
where \(s^b\) solves \(\frac{dD}{dL} = 0\), then there exists a finite debt capacity for the firm.

Proposition (3.1) indicates that once the company reaches its debt capacity, the market value of debt \(D\) is a decreasing function of \(L\). An intuitive explanation for this property is that, before the firm reach its debt capacity, the debt-holders can pass on the entire bankruptcy cost to stockholders by charging an appropriate risk premium. Once the debt level passes \(\bar{D}\), the return from this firm becomes less than the market equilibrium return; therefore, the promised high interest from equity holders is not sufficient to compensate for the debt-holders’ loss due to over-investment. Note also that at the point where the debt capacity is reached, the marginal contribution of an additional unit of promised payment is balanced by the marginal bankruptcy cost.

The interest rate charged by debt-holders and the present market value of debt, must satisfy Equation (4). The following Proposition 3.2 characterizes the relationship between \(r(D)\) and \(D\).

Proposition 3.2 The interest rate charged by the debt-holders, \(r(D)\), is a monotone increasing function of the present value of the debt \(D\).

4 Investment & Capital Structure in an Imperfect Market

To analyze the effects of market imperfections, such as tax and bankruptcy costs, on the firm’s production and financial decisions, we assume corporate profits are taxed at a constant rate \(\tau\) and the debt payments are fully deductible in computing taxable corporate income. If the company’s internal cash is not sufficient to support its optimal production level \(x\), the firm can borrow at an interest rate \(r(D)\), which is a function of the face value of debt borrowed. The firm’s taxable income is \(\max[0, ps - cx - rD]\), where \(s\) is the realization of the demand. Since the company only operates for a single period, we assume gains are taxed at a constant rate \(\tau\), while all tax losses are
not allowed for tax carry-backs or carry-forwards.

The face value of the debt must be fully paid at the end of the period, except in the case of bankruptcy when, as before, all assets are sold and proceeds are distributed to creditors minus a proportional bankruptcy cost. The cash flow to the equity holder is then:

\[
Y_E(x, D) = \begin{cases} 
px - \tau(px - cx - rD) - D(1 + r) & \text{if } x \leq s, \\
ps - \tau(ps - cx - rD) - D(1 + r) & \text{if } s^* \leq s < x, \\
ps - D(1 + r) & \text{if } s^b \leq s < s^*,
\end{cases}
\]

where \( x \) is the production capacity, \( s^* = \frac{cx + rD}{p} \), is the amount of demand for which accounting income equals zero. If the realized demand \( s \) is greater than the break-even point \( s^* \), the company’s operating income is taxed at a constant rate \( \tau \). For \( s^b = \frac{D(1 + r)}{p} \), the bankruptcy point, if demand falls below \( s^b \), the company is forced into bankruptcy and a bankruptcy cost \((1 - \alpha)ps\) is charged.

The debt-holders’ cash flow, \( Y_D \), is the same as in Section 3.

Note that the equity-holders cannot transfer wealth from the debt-holders by the above constraint, maximizing the value of the firm or the value of the equity yields the same results. We can compute the expected future value of the firm at the end of the period by combining the expected values for debt-holders and equity-holders together and deducting the initial investment, i.e., \( V = E(Y_E) + E(Y_D) - cx(1 + r_f) \). The model which maximizes the firm’s value is then:

\[
\text{maximize } V(x, D) = \int_x^{\infty} (px - \tau(px - cx - rD)) f(s) \, ds \\
+ \int_x^{x} (ps - \tau(ps - cx - rD)) f(s) \, ds \\
+ \int_{s^*}^{s^*} ps f(s) \, ds + \alpha \int_0^{s^b} ps f(s) \, ds - cx(1 + r_f), \\
\text{subject to } D(1 + r_f) = D(1 + r)[1 - F(s^b)] + \alpha \int_0^{s^b} ps f(s) \, ds,
\]

\( 0 \leq D \leq cx. \)

Model (5) provides an initial explanation for why we need to make production and financial decisions at the same time. The expected value of the firm depends on both decisions \( x \) and \( D \).
following computation gives the optimal decisions the company should take and the relationship between these two decision variables. Setting derivatives to zero, we first obtain:

\[ 0 = \frac{\partial V}{\partial x} = p(1 - \tau) \int_{x}^{\infty} f(s)ds + c\tau \int_{s^*}^{\infty} f(s)ds - c(1 + r_f). \]  

(6)

The interpretation of equation (6) is that the expected profit of additional capacity \( p(1 - \tau)[1 - F(x)] + c\tau[1 - F(s^*)] \), i.e., the sum of the marginal after tax revenue plus the marginal tax benefit, equals the marginal cost \( c(1 + r_f) \) at time \( T \). We find that the level of debt affects capacity choice because of its impact on the break-even level and, hence, on the probability that debt tax benefits will in fact be used. We can show that, 

\[ \frac{\partial s^*}{\partial D} = \frac{1}{p} \left( r + D \frac{\partial r}{\partial D} \right) \geq 0, \]

which indicates that, as more debt is taken on, \( s^* \) increases and, consequently, the expected marginal production cost increases.

Equation (6) indicates that the optimal production decision should be achieved at the point where marginal production cost equals marginal profit. Since we assume constant marginal cost, the optimal decision is determined by the sum of the marginal after tax revenue and the marginal tax benefit. Both of these are decreasing functions of the production decision \( x \) because an additional capacity unit results in a higher chance of salvage loss and a higher break-even sales level. Notice that the effect of capital structure is illustrated by the marginal tax benefit \( c\tau[1 - F(s^*)] \). Since \( s^* = \frac{cx + rD}{p} \), a higher debt level results in a smaller tax benefit. For the overall effect of debt on firm value, we consider:

\[ \frac{\partial V}{\partial D} = \tau \left( r + D \frac{d r}{d D} \right) \int_{s^*}^{\infty} f(s)ds - (1 - \alpha) \left( 1 + r + D \frac{d r}{d D} \right) s^b f(s^b). \]

Let \( L = D(1 + r) \), we can rewrite \( \frac{\partial V}{\partial D} \) less explicitly and obtain the first order condition:

\[ 0 = \frac{\partial V}{\partial D} = \tau[1 - F(s^*)] \left( \frac{d L}{d D} - 1 \right) - (1 - \alpha)s^b f(s^b) \frac{d L}{d D}. \]  

(7)

An explanation for this equation is that, at optimality, the change in the expected bankruptcy cost is balanced by the expected tax shield benefit of an additional unit of debt. In the extreme
situation of no bankruptcy cost, i.e., $\alpha = 1$, Equation (7) cannot be satisfied; the firm’s optimal decisions are then achieved at an extreme point; the demand amounts corresponding to the bankruptcy and break-even points, \( \frac{c x (1 + r)}{p} \), become identical. Because debt provides a tax-shield for operating income, the firm’s optimal capital structure is all-debt financing.

Equations (6) and (7) indicate that the firm’s production decision and capital structure policy must be made jointly since the break-even point \( s^* \), which is a function of the investment decision, \( x \), and the financing decision, \( D \), appears in both equations. Comparing (3) and (6) we find that the optimal production decision of an all-equity company is a special case of a levered company with the all-equity financing constraint. For the concept of optimal capital structure to be meaningful, it is necessary to show that the firm’s optimal debt level, \( D^* \), is less than its debt capacity \( \bar{D} \). The following proposition gives this result.

**Proposition 4.1** The debt level corresponding to the firm’s optimal capital structure is less than its debt capacity.

**Proposition 4.2** For a firm operating in a market with taxes and bankruptcy cost, the production and financing decisions are interdependent;

(a) the optimal production decision is a decreasing function of financial leverage;

(b) the simultaneous optimal production and financial decisions yield optimal debt \( D^* \) with \( D^* > D^E \) for \( D^E \) the optimal financial decision corresponding to the optimal production level \( x^E \) of an all-equity company.

Proposition (4.2) shows that the production and financial decisions of the firm are inseparable, indicating that the production decisions of an all-equity firm are different from that of a levered firm. Increasing the debt level results in a higher break-even demand realization that decreases the tax shield. Consequently, the optimal production level decreases due to the increase in marginal
production cost. On the other hand, it is not optimal for the company to operate as an all-equity company because of the tax-shield benefit. A small increase in debt level leads to an increase in the cost of capital, and hence, a decrease in production. The optimal decisions are achieved at the point where the marginal benefit of the tax shield, which occurs over the firm’s positive income states of nature, plus the expected marginal profit of additional capacity, are equal to the marginal production cost.

The following proposition discusses the conditions under which the firm has an optimal production and financial solution.

**Proposition 4.3** A solution \((x^*, D^*)\) of Equations (6) and (7) that satisfies \(f'(s^b) > 0\) and 
\[
\frac{1 - \alpha}{2 - \alpha} > \tau
\]
is an optimal solution to (5).

The following proposition gives the effects of corporate tax rate changes on the firm’s optimal production and debt decisions.

**Proposition 4.4** If the conditions in Proposition 4.3 are satisfied, the optimal production decision, \(x^*\), is negatively correlated with the corporate tax rate, \(\tau\); while the optimal debt decision, \(D^*\), is positive correlated with \(\tau\).

### 5 Analysis of the Model

This section presents a comparative statics analysis of the interactions between production and capital structure decisions. We first examine the relationship between these two decisions by conducting a sensitivity analysis for different combinations of operating and environmental variables. We then investigate the effects of misidentifying optimal production and financial leverage decisions on the profit of the company. Our goal is to determine under what conditions making simultaneous operational and financial decisions is most critical.
We consider misidentification of production and financing decisions to determine the impact of not following the optimal policies. We find that the value of the company is a convex function of debt usage and production level. Deviating from optimal production and financial decisions can incur significant losses to the company; therefore, traditional operations management models, such as capacity planning, inventory management, or supply chain management models, that seldom consider the effects of capital structure, tend to undervalue the company or project. Our results also suggest that misidentifying production levels generally has a more significant impact on firm value than misidentifying debt levels but that low-margin firms may suffer significant value loss from misidentifying capital structure.

We choose parameters that are roughly consistent with that of a commodity manufacturing company. For the base case parameters, we assume the selling price, \( p \), of one unit is \$1, and the production cost, expressed as a fraction of \( p \), is \$0.60. The industrial average tax rate, \( \tau \), and risk free interest rate, \( r_f \), are initially set at 35% and 5% respectively. We also assume the market demand of the product approximately follows a log-normal distribution. We suppose the current market demand is 1000 units, the expected market growth rate and volatility are 10% and 40% per year respectively. We also assume the base case debt recovery rate is 30%.

5.1 Sensitivity analysis of optimal production and financial decisions

Our analysis of the interactions between operating decisions and financing policies mainly concentrates on production output level, \( x \), and market leverage ratio, \( D/V \). The market value of the company, \( V \), equals the discounted value of expected future cash flow minus the initial investment. We find that the firm’s optimal leverage ratio is negatively related to its optimal production decisions, which is consistent with our previous analysis in Section 4.

Figure 1 shows financial leverage as a function of production cost \( c \) for different market demand
uncertainty levels. Clearly as the firm’s marginal production cost increases, its optimal leverage ratio increases and optimal production level decreases. This observation suggests that a low-margin company should take a conservative production decision and an aggressive financial decision, while a high-margin company should take aggressive production decisions and conservative debt policy. Because lower margin means higher production cost, the firm’s production output level decreases with decreasing margin. For a company facing uncertain demand, a lower production level decreases the risk of the future cash flow; therefore, the debt holder charges a smaller risk premium to compensate for bankruptcy risk. To take advantage of a lower cost of debt, the firm prefers to use more debt in its capital structure.

For different demand volatility levels, Figure 1 illustrates that both a firm’s investment and financial leverage decisions are negatively correlated with market uncertainty. A rise in the demand volatility causes the probability of bankruptcy to increase; the price of debt therefore rises. As debt becomes more costly, the firm lowers its investment level, hence, decreasing the risk of the cash flow. A negative relationship, therefore, should exist between demand volatility and the firm’s output. Another observation from Figure 1 is that the optimal leverage ratio increases as the market uncertainty diminishes. It suggests that a firm with less volatile cash flow is likely to have a smaller chance of bankruptcy. This pattern confirms the hypothesis of the trade-off theory that firms with less variable earnings have more leverage, as also observed in the empirical work by
5.2 Effects of Mis-specifying Debt & Production Decisions on Firm Valuation

The results in Section 4 indicate that production/investment and financing decisions should be made simultaneously to maximize firm value. Financial leverage decisions can affect investment decisions because debt financing provides a deductible tax shield for operating income and may incur financial distress costs in the case of bankruptcy, thus altering cash flow. We consider the following questions: “How much can the firm increase value by making production and financing decision simultaneously? Is this increase significant?” The following analysis explores these questions by examining the effects of mis-specifying capital structure and production output level on the value of the company.

To analyze the significance of misidentification, we compare the net profit of mis-specified decisions with that corresponding to optimal investment and capital structure decisions. By changing the value of the production cost, demand volatility, bankruptcy recovery rate, and corporate tax rate, we observe the sensitivity of the misidentification effects. For simplicity of comparison, we define the normalized net income, $I(x, l)/I(x^*, l^*)$, as the ratio between the net income associated with a certain production and debt leverage decision pair $(x, l)$, and that corresponding to the optimal decision pair $(x^*, l^*)$. Clearly, the normalized net income ranges from 0 to 1 with maximize value achieved at the optimal decision point. To identify the mis-specification effects of the two contributor factors separately, we analyze the sensitivity of value on the production decision and the debt decision respectively.

We first analyze the effects of debt mis-identification on the value of the company by keeping the production decision at an optimal level. Figure 2 plots the normalized firm value as a function of book leverage ratio for different operating parameters. The sensitivities of the mis-specification
effects of production cost, $c$, demand volatility, $\sigma$, bankruptcy recovery rate, $\alpha$, and corporate income tax rate, $\tau$, are illustrated by Panels A, B, C, and D respectively, controlling all other parameters to be the same as in the base case.

Observe that the firm’s value is a convex function of financial leverage, suggesting the existence of an optimal capital structure as we observed analytically. An under-leveraged firm can increase its profit by raising more debt, taking the benefits of the tax shield. Once the debt usage crosses the optimal leverage ratio, the cost incurred by financial distress cannot be balanced by the tax benefit; so, the company’s value begins to decrease. As illustrated by Panel A, if the production cost is 80% of the selling price, an all-equity financed company can increase its normalized value from 0.9 to 1 by raising debt usage to its optimal leverage ratio, 0.68; however, if the debt leverage rises above the optimal capital structure, more debt issue actually reduces firm value. In this case, one hundred percent debt financing only earns 82% of the value with optimal capital structure.

The above analysis indicates that mis-specifying debt leverage can incur significant losses to the company.
An interesting observation from Figure 2 is that the effect of over-leverage is more severe than under-leverage. The over-leveraged firm faces a higher chance of bankruptcy, which has two effects on the firm value: first, raising financial distress cost; second, decreasing the profit margin, resulting in lower output. Both effects contribute negatively to the performance of the company. Although an under-leveraged firm does not fully take advantage of potential debt service deductibility, the over-leveraged company faces more severe financial distress loss; therefore, for the same ratio of leverage deviation, the effect of over-leverage is more significant than under-leverage. With longer term debt available for lower issuing cost, this observation suggests that firms should have lower leverage ratios than given here to protect against possible future financial distress, which is also consistent with empirical results.

Figure 2 also indicates that the effects of mis-specifying leverage ratios are sensitive to changes in $c$, $\sigma$, $\alpha$, and $\tau$. For example, Panel A indicates that the mis-specification effect becomes more significant as production cost increases. When $c$ is 60% of the selling price, the firm faces a 3% and 7% value loss by taking an all-equity or all-debt financing capital structure respectively. If the production cost accounts for 80% of the price, the value gap between the all-equity financed and the optimally leveraged company rises to 10%, while the all-debt financed company faces an 18% value loss. For a company with high margins, the contribution of the financing decision to the value of the company is small; the high-margin firm’s manager does not need to pay significant attention to the capital structure of the company (relative to the need for careful management of operational decisions). Increasing the production cost lowers the firm’s profitability, hence, increasing the probability of bankruptcy. Both effects make the tax shield play an more important role in asset return; therefore, decreasing the profit margin increases the significance of debt mis-specification.

Another observation from Figure 2 is that, for an under-leveraged company, a decrease in profit margin, market volatility, or an increase in bankruptcy recovery rate or tax rate leads to
a larger value loss; for an over-leveraged company, an increase in market volatility, or a decrease in bankruptcy rate or corporate tax rate results in higher loss. These observations suggest that, to avoid misidentification effects, a company with stable cash flow, low margin, a large volume of fixed assets, and high tax rate, prefers a high debt leverage ratio; a firm with uncertain cash flow, profitable product, small fixed asset, and lower corporate tax rate should tend to use conservative debt policy. This observation is consistent with Graham and Harvey (2001) who find that tax advantage is more important for large, regulated, and dividend-paying companies and that those companies usually have high tax rates and large tax incentives to use debt.

The above analysis suggests that optimal capital structure decision may increase a firm’s asset return and vice versa. Clearly, the company’s initial production decision also plays an important role in determining the cash flow of the company. Corresponding to the previous discussion, Figure 3 displays the firm value as a function of proportional production levels for different production costs, $c$, bankruptcy recovery rates, $\alpha$, and corporate tax rates, $\tau$. To facilitate comparison, the production levels, $x$, are standardized with respect to the optimal production levels, $x^*$, which together with the optimal debt financing decision maximize the value of the company. To identify the effect of production decisions, we hold the debt decision at the optimal leverage ratio.

We summarize our observations from Figure 3 concisely. All four panels illustrate that the value of the company is a concave function of the production level, which suggests that mis-specifying the production decision incurs a loss. Notice also that the firm’s value function is much more sensitive to the change of the production level compared to the previous analysis for the debt decision. In other words, mis-specifying the optimal production decision incurs a higher loss than mis-specifying optimal debt leverage. For example, Panel A illustrates that, when the production cost equals $0.60, if the company’s investment decision decreases to 50% of the optimal production level, the firm’s value is just 64% of that of the optimal decision; while the maximum loss incurred
Figure 3: Effects of output level mis-specification on the normalized net income

by debt mis-specification is only 7% of the optimal profit. The same pattern is repeated in the other three panels. Another observation is that the firm’s value is more sensitive to operating parameters: production cost or market demand volatility, compared to financial parameters, such as bankruptcy recovery rate and corporate tax rate.

6 Conclusions

In this article, we developed models to determine production and financing decisions simultaneously assuming conditions for risk-neutral equivalent distributions. We analyzed the interactions between these sets of decisions and showed when the interactions are most significant. While MM theory demonstrates that, in a perfect capital market, the value of a firm does not depend on its capital structure, and consequently that investment and financial decisions can be made separately, our research provides a characterization of how financial constraints may still affect a firm’s production decisions by capping its maximum output level. Companies with internal financial constraints can improve performance (in terms of maximum value or equity return) by appropriately considering
debt (and the resulting tax shield) along with production quantities.

We find how market imperfections can negatively affect a firm’s production decision and its value. We also find that optimal production decisions are negatively correlated with the optimal debt-to-market-value leverage ratio. An increase in this factor, which contributes to higher net margins, leads to greater production levels and lower market-value leverage ratios. Our numerical results illustrate that both production and financial decisions are sensitive to changing operating and environmental variables, indicating that there may exist large differences in output and capital structure even in the same industries due to differences in those exogenous factors. Our sensitivity analyses also demonstrate the importance of joint production and financial decisions; we observe that mis-specification losses are most severe for production decisions (relative to financing decisions) and, in general, for low-margin compared to high-margin firms.

The models presented here can be extended in a number of ways. To consider the effects of a firm’s growth opportunities and the stochastic properties of the product and financial markets, one possible direction is to model the interactions in a multiple period environment. Since we only consider the interface of financial and production decisions inside a company while ignoring the effects of competition in the industry, another possible approach might be to look at optimal financing for a company to compete with other firms in the industry and to show how to coordinate financing and operations with both up-stream suppliers and down-stream buyers. These considerations remain challenges for future research.

Appendix

Proof of Proposition 3.1 Define $L = D(1 + r)$ as the face value of debt, also let

$$ G(D, L) = L \int_{L/p}^{\infty} f(s) ds + \alpha \int_{0}^{L/p} psf(s) - D(1 + rf). $$  

(8)
Differentiating (8) with respect to $D$ and $L$ yields \( \frac{\partial G}{\partial D} = -(1 + r_f) \), and
\[
\frac{\partial G}{\partial L} = \int_{L/p}^{\infty} f(s)ds - (1 - \alpha) \frac{L}{p} f(L/p).
\]
Substituting the above partial derivatives into \( \frac{dD}{dL} = -\frac{\partial G}{\partial L} / \frac{\partial G}{\partial D} \) gives
\[
\frac{dD}{dL} = \frac{1}{1 + r_f} \left( \int_{L/p}^{\infty} f(s)ds - (1 - \alpha) \frac{L}{p} f(L/p) \right).
\] (9)

To show that the second-order condition is satisfied, we find all the second order derivative terms of $G$ are zero except \( \frac{\partial^2 G}{\partial L^2} = -\frac{2 - \alpha}{p} f(L/p) - \frac{L(1 - \alpha)}{p^2} f'(L/p) \). Substituting the second order derivative terms and (9) into
\[
\frac{d^2D}{dL^2} = -\left( \frac{\partial^2 G}{\partial L^4} + \left( \frac{\partial^2 G}{\partial L\partial D} + \frac{\partial^2 G}{\partial D\partial L} \right) \frac{dD}{dL} + \frac{\partial^2 G}{\partial D^2} \left( \frac{dD}{dL} \right)^2 \right) \frac{\partial G}{\partial D},
\]
and reorganizing items yields \( \frac{d^2D}{dL^2} = -\frac{1}{p(1 + r_f)} \left( (2 - \alpha)f(L/p) + \frac{L(1 - \alpha)}{p} f'(L/p) \right) \). Since
\[
\frac{s^b f'(s^b)}{f(s^b)} < \frac{2 - \alpha}{1 - \alpha},
\]
we have \( \frac{d^2D}{dL^2} < 0 \) which shows the existence of the firm’s debt capacity. \( \blacksquare \)

**PROOF of PROPOSITION 3.2** To prove \( \frac{dr}{dD} > 0 \), let \( G(D, r(D)) = D(1 + r) \int_{s^b}^{\infty} f(s)ds + \alpha \int_{s^b}^{\infty} ps f(s) - D(1 + r_f) \). Taking partial derivatives with respect to $D$ and $r$, we obtain:
\[
\frac{\partial G}{\partial D} = (1 + r) \int_{s^b}^{\infty} f(s)ds - \frac{D(1 + r)^2(1 - \alpha)}{p} f(s^b) - (1 + r_f), \quad \text{and}
\]
\[
\frac{\partial G}{\partial r} = D \int_{s^b}^{\infty} f(s)ds - \frac{D^2(1 + r)(1 - \alpha)}{p} f(s^b).
\]

Substitute the above partial derivatives into \( \frac{dr}{dD} = -\frac{\partial G}{\partial D} / \frac{\partial G}{\partial r} \), we have
\[
\frac{dr}{dD} = \frac{1 + r_f}{D \int_{s^b}^{\infty} f(s)ds - \frac{D^2(1 + r)(1 - \alpha)}{p} f(s^b)} - \frac{1 + r}{D}. \] (11)

To show \( \frac{dr}{dD} > 0 \), we only need \( 1 + r_f > (1 + r) \int_{s^b}^{\infty} f(s)ds - \frac{D(1 + r)^2(1 - \alpha)}{p} f(s^b) \). Notice that
\[
1 \geq \alpha \geq 0 \quad \text{and} \quad 1 + r_f > (1 + r) \int_{s^b}^{\infty} f(s)ds \text{ from Equation (4)}; \quad \text{it follows that} \quad \frac{dr}{dD} > 0. \] \( \blacksquare \)
PROOF of PROPOSITION 4.1 Let $D^\ast$ be the debt level corresponding to the optimal capital structure satisfying $\partial V/\partial D^\ast = 0$. $\partial V/\partial D$ is given by (7). Reorganizing terms, we have:

$$\tau \gamma [1 - F(s^\ast_1)] - (1 - \alpha) s^b_1 f(s^b_1) = 0,$$

where $\gamma = \frac{dL_1/dD - 1}{dL_1/dD}$, $L_1 = D^\ast[1 + r(D^\ast)]$, while $s^b_1 = \frac{D^\ast[1 + r(D^\ast)]}{p}$ and $s^\ast_1 = \frac{c + r(D^\ast)D^\ast}{p}$ are the bankruptcy point and break-even point associated with $D^\ast$. From (9), the company’s debt capacity $\bar{D}$ solves

$$1 - F(s^b_2) - (1 - \alpha) s^b_2 f(s^b_2) = 0,$$

where $s^b_2 = \frac{\bar{D}[1 + r(\bar{D})]}{p}$ is the bankruptcy point corresponding to debt capacity $\bar{D}$.

From Proposition 3.2, we know $\frac{d}{dD} r > 0$. Therefore, $\frac{d}{dD} L_1 = r + D^\ast \frac{d}{dD} r > 0$ leads to $0 < \tau \gamma < \tau < 1$. Comparing (12) and (13), it is easy to show that $s^\ast_1 < s^b_1$. Since $s^b_1 < s^\ast_1$, we have $s^b_1 < s^b_2$. Notice that $r(D)$ is a monotone increasing function of $D$, so that $D^\ast < \bar{D}$, i.e., the optimal capital structure involves less debt financing than the company’s debt capacity.

PROOF of PROPOSITION 4.2 (a) Let $x(D)$ denote the optimal value of production decision $x$ for a given debt level $D$. Total differentiation of $x^\ast(D)$ with respect to $D$ yields $\frac{dx^\ast}{dD} = -\frac{\partial^2 V}{\partial x \partial D} / \frac{\partial V^2}{\partial x^2}$. From (6), we have

$$\frac{\partial^2 V}{\partial x^2} = -p(1 - \tau)f(x) - \frac{c^2 \tau}{p} f(s^\ast) < 0,$$

and

$$\frac{\partial^2 V}{\partial x \partial D} = -c \tau f(s^\ast) \frac{\partial s^\ast}{\partial D} < 0,$$

since $\frac{\partial s^\ast}{\partial D} > 0$. Substituting (14) and (15) into $\frac{dx^\ast}{dD}$, it is clear that $\frac{dx^\ast}{dD} < 0$.

(b) Assume $x^E$ to be the optimal investment level of an all-equity company. The corresponding optimal debt amount $D^E$ is derived for this certain investment level $x^E$ by Equation (7). Let $x^\ast$, $D^\ast$ be the optimal decisions by solving (6) and (7) simultaneously. From (a), the firm determines
an optimal \( x^* < x^E \) since \( D^* \geq 0 \). To satisfy the first order conditions given by Equation (6), the firm must increase its debt level, i.e., \( D^* > D^E \).

**PROOF of PROPOSITION 4.3** From Equation (14), we have \( \frac{\partial^2 V}{\partial x^2} < 0 \); therefore, to demonstrate that the second order condition is satisfied, it is enough to show \( |H| = \frac{\partial^2 V}{\partial D^2} \frac{\partial^2 V}{\partial x \partial D} > 0 \). Taking cross-derivatives with respect to \( x \), \( D \), and \( x \) yields:

\[
\frac{\partial^2 V}{\partial x \partial D} = \frac{\partial^2 V}{\partial D \partial x} = -\frac{c\tau}{p} \left( \frac{dL}{dD} - 1 \right) f(s^*) < 0,
\]

where \( L = D(1 + r) \) is the face value of debt. The second order derivative with respect to \( D \) is given by:

\[
\frac{\partial^2 V}{\partial D^2} = -\frac{d^2 L}{dD^2} \left( (1 - \alpha)s^b f(s^b) - \tau[1 - F(s^*)] \right)
- \frac{\tau}{p} \left( \frac{dL}{dD} - 1 \right)^2 f(s^*) - \frac{1 - \alpha}{p} \left( \frac{dL}{dD} \right)^2 \left( f(s^b) + s^b f'(s^b) \right).
\]

To show \( |H| > 0 \), it is enough to show that

\[
\frac{d^2 L}{dD^2} \left( \tau[1 - F(s^*)] - (1 - \alpha)s^b f(s^b) \right) < \frac{1 - \alpha}{p} \left( \frac{dL}{dD} \right)^2 \left( f(s^b) + s^b f'(s^b) \right).
\]

From Equation (9), we know

\[
\frac{dL}{dD} = \int_{\frac{L}{p}}^{\infty} f(s)ds - (1 - \alpha) \frac{L}{p} f(\frac{L}{p})
\]

We also have

\[
\frac{d^2 L}{dD^2} = \frac{(1 + \tau f)^2 (2 - \alpha)p f(\frac{L}{p}) + L(1 - \alpha) f'(\frac{L}{p})}{p^2} \left( \int_{\frac{L}{p}}^{\infty} f(s)ds - (1 - \alpha) \frac{L}{p} f(\frac{L}{p}) \right)^2
\]

Substituting \( \frac{dL}{dD} \), \( \frac{d^2 L}{dD^2} \), and \( 1 - F(s^*) > 1 - F(s^b) > (1 - \alpha)s^b f(s^b) \) (from Proposition 4.1 and Equation (9) ) into Equation (17), and reorganizing terms yields

\[
\left[ 1 - \tau \frac{2 - \alpha}{1 - \alpha} \right] f(s^b) \int_{s^b}^{\infty} f(s)ds + s^b f^2(s^b) + (1 - \tau) s^b f'(s^b) \int_{s^b}^{\infty} f(s)ds > 0.
\]
By assumption, Condition (18) is satisfied; hence, \( V(x, D) \) is a concave function of \((x, D)\) so that \((x^*, D^*)\) is an optimal decision.

PROOF of PROPOSITION 4.4 From the assumption and Equation 14, we have \( \frac{\partial^2 V}{\partial^2 x} < 0 \) and \(|H| > 0\), hence the first order conditions apply for an optimal decision \((x^*, y^*)\). Taking total differentials of \(x^*\) and \(D^*\) with respect to \(\tau\) yields \( \frac{dx^*}{d\tau} = \left[ \frac{\partial^2 V}{\partial x \partial D} \frac{\partial^2 V}{\partial D \partial \tau} - \frac{\partial^2 V}{\partial x \partial \tau} \right] / |H| \), and \( \frac{dD^*}{d\tau} = \left[ \frac{\partial^2 V}{\partial x \partial \tau} \frac{\partial^2 V}{\partial D \partial x} - \frac{\partial^2 V}{\partial x^2} \right] / |H| \). From Equation (6), the optimal production level is given by

\[
F(x^*) = \frac{p(1 - \tau) - c[1 + r_f - \tau(1 - F(s^*))]}{p(1 - \tau)} < \frac{p - c(1 + r_f)}{p},
\]

which indicates that the optimal production decision in an imperfect market is always less than that of the decision in a perfect market. From Equation (6), we can show that

\[
\frac{\partial^2 V}{\partial x \partial \tau} = -p \int_x^\infty f(s)ds + c \int_{s^*}^\infty f(s)ds < 0
\]
since \( p[1 - F(x^*)] - c(1 + r_f) \geq 0 \). Taking cross-derivative with respect to \(D\) and \(\tau\) yields:

\[
\frac{\partial^2 V}{\partial D \partial \tau} = [1 - F(s^*)] \left( r + D \frac{\partial r}{\partial D} \right) > 0.
\]

From Equation (15) and (16), we have \( \frac{\partial^2 V}{\partial D^2} < 0 \), and \( \frac{\partial^2 V}{\partial x \partial D} = \frac{\partial^2 V}{\partial D \partial x} < 0 \); hence, it is clear that \( \frac{dx^*}{d\tau} < 0 \) and \( \frac{dD^*}{d\tau} > 0 \).

References


