Decomposing the Yield Curve

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January 1, 2010

Objective and motivation

- Yield curve: expected interest rates or risk premiums?

$$\text{Yield:} \quad y_t^{(n)} = \frac{1}{n} E_t \left(y_t^{(1)} + y_{t+1}^{(1)} + \ldots + y_{t+n-1}^{(1)} \right) + rpy_t^{(n)}$$

$$\text{Forward:} \quad f_t^{(n)} = E_t(y_{t+n-1}^{(1)}) + rpf_t^{(n)}$$

$$\text{Returns} \quad : \quad E_t(r_{t+1}^{(n)}) = y_t^{(1)} + rpr_t^{(n)}$$

- Current risk premium or expected future premium? Term structure of risk premiums?

$$rpy_t^{(n)} = \frac{1}{n} \left[E_t \left(rx_{t+1}^{(n)} \right) + E_t \left(rx_{t+2}^{(n-1)} \right) + \ldots + E_t \left(rx_{t+n-1}^{(2)} \right) \right]$$

- $\rightarrow$Affine model with a lot of attention to risk premiums

Affine model structure

$$\text{Factors, e.g.: } X_t = \left[\begin{array}{cccc} x_t & \text{level}_t & \text{slope}_t & \text{curve}_t \end{array}\right]'$$

$$\text{Real factor dynamics: } X_{t+1} = \mu + \phi X_t + v_{t+1};\ E\left(v_{t+1}v_{t+1}'\right) = V$$

$$\text{Affine model: } f_t^{(n)} = E_t^*(y_{t+n-1}^{(1)}) = (\cdot) + \delta_1' \phi^{*n-1} X_t$$

- ϕ^* is easy to fit, pure cross section. Need ϕ for forecasts, premiums

$$\text{Market prices } \lambda\text{: } \phi^* \equiv \phi - V\lambda_1$$

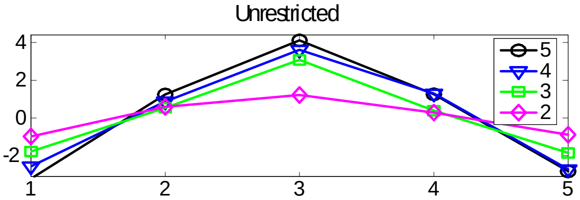
$$\text{Market prices } \lambda\text{: } E_t\left(rx_{t+1}\right) = (\cdot) + cov(rx_{t+1}, v_{t+1}')\left(\lambda_0 + \lambda_1 X_t\right)$$

$$\lambda_t = (\lambda_0 + \lambda_1 X_t) = \left[\begin{array}{c} \lambda_0^{(x)} \\ \lambda_0^{(\text{level})} \\ \lambda_0^{(\text{slope})} \\ \lambda_0^{(\text{curve})} \end{array}\right] + \left[\begin{array}{cccc} \lambda_1^{(x,x)} & \lambda_1^{(x,l)} & \cdot & \cdot \\ \lambda_1^{(l,x)} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array}\right] \left[\begin{array}{c} x_t \\ \text{level}_t \\ \text{slope}_t \\ \text{curve}_t \end{array}\right]$$

- Two issues: 1) How does λ_t vary *over time* (columns)? 2) Covariance with *which shocks* generates a premium (rows)?
- Can we simplify estimation of 20 unknown parameters, please?

CP Review

$$rx_{t+1}^{(n)} = a_n + b_1 y_t^{(1)} + b_2 f_t^{(2)} + ... + b_5 f_t^{(5)} + \varepsilon_{t+1}^{(n)}$$



- →Single-factor model for expected excess returns

$$E_t\left(rx_{t+1}^{(n)}\right) = b_n\left(\gamma' f_t\right) = b_n x_t$$

- Paper: Eigenvalue decompose covariance matrix of *expected* returns; $x_t = \gamma' f_t$ is the dominant (>99%) eigenvector.

CP to affine model

$$\text{Data: } E_t\left(rx_{t+1}^{(n)}\right) = b_n\left(\gamma' f_t\right) = b_n x_t$$

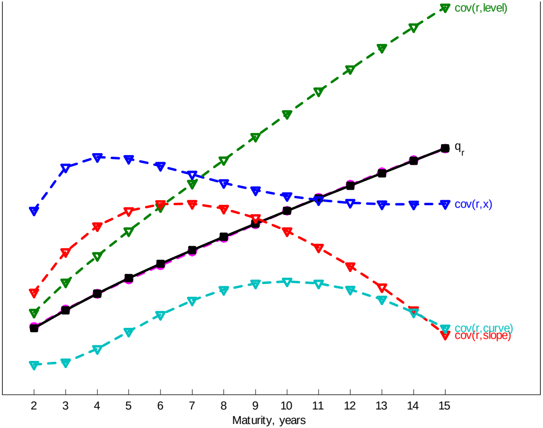
$$\text{Model: } E_t\left(rx_{t+1}\right) = (\cdot) + cov(rx_{t+1}, v_{t+1}')\left(\lambda_0 + \lambda_1 X_t\right)$$

- *All variation through time in market prices of risk is carried by* x_t

$$\lambda_t = (\lambda_0 + \lambda_1 X_t) = \begin{bmatrix} \lambda_0^{(x)} \\ \lambda_0^{(\text{level})} \\ \lambda_0^{(\text{slope})} \\ \lambda_0^{(\text{curve})} \end{bmatrix} + \begin{bmatrix} \lambda_1^{(x,x)} & 0 & 0 & 0 \\ \lambda_1^{(l,x)} & 0 & 0 & 0 \\ \lambda_1^{(s,x)} & 0 & 0 & 0 \\ \lambda_1^{(c,x)} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_t \\ \text{level}_t \\ \text{slope}_t \\ \text{curve}_t \end{bmatrix}$$

$$E_t\left(rx^{(n)}_{t+1}\right) = (\cdot) + cov(rx^{(n)}_{t+1}, v'_{t+1})\,(\lambda_0 + \lambda_1 X_t)$$

$$b_n x_t = cov(rx^{(n)}_{t+1}, v^x_{t+1})\lambda_1^{(x,x)} x_t + cov(rx^{(n)}_{t+1}, v^l_{t+1})\lambda_1^{(l,x)} x_t + ...$$



- *Market prices of risk correspond entirely to covariance with the level shock. You can estimate λ_1 with a "cross-sectional regression"*

Market price of risk summary

- Market price of risk only *varies over time* in response to one state variable, x_t, and *not* to level, slope and curvature.
- Risk premium is only earned in return for exposure to term-structure *level* shocks v_{t+1}^l. The premium for x, slope, curvature risk is zero.
- Dramatic simplification. *Two* parameters to estimate, and cross-sectional regression method to do so!

$$
\begin{aligned}
\phi &= \phi^* + V\lambda_1 \\
E_t\left(rx_{t+1}\right) &= (\cdot) + cov(rx_{t+1}, v_{t+1}^l)\lambda_t \\
\lambda_t &= \lambda_0 + \lambda_1 X_t \\
\lambda_t &= \begin{bmatrix} 0 \\ \lambda_{0l} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ \lambda_{1l} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_t \\ level_t \\ slope_t \\ curve_t \end{bmatrix}
\end{aligned}
$$

Estimation

1. **Find risk-neutral ϕ^* to fit cross section**

$$f_t^{(n)} = (\cdot) + \delta_1' \phi^{*n-1} X_t \quad (+\varepsilon_t)$$

$$\min_{\{\phi^*\}} \sum_{n=1}^{N} \sum_{t=1}^{T} \left((\cdot) + \delta_1' \phi^{*n-1} X_t - f_t^{(n)} \right)^2$$

- No forecasting information in risk-neutral transition matrix ϕ^*.
- As usual, very close fit.

2. **Use cross-sectional regression estimate λ to find real ϕ**

$$\phi = \phi^* + V\lambda_1$$

$$\phi = \phi^* + V \begin{bmatrix} 0 & 0 & 0 & 0 \\ \lambda_{1l} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

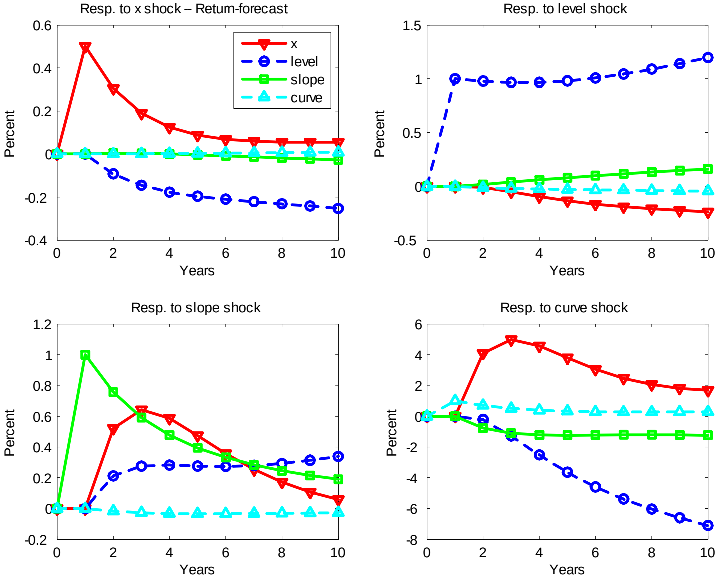
- Zero restrictions mean that all but one column of ϕ is estimated from the cross-section alone!

Transition Matrix Estimates

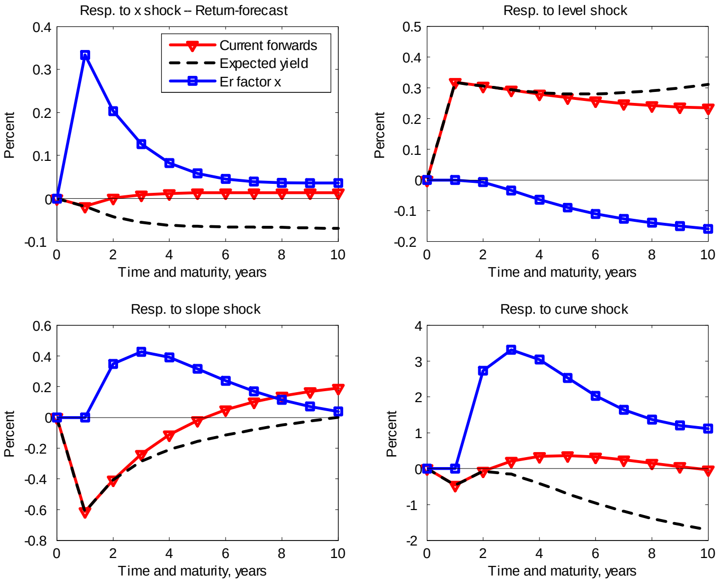
	x	level	slope	curve
Risk-neutral:		ϕ^*		
x	0.35	-0.02	-1.05	8.19
level	0.03	**0.98**	-0.21	-0.22
slope	0.00	-0.02	0.76	0.77
curve	0.00	-0.01	0.02	0.70
Actual:		ϕ		
x	*0.61*	-0.02	-1.05	8.19
level	*-0.09*	**0.98**	-0.21	-0.22
slope	*-0.00*	-0.02	0.76	0.77
curve	*0.00*	-0.01	0.02	0.70

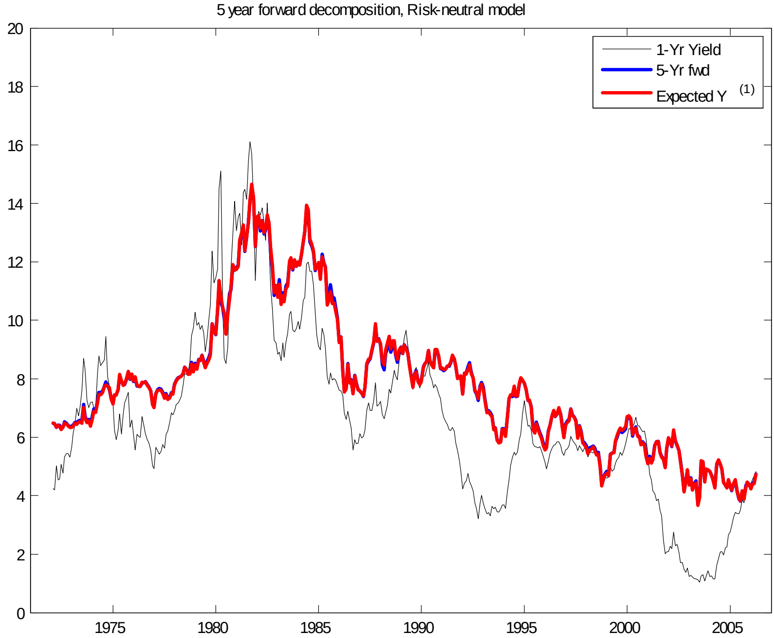
- The risk-neutral ϕ^* from the cross-section = a *lot* of information about the true ϕ!
- 0.98 does not change. Near unit-root estimation problems are solved. The root is identified from the cross section.

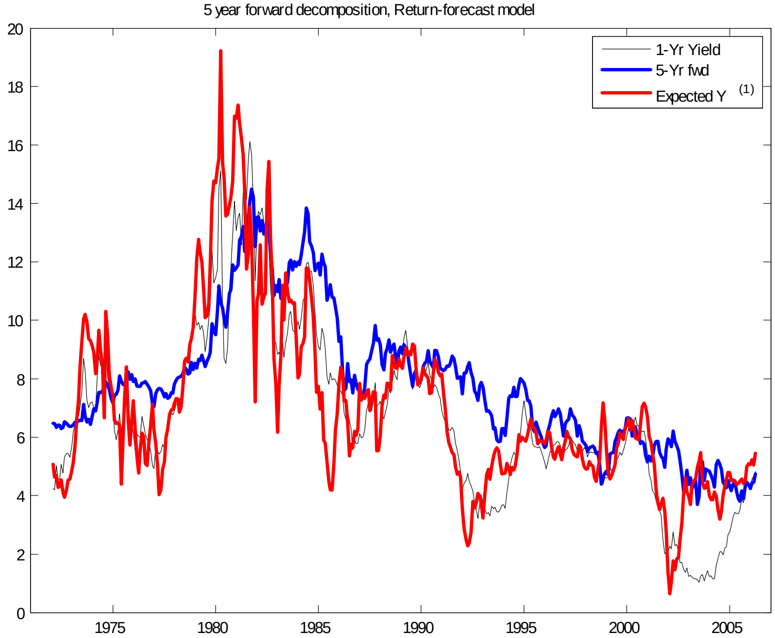
- True dynamics ϕ. *x is not an AR(1)*. Slope, curve $\rightarrow$ x. Can expect *future* risk premium without *current; term-structure of risk premiums*



Term structure of risk premiums




5 year forward decomposition, Risk-neutral model
20
18
16
14
12
10
8
6
4
2
0
1975
1980
1985
1990
1995
2000
2005
1-Yr Yield
5-Yr fwd
Expected Y
(1)


5 year forward decomposition, Return-forecast model
1-Yr Yield
5-Yr fwd
Expected Y (1)
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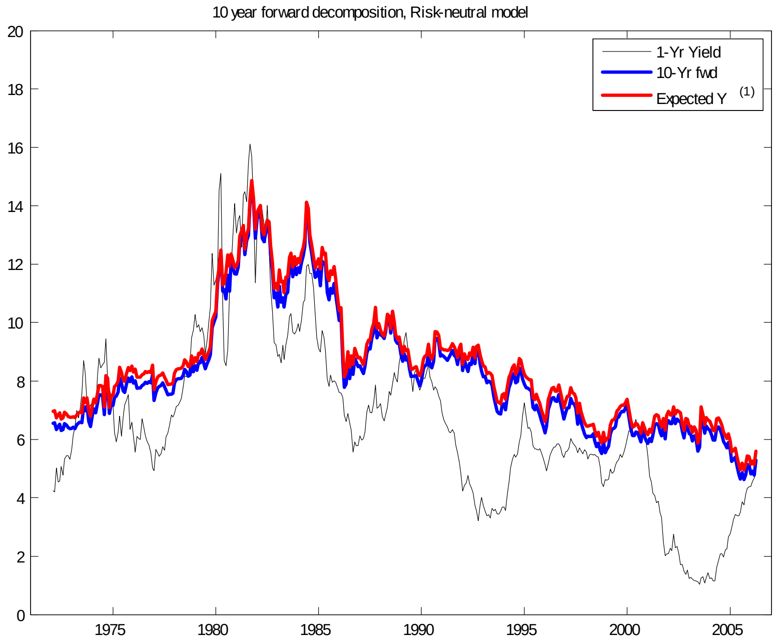
10 year forward decomposition, Risk-neutral model
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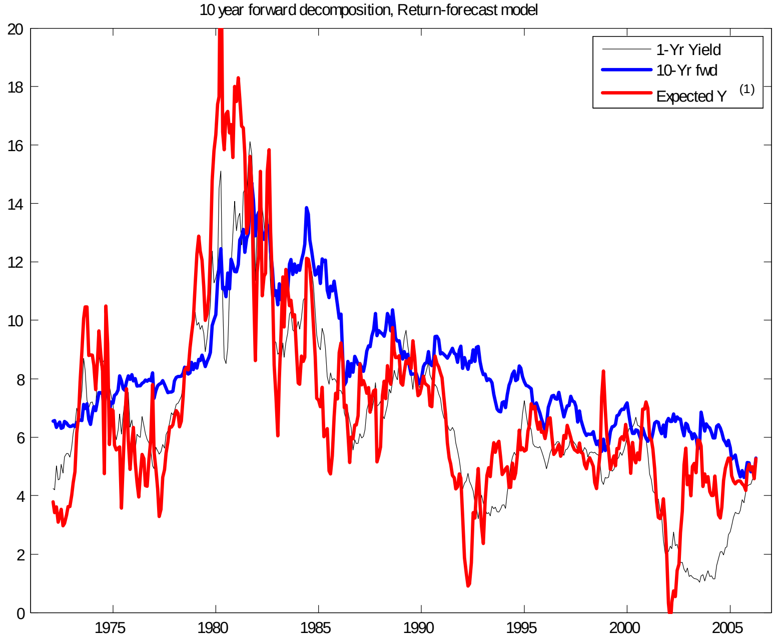
2000
2005
1-Yr Yield
10-Yr fwd
Expected Y (1)


10 year forward decomposition, Return-forecast model
1-Yr Yield
10-Yr fwd
Expected Y (1)
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1975
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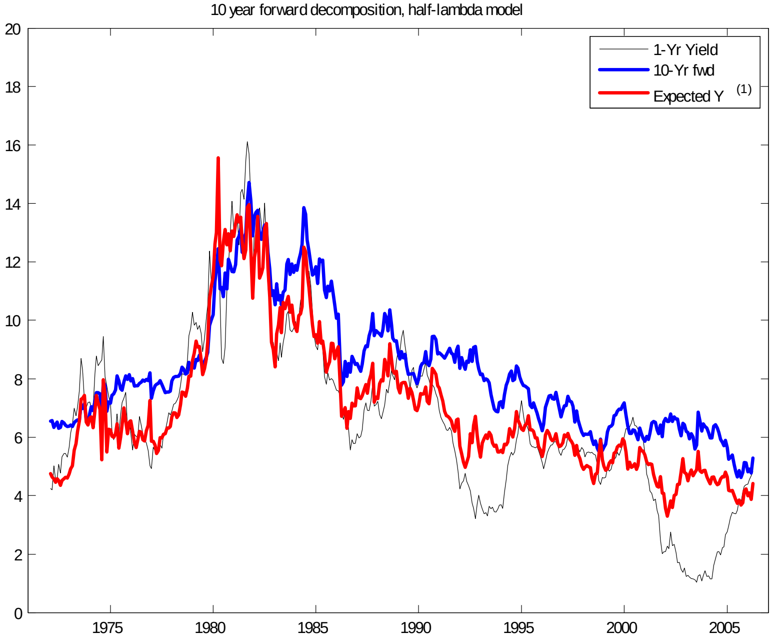
10 year forward decomposition, half-lambda model
1-Yr Yield
10-Yr fwd
Expected Y (1)
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1995
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2005